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# Information, disturbance, and Hamiltonian quantum feedback control

Andrew C. Doherty,[1] Kurt Jacobs,[2] and Gerard Jungman[2]
[1]*Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125*
[2]*T-8, Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
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We consider separating the problem of designing Hamiltonian quantum feedback control algorithms into a measurement (estimation) strategy and a feedback (control) strategy, and we consider optimizing desirable properties of each under the minimal constraint that the available strength of both is limited. This motivates concepts of information extraction and disturbance that are distinct from those usually considered in quantum information theory. Using these concepts, we identify an information tradeoff in quantum feedback control.

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## I. INTRODUCTION

With experimental advances, particularly in the fields of cavity QED [1] and ion trapping [2], it is possible to observe individual quantum systems in real time, and it is therefore natural to consider the possibility of controlling such systems in real time using feedback [3–14]. Feedback control is invaluable in macroscopic applications, and as a consequence there is a vast body of literature devoted to (classical) control, which considers issues of optimality and robustness. The techniques of modern control theory were first applied to the quantum feedback control problem by Belavkin [3–6]. For a recent account, see Belavkin [12], and a recent but less technical account of these ideas may be found in [13] and [14]. In addition, the special case of real-time Markovian quantum feedback has been analyzed [7–10] and implemented experimentally in certain quantum optical systems [11], although this analysis was not concerned with questions of optimal control in the sense of modern control theory. While the quantum optimal feedback control problem may in certain special cases be solved exactly by using techniques developed in classical control theory [4,13], this is not possible in general. This is at least partly because quantum measurement is quite different in nature from classical measurement, in that it has the capacity to disturb the system under observation [14]. As a result, the development of optimal quantum control strategies requires optimizing over possible measurement strategies, which is unnecessary in classical control.

In feedback control, the dynamics of a system is manipulated by using information obtained about the system through measurement. The goal is usually to maintain a desired state or dynamics in the presence of noise. A central problem of feedback control theory is the development of algorithms to achieve this goal. The approach to controller design that we consider here is to examine the measurement and feedback steps separately, thereby splitting the feedback control problem into two parts. One can then consider optimizing desirable properties of these parts separately under suitable constraints. If one allows the strength of either measurement or Hamiltonian feedback to be infinite, then any control objective can be achieved perfectly (this will be shown below once we have made these concepts of strength more precise). A constraint on strength is therefore the minimal constraint under which the problem of quantum feedback control is nontrivial, and this is the constraint we employ here.

The action of optimizing for the feedback and measurement independently ignores the possibility that truly optimal solutions may require considering both together. We will also simplify the problem by considering the optimization at each time step separately. This assumes that it is never desirable to perform worse at the current time in order to perform better at some future time. The approach we take here is therefore not aimed at finding a globally optimal solution given a set of constraints. However, the expectation is that the concepts we introduce here provide a simple systematic approach that one can expect to produce good results, and provide an insight into the kind of measurement processes that are desirable in feedback control.

For the feedback step, we consider the question of the effectiveness of the control by defining a cost function. Since one is interested in controlling the dynamics of a given quantum system (usually in the presence of some unavoidable source of environmental noise), one can specify the objective by specifying the most desired state for the system at each instant. The ''cost'' function is then the sum of the distances of the state of the system from the desired state at each point in time, for some suitable measure of distance. We then find the choice of feedback Hamiltonian (different at each instant) that minimizes this cost function at each time step, under suitable constraints for the strength of the feedback. Note that, as we pointed out above, because each time step is considered separately, while this procedure gives a simple and systematic feedback algorithm, it can be expected to be suboptimal. Note also that this is somewhat different from the standard approach taken in modern classical control theory [16–18], and more similar to the approach taken in the new techniques of ''postmodern'' classical control [19]. In modern classical control (e.g., linear quadratic Gaussian control theory) one usually optimizes a ''total'' cost function obtained from a suitably weighted sum of the cost function defined here, and another cost function intended to capture the cost of feedback strength. We will restrict ourselves to control objectives such that the desired state at each time (the target state) is pure, since impurity (mixing) merely signifies a lack of knowledge of the target state.

In considering the optimality of the measurement step, rather than attempting to find a measurement that explicitly optimizes the cost function, we define concepts of information and disturbance, motivated by the feedback control problem. We then consider finding measurements that maximize the information and minimize the disturbance. We find that in general these two targets are mutually exclusive, in striking contrast to classical control theory. This implies the existence of a tradeoff between information and disturbance in quantum feedback control.

Since we focus on continuous feedback control, and many readers will be familiar with generalized measurements but unfamiliar with the formalism of continuous quantum measurement, we describe in the next section how continuous observation is formulated within the language of generalized measurements. In Sec. III, we define the concept of the strength of a measurement, required as a minimal constraint for the feedback control problem. In Sec. IV, we discuss in detail the division of feedback control into ''pure'' measurement and Hamiltonian feedback, and consider what may be achieved when there is no limitation on the strength of either. We also discuss what may be achieved in this case both without feedback and with measurement-only feedback. In Sec. V, we consider the measurement process, define concepts of information and disturbance, and consider minimizing the disturbance and maximizing the information. In Sec. VI, we examine the Hamiltonian feedback and obtain Hamiltonians that minimize the instantaneous cost function. In Sec. VII, we implement the feedback control of a two-state system, showing how the ideas presented in the previous sections manifest in the performance of the control algorithm. Section VIII summarizes and concludes.

## II. CONTINUOUS OBSERVATION AND GENERALIZED MEASUREMENTS

We will concern ourselves primarily with continuous-time quantum feedback control, in which a system is observed continuously, and the results of the measurements (the *measurement record*) used to continuously alter the Hamiltonian of the system to effect control. We now discuss how continuous observation may be described within the language of generalized quantum measurements, implemented as positive operator valued measures (POVM's).

Continuous measurements on a quantum system generate a measurement record that is a continuous-time stochastic process, which may be either a (Gaussian) Wiener process or a point process [5,6,20,21,29]. For a given physical system, these two kinds of processes will result from making different measurements, for example photon counting (a point process) and homodyne detection (a Wiener process) performed on optical beams.

The key ingredient in describing continuous measurements is that during an infinitesimal time step $dt$, the information obtained by the observer must scale as $dt$, so that one can take the continuum limit and obtain a sensible answer [22–25,5,20,26–29]. This may be realized by defining a POVM, given by $\int\Omega_\alpha^\dagger\Omega_\alpha d\alpha=1$, to describe the result of an observation in the time interval $dt$ by [22,24]

$$\Omega_\alpha=\left(\frac{\pi}{2dt}\right)^{1/4}e^{-kdt(Q-\alpha)^2}, \tag{1}$$

where $Q$ is an arbitrary operator for the system under observation, $\alpha$ takes all values on the real line, and $k$ is a positive real constant. For reasons that will be made clear in the next section, we will only need to be concerned with the case in which $Q$ is Hermitian, so that $Q$ may be referred to as an observable, and we will assume this in what follows. Note that each $\Omega_\alpha$ is a weighted sum of projectors onto the eigenbasis of $Q$, where the weighting is peaked at $\alpha$. Thus each application of the $\Omega$'s provides some information about the observable $Q$. However, as $dt$ tends to zero, this information also tends to zero, since the $\Omega$'s become increasingly broad over the eigenstates of $Q$. Calculating the measurement result in the interval $dt$ at time $t$, and denoting this as $dy(t)$, we have [3,6,29]

$$dy(t)=4k\langle Q\rangle dt+\sqrt{2k}dW, \tag{2}$$

where $dW$ is the Wiener increment for the interval $dt$. Using this, one can obtain the stochastic evolution of the quantum state under this measurement process, referred to as a quantum trajectory, and this is given by the stochastic master equation (SME) [6,20,29]

$$\begin{aligned}d\rho=&-i[H,\rho]dt-k[Q,[Q,\rho]]dt\\&+(Q\rho+\rho Q-2\,\mathrm{Tr}[Q\rho]\rho)\sqrt{2k}dW,\end{aligned} \tag{3}$$

where $H$ gives the system evolution in the absence of the measurement. We can also readily obtain the nonselective evolution, in which the measurement results are ignored, and this is given by

$$\begin{aligned}\rho(t+dt)&=-i[H,\rho]dt+\int\Omega_\alpha\rho\Omega_\alpha^\dagger d\alpha\\&=-i[H,\rho]dt-k[Q,[Q,\rho(t)]]dt.\end{aligned} \tag{4}$$

