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Representations of the quantum doubles of finite group algebras and spectral parameter dependent solutions of the Yang–Baxter equation

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Quantum doubles of finite group algebras form a class of quasitriangular Hopf algebras that algebraically solve the Yang–Baxter equation. Each representation of the quantum double then gives a matrix solution of the Yang–Baxter equation. Such solutions do not depend on a spectral parameter, and to date there has been little investigation into extending these solutions such that they do depend on a spectral parameter. Here we first explicitly construct the matrix elements of the generators for all irreducible representations of quantum doubles of the dihedral groups D_n. These results may be used to determine constant solutions of the Yang–Baxter equation. We then discuss Baxterization ansätze to obtain solutions of the Yang–Baxter equation with a spectral parameter and give several examples, including a new 21-vertex model. We also describe this approach in terms of minimal-dimensional representations of the quantum doubles of the alternating group A_4 and the symmetric group S_4.
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I. INTRODUCTION

Solutions of the Yang–Baxter equation [see (1)] provide a systematic way to construct exactly solvable models of two-dimensional statistical mechanics,[1] integrable quantum systems,[2–5] as well as having applications in other areas such as knot theory.[6] A great impetus to this field was given by Drinfeld,[7] who proposed the *quantum double construction*. This allows for any Hopf algebra A to be embedded in a larger Hopf algebra $D(A)$ in such a way that $D(A)$ is quasitriangular. A consequence of the quasitriangular property is that there exists a canonical element $R \in D(A) \otimes D(A)$, called the *universal R-matrix*, which solves the Yang–Baxter equation algebraically. Thus, for any representation of $D(A)$ a matrix solution of the Yang–Baxter equation is obtained. (Below we will abuse the notation and use R to denote both the universal R-matrix and its matrix representatives.) The seminal examples of quasitriangular Hopf algebras were given by both Drinfeld[7] and Jimbo,[8,9] who independently introduced the notion of quantum algebras, which are deformations of universal enveloping algebras of Lie algebras.

For applications to the areas mentioned above one is generally interested in solutions of the Yang–Baxter equation with a spectral parameter; i.e., for a vector space V one looks for $R(u,v) \in \mathrm{End}(V \otimes V)$ where u,v are complex variables such that

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$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w)=R_{23}(v,w)R_{13}(u,w)R_{12}(u,v), \tag{1}$$

holds on the threefold tensor product space $V\otimes V\otimes V$. The subscripts above refer to the way in which the action of $R(u,v)$ is embedded into the space of endomorphisms on $V\otimes V\otimes V$.

In the context of quantum algebras, the spectral parameter arises naturally when one considers the loop representations of affine algebras.[8,9] In such instances the solutions always satisfy the *difference property* $R(u,v)=R(u-v)$. However, it is worth mentioning that there are solutions that do not have the difference property, including the well-known cases of the solutions giving rise to the Hubbard model,[10] the Bariev model,[11] and the chiral Potts model.[12] Moreover, the spectral parameter need not necessarily be a scalar, but can be a complex vector variable.[13–17] Later we will only concern ourselves with cases of scalar spectral parameters where the difference property does hold.

For later use, we introduce the permutation operator P such that $P(x\otimes y)=y\otimes x$ and set $\check{R}(u)=PR(u)$. Then (1) can equivalently be expressed as

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v)=\check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u), \tag{2}$$

which we will refer to as the *braiding* Yang–Baxter equation. It is in this form that the Yang–Baxter equation is relevant to knot theory.[6] Indeed, setting

$$\check{\mathcal{R}}=\lim_{u\to-\infty}\check{R}(u)$$

gives us

$$\check{\mathcal{R}}_{12}\check{\mathcal{R}}_{23}\check{\mathcal{R}}_{12}=\check{\mathcal{R}}_{23}\check{\mathcal{R}}_{12}\check{\mathcal{R}}_{23}, \tag{3}$$

which can be recognized as a defining relation in the braid group.[6] In terms of

$$\mathcal{R}=\lim_{u\to-\infty}R(u),$$

we have

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{4}$$

which we will refer to as the *constant* Yang–Baxter equation. Finally, we mention that if

$$\check{R}(u)\check{R}(-u)\propto I\otimes I,$$

then the R-matrix is said to satisfy the *unitarity* condition, while if

$$\check{R}(0)\propto I\otimes I, \tag{5}$$

then it is said to satisfy the *regularity* condition. When the regularity condition holds, there is a standard procedure[1–4] for constructing an integrable quantum system on a one-dimensional lattice with periodic boundary conditions. The Hamiltonian is given by

$$H=\sum_{j=1}^{L-1}h_{j,j+1}+h_{L,1}, \tag{6}$$

where the two-site Hamiltonians are given by

$$h=\left.\frac{d}{du}\check{R}(u)\right|_{u=0}.$$

Models constructed in this manner, and other approaches, can be solved exactly using Bethe ansatz methods.[1–6]

One class of quasitriangular Hopf algebras is the set of quantum doubles of the group algebras of finite groups.[18,19] (Throughout we will refer to these as *finite group doubles*). Applications of finite group double representations to knot theory have been addressed in Ref. 20. These algebraic structures also underlie systems of *anyons* in two spatial dimensions. In cases where the global symmetry of the system is spontaneously broken to a discrete gauge group, the finite group double is the appropriate structure to describe the fusion properties and statistics. The fusion properties are essentially determined by the Clebsch-Gordan decomposition of tensor products into irreducible representations. The statistics associated with the interchange of two anyons is described by braiding. The consistency condition for the two ways in which three anyons may be interchanged by a sequence of three two-anyon exchanges is precisely (3), where $\check{R}_{jk}$ is the operation that interchanges the jth and kth anyons. For a comprehensive review of the salient features we refer to Refs. 21 and 22.

Such systems may exhibit topological order,[23] where quantum numbers are conserved for topological reasons, as opposed to the manifest symmetry. Due to the topological nature, excitations are resistant to decoherence. This property forms the basis of topological quantum computation that was first put forth by Kitaev[24] (see also, e.g., Refs. 25–28). When the symmetry is described by a finite group double, the braiding $\check{R}_{jk}$ is a unitary operator that can be employed as a quantum gate.

