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Asymptotic behavior of small eigenvalues, short geodesics and period matrices on degenerating hyperbolic Riemann surfaces

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Abstract. Consider $\{M_t\}$, a semi-stable family of compact, connected algebraic curves which degenerate to a stable, noded curve M_0. The uniformization theorem allows us to endow each curve M_t in the family, as well as the limit curve M_0 (after its nodes have been removed), with its natural complete hyperbolic metric (i.e. constant negative curvature equal to -1), so that we are considering a degenerating family of compact hyperbolic Riemann surfaces. Assume that M_0 has k components and n nodes, so there are n families of geodesics whose lengths approach zero under degeneration and $k-1$ families of eigenvalues of the Laplacian which approach zero under degeneration. A problem which has received considerable attention is to compare the rate at which the eigenvalues and the lengths of geodesics approach zero. In this paper, we will use results from complex algebraic geometry and from heat kernel analysis to obtain a precise relation involving the small eigenvalues, the short geodesics, and the period matrix of the underlying complex curve M_t. Our method leads naturally to a general conjecture in the setting of an arbitrary degenerating family of hyperbolic Riemann surfaces of finite volume.

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§1 Introduction

For a compact hyperbolic Riemann surface M, the spectrum $\mathrm{Spec}(\Delta_M)$ of the associated Laplacian Δ_M which acts on the space of smooth functions on M is discrete; the sign of the Laplacian is chosen so that all eigenvalues are non-negative. If M is a non-compact hyperbolic Riemann surface of finite volume, it is well-known (see, for example, Chapter 6.9 of [He 83]) that the essential spectrum of the Laplacian Δ_M is contained in $[1/4, \infty)$, so the set $\mathrm{Spec}(\Delta_M) \cap [0, 1/4)$ consists only of discrete eigenvalues. Further, for any finite volume surface M, the number of eigenvalues of Δ_M in $[0, 1/4)$, known as small eigenvalues, can be bounded by a constant depending

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only on the topological data of M (see Chapter X of [Ch 84]). Since one can continuously deform compact hyperbolic Riemann surfaces by pinching a finite number of geodesics to zero so that the limiting surface is non-compact, it is natural to consider the asymptotics of the spectrum of the associated family of Laplacians.

In [CC 89], it was shown that, in slightly imprecise terms, the family of small eigenvalues of a degenerating family $\{M_t\}$ of either compact or non-compact finite volume surfaces varies continuously through degeneration (see also [He 90] and, for a proof using heat kernels, see [HJL 97]). Since the dimension of the zero eigenspace of Δ_M is equal to the number of connected components of M, the dimension of this space can increase through degeneration. Therefore, a particular problem is to consider the rate at which the small eigenvalues approach zero under degeneration and, when possible, compare the rate at which these small eigenvalues approach zero to other topological data associated to the family of Riemann surfaces.

The purpose of this note is to study the rate at which the small eigenvalues approach zero through degeneration. Our starting point is to utilize the "double-pole" conjecture of the Polyakov string integrand, as proved in [BM 86]. From this, we shall define a function involving lengths of geodesics which are tending to zero under degeneration, eigenvalues of the Laplacian which approach zero under degeneration, and data from complex algebraic geometry, namely the determinant of the imaginary part of the period matrix relative to a particular basis of holomorphic 1-forms and the first homology group. The main result of this paper is that our function has a non-zero finite limit when considering a family of compact hyperbolic Riemann surfaces whose limit is a non-compact surface.

Since the problem in hand has received considerable attention, it is not possible, due to space considerations, to give a complete discussion of other results which have been obtained in this area. Instead, let us focus on related work yielding results most similar to our main theorem. One of the first results in the investigation of the question under study is from [SWY 80], where the authors prove the following theorem. Let M be a connected, compact hyperbolic Riemann surface of genus $g \geq 2$. For $1 \leq m \leq 2g - 3$, let $\mathscr{C}_m$ denote the class of all disjoint unions of simple closed curves which divide a given compact surface M into $m + 1$ components. Let

$$\mathscr{L}_m(M) = \min\{\ell(C) : C \in \mathscr{C}_m\},$$

where $\ell(C)$ is the sum of the lengths of the component curves of C; let $\{\lambda_j(M)\}$ be the sequence of eigenvalues of the Laplacian Δ_M which acts on the space of smooth functions on M, and order the eigenvalues so that

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots.$$

The main result of [SWY 80] asserts the existence of positive constants α_1 and α_2 depending solely on g such that

$$\alpha_1 \mathscr{L}_m(M) \leq \lambda_m(M) \leq \alpha_2 \mathscr{L}_m(M) \quad \text{for } 1 \leq m \leq 2g - 3,$$

and

$$\alpha_1 \leq \lambda_{2g-2}(M) \leq \alpha_2;$$

see also Theorem 4.6, Chapter X, of [Ch 84]. Hence, the authors have stated upper and lower bounds for small eigenvalues in terms of lengths of families of geodesics.

