

Irreversibility in Asymptotic Manipulations of Entanglement

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We show that the process of entanglement distillation is irreversible by showing that the entanglement cost of a bound entangled state is finite. Such irreversibility remains even if extra pure entanglement is loaned to assist the distillation process.

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The appearance of irreversibility in physical processes can be regarded as one of the most fundamental and studied problems in the history of physics. In the context of quantum information, it was pointed out that an irreversible loss of entanglement might be present in the process of entanglement distillation [1,2]. That is, the amount of pure entanglement that can be distilled out of N copies of some state ρ might be strictly smaller than the one needed to create those copies if only local operations and classical communication (LOCC) are allowed and in the asymptotic limit ($N \rightarrow \infty$). Although when ρ represents a pure state this process is known to be reversible [3], it is generally believed that for mixed states this is not the case [4]. This last statement has not been proved so far [5,6]. In this Letter we prove it, i.e., we show that, by means of an example, the process of entanglement distillation is inherently irreversible. We will also extend this result to a broader context set by catalytic local operations and classical communication (LOCCc) [7], where pure entanglement—to be subsequently returned—is loaned to assist the distillation process.

Perhaps, the strongest indication that we have so far of the irreversibility of entanglement distillation is given by the existence of so-called bound entangled states [9]. Those are states from which no entanglement can be distilled but for which, in order to create a single copy, entanglement is required. Notice that, in spite of being very suggestive, this indication is not conclusive. It does not rule out the possibility that, in order to create a larger number of copies, the amount of entanglement needed per copy vanishes in the asymptotic limit. Although this seems unlikely, it has not been disproved so far. On the other hand, it is also not clear yet whether bound entangled states can be activated, and ultimately distilled, with the help of some borrowed pure entanglement. This would still leave an open door for some form of catalytic reversibility.

In this Letter we will show that the bound entangled state with positive partial transposition (PPT) introduced in Ref. [10] has a nonvanishing entanglement cost in the asymptotic limit. We will also show that more pure entanglement cannot be distilled from PPT states than just the amount that may have been used in order to assist the distillation process. In this way, the irreversibility of the asymptotic manipulation of entanglement in the context of

LOCC—and also in that of LOCCc—will immediately follow.

Let us formulate more precisely the problem. We consider two parties located in spatially separated regions and possessing N copies of the state ρ . Let us consider a transformation $\rho^{\otimes N} \rightarrow \varrho_N$ which fulfills

$$\lim_{N \rightarrow \infty} D(\varrho_N, |\Psi\rangle\langle\Psi|^{\otimes M}) \rightarrow 0, \quad (1)$$

for some integer M depending on N , where $|\Psi\rangle \equiv (|0,1\rangle - |1,0\rangle)/\sqrt{2}$ is the two-qubit singlet state and D is a properly chosen distance measure [11]. The entanglement of distillation $E_D(\rho)$ is defined as the maximal asymptotic ratio M/N with respect to all possible transformations which consist of LOCC [12]. On the other hand, let us assume now that the parties possess M two-qubit singlet states and they are able to transform them into the state ρ_M fulfilling

$$\lim_{M \rightarrow \infty} D(\rho_M, \rho^{\otimes N}) \rightarrow 0, \quad (2)$$

for some integer N . The entanglement cost $E_C(\rho)$ is defined as the minimal asymptotic ratio M/N also with respect to all LOCC [13]. The distillation process of a state ρ is irreversible if $E_D(\rho) < E_C(\rho)$.

Let us consider a density operator ρ acting on $H_A \otimes H_B$ and let us call P the projector onto the range of ρ . Then, we have that the entanglement cost of ρ can be bounded below as follows:

Theorem 1.—If $\langle e, f | P^{\otimes N} | e, f \rangle \leq \alpha^N$ for all normalized product vectors $|e, f\rangle \in (H_A)^{\otimes N} \otimes (H_B)^{\otimes N}$, then $E_C(\rho) \geq -\log_2 \alpha$.