In view of the previous literature, it is surprising that there has been very little study on the role of finite group doubles in obtaining solutions for the spectral parameter dependent Yang–Baxter equation (1). Integrable systems constructed from such solutions via (6) realize models for interacting anyons with internal symmetries described by the finite group double. Even though the models are one-dimensional, there is a precedent, the Hubbard model, which leads us to believe that such models may have applications for understanding two-dimensional systems. One property that is evident from the analysis of the Bethe ansatz solution of the one-dimensional Hubbard model is spin-charge separation. The Hubbard model has an $so(4)\cong so(3)\oplus so(3)$ symmetry, where the two quantum numbers associated with the two copies of $so(3)$ can be assigned to spin and charge degrees of freedom. From this symmetry and the Bethe ansatz solution, it can be concluded that in one-dimension there exist excitations that carry spin but not charge, and *vice versa*, so spin-charge separation occurs.[29] It has been proposed that spin-charge separation is the mechanism responsible for high temperature superconductivity in two-dimensions.[30] Likewise, there may be insights gained into the properties of interacting anyons by studying one-dimensional models that can be solved exactly.

Our aim is to investigate the extent to which solutions of the spectral parameter dependent Yang–Baxter equation can be obtained using the Hopf algebra structure of finite group doubles. This is not straightforward, as there appears to be no obvious manner in which to consider the affine extension of a finite group double that affords loop representations. Using a different approach, some preliminary results in this regard have been obtained by Yang *et al.*[31] We believe these represent the tip of an iceberg, and there is ample scope for further work. Our aim here is to continue the advances in this direction.

Our starting point is to consider the quantum doubles of the dihedral groups D_n. Of all non-Abelian finite groups, the series of dihedral groups has the simplest representation theory. We will show that the representation theory for the quantum doubles is also readily tractable. Using the general results on the representation theory of finite group doubles given in Refs. 18 and 19, we begin by explicitly constructing *all* irreducible representations for the doubles $D(D_n)$[1] From these results it is straightforward to explicitly construct solutions R for the constant Yang–Baxter equation (4) that do not depend on the spectral parameter.

Our next goal is to determine if these constant solutions of the Yang–Baxter equation can be extended to spectral parameter dependent solutions. This is a procedure colloquially known as

[1]After completing this work, we learned that for odd n these representations have been constructed in the thesis by de Wild Propitius.[22] Results for even n have independently been obtained by Slingerland.[32]

Baxterization, as coined by Jones,[33] and there is sizable literature on this topic.[34–41] We begin by studying the case of the two-dimensional irreducible representations of $D(D_n)$ and find that Baxterization can be performed successfully. Our approach is based on an ansatz taken from Ref. 35, which is chosen by symmetry considerations. In all these cases we find that the resulting solution of the Yang–Baxter equation is a particular case of the well-known trigonometric six-vertex model in the symmetric gauge at a specific root of unity. We mention that this result is not obvious in the sense that the Baxterization is not underpinned by a Hecke algebra representation.

We then turn our attention to the three-dimensional irreducible representations of $D(D_n)$. We find that the only cases for which an irreducible three-dimensional representation exists are $D(D_3)$ and $D(D_6)$. All instances give unitarily equivalent constant solutions of the Yang–Baxter equation. Our Baxterization ansatz leads to a 21-vertex solution of the spectral parameter dependent Yang–Baxter equation, which as far as we can ascertain is new.

Rather than continuing on to investigate higher dimensional representations of the $D(D_n)$ series, we finish by considering minimal-dimensional representations of the double of the alternating group A_4 and the symmetric group S_4. Neither of these cases admit irreducible two-dimensional representations, but they both admit three-dimensional ones. Our ansatz for Baxterizing the constant solutions does lead us to spectral parameter dependent solutions. Remarkably though, we find in these latter examples that the infinite spectral parameter limit of the Baxterized solutions do not necessarily reproduce the original solutions of the constant Yang–Baxter equation.

II. THE DIHEDRAL GROUP D_n

Consider the dihedral group D_n. This has two generators σ,τ satisfying

$$\sigma^n = e, \quad \tau^2 = e, \quad \tau\sigma = \sigma^{n-1}\tau.$$

The properties of D_n vary according to whether n is odd or even, with the odd case being slightly simpler.

A. D_n where n is odd

When n is odd, there are $(n+3)/2$ conjugacy classes divided into three families, given by

$$\{e\},$$

$$\{\sigma^k,\sigma^{-k}\}, \quad \text{for } 1 \le k \le \frac{n-1}{2},$$

$$\{\sigma^i\tau, 0 \le i \le n-1\}.$$

There are $(n+3)/2$ irreducible representations (irreps), two of which are one-dimensional and the remaining $(n-1)/2$ which are two-dimensional. They are given by

$$\pi_\pm(\sigma) = 1, \quad \pi_\pm(\tau) = \pm 1$$

and

$$\pi_k(\sigma) = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix}, \quad \pi_k(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \omega = \exp\left(\frac{2\pi i}{n}\right), \quad 1 \le k \le \frac{n-1}{2}.$$

As required, the sum of the squares of the dimensions of the irreps is $2\times 1^2 + (n-1)/2 \times 2^2 = 2n = |D_n|$.[42]

B. D_n where n is even

When n is even, there are $(n+6)/2$ conjugacy classes divided into five families, given by

$$\{e\},$$

$$\{\sigma^{n/2}\},$$

$$\{\sigma^k,\sigma^{-k}\}, \quad \text{for } 1 \le k \le (n-2)/2,$$

$$\{\sigma^{2j}\tau, 0 \le j \le (n-2)/2\},$$

$$\{\sigma^{(2j+1)}\tau, 0 \le j \le (n-2)/2\}.$$

The $(n+6)/2$ irreps consist of 4 one-dimensional irreps and $(n-2)/2$ two-dimensional irreps. They are given by

$$\pi(\sigma) = (-1)^a, \quad \pi(\tau) = (-1)^b, \quad \text{for } a,b \in \{0,1\}$$

and

$$\pi_k(\sigma) = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix}, \quad \pi_k(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \omega = \exp\left(\frac{2\pi i}{n}\right), \quad 1 \le k \le \frac{n-2}{2}.$$

Again, the sum of the squares of the dimensions of the irreps is $2n$.