Different approaches to this problem of were employed in [DPRS 87] and [Bu 90], where the authors related the problem in hand to eigenvalues on graphs. The reader is referred to these articles for statements of results and discussion of techniques.

§2 Statement of the main result

We shall consider a family $\{M_t\}$ of compact, connected, hyperbolic Riemann surfaces of genus g which degenerates to a stable, noded Riemann surface M_0 with n nodes. Explicit constructions of a holomorphic family $\{M_t\}$ of compact Riemann surfaces of genus g which degenerates to a given limit surface M_0, which is a stable, noded algebraic curve in the sense of Deligne-Mumford, are discussed throughout the literature; see, for example, Chapter 3 of [Fa 73] or Section 2 of [Ma 76]. These constructions yield holomorphic families parameterized over n copies of a slit punctured disc $\mathcal{D}_s$, which is obtained by taking the punctured disc $\mathcal{D}$ and removing a ray connecting the removed origin to a boundary point, i.e.

$$\mathcal{D}_s = \{z \in \mathbb{C} \mid 0 < |z| < 1 \text{ and } \operatorname{Arg}(z) \neq \vartheta\}$$

for a given, fixed $\vartheta \in (-\pi, \pi]$. By '$t \to 0$', we mean that no component t_j of t is equal to zero and that the vector t approaches zero along a ray in $(\mathcal{D}_s)^n$. (In particular, to fix ideas, the reader could consider the case when all t_j's are equal.)

Holomorphic families of Riemann surfaces can of course be defined over n copies of the punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Indeed, they can be parametrized in a more general way, described as follows. Let S be an open domain in $\mathbb{C}^n$ which admits an analytic embedding into the stably compactified moduli space having the following properties:

a) There is a point $p \in S$ such that the set $S \backslash \{p\}$ embeds into the interior of the moduli space, that is, $S \backslash \{p\}$ embeds into the moduli space of non-singular Riemann surfaces;

b) The domain S and the Deligne-Mumford boundary of the moduli space intersect transversally at p.

One such explicit construction is given in Chapter 3 of [Fa 73] in the case when S is the unit disc in $\mathbb{C}$. This construction extends immediately to the setting where S is a product of unit discs. A family of Riemann surfaces parameterized over the base S is obtained by restricting the universal curve to S.

As a prerequisite to our arguments, we need to choose and fix a basis of $H_1(M_t, \mathbb{Z})$ for each $t \in S$. In general, there is a well-defined basis of $H_1(M_t, \mathbb{Z})$ only if the number of components of the singular surface over $p \in S$ is one more than the dimension of S. To avoid this complication, we restrict our attention to a sub-family of Riemann surfaces defined over an open, simply connected subset S' of $S\backslash\{p\}$ such that p is in the closure (in S) of S'. In particular, we restrict attention to the situation that S is a product of unit discs $\mathscr{D}^n$ with p corresponding to the origin, and S' is a product of slit punctured discs $(\mathscr{D}_s)^n$. We note here that a different choice of S' would, in general, affect the basis of $H_1(M_t, \mathbb{Z})$ under degeneration, and hence would alter the constants C and C_1 appearing in Lemma 5 and Lemma 8 respectively.

Given a holomorphic family $\{M_t\}$ parametrized over $(\mathscr{D}_s)^n$ with a natural basis of $H_1(M_t, \mathbb{Z})$ and corresponding dual basis of holomorphic 1-forms $\{\zeta_i\}$ let Ω_t be the corresponding period matrix of M_t. We note that by a standard application of Riemann's bilinear relations, the imaginary part of the period matrix $\mathrm{Im}(\Omega_t)$ is symmetric and positive definite (see page 63 of [FK 92]).