Proof.—We will use the results of Ref. [13], where it is shown that

$$E_C(\rho) = \lim_{N \rightarrow \infty} \frac{E_f(\rho^{\otimes N})}{N}, \quad (3)$$

where the limit exists. The entanglement of formation E_f can be determined by considering decompositions of the form [1,14]

$$\rho^{\otimes N} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|, \quad (4)$$

and minimizing the quantity $\sum_i p_i E(\Psi_i)$ with respect to all possible decompositions. Here, E denotes the entropy

of entanglement [3]. By writing the Schmidt decomposition $|\Psi_i\rangle = \sum_k c_{i,k}|e_{i,k}, f_{i,k}\rangle$, we see that

$$|c_{i,k}|^2 \leq \langle e_{i,k}, f_{i,k} | P^{\otimes N} | e_{i,k}, f_{i,k} \rangle \leq \alpha^N, \quad (5)$$

where the first inequality is a consequence of the fact that $|\Psi_i\rangle\langle\Psi_i| \leq P^{\otimes N}$, since all the vectors $|\Psi_i\rangle$ must be in the range of $P^{\otimes N}$. Given the fact that $\sum_k |c_{i,k}|^2 = 1$, we obtain $E(\Psi_i) \geq -N \log_2(\alpha)$ for all i and therefore $\sum_i p_i E(\Psi_i) \geq -N \log_2(\alpha)$ for all decompositions. \square

Let us consider the bound entangled state ρ_b introduced in Ref. [10], where $H_A = H_B = \mathcal{Q}^3$. It is defined as $\rho_b \equiv P_b/4$, where P_b is a projector operator onto the orthogonal complement to the subspace spanned by the following vectors:

$$\begin{aligned} & |0\rangle \otimes (|0\rangle + |1\rangle), \\ & (|0\rangle + |1\rangle) \otimes |2\rangle, \\ & |2\rangle \otimes (|1\rangle + |2\rangle), \\ & (|1\rangle + |2\rangle) \otimes |0\rangle, \\ & (|0\rangle - |1\rangle + |2\rangle) \otimes (|0\rangle - |1\rangle + |2\rangle). \end{aligned}$$

This state has a positive partial transposition, and therefore it is not distillable [$E_D(\rho_b) = 0$] [9]. Later we will elaborate on this result. Our goal for now is to show that $E_C(\rho_b) > 0$.

We begin by proving the following two properties of the operator P_b :

$$\mathbb{1} + P_b = \sum_k |a_k, b_k\rangle\langle a_k, b_k|, \quad (6a)$$

$$\alpha_1 \equiv \sup_{|e,f\rangle \neq 0} \langle e, f | P_b | e, f \rangle < 1. \quad (6b)$$

Equation (6a) indicates that the operator $\mathbb{1} + P_b$ is separable. This can be proven by showing that the projector operators $P_1 \equiv \mathbb{1} - |a_0, a_0\rangle\langle a_0, a_0|$ and $P_2 \equiv P_b + |a_0, a_0\rangle\langle a_0, a_0|$ are both separable, where $|a_0\rangle = (|0\rangle - |1\rangle + |2\rangle)/\sqrt{3}$. By choosing $|a_1\rangle$ and $|a_2\rangle$ in such a way that $\{|a_k\rangle\}_{k=0}^2$ forms an orthonormal basis, one immediately sees that the range of P_1 is spanned by the mutually orthogonal product vectors $|a_{k_1}, a_{k_2}\rangle$, where $k_1, k_2 = 0, 1, 2$, except for $k_1 = k_2 = 0$, and therefore P_1 is separable. Analogously, the range of P_2 is spanned by the following mutually orthogonal product vectors

$$\begin{aligned} & |0\rangle \otimes (|0\rangle - |1\rangle), \\ & (|0\rangle - |1\rangle) \otimes |2\rangle, \\ & |2\rangle \otimes (|1\rangle - |2\rangle), \\ & (|1\rangle - |2\rangle) \otimes |0\rangle, \\ & |1\rangle \otimes |1\rangle. \end{aligned}$$

On the other hand, Eq. (6b) is a direct consequence of the fact that the range of P_b contains no product vectors and that $\langle e, f | P_b | e, f \rangle$ is a continuous function of $|e, f\rangle$ defined on a compact set so that it reaches its supremum [15].