When $H$ commutes with $Q$, this evolution leads to a diagonalization of $\rho$ in the basis of $Q$, as one would expect for measurements of $Q$. Similarly, integrating the SME in this case, one finds that the result in the long-time limit is a projection onto one of the eigenstates of $Q$. Such a POVM realizes a continuous measurement of the operator $Q$, such that the measurement record is a Wiener process.

One can also define a POVM to provide continuous observation in which the measurement record is a Poisson process. Since this requires only one of the two possible outcomes at each interval $dt$, the POVM consists of only two measurement operators:

$$\Omega_0=1-\tfrac{1}{2}kQ^2dt, \tag{5}$$

$$\Omega_1=Q\sqrt{kdt}. \tag{6}$$

That this gives a Poisson process can be seen by considering the probabilities for the outcomes 0 and 1, which are $1-k\langle Q^2\rangle dt$ and $k\langle Q^2\rangle dt$, respectively. Result 1 therefore corresponds to a Poisson ''event,'' which happens occasionally, and 0 to the absence of one. The SME corresponding to the measurement process is different from that corresponding to the Wiener measurement, but the nonselective evolution is identical. Physically, the nonselective evolution is fixed by choosing the interaction of the system with the environment that is mediating the measurement, and the trajectory, whether Poisson or Wiener, is selected by how one chooses to measure the environment so as to extract the information about the system. In fact, by taking a suitable unitary transformation of the Poisson measurement operators, and taking the appropriate limit in which there are many events in each interval $dt$, one can obtain the Wiener process measurement from the Poisson measurement, and so the first can be regarded as a special case of the second [20,29]. This is also discussed in detail in [15,30].

The point we wish to note here is that regardless of how one chooses the trajectory, a continuous measurement of an observable $Q$ is given by a POVM in which all the measurement operators $\Omega_\alpha$ are positive operators, diagonal in the basis of $Q$, and one must merely be careful to choose the form of these operators with respect to $dt$ so as to provide a sensible continuum limit.

## III. THE STRENGTH OF A MEASUREMENT

Clearly the more accurate the measurements of the observer, the more information she is able to obtain, and the better able she is to choose feedback to effectively control the system. However, in general, more accurate measurements require more resources. A particular example is the measurement of position by the reflection of a laser beam [13,34], a technique used in the atomic force microscope. In that case, it is the laser power on which the measurement accuracy depends. In treating quantum feedback control, it is sensible to consider a restriction on available resources, and hence a restriction on measurement accuracy. To treat this quantitatively, one must introduce a sufficiently precise notion of the accuracy, or *strength*, of a quantum measurement.

For the purposes of feedback control, since it is the final state resulting from measurement that the observer must act upon with feedback, it is the observer's information about this *final* state that is relevant. Intuitively, one can therefore think of stronger measurements as providing, on average, final states that are more pure (or, alternatively, have a smaller von Neumann entropy) than weaker measurements. When considering continuous observation, in the absence of any noise sources, an initially impure state is continually purified. In this case, the strength of the measurement can be thought of as being proportional to the *rate* of this purification. Note that this concept of information extraction by a measurement is quite different from that usually considered in quantum information theory. There, authors have been concerned about the information that a measurement provides about the initial state of the system (the state immediately before the measurement) [31,32], whereas in our case it is the information about the final state that is important.

We will not need an explicit definition for measurement strength here, since we will only require two properties of measurement strength that we will motivate below. However, we will give an example of an explicit definition that satisfies these two properties. To motivate the first property, we note that as we have defined it so far, it is clear that the strength of a measurement in some sense characterizes the average rank of the operators $\Omega_m$ that make up the associated POVM ($\Sigma_m\Omega_m^\dagger\Omega_m=1$). If all the $\Omega_m$ are rank 1, then one always obtains a pure final state, and therefore complete information, regardless of the initial state. The higher the rank of the projectors, the higher in general will be the von Neumann entropy for a fixed initial state. The first property we will require is that measurements that consist of rank 1 projectors should have maximum strength (for measurements on a system of a given dimension).

For the remainder of this paper, we will refer to measurements for which at least one of the $\Omega_m$ are rank 1 as infinite strength measurements. This terminology is natural in the context of continuous observation, since in order to provide rank 1 projections in a finite time from a continuous measurement, one would have to take the limit $k\rightarrow\infty$ in Eq. (3). However, we wish to stress that our use of this terminology is not intended to imply that any explicit definition of measurement strength should necessarily take this value for these kinds of measurements.

The second property we wish to impose is that strength be invariant under unitary transformations of the measurement operators. To motivate this property, one can consider a device that measures the spin of a two-state system. One would expect such a device to provide the same strength of measurement regardless of how it is oriented in space. Since spatial rotation covers all unitary transformations for a spin-half, for this system strength should be invariant under all unitary transformations of the $\Omega_n$. We will explicitly consider the spin-half system later.

To provide an example of an explicit definition of measurement strength for single-shot measurements on finite-dimensional systems, one can first consider the average uncertainty after the measurement result is known. Using the von Neumann entropy, for a measurement described by $\Sigma_n\Omega_n^\dagger\Omega_n=1$, this is

$$u_V(\rho)=\sum_n \mathrm{Tr}[\Omega_n\rho\Omega_n^\dagger \ln(\Omega_n\rho\Omega_n^\dagger/\mathrm{Tr}[\Omega_n\rho\Omega_n^\dagger])], \quad (7)$$

where $\rho$ is the initial state of the system. Using the purity as an alternative measure of uncertainty, we have

$$u_p(\rho)=1-\sum_n \frac{\mathrm{Tr}[(\Omega_n\rho\Omega_n^\dagger)^2]}{\mathrm{Tr}[\Omega_n\rho\Omega_n^\dagger]}. \quad (8)$$

We can define the strength of a measurement to be the difference between the uncertainty in the initial state and the average uncertainty after the measurement (i.e., the average change in uncertainty) for some fixed initial state. If we choose the initial state to be $I/N$, then this definition satisfies our two properties. Using the von Neumann entropy this gives

$$s_V=\ln(N)-u_V(I/N), \quad (9)$$

and using the purity

$$s_p=(1-1/N)-u_p(I/N), \tag{10}$$

in which $N$ is the dimension of the system being measured.