If the limit surface M_0 has k components, then the $k-1$ smallest non-zero eigenvalues $\{\lambda_j(t)\}$ of the hyperbolic Laplacian Δ_{M_t} on M_t approach zero. Let $\Pi_{\mathrm{sev}}(M_t)$ be the product of the $k-1$ small eigenvalues which approach zero; if $k=1$, then define $\Pi_{\mathrm{sev}}(M_t)$ to equal 1. Similarly, let $\Pi_{\mathrm{sge}}(M_t)$ be the product of the lengths of the n geodesics whose lengths are approaching zero, which we call the pinching, or short, geodesics.

The main result of the paper is the following.

Theorem 1. *Let $\{M_t\}$ be a family of compact, connected, hyperbolic Riemann surfaces of genus g which degenerates to a stable, noded Riemann surface M_0 with n nodes and k components. Let $\Pi_{\mathrm{sev}}(M_t)$ be the product of the small eigenvalues on M_t, and let $\Pi_{\mathrm{sge}}(M_t)$ be the product of the lengths of pinching geodesics on M_t. Let Ω_t be the family of period matrices on M_t. Then the limit*

$$\lim_{t\to 0}\frac{\Pi_{\mathrm{sev}}(M_t)}{\Pi_{\mathrm{sge}}(M_t)}[\det \mathrm{Im}(\Omega_t)]^{-1}$$

exists and is non-zero.

We remark here that our proof shows that the limiting value depends solely on the limit surface M_0 and the limit basis of $H_1(M_0, \mathbb{Z})$, and not on the particular degeneration $t\to 0$ under consideration, cf. also the remarks after Corollary 2. In Conjecture 9 we state precisely a conjecture as to the structure of the limiting value.

Note also that if $n=k-1$, then the results from Chapter 3 of [Fa 73], specifically page 41, implies that $\det \mathrm{Im}(\Omega_t)$ has a non-zero finite limit. In this case, Theorem 1 immediately implies the following corollary (cf. Theorem 1.1 of [Bu 90] as well as the main result of [SWY 80]):

Corollary 2. *Under the conditions of Theorem 1, assume M_0 has n nodes and $n+1$ components. Then the limit*

$$\lim_{t\to 0}\frac{\Pi_{\mathrm{sev}}(M_t)}{\Pi_{\mathrm{sge}}(M_t)}$$

exists and is non-zero.

The factor $\det \operatorname{Im}(\Omega_t)$ is the only term in Theorem 1 which depends on the choice of the slit disc $\mathscr{D}_s$. However, the structure of the asymptotic behavior in t of $\log(\det \operatorname{Im}(\Omega_t))$ is independent of the choice of the $\mathscr{D}_s$. If one were to make a different choice of slit discs, the change in the limit asserted in Theorem 1 would be calculable in terms of the action of an associated element of the symplectic group $\mathrm{Sp}_g(\mathbb{Z})$ which acts on the first homology group of (possibly disconnected) the limit surface M_0 (see page 60 of [FK 92]).

Although we have defined all the terms used in the statements of the various results necessary for our proof of Theorem 1, it is not possible to give a brief, thorough discussion of all the necessary background material. We refer the reader to the works cited in the bibliography for further details.

§3 Proof of main theorem

Theorem 1 will be proved by combining various results in the literature involving asymptotic behavior of determinants of Laplacians, determinants of bases of holomorphic forms, the holomorphicity of the Polyakov string integrand, and the "double-pole" conjecture first stated in [BK 86a] and [BK 86b], as proved in detail in [BM 86]. For the convenience of the reader, we shall isolate the necessary results and asymptotic formulae into separate lemmas. To begin, we need to establish necessary notation.

Let M be a compact Riemann surface of genus $g \geq 2$ endowed with the unique Riemannian metric of constant curvature equal to -1 which is compatible with the underlying complex structure. Let us consider a local variation in $\mathscr{M}_g$. Let $\{\zeta_i\}$ be a holomorphically varying family of holomorphic 1-forms, and let $\{\varphi_i\}$ be a holomorphically varying family of holomorphic 2-forms on $\mathscr{M}_g$ in a neighborhood of M. Let $\Delta_{M,k}$ be the Laplacian which acts on the space of smooth k-forms on M, relative to the hyperbolic metric, and let $\det^*\Delta_{M,k}$ be the determinant of the Laplacian obtained through zeta function regularization, whose definition we now recall. Let $\{\lambda_{M,k;m}\}$ be the sequence of non-zero eigenvalues of the Laplacian $\Delta_{M,k}$, and define the zeta function