III. THE QUANTUM DOUBLE ALGEBRA $D(G)$

Here we give a brief survey of finite group doubles $D(G)$. Throughout, our approach follows that of Ref. 19. Let A be the group algebra of a finite group G over the complex field $\mathbb{C}$. Then A becomes a co-commutative Hopf algebra with coproduct, antipode, and counit, respectively, defined by

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = e, \quad \forall\, g \in G.$$

Let A^* be the dual space of A, so $A^* = \{f | f: A \to \mathbb{C}\}$. Then A^* becomes an algebra on the dual elements g^* defined by

$$g^*(h) = \delta(g,h), \quad \forall\, g,h \in G,$$

which have the property

$$g^* h^* = \delta(g,h) h^*.$$

The resultant dual algebra is commutative and does not have an interesting representation theory. Now we follow the quantum double construction to obtain $D(G)$, which is a $|G|^2$-dimensional algebra spanned by the free products,

$$gh^*, \quad g,h \in G.$$

The elements $h^* g$ are calculated using

$$h^* g = g(g^{-1} h g)^*.$$

Then $D(G)$ becomes a quasitriangular Hopf algebra with coproduct $\bar{\Delta}$, antipode $\bar{S}$, and counit $\bar{\varepsilon}$ given by

$$\bar{\Delta}(gh^*) = \sum_{k\in G} g(k^{-1}h)^* \otimes gk^* = \sum_{k\in G} gk^* \otimes g(kh^{-1})^*,$$

$$\bar{S}(gh^*) = g^{-1}(gh^{-1}g^{-1})^*,$$

$$\bar{\varepsilon}(gh^*) = \delta(h,e).$$

Note that we identify $g\varepsilon$ with g and eg^* with g^* for all $g \in G$. The universal R-matrix is given by

$$R = \sum_{g\in G} g \otimes g^*,$$

which can easily be shown to satisfy the defining relations for a quasitriangular Hopf algebra:

$$R\bar{\Delta}(a) = \bar{\Delta}^T(a)R, \quad \forall\, a \in D(G), \tag{7}$$

$$(\bar{\Delta} \otimes \mathrm{id})R = R_{13}R_{23}, \tag{8}$$

$$(\mathrm{id} \otimes \bar{\Delta})R = R_{13}R_{12}, \tag{9}$$

where $\bar{\Delta}^T$ is the opposite coproduct,

$$\bar{\Delta}^T(gh^*) = \sum_{k\in G} gk^* \otimes g(k^{-1}h)^* = \sum_{k\in G} g(kh^{-1})^* \otimes gk^*.$$

It follows from the relations (7)–(9) that R is a solution of the constant Yang–Baxter equation. We note that in any tensor product representation $\pi \otimes \pi$ we have that $\check{R} = PR$ commutes with the action of the finite group double; i.e.,

$$[\check{R}, (\pi \otimes \pi)\bar{\Delta}(a)] = 0, \quad \forall\, a \in D(G).$$

IV. REPRESENTATION THEORY OF *D*(*G*)

Several properties of group algebras extend to the quantum double. For example, the set

$$Q = \{gh^* | g,h \in G \text{ with } gh = hg\},$$

is stable under the adjoint action of G, i.e., $gQg^{-1} = Q$. Hence, Q can be partitioned into G-conjugacy classes, which implies[19] the following.

Theorem 4.1: *The number of nonisomorphic $D(G)$-modules equals the number of G-equivalence classes of Q.*

Moreover, a construction for these modules is known.[19] The general form is included in this section, with the explicit results for the odd and even dihedral groups given in the following two sections, respectively.

First, partition G into conjugacy classes,

$$G = \bigcup_k C_k.$$

Recall that the centralizer subgroup of an element h is defined by

$$Z(h)=\{g\in G|gh=hg\}.$$

Then, for each conjugacy class C_k choose a representative $g_k\in C_k$ and set $Z_k=Z(g_k)$ to be the centralizer subgroup of g_k, noting that $|Z_k||C_k|=|G|$. Denote the group algebra of Z_k by A_k. Also, for each $s\in C_k$ choose a fixed element $\alpha_s\in G$ satisfying

$$s=\alpha_s g_k \alpha_s^{-1}.$$

For simplicity, choose $\alpha_{g_k}=e, \forall g_k$.

Lemma 4.1: *We have the following properties of* α_s:

1. $G=\cup_{s\in C_k}\alpha_s Z_k$
2. *Given* $g\in G, s\in C_k, \exists t\in C_k$ *unique with the property* $\alpha_t^{-1}g\alpha_s\in Z_k$; *explicitly* $t=gsg^{-1}$.

Again, a proof can be found in Ref. 19.

The irreducible modules of $D(G)$ can be constructed from modules of the group algebras A_k. Let V_β^k denote an irreducible A_k-module. Then there is a corresponding induced A-module,[42]

$$V_{k,\beta}\subseteq A\otimes_{A_k}V_\beta^k,$$

spanned by vectors

$$v(s)=\alpha_s\otimes v, \quad v\in V_\beta^k, \quad s\in C_k,$$

where the action of G is given by

$$g(\alpha_s\otimes v)=\alpha_{gsg^{-1}}\otimes(\alpha_{gsg^{-1}}^{-1}g\alpha_s)v,$$

or, equivalently,

$$gv(s)=(\alpha_{gsg^{-1}}^{-1}g\alpha_s v)(gsg^{-1}).$$

Note $\dim V_{k,\beta}=|C_k|\dim V_\beta^k$. It follows from Lemma 4.1 that $V_{k,\beta}$ is an A-module under this definition.

The module $V_{k,\beta}$ can be decomposed according to

$$V_{k,\beta}=\bigoplus_{s\in C_k}V_{k,\beta}(s),$$

where

$$V_{k,\beta}(s)=\{v(s)|v\in V_\beta^k\}.$$

The latter becomes an irreducible module over the group algebra of $Z(s)=\alpha_s Z_k\alpha_s^{-1}$. When $s=g_k$, the module is isomorphic to V_α^k. Then $V_{k,\beta}$ becomes an irreducible $D(G)$-module with the action

$$h^*v(s)=\delta(h,s)v(s), \quad \forall h\in G.$$

Moreover, two $D(G)$-modules of this form, $V_{k,\beta},V_{l,\gamma}$, are isomorphic iff $k=l$ and V_β^k,V_γ^k are isomorphic. Then using counting arguments the following can be shown.

Theorem 4.2: *Every irreducible* $D(G)$*-module is isomorphic to one of the* $V_{k,\beta}$.