Definitions of measurement strength for single-shot measurements may be extended to continuous measurements by using the initial *rate* of uncertainty reduction. Using the explicit definitions given above [Eq. (9) and Eq. (10)], it is straightforward to calculate this rate from Eq. (3) and the Ito rules for stochastic differential equations [33]:

$$\left.\frac{d}{dt}s_V\right|_{t=0}=8kV_Q, \tag{11}$$

$$\left.\frac{d}{dt}s_p\right|_{t=0}=\frac{8k}{N}V_Q, \tag{12}$$

where $V_Q$ is the variance of $Q$ in the initial state, being

$$V_Q\equiv \mathrm{Tr}[Q^2/N]-\mathrm{Tr}[Q/N]^2. \tag{13}$$

## IV. MEASUREMENT AND FEEDBACK

In classical feedback control, it is natural to consider the measurement process as being qualitatively different from the feedback process. In particular, they may be distinguished by the fact that the measurement in each time step involves no change to the system Hamiltonian, and the feedback step provides no information. In quantum feedback, since measurement has the ability to affect the dynamics in ways that in classical mechanics would have to be attributed to a Hamiltonian, the distinction is not as fundamental. However, in the vast majority of quantum feedback schemes considered to date, it is some set of parameters describing the system Hamiltonian that are under the observer's control. This is motivated by practical considerations, since it is as of yet easiest experimentally to externally control aspects of the Hamiltonian. In this case, the feedback step involves no measurement, and the observation and feedback processes may be regarded as qualitatively different, as in the classical theory. In view of this, the polar decomposition theorem motivates some definitions.

By Kraus's representation theorem [35], every valid quantum evolution (a quantum operation) may be written as a POVM given by a set of operators $\Omega_n$, where the probability of each outcome is $P(n)=\mathrm{Tr}[\Omega_n^\dagger\Omega_n\rho]$ and the state resulting from each outcome is $\rho_n=\Omega_n\rho\Omega_n^\dagger/P(n)$. The only constraint on the $\Omega_n$'s is that $\Sigma_n\Omega_n^\dagger\Omega_n=1$. However, from the polar decomposition theorem, each of the operators $\Omega$ may be written as the product of a unitary operator and a positive operator, so that

$$\Omega_n=U_n\sqrt{\Omega_n^\dagger\Omega_n}. \tag{14}$$

This provides a natural decomposition of a general quantum operation in terms of measurement and feedback. Consider first the action of the unitary operators. By themselves they do not describe the acquisition of information, and in that sense they do not describe a measurement. This can be seen from the fact that a unitary operator does not change the von Neumann entropy of any state it acts upon, and consequently extracts no information. However, unitary operations are precisely the kind that can be applied by Hamiltonian feedback. Hence, the unitary operators appearing in the polar decomposition may be thought of as characterizing purely the feedback part of the quantum operation. Note that we have written the polar decomposition so that the action of the unitary operator follows after the action of the positive operator, being a necessary condition for feedback.

Conversely, the positive operators characterize the acquisition of information. They may always be written as a weighted sum of projectors, and therefore thought of as providing partial information about the states in the basis in which they are diagonal. When they correspond to rank 1 projectors, they provide complete information, in that the final state is pure. Since the unitary part has been factored out to obtain the positive operators, we may regard these operators as representing pure measurement; the change induced in the quantum state is only that which is strictly necessary in order provide the information obtained during the measurement. We note that this decomposition of measurements into unitary and positive operators has been considered before in the context of measurements of the first and second kind [36].

From this it is clear that *every* quantum evolution can be realized by a measurement in which the measurement operators are positive, followed by a feedback step in which the Hamiltonian is chosen to depend upon the measurement result. We see that the observation of a single observable, considered in Sec. II, corresponds to the special case in which all the positive operators forming the POVM are mutually commuting.

Under the above definitions, damping processes, such as cavity decay and Brownian motion, are not considered pure measurements; they are viewed as equivalent to a fixed combination of measurement and feedback. Since the object of feedback control is to limit the deviations of a system from a desired state (or more generally, from a particular evolution, which means merely that the target state changes with time), feedback control is essentially a damping process (toward the target state).

The polar decomposition theorem therefore fits snugly with the structure of Hamiltonian feedback, but it is nevertheless important to realize that this is not the only feedback process that may be considered in quantum mechanics. First note that the product of two positive operators need not be positive. Hence the evolution resulting from a sequence of pure measurements as defined above will in general be equivalent to a single pure measurement followed by some Hamiltonian evolution (i.e., both measurement and Hamiltonian feedback). This is an illustration of the fact that quantum measurements involve ''active'' transformations of the states, as opposed to the ''passive'' measurements of classical physics [14].

Consider now the full evolution of a system under Hamiltonian feedback control in a single infinitesimal time step $dt$, with initial state $\rho$. Since all dynamical processes commute to first order, one can treat even continuous feedback control as alternating steps consisting of measurement and feedback. This is consistent with the general approach of this paper, which is to consider the two steps separately. The system evolves under its own ''free'' Hamiltonian, $H_0$ (which in many cases will be the desired evolution), and is affected by a source of environmental noise, which can be described by the nonselective evolution generated by a POVM. The measurement is also performed, and the feedback evolution applied. For a given measurement result $n$, we may write the full evolution as

$$\tilde{\rho}_n = e^{-i(H_n+H_0)dt} P_n \left( \sum_j W_j \rho W_j^\dagger \right) P_n e^{i(H_n+H_0)dt}, \quad (15)$$

where the tilde indicates that we have not bothered to normalize the final state, and $W_j$ are the operators describing the (undesirable) action of the environment. Since all the operators always commute to first order in $dt$, we have combined the free Hamiltonian with the feedback Hamiltonian in the exponential. The task of feedback control is to choose operators $P_n$ and $H_n$ such that the evolution is closest to the desired evolution. Before we consider this for Hamiltonian feedback, let us examine what can be done in the absence of the conditional unitaries, using measurement alone, and the difference between the two kinds of feedback.

By the definition above, using measurement alone, one is restricted to POVM's in which all the measurement operators are positive, along with some overall unitary evolution independent of the measurement results. Now, to evaluate the efficacy of the control procedure, we must have a ''cost function'' that measures how well we have achieved the control objective, as discussed above in the Introduction. Since we have a desired ''target'' state $\sigma = |\psi_T\rangle\langle\psi_T|$ in mind at some final time (to be achieved following a single measurement, or a series of measurements), sensible cost functions will provide a measure of how close the final state $\rho_f$ is to the target state. A number of measures are possible, such as the inner product (Tr$[\rho_f \sigma]$), the fidelity (Tr$[\sqrt{\sigma^{1/2}\rho_f\sigma^{1/2}}]$), or the distinguishability ((1/2)Tr$[\rho_f - \sigma]$). Since we are interested only in target states that are pure, the fidelity is simply the square root of the inner product, so that they provide equivalent optimization problems. Throughout this paper, we will use these as the quantities to be optimized.

Now, the final state resulting from averaging the results of a single pure measurement is given by

$$\rho_f = \sum_n P_n \rho P_n . \quad (16)$$

Since $P_n = P_n^\dagger$, Ando's result [37] states that $\rho_f$ is always majorized by $\rho$, which means that the eigenvalues of $\rho_f$ are at least as evenly distributed as the eigenvalues of $\rho$. This means that the von Neumann entropy of $\rho_f$ is always at least as large as the entropy of $\rho$. Another way of putting this is that each eigenvalue of $\rho_f$ is some weighted average of one or more of the eigenvalues of $\rho$.