$$\zeta_{M,k}(s) = \sum_m \lambda_{M,k;m}^{-s}.$$

The above series converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s)$ sufficiently large. Using small time asymptotics of the heat kernel associated to $\Delta_{M,k}$, one can prove that the zeta function $\zeta_{M,k}(s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ which is holomorphic at $s = 0$ (see [MP 49]). With this, one defines the logarithm of the determinant of the Laplacian by

$$\log(\det^* \Delta_{M,k}) = -\zeta'_{M,k}(0).$$

Directly from formulae (15), (18) and the discussion after (19) in [BK 86a] we have the following variational formula, which holds locally on $\mathscr{M}_g$.

Lemma 3. *Let ∂ denote the holomorphic derivative on $\mathcal{M}_g$ with respect to the natural complex structure. Then, with notation as above, we have the local variational formula*

$$\partial\bar{\partial}\log\left[\left(\frac{\det^*\Delta_{M,1}}{\det(\langle\zeta_i,\zeta_j\rangle)}\right)^{-13}\left(\frac{\det^*\Delta_{M,2}}{\det(\langle\varphi_i,\varphi_j\rangle)}\right)\right]=0.$$

Let $Z_M(s)$ be the Selberg zeta function associated to the finite volume hyperbolic Riemann surface M, which is defined for $\mathrm{Re}(s)>1$ by the Euler product

$$Z_M(s)=\prod_{v=0}^{\infty}\prod_{\ell\in P(M)}(1-e^{-(s+v)\ell}),$$

where $P(M)$ is the set of lengths of inconjugate, primitive closed geodesics. One obtains a holomorphic continuation of $Z_M(s)$ to all $s\in\mathbb{C}$ by the Selberg trace formula (see [He 83]). Details concerning properties of the Selberg zeta function can be found in a number of sources, beginning with [He 83] and the references therein.

Lemma 4. *There exist universal constants c_1 and c_2 such that, with notation as above, we have the relations*

$$Z_M'(1)=c_1^{g-1}\det{}^*\Delta_{M,1}\quad\text{and}\quad Z_M(2)=c_2^{g-1}\det{}^*\Delta_{M,2}.$$

Proof. These relations were first proved in [DP 86]: see formulae (3.11) and (3.8) of that paper.

By combining the formulae in Lemma 3 and Lemma 4, we obtain the differential equation

$$\partial\bar{\partial}\log\left[\left(\frac{Z_M'(1)}{\det(\langle\zeta_i,\zeta_j\rangle)}\right)^{-13}\left(\frac{Z_M(2)}{\det(\langle\varphi_i,\varphi_j\rangle)}\right)\right]=0. \tag{1}$$

Our main theorem will follow by studying (1).

In order to study (1) locally in $\mathcal{M}_g$, one needs to choose a locally holomorphically varying basis of holomorphic 1-forms $\{\zeta_i\}$, which, by the Riemann-Roch theorem, has (complex) dimension g, and a locally holomorphically varying basis of holomorphic 2-forms $\{\varphi_i\}$, which has dimension $3g-3$ (see page 80 of [FK 92]). We are interested in solving (1) over the simply connected region $(\mathcal{D}_s)^n$, which parameterizes a degenerating family of compact Riemann surfaces $\{M_t\}$, and then studying the behavior of the solution as $t\to 0$.

Lemma 5. *Let $(\mathcal{D}_s)^n$ parameterize a semi-stable degenerating family of compact hyperbolic Riemann surfaces. Then, with the above described families of holomorphic 1-forms and holomorphic 2-forms, there is a non-vanishing holomorphic function F on $(\mathcal{D}_s)^n$ such that*

$$\left(\frac{Z'_M(1)}{\det(\langle\zeta_i,\zeta_j\rangle)}\right)^{-13}\left(\frac{Z_M(2)}{\det(\langle\varphi_i,\varphi_j\rangle)}\right) = |F(t)|^2.$$

Furthermore, there is some constant C (which depends on M_0, on the choice of the basis of holomorphic 2-forms $\{\varphi_i\}$ and the choice of local coordinates $\{t_j\}$, but is independent of the way $t \to 0$ in $(\mathscr{D}_s)^n$) such that

$$\log|F(t)|^2 = -2\sum_{j=1}^{n}\log|t_j|^2 + C + o(1).$$

as $t \to 0$ in $(\mathscr{D}_s)^n$.