V. REPRESENTATIONS OF $D(G)$, WHERE $G=D_n$, n EVEN

The conjugacy classes C_k of $G=D_n$, chosen representatives g_k, corresponding centralizer subgroups and the elements $\alpha_s, \forall s\in C_k$, are given in Table I.

Throughout the remainder of this paper, E_j^i denotes an elementary matrix with a 1 in the (i,j) position and zeros elsewhere. We also abuse notation by using g to denote both an element of the algebra $D(G)$ and its matrix representative in a given irrep, which should be clear from the context.

TABLE I. C_k, g_k, Z_k, and α_s for $G=D_n$, n even.

C_k	g_k	$Z_k=Z(g_k)$	$\alpha_s, \forall s \in C_k$
$\{e\}$	e	D_n	$\alpha_e=e$
$\{\sigma^{n/2}\}$	$\sigma^{n/2}$	D_n	$\alpha_{\sigma^{n/2}}=e$
$\{\sigma^k,\sigma^{-k}\}, 1\le k<n/2$	σ^k	$\{\sigma^i \mid 0\le i<n\}$	$\alpha_{\sigma^k}=e, \alpha_{\sigma^{-k}}=\tau$
$\{\sigma^{2i}\tau \mid 0\le i<n/2\}$	τ	$\{e,\tau,\sigma^{n/2},\sigma^{n/2}\tau\}$	$\alpha_{(\sigma^{2i}\tau)}=\sigma^i$
$\{\sigma^{2i+1}\tau \mid 0\le i<n/2\}$	$\sigma\tau$	$\{e,\sigma\tau,\sigma^{n/2},\sigma^{n/2+1}\tau\}$	$\alpha_{(\sigma^{2i+1}\tau)}=\sigma^i$

Representations induced by $C_k=\{e\}$. The module elements are of the form $e\otimes v$ where $v \in V$, V a D_n-module. In representation terms, there are 4 one-dimensional irreps and $(n/2-1)$ two-dimensional irreps. They are

$$\sigma=(-1)^a, \quad \tau=(-1)^b, \quad g^*=\delta(g,e)$$

where $a,b\in\{0,1\}$, and

$$\sigma=\begin{bmatrix}\omega^k & 0\\ 0 & \omega^{-k}\end{bmatrix}, \quad \tau=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad g^*=\delta(g,e)I_2,$$

where $1\le k<n/2$.

Representations induced by $C_k=\{\sigma^{n/2}\}$. The module elements are of the form $e\otimes v$, where $v \in V$, V a D_n-module. Again, there are 4 one-dimensional irreps and $(n-2)/2$ two-dimensional irreps. They are

$$\sigma=(-1)^a, \quad \tau=(-1)^b, \quad g^*=\delta(g,\sigma^{n/2}),$$

where $a,b\in\{0,1\}$, and

$$\sigma=\begin{bmatrix}\omega^k & 0\\ 0 & \omega^{-k}\end{bmatrix}, \quad \tau=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad g^*=\delta(g,\sigma^{n/2})I_2,$$

where $1\le k<n/2$.

Representations induced by $C_k=\{\sigma^k,\sigma^{-k}\}$, $1\le k<n/2$. The module elements are of the form $e\otimes v, \tau\otimes v$, where $v\in V$, V a module of the group algebra of $Z_k=\{\sigma^j \mid 0\le j<n\}$. There are n such A_k-modules, with the corresponding representations given by $\pi(\sigma)=\omega^j$, $0\le j<n$, where $\omega=\exp(2\pi i/n)$. Thus, we have $n(n-2)/2$ different irreducible representations of $D(D_n)$ induced by these conjugacy classes, given by

$$\sigma=\begin{bmatrix}\omega^j & 0\\ 0 & \omega^{-j}\end{bmatrix}, \quad \tau=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad (\sigma^k)^*=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad (\sigma^{-k})^*=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad g^*=0, \quad \text{otherwise},$$

where $0\le j<n$, $1\le k<n/2$.

Representations induced by $C_k=\{\sigma^{2j}\tau \mid 0\le j<n/2\}$. The module elements are of the form $\sigma^j\otimes v$, $0\le j<n/2$, where $v\in V$, V a module of the group algebra of $Z_k=\{e,\tau,\sigma^{n/2},\sigma^{n/2}\tau\}$. Hence, there are 4 $(n/2)$-dimensional irreps of this form. They are

$$\sigma=A\in M_{n/2\times n/2}, \quad \text{where } [A]_{ij}=(-1)^{a\delta(i,1)}\delta(i,j+1), \quad \text{addition mod } n/2,$$

$$\tau=(-1)^{a+b}B\in M_{n/2\times n/2}, \quad \text{where } [B]_{ij}=(-1)^{a\delta(i,1)}\delta(i+j,2), \quad \text{addition mod } n/2,$$

$$(\sigma^i)^*=0, \quad (\sigma^{2j}\tau)^*=E_{j+1}^{j+1}, \quad (\sigma^{(2j+1)}\tau)^*=0, \quad 0\le i<n, \quad 0\le j<n/2,$$

where $a,b\in\{0,1\}$.

Example 5.1: *In* $D(D_6)$, σ *and* τ *are as follows:*

$$\sigma = \begin{bmatrix} 0 & 0 & (-1)^a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tau = (-1)^b \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & (-1)^a \\ 0 & (-1)^a & 0 \end{bmatrix},$$

where $a,b \in \{0,1\}$, *giving 4 three-dimensional irreps.*

Representations induced by $C_k = \{\sigma^{2j+1}\tau | 0 \leq j < n/2\}$. The module elements are of the form $\sigma^j \otimes v$, $0 \leq j < n/2$, where $v \in V$, V a module of the group algebra of $Z_k = \{e, \sigma\tau, \sigma^{n/2}, \sigma^{(n+2)/2}\tau\}$. Hence, there are 4 $(n/2)$-dimensional irreps of this form. They are as follows:

$$\sigma = A \in M_{n/2\times n/2} \quad \text{where } [A]_{ij} = (-1)^{a\delta(i,1)}\delta(i,j+1), \quad \text{addition mod } n/2,$$

$$\tau = (-1)^{a+b}B \in M_{n/2\times n/2} \quad \text{where } [B]_{ij} = \delta(i+j, n/2+1), \quad \text{addition mod } n/2,$$

$$(\sigma^i)^* = 0, \quad (\sigma^{2j}\tau)^* = 0, \quad (\sigma^{(2j+1)}\tau)^* = E^{j+1}_{j+1}, \quad 0 \leq i < n, \quad 0 \leq j < n/2,$$

where $a,b \in \{0,1\}$.