It follows almost immediately from the above results that the fidelity of the final state cannot be any larger than the maximum eigenvalue, $\lambda_{\max(\rho)}$, of the initial state $\rho$. To see this, we first note that since all the eigenvalues of the final state, $\lambda_j$, are a weighted average of the eigenvalues of $\rho$, none can be larger than the largest eigenvalue of $\rho$. Now, writing the fidelity in terms of the eigenvectors of $\rho_f$, $|\phi_j\rangle$ we have

$$\langle\psi_T|\rho_f|\psi_T\rangle = \sum_j \lambda_j |\langle\phi_j|\psi_T\rangle|^2. \quad (17)$$

Since $\Sigma_j |\langle\phi_j|\psi_T\rangle|^2 = 1$, the fidelity is merely a weighted average of the eigenvalues of $\rho_f$, which proves the result. In fact, choosing any basis $|\psi_i\rangle$, we obtain the probability distribution over these states as

$$\mu_i = \langle\psi_i|\rho_f|\psi_i\rangle = \sum_j \chi_{ij}\lambda_j , \quad (18)$$

where $\chi_{ij} = |\langle\phi_j|\psi_i\rangle|^2$. Since the matrix $\chi_{ij}$ satisfies $\Sigma_i \chi_{ij} = 1$ and $\Sigma_j \chi_{ij} = 1$, it is a doubly stochastic map, with the result that the vector $\{\mu_i\}$ is majorized by the vector $\{\lambda_j\}$, and hence the von Neumann entropy of the distribution over any set of basis states is always at least as large as the distribution over the eigenvectors. Another way of saying this is that diagonal elements of a matrix resulting from a unitary transformation performed on a diagonal matrix are always at least as uniformly distributed as the original elements (and almost always more so).

Clearly this result for the upper bound on the final fidelity also holds for repeated measurements, in which subsequent measurements are *not* conditioned on the results of previous measurements (i.e., for pure measurements with no feedback). However, it does not hold for sequences of conditional measurements. In this case, the initial state seen by subsequent measurements cannot be written as the state given by averaging over the results of previous measurements, since each final state may have a different measurement performed on it.

It turns out that if we allow ourselves an infinite measurement strength, then the upper bound on the final entropy derived above can always be achieved in the limit of an infinite number of measurements. To see this, one simply follows the procedure of Aharonov and Vardi, referred to as the ''inverse quantum Zeno effect,'' developed in Ref. [38]. Consider first an initial pure state, $|\psi\rangle$. We can always write the target state as a superposition of the initial state and a state orthogonal to the initial state, $|\psi\rangle_\perp$. That is, we can write

$$|\psi_T\rangle = \cos(\theta)|\psi\rangle + \sin(\theta)e^{i\phi}|\psi\rangle_\perp \quad (19)$$

for some value of $\theta$ and $\phi$. Now consider the projector $P_\varepsilon = |\varepsilon\rangle\langle\varepsilon|$ onto the state

$$|\varepsilon\rangle = \cos(\varepsilon)|\psi\rangle + \sin(\varepsilon)e^{i\phi}|\psi\rangle_\perp . \quad (20)$$

For $\varepsilon = 0$, this is the initial state, and for $\varepsilon = \theta$, this is the final state. For any value in between, this state represents a rotation through an angle $\varepsilon$ from the initial state to the final state. If $P_\varepsilon = |\varepsilon\rangle\langle\varepsilon|$ makes up one of the operators describing the measurement, the probability of *failing* to obtain the state $|\varepsilon\rangle$ is

$$P(\varepsilon) = \sin^2(\varepsilon) = \varepsilon^2 + \cdots. \tag{21}$$

If we succeed in obtaining the state $|\varepsilon\rangle$, then in that measurement step we have succeeded in rotating the state through $\varepsilon$ toward the desired state. We can attempt to rotate the state through the full $\theta$ radians by choosing $\varepsilon = \theta/M$ and repeating the process using $M$ measurements. Since the probability of failure at each step is then second order in $(1/M)$, while the number of steps scales only as $M$, as the number of steps tends to infinity, the total probability of failure tends to zero, and in this limit one achieves the desired rotation. To see that this achieves the upper bound when the initial state is mixed, we choose the projector so as to rotate the eigenvector of $\rho$ corresponding to the maximum eigenvalue to the desired state. For each measurement, the corresponding POVM is then given, in general, by $P_{\varepsilon_i} + \Sigma_l \Omega_l^\dagger \Omega_l = 1$, where $P_{\varepsilon_i}$ is the projector for the $i$th measurement, and the $\Omega_l$ are arbitrary. The final state may then be written

$$\rho_f = \prod_i^M \mathcal{L}_{\varepsilon_i}[\lambda_1|1\rangle\langle 1|] + \prod_i^M \mathcal{L}_{\varepsilon_i}\left[\sum_{n=2}^N \lambda_n|n\rangle\langle n|\right], \tag{22}$$

$$\mathcal{L}_{\varepsilon_i}[A] = P_{\varepsilon_i} A P_{\varepsilon_i} + \sum_l \Omega_l A \Omega_l^\dagger, \tag{23}$$

where $\rho|n\rangle = \lambda_n|n\rangle$, with $\lambda_1$ the maximum eigenvalue, and $A$ is an arbitrary operator. As the number of measurements tends to infinity, the first term on the right-hand side of Eq. (22) becomes

$$\lim_{M\to\infty} \prod_i^M \mathcal{L}_{\varepsilon_i} \lambda_1 |1\rangle\langle 1| = \lambda_1 |\psi_T\rangle\langle\psi_T|. \tag{24}$$

Since this term contributes $\lambda_{\max}(\rho)$ to the fidelity, and since the other pure states making up the final density matrix cannot contribute negatively, the upper bound is achieved.

What happens when we allow ourselves infinite measurement strength, and sequences of measurements in which subsequent measurements are *conditioned* on previous results (i.e., measurement-only feedback)? In that case it is clear that the desired state can always be obtained with certainty; one begins by making a projection measurement in an arbitrary basis, which results in a set of pure states. Then the above procedure is used to rotate the resulting state to the target.

When infinite strength is available, measurement-only feedback is equivalent to Hamiltonian feedback, since both allow any state to be created. However, in many continuous feedback control applications, the strength of the measurement is unlikely to be so much stronger than either the environmental noise or the free system dynamics that it can be used in this fashion in place of Hamiltonian feedback [1,39,40]. With measurements of finite strength, the outcomes are necessarily random, so that Hamiltonian feedback cannot be simulated reliably. One can expect therefore that real applications will find the use of Hamiltonian feedback invaluable.

## V. MEASUREMENT: MAXIMAL INFORMATION AND MINIMAL DISTURBANCE

In Sec. III, we introduced the concept of the information provided about the system by a measurement, and this involved specifically the information regarding the state *resulting* from the measurement, a definition motivated by feedback control. This in turn motivated the definition of the strength of a measurement (e.g., $s_v$ or $s_p$), important because it constitutes a natural constraint when considering the optimization of control strategies. However, the actual information provided by a given measurement is not only a function of the measurement strength, but also the state of the system immediately prior to the measurement. As a result, once the available measurement strength is known, one can ask how to optimize the information provided by the measurement given the current state of the system. This defines the concept of a measurement returning maximal information (for a fixed measurement strength).