Proof. As stated above, results from Chapter 3 of [Fa 73] and [Ma 76] assert the existence of bases of holomorphic 1-forms $\{\zeta_i\}$ and holomorphic 2-forms $\{\varphi_i\}$ which vary holomorphically over $(\mathscr{D}_s)^n$ such that the functions

$$\det(\langle\zeta_i,\zeta_j\rangle) \quad \text{and} \quad \det(\langle\varphi_i,\varphi_j\rangle)$$

are well-defined over $(\mathscr{D}_s)^n$. Since the region $(\mathscr{D}_s)^n$ is simply connected, one can solve the differential equation (1) over all of $(\mathscr{D}_s)^n$, and the solution of the equation is as asserted, namely

$$\left(\frac{Z'_M(1)}{\det(\langle\zeta_i,\zeta_j\rangle)}\right)^{-13}\left(\frac{Z_M(2)}{\det(\langle\varphi_i,\varphi_j\rangle)}\right) = |F(t)|^2, \tag{2}$$

i.e., any pluri-harmonic function on simply connected region is the logarithm of the square modulus of a non-vanishing holomorphic function. What remains is to verify the claimed behavior of F as $t \to 0$ in $(\mathscr{D}_s)^n$. This result is the "double-pole" conjecture, first stated in [BK 86a], [BK 86b] and later proved in detail in [BM 86]. Precisely, the assertion follows directly from (2.19), (3.4) and (4.27) of [BK 86b]; cf. (2), (4), (5) in [BM 86].

Lemma 5 leads us to study the asymptotic behavior of the other functions which appear in (2). We recall the asymptotic behavior of the Selberg zeta function over $(\mathscr{D}_s)^n$, a problem first studied in [Wo 86] and [He 90]. in the next lemma we use the sharper form of these estimates obtained in [JL 97].

Lemma 6. *With notation as above, there exist universal constants c_1 and c_2 such that as $t \to 0$ in $(\mathscr{D}_s)^n$ we have the asymptotic formulae:*

a) $\log Z'_{M_t}(1) = -\sum_{j=1}^{n}\frac{\pi^2}{3\ell_j} - \log\Pi_{\mathrm{sge}}(M_t) + \log\Pi_{\mathrm{sev}}(M_t) + \log Z'_{M_0}(1) + c_1 + o(1);$

b) $\log Z_{M_t}(2) = -\sum_{j=1}^{n}\frac{\pi^2}{3\ell_j} - 3\log\Pi_{\mathrm{sge}}(M_t) + \log Z_{M_0}(2) + c_2 + o(1).$

Proof. Let $PD(M_t)$ denote the set of lengths of inconjugate primitive pinching geodesics, counted with multiplicity. Set

$$\mathscr{Z}_{M_t}(s) = \prod_{v=0}^{\infty} \prod_{\ell \in PD(M_t)} (1 - e^{-(s+v)\ell}).$$

Then for $\mathrm{Re}(s^2 - s) > -1/4$ we have, after taking logarithms and integrating the formula given in Theorem 4.5 of [JL 97]

$$\lim_{t \to 0} \left(\frac{Z_{M_t}(s)}{\mathscr{Z}_{M_t}(s) \Pi_{\mathrm{sev},\, M_t}(s^2 - s - \lambda_{M_t,k})} \right) = \frac{Z_{M_0}(s)}{\Pi_{\mathrm{sev},\, M_0}(s^2 - s - \lambda_{M_0,k})},$$

where $\Pi_{\mathrm{sev},\, M_t}$ denotes a product over the small eigenvalues on the surface M_t. To complete the proof of Lemma 6, it suffices to study the behavior of $\mathscr{Z}_{M_t}(s)$ as $t \to 0$ for $s = 1$ and $s = 2$. To do this, we recall the definition of the Dedekind Delta function (see e.g. [Se 73])

$$\Delta(\tau) = e^{2\pi i \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})^{24}$$

which is valid whenever $\mathrm{Im}(\tau) > 0$ and which satisfies the functional equation

$$\Delta(\tau) = \tau^{-12} \Delta(-1/\tau).$$

Thus one can obtain the asymptotics of $\mathscr{Z}_{M_t}(1)$ and $\mathscr{Z}_{M_t}(2)$ via this functional equation, taking $\tau = i\ell/2\pi$. (Note that for each pinching geodesic, there are two inconjugate, primitive homology classes corresponding to the two different possible orientations of the geodesic, so there are two such factors, i.e. two different Dedekind Delta functions, which appear for each ℓ_j in ℓ. This factor of 2 is necessary in the calculations outlined above.)