Example 5.2: *In* $D(D_6)$, σ *and* τ *are as follows:*

$$\sigma = \begin{bmatrix} 0 & 0 & (-1)^a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tau = (-1)^{a+b} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where $a,b \in \{0,1\}$, *giving 4 three-dimensional irreps.*

Thus when n is even $D(D_n)$ has 8 one-dimensional irreps, 8 $(n/2)$-dimensional irreps, and $(n+2)(n/2-1)$ two-dimensional irreps. The sum of the squares of the dimension of the irreps is $|D_n|^2 = |D(D_n)|$, as required.[19]

Analogous results for odd n are given in the Appendix .

VI. SOLUTIONS OF THE YANG–BAXTER EQUATION ASSOCIATED WITH THE TWO-DIMENSIONAL IRREPS OF $D(D_n)$

As stated earlier, a solution to the constant Yang–Baxter equation in $D(G)$ is given by

$$R = \sum_{g \in G} g \otimes g^*.$$

In this section we will only consider the two-dimensional irreps of $D(G)$. Then, by inspection, the R-matrix will always be of the form

$$R = \text{diag}(\omega^j, \omega^{-j}, \omega^{-j}, \omega^j),$$

where $\omega = \exp(2\pi i/n)$ and $0 \leq j < n$, with the cases $j=0$ and $j=n/2$ being trivial. Thus $\check{R} = PR$ is given by

$$\check{R} = \begin{bmatrix} \omega^j & 0 & 0 & 0 \\ 0 & 0 & \omega^{-j} & 0 \\ 0 & \omega^{-j} & 0 & 0 \\ 0 & 0 & 0 & \omega^j \end{bmatrix}. \tag{10}$$

As remarked earlier, $\check{R}$ commutes with the action of $D(D_n)$. We seek a solution $\check{R}(x)$ of the braiding Yang–Baxter equation that has this symmetry. Now the matrix (10) has 3 different eigenvalues, namely $w^j, \pm w^{-j}$. Hence, for any k, $\check{R}^k$ can be written as a linear combination of $I \otimes I, \check{R}$ and $\check{R}^{-1}$. Making a change of variables $x = \exp(u)$, $z = \exp(v)$ in Eq. (2), we look for $\check{R}(x)$ in the following form:

$$\check{R}(x) = f(x)I \otimes I + g(x)\check{R} + h(x)\check{R}^{-1}.$$

Then

$$\check{R}(x) = \begin{bmatrix} A(x) & 0 & 0 & 0 \\ 0 & f(x) & B(x) & 0 \\ 0 & B(x) & f(x) & 0 \\ 0 & 0 & 0 & A(x) \end{bmatrix},$$

where

$$A(x) = f(x) + \omega^j g(x) + \omega^{-j} h(x),$$

$$B(x) = \omega^{-j} g(x) + \omega^j h(x).$$

We directly apply the braiding Yang–Baxter equation in the form

$$\check{R}_{12}(x)\check{R}_{23}(xz)\check{R}_{12}(z) = \check{R}_{23}(z)\check{R}_{12}(xz)\check{R}_{23}(x). \tag{11}$$

Although there are 20 nonzero entries on each side of the equation, there are only two independent nontrivial relations that must be satisfied. These are

$$A(z)f(xz)A(x) = f(x)A(xz)f(z) + B(x)f(xz)B(z), \tag{12}$$

$$A(z)B(xz)f(x) = f(x)A(xz)B(z) + B(x)f(xz)f(z). \tag{13}$$

Note that $f(x)=0$, $\forall x$ trivially satisfies the Yang–Baxter equation.

A proposal for constructing $\check{R}(x)$ when $\check{R}$ has three distinct eigenvalues λ_1, λ_2, and λ_3 has been discussed in Refs. 31 and 35, but it has not been proven to always be true. The conjecture is

$$\check{R}(x) = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_1\lambda_3\lambda_2^{-1})xI \otimes I - (x-1)\check{R} + \lambda_1\lambda_3 x(x-1)\check{R}^{-1}.$$

Three distinct solutions are obtained by changing the ordering of the eigenvalues. We note that when the ansatz holds we have

$$\check{\mathcal{R}} = \lim_{x\to 0} R(x) = R.$$

Applying this ansatz to (10), we find that if $\lambda_2 = \pm\omega^{-j}$, then $f(x)=0$, which we have already shown gives a trivial result. Hence we consider the case when $\lambda_2 = \omega^j$. This gives

$$f(x) = (\omega^j - \omega^{-3j})x, \quad g(x) = -(x-1), \quad h(x) = -w^{-2j}x(x-1),$$

$$\Rightarrow A(x) = \omega^j - \omega^{-3j}x^2, \quad B(x) = -\omega^{-j}(x^2-1).$$

It can be easily shown that $f(x)$, $A(x)$, and $B(x)$ satisfy relations (12) and (13). Hence, we have solutions to the braiding Yang–Baxter equation, which are

$$\check{R}(x) = \begin{bmatrix} \omega^j - \omega^{-3j}x^2 & 0 & 0 & 0 \\ 0 & (\omega^j - \omega^{-3j})x & -\omega^{-j}(x^2-1) & 0 \\ 0 & -\omega^{-j}(x^2-1) & (\omega^j - \omega^{-3j})x & 0 \\ 0 & 0 & 0 & \omega^j - \omega^{-3j}x^2 \end{bmatrix},$$

where $\omega = \exp(2\pi i/n)$ and $0 \le j < n$. Rescaling by a factor of $\omega^j x^{-1}$, we can write

TABLE II. Possible solutions for $f(x)$, $g(x)$, and $h(x)$.