In addition to providing information, quantum measurements can also introduce noise, a statement that we will now make precise. Consider first a classical system driven by noise. One can characterize the extent of the noise in some time interval by the increase in the entropy of the phase-space probability distribution for the system state that is given by averaging over the noise realizations. This tells us how much we expect the noise to spread out the system in phase space in that time interval, and characterizes our uncertainty about the future state of the system resulting from the noise. Now consider a classical measurement. Since the initial state is uncertain (or else we would not need to make the measurement), the result of the measurement is random, and as a result one's state of knowledge changes in a random fashion. However, this random change should not be considered noise, since if one averages over all the possible measurement results (all the possible random changes), the probability distribution for the state of the system remains unchanged. This is the sense in which classical measurement introduces no noise into the system.

Now consider a quantum system driven by noise. The equivalent of the phase-space distribution is the density matrix. In the same manner, one can characterize noise by the resulting increase in the von Neumann entropy of the density matrix resulting from averaging over the possible noise realizations. One can therefore characterize the noise introduced by a quantum measurement by calculating the increase in the von Neumann entropy (or alternatively the decrease in the purity) of the density matrix that results from averaging over the possible measurement results. While we saw above that in the classical case the measurement introduces no noise, this is not, in general, true for quantum measurement. In terms of the von Neumann entropy, the excess noise introduced by a measurement is

$$N_e^V = S_V(\rho_f) - S_V(\rho). \tag{25}$$

Defining it in terms of the purity, we have

$$N_e^p = \mathrm{Tr}[\rho^2] - \mathrm{Tr}[\rho_f^2]. \tag{26}$$

This makes precise the intended meaning of our initial statement that quantum measurements can introduce noise. Note that this has nothing to do with the Heisenberg uncertainty principle, what concerns us here is the uncertainty of the future quantum state, and not the uncertainty of some set of observables for a given state. Recall that this is because the object of the control is the state of the system, and it is up to the observer to decide what the desired state is. Whether it be a minimal uncertainty state in the sense of the Heisenberg uncertainty principle is immaterial.

Let us consider first the question of minimizing the disturbance due to the measurement. Recall that for a pure measurement, the evolution given by averaging over the measurement results is given by Eq. (16), where all the $P_n$ are positive operators. Once again invoking Ando's result, we have that the von Neumann entropy of the final state is never decreased by the measurement. Measurements that minimize noise are therefore the measurements that leave the von Neumann entropy unchanged. These measurements are in this sense most like classical measurements. A set of measurements satisfying this criterion are those in which all the $P_n$ commute with the initial density matrix. In this case we have

$$\rho_f = \sum_n P_n \rho P_n = \sum_n P_n^2 \rho = \rho. \tag{27}$$

In the language of continuous measurements, since the operators $P_n$ are diagonal in the eigenbasis of the observable, this means choosing to measure an observable that shares an eigenbasis with the density matrix.

On a practical note, for continuous observation, measuring in the eigenbasis of the density matrix involves continuously changing the measured observable (note that such a process has been considered previously in the context of adaptive measurements [41]). In many situations, this flexibility may be only partially available, or not at all. However, the above analysis indicates that for the purposes of noise minimization, one should choose the measured observable to be that in which the system is diagonal, or nearly diagonal, for the longest time during the period of control. In fact, this introduces the possibility that in certain cases it may be desirable to turn off measurement for periods in which the system occupies states that have large off-diagonal elements in the eigenbasis of the observable. Of course, the resulting noise reduction would have to be balanced against the accompanying loss of information.

Maximizing the information for a fixed measurement strength is a much more difficult problem. Here we will examine a specific example for the continuous measurement of a two-state system. In the formulation of continuous measurements that was discussed in Sec. II, we used measurement operators where each was a sum over an infinite number of projectors. For a two-state system, it is possible to obtain the same result (i.e., the same continuous measurement driven by Gaussian noise) by using a formulation with only two measurement operators where each is the sum over only two projectors. To obtain a continuous measurement of a given observable, the POVM is given by $\Omega_0^2 + \Omega_1^2 = 1$, where the measurement operators are

$$\Omega_0 = \sqrt{\kappa}|0\rangle\langle 0| + \sqrt{1-\kappa}|1\rangle\langle 1|, \tag{28}$$

$$\Omega_1 = \sqrt{\kappa}|1\rangle\langle 1| + \sqrt{1-\kappa}|0\rangle\langle 0| \tag{29}$$

in which $\kappa = 1/2 + \sqrt{kdt}$, and $|0\rangle$ and $|1\rangle$ are the eigenstates of the observable. In each time step $dt$, this produces one of two results. The sum of these, in a time interval $\Delta t = Ndt$ in which $N$ results are obtained, is naturally governed by the binomial distribution. In the limit of large $N$ (and infinitesimal $\Delta t$), this tends to a Gaussian, and one obtains the measurement record [Eq. (2)] and SME [Eq. (3)] given in Sec. II, where the measured observable $Q = |0\rangle\langle 0| - |1\rangle\langle 1|$. We can alternatively think of this measurement as a single-shot measurement, and in that case $\kappa$ can take any value between 0 and 1. Note that when $\kappa = 0$ or $\kappa = 1$, the measurement is one of infinite strength. As $\kappa$ becomes closer to $\frac{1}{2}$, the strength reduces, and for $\kappa = \frac{1}{2}$ the measurement provides no information.

We can obtain measurements of all possible observables by applying to the measurement operators an arbitrary rotation over the Bloch sphere, given by the unitary transformation

$$U(\theta,\phi)|0\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle, \tag{30}$$

$$U(\theta,\phi)|1\rangle = \cos(\theta/2)|1\rangle - e^{-i\phi}\sin(\theta/2)|0\rangle. \tag{31}$$

Recall that this unitary transformation of the measurement operators preserves the measurement strength as defined in Sec. III. Without loss of generality, we can choose the initial density matrix to be diagonal, and write it as $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$. One can then obtain an analytic expression for the final average purity, which is given by

$$I_f^p \equiv 1 - u_p = \sum_{n=0}^{1} \frac{\mathrm{Tr}[(U\Omega_n^\dagger U^\dagger \rho U \Omega_n U^\dagger)^2]}{\mathrm{Tr}[U\Omega_n^\dagger U^\dagger \rho U \Omega_n U^\dagger]}. \tag{32}$$

This expression is fairly complex, and we will not need it here. (For a detailed analysis of this expression, including analytic expressions for general two-outcome measurements on two-state systems, the reader is referred to [45].) It is explicitly independent of $\phi$, as one would expect, since it is $\theta$ alone that gives the angle (on the Bloch sphere) between the basis in which the density matrix is diagonal and the basis of the measured observable. The final average purity is then explicitly dependent on the three parameters $p$, $k$, and $\theta$, and we are concerned with maximizing this with respect to $\theta$. When the measurement is nontrivial ($\kappa \neq 0.5$), the strength of the measurement is finite ($\kappa \neq 0,1$), and the initial state is impure [$p \in (0,1)$], one finds that the location of the maximum is independent of $p$ and $k$, and occurs for $\theta = \pi/2$. This means that on average the maximum information is obtained about the final state (for fixed measurement strength) when the basis of the measured observable is maxi-