In order to compare the asymptotic formulae in Lemma 5 and Lemma 6 we need the following result.

Lemma 7. *Under the degeneration $t \to 0$ given above, we have*

$$\ell_j = \frac{(2\pi)^2}{-\log|t_j|^2} + O\left(\frac{1}{(\log|t_j|)^2} \sum_k 1/(\log|t_k|)^2 \right),$$

from which we obtain the asymptotic formula

$$\frac{(2\pi)^2}{\ell_j} = -\log|t_j|^2 + o(1).$$

Proof. The result is quoted directly from Example 4.3 of [Wo 90].

To complete the study of all quantities which appear in Lemma 3, we need the following formulae.

Lemma 8. *With notation as above, we have:*

a) *the equality*

$$\det(\langle\zeta_i, \zeta_j\rangle)_t = \det \operatorname{Im}(\Omega_t);$$

b) *the asymptotic behavior under degeneration* $t \to 0$

$$\log\det(\langle\varphi_i, \varphi_j\rangle)_t = -\sum_{j=1}^{n} \frac{(2\pi)^2}{\ell_j} - 3\log\Pi_{\mathrm{sge}}(M_t) + C_1 + o(1),$$

where C_1 *depends on the basis of* 2*-forms* $\{\varphi_i\}$ *and the limit surface* M_0.

Proof. The first assertion is a direct consequence of the classical Riemann bilinear relations. The first two terms in (b) follow directly from p. 634 of [Ma 76], where the author obtained a partial asymptotic expansion (out to $O(1)$) of the determinant of the matrix of inner products of holomorphic 2-forms, viz.

$$\log\det(\langle\varphi_i, \varphi_j\rangle)_t = -\sum_{j=1}^{n} \frac{(2\pi)^2}{\ell_j} - 3\log\Pi_{\mathrm{sge}}(M_t) + O(1). \tag{3}$$

The limitation of the method in [Ma 76] was the lack of a sufficiently precise asymptotic expansion of the hyperbolic metric in neighborhoods of the pinching geodesics, which was later obtained in [Wo 90] (see Expansion 0.1 and 4.2). Thus, by combining these results, one can replace the error $O(1)$ in (3) with the term $C_1 + o(1)$, which completes the proof of the lemma.

Proof (of Main Theorem). If we write Lemma 5 in terms of the length parameters via Lemma 7, we obtain the formula

$$\log|F(t)|^2 = \sum_{j=1}^{n} \frac{8\pi^2}{\ell_j} + C_1 + o(1),$$

for some constant C_1. In other words, the quantity

$$\log|F(t)|^2 - \sum_{j=1}^{n} \frac{8\pi^2}{\ell_j}$$

has a finite limit under degeneration. From Lemma 6(a) and Lemma 8(a) we get

$$\log\left(\frac{Z'_M(1)}{\det(\langle\zeta_i,\zeta_j\rangle)}\right) = -\sum_{j=1}^{n}\left(\frac{\pi^2}{3\ell_j}\right) - \log\Pi_{\mathrm{sge}}(M_t) + \log\Pi_{\mathrm{sev}}(M_t)$$

$$- \log\det\mathrm{Im}(\Omega_t) + C_2 + o(1)$$

for some constant C_2. From Lemma 6(b) and Lemma 8(b) we get

$$\log\left(\frac{Z_{M_t}(2)}{\det(\langle\varphi_i,\varphi_j\rangle)_t}\right) = \sum_{j=1}^{n}\left(\frac{11\pi^2}{3\ell_j}\right) + C_3 + o(1)$$

for some constant C_3. Theorem 1 follows by combining these asymptotic formulae and Lemma 5.