We will now show that, for any normalized product vector $|e^N, f^N\rangle$ where $|e^N\rangle, |f^N\rangle \in (\mathcal{Q}^3)^{\otimes N}$,

$$\langle e^N, f^N | P_b^{\otimes N} | e^N, f^N \rangle < \beta^N, \quad (7)$$

where $\beta \equiv (1 + \alpha_1)/2 < 1$. Then, the above theorem readily implies that the entanglement cost E_C of the bound entangled state $\rho_b = P_b/4$ (and of any mixed state with the same support P_b) is finite. We will use induction over the number of copies N to show that Eq. (7) holds. For $N = 1$ it is true because of Eq. (6b) and $\alpha_1 < \beta$. Now, let us assume that it is true for a given N . Then, for any product vector $|e^{N+1}, f^{N+1}\rangle \in (\mathcal{Q}^3)^{\otimes N+1} \otimes (\mathcal{Q}^3)^{\otimes N+1}$ we have

$$\begin{aligned} & \langle e^{N+1}, f^{N+1} | (\mathbb{1} + P_b) \otimes \\ & \left[\mathbb{1} - \frac{1}{\beta^N} P_b^{\otimes N} \right] | e^{N+1}, f^{N+1} \rangle \geq 0. \end{aligned} \quad (8)$$

The reason is that by using (6a) and defining $|e_k^N\rangle \equiv \langle a_k | e^{N+1} \rangle$ and $|f_k^N\rangle \equiv \langle b_k | f^{N+1} \rangle$ we can write the left-hand side of this equation as

$$\sum_k \langle e_k^N, f_k^N | \left[\mathbb{1} - \frac{1}{\beta^N} P_b^{\otimes N} \right] | e_k^N, f_k^N \rangle, \quad (9)$$

where all the terms in the sum are positive according to the induction hypothesis (7). Now, we can write

$$(\mathbb{1} + P_b) \otimes \left[\mathbb{1} - \frac{1}{\beta^N} P_b^{\otimes N} \right] \leq \mathbb{1} + P_b \otimes \mathbb{1} - \frac{2}{\beta^N} P_b^{\otimes N+1}. \quad (10)$$

By substituting this expression in Eq. (8), we arrive at

$$\begin{aligned} & \langle e^{N+1}, f^{N+1} | P_b^{\otimes N+1} | e^{N+1}, f^{N+1} \rangle \leq \\ & \beta^{N\frac{1}{2}} (1 + \langle e^{N+1}, f^{N+1} | P_b \otimes \mathbb{1} | e^{N+1}, f^{N+1} \rangle) \leq \beta^{N+1}, \end{aligned} \quad (11)$$

as we wanted to prove.

The very same techniques can be applied to obtain lower bounds for the entanglement cost E_C also for more general mixed states. Notice that the relevant ingredients we have used are Eqs. (6a) and (6b), and that both conditions are only concerned with the support of the state ρ_b . Therefore, for any projector P satisfying Eqs. (6a) and (6b), we get a nontrivial bound for the entanglement cost of any state supported on it [16].

We move now to consider the distillability of the PPT state ρ_b and the extension of the irreversibility result to catalytic (i.e., LOCC-based) distillation. In [9] it was shown that inseparable states σ_b with PPT cannot be distilled into two-qubit singlet states $|\Psi\rangle$ using LOCC. The original proof relies on the fact that singlet states have a negative partial transposition (NPT), and LOCC cannot transform a PPT state into a NPT state. Notice, however, that if the parties initially share, in addition to the N copies of the state σ_b , L two-qubit singlet states $|\Psi\rangle$, then the original argument cannot be applied, because $\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L}$ is a NPT state for any $L \geq 1$. And, then, maybe the transformation

$$\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L} \rightarrow |\Psi\rangle\langle\Psi|^{\otimes M+L} \quad (12)$$

is asymptotically possible with some finite ratio M/N , in what would be a LOCCc distillation. Thus, our previous results for the state ρ_b do not yet exclude the possibility that in the large N limit the equivalence

$$\rho_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L} \approx |\Psi\rangle\langle\Psi|^{\otimes M+L} \quad (12)$$

under LOCC holds, i.e., the distillation of ρ_b can be turned into a reversible process using entanglement catalysis.

The following general result on bound entanglement readily implies that ρ_b is not distillable even with LOCCc, thereby providing an example of asymptotic irreversibility also in a broader sense than that of LOCC. While the proof we present here is original, the theorem also follows from results originally derived in [17].