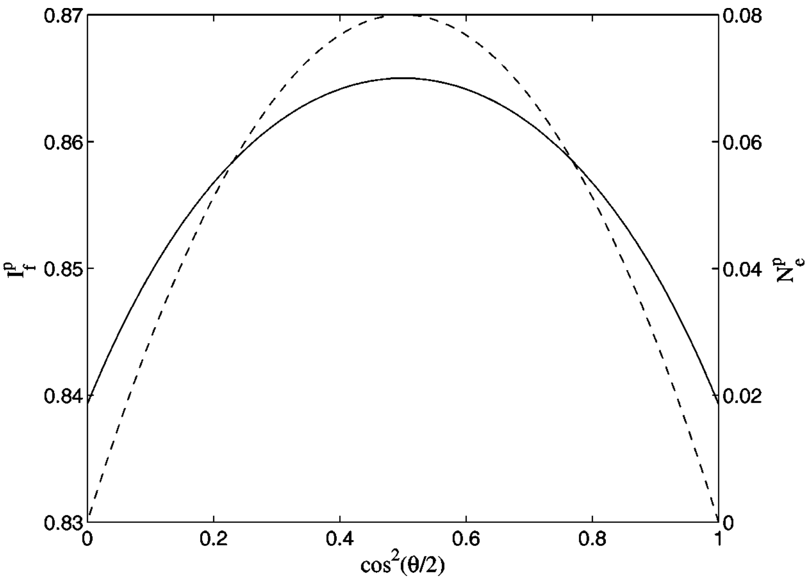


FIG. 1. Here we plot the information obtained about the final state (solid line), characterized by the final average purity, $I_f^p$, and the excess noise introduced by the measurements (dashed line), $N_e^p$, against the measured observable, parametrized by the angle $\theta$. The parameters are $p=0.1$ and $\kappa=0.75$

mally *different* from the eigenbasis of the density matrix (i.e., if the density matrix is a mixture of $\sigma_z$ eigenstates, then one should measure $\sigma_x$ or $\sigma_y$).

We see then that at least for a two-state system, the minimal disturbance is obtained when the measured observable has the same eigenbasis as the density matrix, and the maximal information is obtained when the two bases are maximally different. Thus we obtain the result that, at least for a two-state system, there is a tradeoff between information and disturbance in quantum feedback control (in contrast to classical feedback control). This tradeoff for finite strength quantum measurements is also of interest from a purely fundamental point of view, and this is explored in detail in Ref. [45]. We plot both the excess noise introduced by the measurement and the average final purity resulting from the measurement as a function of $\theta$ in Fig. 1. For a fixed measurement strength, one therefore has the choice between choosing a measurement to minimize the noise, and consequently obtain better control of the system (in that the system will fluctuate less around the desired value), or obtain a more accurate knowledge of the system at the expense of increased noise. Which is most desirable may well depend upon the current state of knowledge. For example, if the state of the system is poorly known, perhaps early on in the control process, then it may prove desirable to obtain information more quickly, at the expense of introducing extra noise, since the large uncertainty will be the major factor in reducing the effectiveness of the feedback. However, once the observer's knowledge is sufficiently sharp, it may prove more effective to reduce the noise at the expense of some added uncertainty. In Sec. VII, we will present simulations to show how this information tradeoff affects the performance of feedback control in a two-state system.

## VI. A SUBOPTIMAL HAMILTONIAN FEEDBACK ALGORITHM

Here we consider a feedback algorithm in which the feedback Hamiltonian will, at each point in time, be chosen to be a function of the observer's estimated state (and the target state) at that time. Since the estimated state, in general, depends upon the measurement record up until that time, the feedback Hamiltonian depends ultimately upon the measurement record, and this is what makes the procedure *feedback* control. The observer's estimate of the state of the system at time $t$ is some density matrix, $\rho(t)$. To obtain the true best estimate, the observer will integrate the SME for the system under observation (naturally including the dynamics given by the feedback history) using the measurement record. Alternatively, if this is too time-consuming, the observer might use some suboptimal estimation procedure [14].

We will chose the feedback Hamiltonian at time $t$ to be that which will, in the next infinitesimal time step, maximize the fidelity of the current state estimate with the current target state. Naturally, this will define a continuously changing feedback Hamiltonian. Denoting the state at time $t$ simply as $\rho$, the state after an infinitesimal time step is given by

$$\rho_f=\rho-i[H,\rho]\Delta t-\tfrac{1}{2}[H,[H,\rho]](\Delta t)^2+\cdots, \qquad (33)$$

where $H$ denotes the feedback Hamiltonian at time $t$ (we will usually suppress the time argument for simplicity). In what follows, we will refer to the state after the infinitesimal step as the final state, although, naturally, this is just another state that the system passes through during the period of control. The fidelity of the final state with the target state is then

$$\langle\psi_T|\rho_f|\psi_T\rangle=\langle\psi_T|\rho|\psi_T\rangle-i\langle\psi_T|[H,\rho]|\psi_T\rangle\Delta t - \tfrac{1}{2}\langle\psi_T|[H,[H,\rho]]|\psi_T\rangle(\Delta t)^2+\cdots. \qquad (34)$$

The first term is fixed, so to maximize the fidelity we should maximize the coefficient of $\Delta t$, being the dominant term. If the target state commutes with $\rho$, then this term vanishes for all $H$, so that we cannot choose a Hamiltonian that will cause an increase in the fidelity that is first order in time. If this situation occurs only for vanishingly small times, then it will make effectively no difference to the feedback performance. However, in those special situations in which the intrinsic dynamics preserves the commutivity of $\rho(t)$ and the target state, it can be important to choose a Hamiltonian that maximizes the term that is second order in time. One should note, however, that if one has freedom to choose the measurement basis, one can always choose a basis that disturbs the state so as to break the commutivity of $\rho(t)$ with the target state, eliminating the need to consider the second-order term in the time evolution.

The maximization must be performed under a reasonable constraint on the eigenvalues of $H$ (i.e., a constraint that captures the concept of a limitation on the strength of feedback). A number of suitable constraints are possible, such as a restriction on the maximum eigenvalue of $H$, the sum of the norms of the eigenvalues, the sum of the squares of the eigenvalues, etc. Here we choose to use the last of these constraints, namely

$$\sum_n \lambda_n(H)^2 \leqslant \mu. \tag{35}$$

To maximize the coefficient of $\Delta t$ in Eq. (34), we first note that it may be written as the operator inner product $\mathrm{Tr}[HA]$, where

$$A = i|\psi_T\rangle\langle v| - i|v\rangle\langle\psi_T|,$$
$$|v\rangle = \rho|\psi_T\rangle. \tag{36}$$

The maximum of the inner product, under the condition that the norms of the operators are constrained, occurs when the operators are aligned: $H = cA$, where $c$ is in general a complex number, but real in this case to preserve the Hermiticity of $H$. With this inner product, the norm of $H$ is

$$\mathrm{Tr}[H^2] = \sum \lambda_n(H)^2. \tag{37}$$

Naturally, we take the maximum value allowed under the constraint, setting $\mathrm{Tr}[H^2] = \mu$. This fixes the magnitude of the proportionality constant $c$, and results in the following explicit construction for the feedback Hamiltonian:

$$H(\rho(t),|\psi_T(t)\rangle) = i\chi[|\psi_T(t)\rangle\langle\psi_T(t)|,\rho(t)], \tag{38}$$

where we have included the time dependence explicitly, and where

$$\chi = \sqrt{\frac{\mu}{a(t) - b^2(t)}}, \tag{39}$$

with

$$a(t) = \langle\psi_T(t)|\rho^2(t)|\psi_T(t)\rangle, \tag{40}$$

$$b(t) = \langle\psi_T(t)|\rho(t)|\psi_T(t)\rangle. \tag{41}$$

It now remains to maximize the coefficient of $(\Delta t)^2$ in Eq. (34). Recall this is only required under the condition that the first term is zero, which implies that $\rho|\psi_T\rangle = \lambda_T|\psi_T\rangle$. In this case, the expression for the coefficient may be written as

$$\langle\psi_T|H(\rho - \lambda_T I)H|\psi_T\rangle. \tag{42}$$

Now, denoting the eigenvalues of $\rho$ by $\lambda_n$ (ordered in decreasing order), the eigenstates by $|n\rangle$, and denoting the target state as the eigenvector with $n = M$, the above expression becomes

$$\langle\psi_T|H\rho H|\psi_T\rangle = \sum_{n=1}^{N} (\lambda_n - \lambda_M)|\langle n|H|M\rangle|^2. \tag{43}$$