§4 Concluding remarks

A crucial ingredient in our proof of Theorem 1 is the "double-pole" conjecture (Lemma 5). Although we quote the proof given in [BM 86], one should note that the first proof of this conjecture was given in [Wo 87]; in that paper, the author established the (weaker) asymptotic formula

$$\log|F(t)| = -2\sum_{j=1}^{n}\log|t_j|^2 + o\left(\sum_{j=1}^{n}\log|t_j|^2\right)$$

by proving Lemma 6 and then using the known asymptotic formula for the matrix of inner product of quadratic differentials as given in [Ma 76] (see (3)) together with previously quoted results from Chapter 3 of [Fa 73] concerning the order of growth of the period matrix and the eigenvalue growth estimate from [SWY 80] quoted in the introduction. The proof of Lemma 5 from [BM 86] involves methods from algebraic geometry. In essence, our proof of Theorem 1 is based on the improvement in the error term in Lemma 5 provided by [BM 86] over that which was proved in [Wo 87].

In addition, let us comment on the role of [Wo 90] in our paper. As noted in the proof of Lemma 8(b), the results of [Ma 76] yield an error term of $O(1)$ in Lemma 8(b), from which one can conclude that our function is simply bounded through degeneration. However, [Wo 90] allows for the sharpening of Lemma 8(b), namely that an asymptotic expansion out to $o(1)$ exists, from which we conclude the same for the logarithm of our function. It is interesting to note that this weaker version of Theorem 1 (bounded through degeneration, not necessarily having a limit) is itself a surprising relation between the short geodesics, the small eigenvalues, and the determinant of the imaginary part of the period matrix.

A problem which remains is to consider Theorem 1 in the setting of degenerating *non-compact* surfaces. Although the proof of Lemma 6 from [Wo 87] and [He 90] holds only for degenerating families of compact hyperbolic Riemann surfaces, the work in [JL 97] establishes asymptotic behavior of the Selberg zeta function for degenerating non-compact surfaces. We believe that one can readily extend the remainder of the above lemmas to the non-compact setting to obtain a generalization of Theorem 1 to the case of degenerating hyperbolic Riemann surfaces of finite volume, not necessarily non-compact. To go further, based on formal calculations when considering families of both compact and non-compact surfaces, we can state the following conjecture.

Conjecture 9. *With notation as in Theorem 1,*

$$\lim_{t\to 0}\frac{\Pi_{\mathrm{sev}}(M_t)}{\Pi_{\mathrm{sge}}(M_t)}(\det\mathrm{Im}(\Omega_t))^{-1}=\frac{\pi\,\mathrm{vol}(M_0)}{\prod(\pi\,\mathrm{vol}(M_j))}(\det\mathrm{Im}(\Omega_0))^{-1}.$$

In particular, in the case $n=k-1$,

$$\lim_{t\to 0}\frac{\Pi_{\mathrm{sev}}(M_t)}{\Pi_{\mathrm{sge}}(M_t)}=\frac{\pi\,\mathrm{vol}(M_0)}{\prod(\pi\,\mathrm{vol}(M_j))}.$$

Consider the case where $n=k-1=1$, so that the single pinching geodesic separates each surface in the family M_t. The limit surface M_0 consists of two components, of genra g_1 and g_2, and there is a single node obtained by identifying two points, one from each component. Using the results in [Bu 90], it was computed in [Ji 93] that one has the limit

$$\lim_{t\to 0}\frac{\lambda_1(t)}{\ell(t)}=\frac{1}{2\pi^2}\,\frac{2g-2}{(2g_1-1)(2g_2-1)},$$

which verifies Conjecture 9 in this case.

Finally, we remark that in [SWY 80], the authors obtain bounds on small eigenvalues not just for the hyperbolic metric, but for general metrics of negative curvature. In our setting, we can prove analogues of Theorem 1 and Corollary 2 for any family of metrics obtained by a compactly supported conformal perturbation of the hyperbolic metric such that the support of the perturbation remains bounded away from the pinching geodesics. To do so, one simply uses the result obtained in the hyperbolic case, namely Lemma 6 and the relation to the hyperbolic determinant of the Laplacian as stated above, together with the Polyakov variation formula which relates two determinants of the Laplacian under a conformal change of metric. Elementary considerations immediately yield an extension of Lemma 6 to the non-hyperbolic consideration (although we lose the interpretation of the determinant of the laplacian in terms of special values of zeta functions, other than the spectral zeta function). The other steps in the above arguments are identical to those in the hyperbolic case.

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