Theorem 2.—Given N copies of a PPT state σ_b and K copies of some other state σ , the number of singlets that can be asymptotically distilled from them are, at most, the number of singlets required to create $\sigma^{\otimes K}$:

$$E_D(\sigma_b^{\otimes N} \otimes \sigma^{\otimes K}) \leq KE_C(\sigma), \quad (14)$$

which in particular means that

$$E_D(\sigma_b \otimes \sigma) \leq E_C(\sigma), \quad E_D(\sigma_b) = 0. \quad (15)$$

Proof.—Consider the upper bound on distillability [18] given by the logarithmic negativity $E_{\mathcal{N}}(\rho) \equiv \log_2 \|\rho^{T_B}\|$, where T_B stands for partial transposition and $\|A\|_1 \equiv \text{tr}\sqrt{A^\dagger A}$ is the trace norm of A . $E_{\mathcal{N}}$ is an additive function which vanishes for PPT states and is 1 for singlet states. Therefore, for general N, L we have

$$E_{\mathcal{N}}(\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L}) = L, \quad (16)$$

which implies that, at most, L singlet states can be distilled from $\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L}$. Setting $L \equiv KE_C(\sigma)$, and observing that such a number of singlets is sufficient to create $\sigma^{\otimes K}$ locally, so that no more singlets can be distilled from $\sigma_b^{\otimes N} \otimes \sigma^{\otimes K}$ than from $\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L}$, we obtain Eq. (14). \square

Thus, an optimal LOCC transformation of the form of Eq. (12) has $M = 0$, and pure entanglement does not help at distilling PPT states for LOCCc transformations [19]. On the other hand, it is easy to see that the entanglement cost of creating a mixed and a pure state is additive, so that [20]

$$E_C(\sigma_b^{\otimes N} \otimes |\Psi\rangle\langle\Psi|^{\otimes L}) = NE_C(\sigma_b) + L. \quad (17)$$

Then, when considering both the upper bound on distillability of Eq. (16) and the entanglement cost of Eq. (17), both applied to the PPT state ρ_b for which we have proved that $E_C(\rho_b) > 0$, we readily conclude that the asymptotic LOCCc manipulation of entanglement is also irreversible.

In summary, we have shown that the asymptotic entanglement cost E_C for locally preparing a given bound entangled state ρ_b is finite. Since no pure-state entanglement can be distilled from the state ρ_b even in the asymptotic limit [i.e., $E_D(\rho_b) = 0$], this result implies that the asymptotic interconversion, by means of LOCC,

between pure- and mixed-state entanglement is in general not a reversible process. We have finally proved that such an irreversibility also occurs for asymptotic LOCCc transformations.

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- [16] Consider, as an alternative—i.e., not using Eq. (6a)—example of how to get bounds for E_C , the projector P spanned by the vectors

$$\sqrt{\frac{2}{3}}|0, 2\rangle + \sqrt{\frac{1}{3}}|1, 0\rangle, \quad \sqrt{\frac{2}{3}}|2, 0\rangle + \sqrt{\frac{1}{3}}|1, 2\rangle,$$

of a $\mathcal{C}^3 \otimes \mathcal{C}^3$ system. For any product vector $|e, f\rangle \in \mathcal{C}^3 \otimes \mathcal{C}^3$ we have that

$$\langle e, f|P|e, f\rangle \leq \langle e|\text{tr}_B P|e\rangle = \langle e|\frac{2}{3}\div|e\rangle = \frac{2}{3},$$

and, similarly, for any product vector $|e^N, f^N\rangle$, where $|e^N\rangle, |f^N\rangle \in (\mathcal{C}^3)^{\otimes N}$,

$$\langle e^N, f^N|P^{\otimes N}|e^N, f^N\rangle \leq \left(\frac{2}{3}\right)^N.$$

Then the above theorem implies that the entanglement cost E_C of any state ρ supported on P is at least $-\log_2\left(\frac{2}{3}\right)$.

- [17] See proposition 3 in section VI of D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, *quant-ph/9908070*. See also D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, *Phys. Rev. A* **61**, 062312 (2000).
- [18] This bound has been derived by R. F. Werner [see G. Vidal and R. F. Werner *quant-ph/0102117*] and by the Horodecki family in [5].
- [19] Let us marginally comment on a weaker result that also follows from Eq. (16). Namely, that PPT states cannot be distilled in the so-called LOCCq [8] setting, when an asymptotically vanishing amount (per copy to be distilled) of pure entanglement is used in the distillation process. In a sense, this provides robustness to the original proof [9] of indistillability of PPT states.
- [20] Since E_C is defined in an asymptotic sense, it is also an additive for identical copies of a state, and thus $E_C(\sigma_b^{\otimes N}) = NE_C(\sigma_b)$.