Now, the constraint on the feedback Hamiltonian may be written

$$\mathrm{Tr}[H^2] = \mathrm{Tr}\left(\sum_n |n\rangle\langle n|H\sum_m |m\rangle\langle m|H\right)$$
$$= \sum_{nm} |\langle n|H|m\rangle|^2 = \mu, \tag{44}$$

being a constraint on the sum of the square magnitudes of the elements of $H$. Only the subset $|\langle n|H|M\rangle|^2$ of these, with $n \neq M$, contributes to the expression to be maximized. To obtain the maximum value for the expression, we must therefore set all the elements of $H$ that do not contribute to it to zero, this allowing the contributing elements to be as large as possible. The constraint then becomes

$$\sum_{n \neq M}^{N} |\langle n|H|M\rangle|^2 = \mu/2, \tag{45}$$

where the factor of one-half is enforced by the Hermiticity of $H$. The expression can now be seen as an average of the eigenvalues of $\rho$ over the ''distribution'' $P(n) = |\langle n|H|M\rangle|^2$, which is normalized to $\mu$ by the constraint. The maximum value is therefore achieved when all the weight of the distribution is placed on the term with the largest eigenvalue. The solution is therefore

$$|\langle 1|H|M\rangle|^2 = |\langle M|H|1\rangle|^2 = \mu/2, \tag{46}$$

with all other elements zero. The explicit construction for the resulting feedback Hamiltonian is

$$H(\rho(t),|\psi_T(t)\rangle) = \sqrt{\mu/2}[|1\rangle\langle\psi_T(t)| + |\psi_T(t)\rangle\langle 1|]. \tag{47}$$

Note that this assumes that the target state is orthogonal to $|1\rangle$, being the eigenvector with the largest eigenvalue. If the $|1\rangle$ is the target state, then there exists no Hamiltonian evolution that will increase the fidelity, since the fidelity is the maximum it can be given the current purity of $\rho$. In that case, we are free to set $H = 0$ for that time step.

It is worth noting that since the magnitude of the feedback Hamiltonian and the strength of the continuous observation are uniformly bounded, the evolution of the system is continuous. Given this, since the feedback Hamiltonian is a continuous function of the system state, it is intuitively clear that the feedback algorithm is well-defined (and continuous) in the continuum limit for almost all sample paths.

We now have a feedback algorithm that can be used in conjunction with a measurement strategy for feedback control. In the next section, we will implement such a strategy for the control of a two-state system.

Having found a continuous feedback algorithm by maximizing the fidelity at each time step, it is natural to ask what one should do given a *finite* time in which to perform the feedback. That is, we can imagine a situation where one observes the system for a finite time, and then has a finite time in which to perform a unitary transformation to bring the system as close as possible to the target state.

Denoting the state of the system at the beginning of the finite feedback step as $\rho$, the fidelity with respect to the target state at the end of the feedback step is given by

$$F(\rho_f,\sigma)=\mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}], \tag{48}$$

where $\sigma=|\psi_T\rangle\langle\psi_T|$ is the target state and $U$ is the unitary transformation constituting the feedback. We wish to find $U$ to maximize $F(\rho_f,\sigma)$. Observing first that, for arbitrary $A$ and unitary $V$,

$$\begin{aligned}\max_V|\mathrm{Tr}[AV]| &= \max_V|\mathrm{Tr}[\sqrt{A^\dagger A}V'V]|\\ &= \max_V\left|\sum_j \sigma_j(A)e^{i\theta_j}\right|\\ &= \mathrm{Tr}[\sqrt{A^\dagger A}],\end{aligned} \tag{49}$$

where we have used the polar decomposition theorem for $A(A=\sqrt{A^\dagger A}V')$, and the $\sigma_j(A)$ are the eigenvalues of $\sqrt{A^\dagger A}$. Setting $A=\rho_f^{1/2}\sigma^{1/2}$, this gives

$$\begin{aligned}F(\rho_f,\sigma) &= \max_V|\mathrm{Tr}[(U\rho U^\dagger)^{1/2}\sigma^{1/2}V]|\\ &= \max_V|\mathrm{Tr}[U\rho^{1/2}U^\dagger\sigma^{1/2}V]| \le \sum_j \lambda_j(\rho)^{1/2}\lambda_j(\sigma)^{1/2},\end{aligned} \tag{50}$$

where the final inequality uses the result by von Neumann [42]. In the last line, $\lambda_j(\rho)$ and $\lambda_j(\sigma)$ are the eigenvalues of $\rho$ and $\sigma$, respectively, ordered such that the largest eigenvalue of $\rho$ multiplies the largest eigenvalue of $\sigma$, the second largest the second largest, and so on down to the smallest eigenvalue of both states. Now we need merely realize that we can achieve the upper bound by choosing $U$ so as to diagonalize $\rho$ in the basis of $\sigma$, reordering the basis states such that the largest eigenvalue of $\rho$ is attached to the eigenstate of $\sigma$ with the largest eigenvalue. Writing the eigenstates of $\sigma$ as $|\sigma_j\rangle$ (with eigenvalues ordered by size) and those of $\rho$ as $|\rho_j\rangle$ (similarly ordered), then the explicit construction for the optimal $U$ is

$$U=\sum_j |\sigma_j\rangle\langle\rho_j|. \tag{51}$$

## VII. FEEDBACK CONTROL OF A TWO-STATE SYSTEM

In the previous sections we have considered the measurement and Hamiltonian feedback parts of the control problem separately. This resulted in a straightforward choice for a Hamiltonian feedback algorithm, but did not result in a clear choice for the measurement strategy. This was because we were able to identify in the measurement process a tradeoff between information and disturbance. Because of this, the optimal measurement strategy for a given application is likely to depend upon the relative strengths available for the measurement and feedback. For example, if the feedback Hamiltonian is relatively strong, then it is likely that it will be able to effectively counter the disturbance introduced by the measurement, and therefore the measured observable should be chosen to provide maximal information and the expense of maximal disturbance. When this is not the case, the most desirable measurement is likely to be that which introduces less disturbance at the expense of providing reduced information.

To examine the performance of a feedback control algorithm, we must run the algorithm many times in order to obtain the average behavior. This is computationally very expensive, and so we use massively parallel supercomputers that are ideal for this task. The results we present here are obtained by averaging 1000 realizations of the control algorithm.