$f(x)$	$g(x)$	$h(x)$
x	$1-x$	$x(x-1)$
ωx	$1-x$	$\omega^2 x(x-1)$
$\omega^2 x$	$1-x$	$\omega x(x-1)$

$$\check{R}(x) = \begin{bmatrix} \omega^{2j}x^{-1} - \omega^{-2j}x & 0 & 0 & 0 \\ 0 & \omega^{2j} - \omega^{-2j} & x^{-1} - x & 0 \\ 0 & x^{-1} - x & \omega^{2j} - \omega^{-2j} & 0 \\ 0 & 0 & 0 & \omega^{2j}x^{-1} - \omega^{-2j}x \end{bmatrix}. \tag{14}$$

Note that the unitarity condition $\check{R}(x)\check{R}(x^{-1}) = [\omega^{4j} + \omega^{-4j} - (x^2 + x^{-2})]I \otimes I$ is satisfied. We can recognize (14) as specific cases of the six-vertex solution in the symmetric gauge, where the parameter q in the general solution is constrained to be a root of unity. We remark that the choice of gauge is related to the gradation chosen for the affine algebra (e.g., see Ref. 43). It is interesting to note that in the nonsymmetric gauge the constant solution $\check{\mathcal{R}} = \lim_{x\to 0} R(x)$ can be used to give rise to a representation of the Temperley-Lieb algebra. In this case there is a well known procedure for Baxterizing $\check{\mathcal{R}}$ to recover $\check{R}(x)$.[33,34] This is not the case for the symmetric gauge case described previously.

VII. SOLUTIONS OF THE YANG–BAXTER EQUATION ASSOCIATED WITH THE THREE-DIMENSIONAL IRREPS OF $D(D_3)$ AND $D(D_6)$

From the construction of the irreps of $D(D_n)$ given explicitly in Sec. V and the Appendix, we find that three-dimensional irreps only occur for $D(D_3)$ and $D(D_6)$. Moreover, the 2 three-dimensional irreps of $D(D_3)$ are also representations for the $D(D_3)$ subalgebra of $D(D_6)$, so these cases give identical R matrices. For any three-dimensional irrep of $D(D_6)$, we find $\check{R} = PR = P\Sigma_g g \otimes g^*$ is of the following form:

$$\check{R} = (-1)^b \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1)^a & 0 \\ 0 & 0 & (-1)^a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (-1)^a & 0 & 0 \\ 0 & (-1)^a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $a, b \in \{0,1\}$. Without loss of generality we take $b=0$, and find the eigenvalues of $\check{R}$ are 1, ω, and ω^2 with multiplicities 5, 2, and 2, respectively, where $\omega = \exp(2\pi i/3)$. As the two values of a give unitarily equivalent R matrices, we choose to take $a=0$. We can then write

$$\check{R}(x) = f(x) I \otimes I + g(x)\check{R} + h(x)\check{R}^{-1}.$$

Again we follow the procedure outlined in Refs. 31 and 35 to find possible solutions, which gives the following:

Using MATHEMATICA, we find only the first of these possible solutions satisfies the braiding Yang–Baxter equation (11) (see Table II):

$$\check{R}(x)=\begin{bmatrix} x^2-x+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 1-x & x(x-1) & 0 & 0 \\ 0 & 0 & x & x(x-1) & 0 & 0 & 0 & 1-x & 0 \\ 0 & 0 & 1-x & x & 0 & 0 & 0 & x(x-1) & 0 \\ 0 & 0 & 0 & 0 & x^2-x+1 & 0 & 0 & 0 & 0 \\ 0 & x(x-1) & 0 & 0 & 0 & x & 1-x & 0 & 0 \\ 0 & 1-x & 0 & 0 & 0 & x(x-1) & x & 0 & 0 \\ 0 & 0 & x(x-1) & 1-x & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-x+1 \end{bmatrix}.$$

The corresponding R-matrix $R(x)=P\check{R}(x)$ is given by

$$R(x)=\begin{bmatrix} x^2-x+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-x & x & 0 & 0 & 0 & x(x-1) & 0 \\ 0 & 1-x & 0 & 0 & 0 & x(x-1) & x & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 1-x & x(x-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & x^2-x+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x(x-1) & 1-x & 0 & 0 & 0 & x & 0 \\ 0 & 0 & x & x(x-1) & 0 & 0 & 0 & 1-x & 0 \\ 0 & x(x-1) & 0 & 0 & 0 & x & 1-x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-x+1 \end{bmatrix}. \qquad (15)$$

Note $\check{R}(x)\check{R}(x^{-1})=(x-1+1/x)^2 I\otimes I$, so the unitarity property holds.

The previous solution gives rise to a 21-vertex model, which appears to be new. It does not belong to the class of 21-vertex models discussed in Ref. 44. Viewed as a two-dimensional lattice statistical mechanics model though, it does not have real, non-negative Boltzmann weights. Since the regularity property (5) holds, we can, however, construct an integrable one-dimensional model. Even though $\check{R}(x)$ is not Hermitian, we obtain a Hermitian Hamiltonian in the following manner. We rescale $\check{R}(x)$ by a factor of i/x and define the two-site Hamiltonian h as

$$h=\frac{d}{dx}\left.\frac{i\check{R}(x)}{x}\right|_{x=1}=i\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which can also be written as

$$h=\sum_{\gamma\in D_3} i(E^{\gamma(1)}_{\gamma(2)}\otimes E^{\gamma(2)}_{\gamma(3)}-E^{\gamma(2)}_{\gamma(3)}\otimes E^{\gamma(1)}_{\gamma(2)}), \qquad (16)$$

where the elements γ of D_3 are written as permutations of $\{1,2,3\}$. The above integrable system describes a one-dimensional lattice of anyons with $D(D_3)$ or $D(D_6)$ symmetry and local interactions given by (16). The two-site Hamiltonian may also be expressed in terms of spin-1 operators:

$$-ih=-S_+^2\otimes(S_zS_-+S_-S_z)+(S_zS_-+S_-S_z)\otimes S_+^2$$

$$+S_-^2\otimes(S_zS_++S_+S_z)-(S_zS_++S_+S_z)\otimes S_-^2$$

$$+S_+S_z \otimes S_zS_+ - S_zS_+ \otimes S_+S_z$$

$$+S_-S_z \otimes S_zS_- - S_zS_- \otimes S_-S_z,$$

where $S_\pm = 1/2(S_x \pm iS_y)$. This Hamiltonian is not the same as other known integrable spin-1 Hamiltonians.[45–49]

In principle, the integrability of the previous model implies that algebraic Bethe ansatz methods may be used to obtain the exact solution. In the present case, however, there is no simple choice of reference state needed for the Bethe ansatz calculation, and a generalized algebraic Bethe ansatz similar to that employed for the XYZ model[50] is required. Because of the technical nature of such a calculation we defer it to a future publication.