To provide a simple example of feedback control, we consider a spin-half system precessing in a magnetic field aligned along the $z$ axis [43,44]. In the absence of any noise, a spin aligned originally along the $x$ axis would rotate at a constant angular velocity around the $z$ axis, and we take this to be the desired (target) behavior. To provide the control problem, we subject the spin to noise that dephases it around the $z$ axis (this could arise from fluctuations in the magnetic field). The master equation describing the free (but noisy) evolution of the spin is thus given by

$$\dot{\rho}=-i\hbar\omega[\sigma_z,\rho]-\beta[\sigma_z,[\sigma_z,\rho]], \tag{52}$$

where $\omega$ is the precession frequency in the magnetic field and $\beta$ is the strength of the dephasing noise. To implement feedback control, we allow the observer (who is also naturally the controller) to measure the spin along an arbitrary spin direction $\mathbf{v}(t)$, with measurement constant $k$, and apply a feedback Hamiltonian, $H_{\mathrm{fb}}(t)$, obtained using the algorithm presented in the preceding section. The full evolution of the controllers' state of knowledge, including the measurement and feedback, is therefore

$$\begin{aligned}d\rho &= -i\hbar[\sigma_z+H_{\mathrm{fb}}(t),\rho]dt-\beta[\sigma_z,[\sigma_z,\rho]]dt\\ &\quad -k[\sigma_{\mathbf{v}(t)},[\sigma_{\mathbf{v}(t)},\rho]]dt\\ &\quad +\sqrt{2k}(\sigma_{\mathbf{v}(t)}\rho+\rho\sigma_{\mathbf{v}(t)}-2\,\mathrm{Tr}[\sigma_{\mathbf{v}(t)}\rho]\rho)dW.\end{aligned} \tag{53}$$

We now simulate the dynamics resulting from the feedback control loop for different values of $\theta$, being the angle between the eigenbasis of the instantaneous system density matrix and the instantaneous measured observable, as discussed in Sec. V. For these simulations, the strength of the

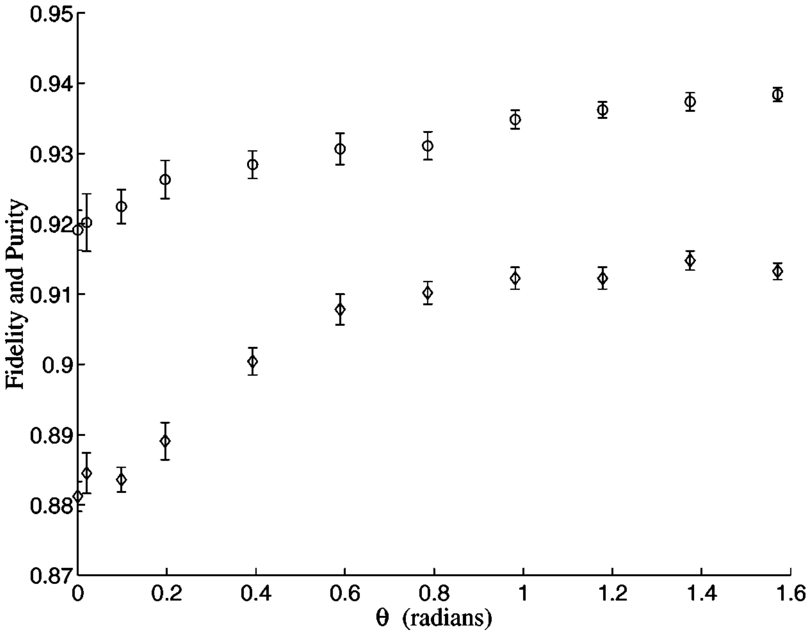


FIG. 2. The average purity (diamonds) and fidelity (circles) of a feedback control algorithm for different measurement angles $\theta$. The parameters are precession frequency $\hbar\omega=\pi$, measurement constant $k=2$, noise strength $\beta=0.4$, and feedback strength $\mu=10$.

magnetic field is such that $\hbar\omega=\pi$, so that the spin rotates once in a time interval $t=1$. The noise strength is $\beta=0.4$, and the parameters for the control loop are $k=2$ and feedback strength $\mu=10$. We start the system in a pure state with the spin pointing along the $x$ direction, and evolve the controlled dynamics for a duration of $t=2$ (the purity and fidelity settle down to their steady-state behavior by approximately $t=0.8$). Averaging the fidelity and purity over the full length of the run, for different values of $\theta$ we obtain Fig. 2. Examining the dependence of the purity on $\theta$, we find what we expect from the discussion in Sec. V. That is, the average purity of the system increases with $\theta$, achieving a maximum at $\theta=\pi/2$. This reflects the fact that, on average, measurements with a larger value of $\theta$ extract information from the system at a faster rate.

The behavior of the fidelity, in this case, is similar to that of the purity. As $\theta$ increases, the feedback is sufficient to ensure that even though we can expect the noise to increase with $\theta$, the increase in purity has more of an effect on the fidelity than the noise. The result is that, with these resources, it is best to choose $\theta=\pi/2$ (so as to measure in a basis maximally different from that which diagonalizes the density matrix). However, from our previous analysis of the tradeoff between information and disturbance, we cannot always expect this to be the case.

## VIII. CONCLUSION

In this paper, we have considered the problem of controlling a quantum system in real time using feedback conditioned on information obtained by continuous observation. The question of how to effect the best control given the system dynamics (including environmental noise) and constraints on available resources is highly nontrivial. Here we considered a simplified problem in which we examine the measurement process and the resulting Hamiltonian feedback separately. Our purpose was both to examine what concepts are motivated by the feedback control problem and to explore the question of optimization in this simplified problem. A concept that arises immediately in considering feedback control is the strength of a measurement. This strength quantifies the amount of information that a measurement provides. Previous definitions of the information provided by quantum measurements have focused on information regarding the state prior to the measurement. Here we have argued that it is the information regarding the state *resulting* from the measurement that is relevant to quantum feedback control, and we introduced a concept of measurement strength accordingly.

Since measurements disturb quantum systems, it is important to understand how this relates to feedback control. We showed how it is possible to quantify the concept of the noise introduced by measurements in a way that is relevant to feedback control. One finds that while classical measurements do not introduce noise, quantum measurements in general do, although it is possible, at least in principle, to make continuous quantum measurements that are noise-free.

Having arrived at precise concepts of information and disturbance, we examined the special case of continuous measurements performed on a two-state system, and found that maximization of information and minimization of noise were mutually exclusive goals, implying the existence of an information-disturbance tradeoff in quantum feedback control. This highlights the complexity of the control problem.

We also considered the Hamiltonian feedback part of the control process. Defining the cost function as the fidelity with a target state, and the feedback strength as the norm of the Hamiltonian, we were able to obtain the Hamiltonian generating the optimal instantaneous feedback.

Here we explicitly consider control realized by choosing dynamics conditional upon a measurement process. Because of this, one can refer to this technique as using a classical controller, since it works by taking a classical process (the measurement record) and altering the system Hamiltonian accordingly, all of which can be achieved by a classical system. It is therefore worth noting that, so long as we are considering the dynamics of the controlled system alone to be the important quantity, this is equivalent to control that is realized by connecting the system, via an interaction Hamiltonian, to another quantum system, where this second system is large enough to be treated as a bath [15]. In general, using a second quantum system in this fashion may be referred to as using a quantum controller. When the quantum controller is finite-dimensional and restricted in its dynamical response time, one can expect the performance of classical and quantum controllers to be somewhat different, and this is an interesting area for future work.

The question of how best to design feedback strategies to control noisy quantum systems is a complex one. However, the study of this problem will help us to understand better how quantum measurement may be exploited in the manipulation of quantum systems, and as quantum technology advances, we can expect that this question will become increasingly important in practical applications.

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