VIII. SOLUTIONS OF THE YANG–BAXTER EQUATION ASSOCIATED WITH $D(A_4)$

The same procedure can be applied to the symmetric and alternating groups. The *symmetric group* S_n is the group of permutations of $\{1,2,\ldots,n\}$ where the operation is composition. The subgroup of S_n consisting of permutations that can be written as the product of an even number of transpositions is known as the *alternating group* and denoted A_n. Now $S_3 \cong D_3$ and $A_3 \cong Z_3$, so we only consider $n \geq 4$. In A_n the only conjugacy class with only one element is $\{e\}$, which always gives rise to the trivial R-matrix $R = I \otimes I$. Moreover, there are no conjugacy classes with two elements and only A_4 has a conjugacy class with three elements. Therefore only A_4 can give rise to a three-dimensional irrep, and we can never obtain a two-dimensional irrep.

Consider A_4, using the convention $(12)\circ(13) = (132)$. The relevent conjugacy class C_k and the details required to construct the representations are

$$C_k = \{(12)(34),(13)(24),(14)(23)\}, \quad g_k = (12)(34),$$

$$Z_k = \{e,(12)(34),(13)(24),(14)(23)\},$$

$$\alpha_{(12)(34)} = e, \quad \alpha_{(13)(24)} = (132), \quad \alpha_{(14)(23)} = (123).$$

This time $Z_k \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, with the 4 one-dimensional irreps given by $(12)(34) = (-1)^a$, $(13)(24) = (-1)^b$, $a,b \in \{0,1\}$. We obtain

$$R = \mathrm{diag}\big((-1)^a,(-1)^{a+b},(-1)^b,(-1)^b,(-1)^a,(-1)^{a+b},(-1)^{a+b},(-1)^b,(-1)^a\big).$$

Applying the permutation operator, we find

$$\check{R} = \begin{bmatrix} (-1)^a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (-1)^{a+b} & 0 & 0 \\ 0 & (-1)^{a+b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (-1)^a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1)^b & 0 \\ 0 & 0 & (-1)^b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)^{a+b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1)^a \end{bmatrix},$$

which has eigenvalues $1,-1$ with multiplicities $6,3$, respectively, when $a=0$, and eigenvalues $-1,i,-i$, each with multiplicity 3 when $a=1$. When $a=b=0$ this is the permutation matrix and gives rise to a representation of the Hecke algebra. Baxterization then leads to the known $su(3)$ invariant solution. In the case $a=0,b=1$, $\check{R}$ is again a Hecke algebra representation, which can be Baxterized as $\check{R}(u) = I \otimes I + u\check{R}$. This solution corresponds to the rational 15-vertex solution with a

Reshetikhin twist.[51] The last case is when $b=1$, in which case we can write without loss of generality:

$$\check{R}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We recognize this solution as belonging to the class of *nonstandard* solutions of the Yang–Baxter equation. However, it is curious that the origin of the nonstandard structure cannot be explained in terms of an underlying Lie superalgebra or colour Lie algebra structure,[52,53] nor is it due to a Reshetikhin twist.[51]

As $\check{R}$ has three eigenvalues, we try the ansatz $\check{R}(x)=f(x)I\otimes I+g(x)\check{R}+h(x)\check{R}^{-1}$. Applying the conjecture given in Refs. 31 and 35, we obtain three possible solutions. Two of these, however, have $f(x)=0$, which is undesirable if we want the regularity property to hold. The third possible case can be shown to not satisfy the braiding Yang–Baxter equation (11). Hence, we attempt to find another way to introduce a spectral parameter.

First we return to the original variables u,v instead of x,z. Writing $a(u)=f(u)+g(u)+h(u)$ and $b(u)=g(u)-h(u)$, we note $\check{R}(u)$ is

$$\check{R}(u)=\begin{bmatrix} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f(u) & 0 & 0 & 0 & -b(u) & 0 & 0 \\ 0 & -b(u) & 0 & f(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f(u) & 0 & b(u) & 0 \\ 0 & 0 & b(u) & 0 & 0 & 0 & f(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b(u) & 0 & f(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{bmatrix}.$$

Substituting $\check{R}(u)$ into the braiding Yang–Baxter equation (2), we find $\check{R}(u)$ satisfies the Yang–Baxter equation if and only if the following conditions are met:

$$b(u+v)f(u)f(v)=f(u+v)[b(u)f(v)+b(v)f(u)], \tag{17}$$

$$a(u)b(u+v)f(v)=b(v)f(u)f(u+v)+a(u+v)b(u)f(v), \tag{18}$$

$$a(u+v)f(u)f(v)=f(u+v)[a(u)a(v)+b(u)b(v)]. \tag{19}$$

First, consider the case when $b(u)=0$. Then equations (17), (18) are automatically satisfied, and we need only consider Eq. (19). The solution $\check{R}(u)\propto I\otimes I$ is uninteresting, so we instead choose $f(u)=1$ and $a(u)=e^u$, giving

$$\check{R}(u) = \mathrm{diag}(e^u, 1, 1, 1, e^u, 1, 1, 1, e^u).$$

Observe that this solution has the following peculiar property:

$$\check{R} \neq \lim_{u \to -\infty} \check{R}(u). \tag{20}$$

Next consider $b(u) \neq 0$. We begin by choosing $f(u)=1$. Then we see that $b(u)=b_0 u$ is the only solution to (17). We substitute these into Eq. (18) to obtain

$$(u+v)a(u) = v + ua(u+v),$$

$$(u+v)a(v) = u + va(u+v).$$

Eliminating $a(u+v)$, we find

$$(u+v)\left(\frac{a(u)}{u} - \frac{a(v)}{v}\right) = \frac{v}{u} - \frac{u}{v} = \frac{v^2 - u^2}{uv}$$

$$\Rightarrow \left(\frac{a(u)}{u} - \frac{a(v)}{v}\right) = \frac{v-u}{uv} = \frac{1}{u} - \frac{1}{v}$$

$$\Rightarrow \frac{a(u)-1}{u} = \frac{a(v)-1}{v} = c$$

$$\Rightarrow a(u) = 1 + cu.$$

We find that this satisfies (19), provided $c = \pm i b_0$, so we have found a solution to the braiding Yang–Baxter equation (2). Note that b_0 is just a scaling factor on u, so we can choose any nonzero complex number. Choosing $b_0 = i$ and $c=1$, we have

$$\check{R}(u) = \begin{bmatrix} 1+u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & iu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -iu & 0 & 0 \\ 0 & -iu & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & iu & 0 \\ 0 & 0 & iu & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -iu & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+u \end{bmatrix}.$$

The previous solution again corresponds to the rational 15-vertex solution with a Reshetikhin twist.[51] We also note that the property (20) also holds for this solution.

IX. SOLUTIONS OF THE YANG–BAXTER EQUATION ASSOCIATED WITH $D(S_4)$

As with $D(A_n)$, the algebra $D(S_n)$ has no nontrivial two-dimensional irreps, and the only nontrivial three-dimensional irrep occurs when $n=4$. Then C_k, g_k, Z_k, and α_s, $s \in C_k$ are given by

$$C_k = \{(12)(34), (13)(24), (14)(23)\}, \quad g_k = (12)(34),$$

$$Z_k = \{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\},$$

$$\alpha_{(12)(34)} = e, \quad \alpha_{(13)(24)} = (14), \quad \alpha_{(14)(23)} = (13).$$

Note that $Z_k \cong D_4$ with generators $\{(12),(1324)\}$, so we know it has exactly 4 one-dimensional reps given by $(12)=(-1)^a$, $(1324)=(-1)^b$, $a,b \in \{0,1\}$. Following the same procedure as earlier, we obtain

$$R = \sum_{g \in S_4} g \otimes g^* = \mathrm{diag}(1,(-1)^{a+b},(-1)^{a+b},(-1)^{a+b},1,(-1)^{a+b},(-1)^{a+b},(-1)^{a+b},1).$$

Both these solutions arose in $D(A_4)$ and were discussed in the previous section.

X. CONCLUSION

Our results show that for certain constant solutions of the Yang–Baxter equation obtained by using representations of finite group doubles, it is possible to Baxterize them to yield solutions of the spectral parameter Yang–Baxter equation. We have considered several examples where this is true and, in particular, we have found a new 21-vertex solution (15) from which we obtained an integrable model for a system of anyons with $D(D_3)$ or $D(D_6)$ symmetry. It is clearly of interest to determine if all constant solutions may be Baxterized. In contrast to the case of affine quantum algebras, where the spectral parameter has its origins in the loop representation, the origin of the spectral parameter for the above instances is unknown.

In all our examples we have only looked for cases where the spectral parameter has the difference property. For the case of the generalized chiral Potts model in Ref. 15, which does not have the difference property, an underlying finite group structure appears. This suggests that a Baxterization ansatz without the assumption of the difference property may also be fruitful. Certainly more work is needed to fully realize the potential of finite group doubles in solving the Yang–Baxter equation with spectral parameter.

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APPENDIX: REPRESENTATIONS OF $D(G)$, WHERE $G=D_n$, n ODD

The conjugacy classes C_k of $G=D_n$, chosen representatives g_k, corresponding centralizer subgroups Z_k and the elements α_s, $\forall s \in C_k$, are given in Table III.

1. Representations induced by $C_k=\{e\}$

The module elements are of the form $e \otimes v$, where $v \in V$, V a D_n-module. In representation terms, there are 2 one-dimensional irreps and $(n-1)/2$ two-dimensional irreps. They are as follows:

$$\sigma = 1, \quad \tau = \pm 1, \quad g^* = \delta(g,e)$$

and

TABLE III. C_k, g_k, Z_k and α_s for $G=D_n$, n odd.

C_k	g_k	$Z_k=Z(g_k)$	$\alpha_s, \forall s \in C_k$
$\{e\}$	e	D_n	$\alpha_e=e$
$\{\sigma^k,\sigma^{-k}\}, 1 \le k \le n-1/2$	σ^k	$\{\sigma^i \mid, 0 \le i < n\}$	$\alpha_{\sigma^k}=e, \alpha_{\sigma^{-k}}=\tau$
$\{\sigma^i\tau \mid 0 \le i < n\}$	τ	$\{e,\tau\}$	$\alpha_{\sigma^i\tau}=\sigma^{(n+1/2)i}$

$$\sigma = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad g^* = \delta(g,e) I_2,$$

where $1 \le k \le (n-1)/2$.

2. Representations induced by $C_k = \{\sigma^k, \sigma^{-k}\}$, $1 \le k \le (n-1)/2$

The module elements are of the form $e \otimes v, \tau \otimes v$, where $v \in V$, V a module of the group algebra of $Z_k = \{\sigma^i | 0 \le i < n\}$. There are n such A_k-modules, with the corresponding representations π_j given by $\pi_j(\sigma) = \omega^j$, $0 \le j < n$, where $\omega = \exp(2\pi i/n)$. Thus, there are $n(n-1)/2$ different irreps of $D(D_n)$ induced by these conjugacy classes, given by

$$\sigma = \begin{bmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\sigma^k)^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (\sigma^{-k})^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad g^* = 0 \text{ otherwise},$$

where $0 \le j < n$, $1 \le k \le (n-1/2)$.

3. Representations induced by $C_k = \{\sigma^i \tau | 0 \le i < n\}$

The module elements are of the form $\sigma^{j(n+1)/2} \otimes v$, $0 \le j < n$, where $v \in V$, V a module of the group algebra of $Z_k = \{e, \tau\}$. Hence, there are two n-dimensional irreps of this form. They are as follows:

$$\sigma = A \in M_{n \times n}, \quad \text{where } [A]_{ij} = \delta(i, j+2), \quad \text{addition mod } n,$$

$$\tau = \pm B \in M_{n \times n}, \quad \text{where } [B]_{ij} = \delta(i+j, 2), \quad \text{addition mod } n,$$

$$(\sigma^i)^* = 0, \quad (\sigma^i \tau)^* = E_{i+1}^{i+1}, \quad 0 \le i < n.$$

Example 10.1: *In $D(D_3)$, σ and τ are as follows:*

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \tau = \pm \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence, when n is odd $D(D_n)$ has 2 one-dimensional irreps, 2 n-dimensional irreps and $(n^2-1)/2$ two-dimensional irreps, all of which are given above. Note that the sum of the squares of the dimensions is $4n^2 = |D_n|^2 = |D(D_n)|$, as we expect.[19]

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