Irreversibility in asymptotic manipulations of a distillable entangled state

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We provide an example of a distillable bipartite mixed state such that, even in the asymptotic limit, more pure-state entanglement is required to create it than can be distilled from it. Thus, we show that the irreversibility in the processes of formation and distillation of bipartite states, recently proved in [G. Vidal and J. I. Cirac, Phys. Rev. Lett. **86**, 5803 (2001)], is not limited to bound-entangled states.

Distillation is one of the basic concepts in entanglement theory. As shown in the pioneering works on entanglement transformations [1,2], it is possible to use local operations and classical communication (LOCC) to convert, in the asymptotic limit ($N \rightarrow \infty$), N copies of some bipartite mixed state ρ into M of copies of some reference pure state, the maximally entangled state

$$|\Phi\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad (1)$$

of a two-qubit system, which is said to contain one ebit (entangled bit). Moreover, the distillable entanglement $E_D(\rho)$, defined as the maximal achievable yield M/N, was shown to be often finite. This is a remarkable result with important implications in quantum information theory. It says, for instance, that a noisy channel can be used to establish perfect quantum communication between two distant parties, if these are allowed to perform LOCC. Indeed, the imperfect channel can be used to create many copies of some mixed entangled state ρ, which can then be purified into fewer copies of $|\Phi\rangle$ and subsequently used to achieve perfect quantum communication through teleportation [3].

A notion dual to distillation is that of preparation of ρ using pure-state entanglement and LOCC [2]. Now M copies of $|\Phi\rangle$ are transformed into N copies of ρ. The entanglement cost $E_C(\rho)$ [4] (asymptotic version of the entanglement of formation $E_F(\rho)$ [2,5]) is defined as the minimal ratio M/N asymptotically achievable by LOCC. $E_C(\rho)$ quantifies the amount of pure-state entanglement required to create a copy of ρ, in the above asymptotic sense.

Notice that the processes of formation and distillation can be concatenated into a cycle. Starting from $NE_C(\rho)$ copies of $|\Phi\rangle$, two distant parties can use LOCC to prepare N copies of ρ; and the N copies of ρ can be subsequently distilled back into $NE_D(\rho)$ copies of $|\Phi\rangle$,

$$|\Phi\rangle^{\otimes NE_C(\rho)} \Rightarrow \rho^{\otimes N} \Rightarrow |\Phi\rangle^{\otimes NE_D(\rho)}. \qquad (2)$$

Already in the early contributions it was suggested that maybe sometimes this cycle cannot be closed completely, in that perhaps not all the initial pure-state entanglement used in the preparation process can be recovered through distillation. That is, maybe an irreversible loss of quantum correlations takes place during the mixing of pure-state entangle-

ment into $\rho^{\otimes N}$ and, accordingly, the distillable entanglement $E_D(\rho)$ is smaller than the entanglement cost $E_C(\rho)$.

Very recently this phenomenon has been proved to indeed occur [6]. In particular, it has been shown that some undistillable bipartite state ρ_b—i.e., with $E_D(\rho_b) = 0$ ebits—has nonvanishing entanglement cost. Notably, the irreversibility observed in the asymptotic preparation and distillation of ρ_b remains even when LOCC are supplemented with loaned pure-state entanglement, to be returned after the manipulation, in the so-called catalytic LOCC setting [7].

The results in Ref. [6] still leave, however, an important question open. One could associate the irreversibility demonstrated there to the fact that the state ρ_b is bound entangled, that is, to the remarkable property that no pure-state entanglement at all can be distilled from it [8]. It could well be the case that the gap observed between E_C and E_D is just a characteristic feature of some bound entangled states, whereas $E_C = E_D$ always holds for distillable states. After all, this is the case for bipartite pure states [9] and some simple cases of mixed state [10], which exhaust all the cases where E_C and E_D have been computed.

In this paper we will present an example of a bipartite mixed state σ that can be distilled, that is, $E_D(\sigma) > 0$, and such that $E_C(\sigma) > E_D(\sigma)$. We extend, thereby, the irreversibility result of Ref. [6] to the case of distillable states. In particular, the extension also holds for catalytic LOCC transformations [7].

A widely recognized, major problem concerning the study of mixed-state entanglement is that it is very difficult to compute the asymptotic measures E_C and E_D. Here, however, we are not interested in the actual values of $E_C(\sigma)$ and $E_D(\sigma)$. For the present purposes it is sufficient to show that σ can be distilled, and to bound $E_D(\sigma)$ and $E_C(\sigma)$ tight enough from above and from below, respectively, so that the bounds already imply a gap between the two quantities. We start by collecting an amalgam of useful facts.

(i) A sufficient condition for a mixed state ρ to be distillable is that a projector P into a $\mathcal{C}^2 \otimes \mathcal{C}^2$ subspace (that is, a subspace, which is the tensor product of two-dimensional subspaces for each of the two separated parts of the composite system) exists such that the projection $P\rho P^\dagger$ is still entangled [8], that is, such that the partial transposition of $P\rho P^\dagger$ has a negative eigenvalue.

(ii) The logarithmic negativity $E_N(\rho) \equiv \log_2[1 + 2\mathcal{N}(\rho)]$ [11], where $\mathcal{N}(\rho)$ is the absolute value of the sum of nega-

tive eigenvalues of partial transposition of ρ, is an upper bound to the distillable entanglement,

$$E_D(\rho) \leqslant E_N(\rho). \qquad (3)$$

In addition, E_N is an additive function,

$$E_N(\rho_1 \otimes \rho_2) = E_N(\rho_1) + E_N(\rho_2), \qquad (4)$$

which only vanishes for states with positive partial transposition (PPT) states,

$$E_N(\rho_{PPT}) = 0. \qquad (5)$$

Finally, and very important to us, $E_N(\rho)$ is a continuous function of ρ.

(iii) The entanglement of formation E_F [2] of ρ is bounded below by [6]

$$E_F(\rho) \geqslant -\log_2 \alpha, \qquad (6)$$

where α is the maximal overlap of a product state $|ab\rangle$ with the projector Π onto the support of ρ,

$$\alpha \equiv \max_{|ab\rangle} \langle ab|\Pi|ab\rangle. \qquad (7)$$

Accordingly, the entanglement cost $E_C(\rho)$ is bounded below by [6]

$$E_C(\rho) \geqslant -\log_2 \beta, \qquad (8)$$

if for all N the maximal overlap of a normalized product vector $|a_N b_N\rangle$ with the N-fold tensor product of Π is at most β^N,

$$\max_{|a_N b_N\rangle} \langle a_N b_N|\Pi^{\otimes N}|a_N b_N\rangle \leqslant \beta^N. \qquad (9)$$

(iv) The four-dimensional subspace $V \subset \mathcal{C}^3 \otimes \mathcal{C}^3$ orthogonal to the five product vectors

$$|0\rangle \otimes (|0\rangle + |1\rangle),$$

$$(|0\rangle + |1\rangle) \otimes |2\rangle,$$

$$|2\rangle \otimes (|1\rangle + |2\rangle),$$

$$(|1\rangle + |2\rangle) \otimes |0\rangle,$$

$$(|0\rangle - |1\rangle + |2\rangle) \otimes (|0\rangle - |1\rangle + |2\rangle) \qquad (10)$$

does not contain product vectors [12]. The projector Π_b onto V satisfies the following: (a) it has a PPT [12]; and (b) it fulfills Eq. (9) with $\beta < 0.99$ [6].

We introduce now a one-parameter family of states

$$\sigma(p) \equiv (1-p)\rho_b + p|\psi\rangle\langle\psi|, \qquad (11)$$

where $\rho_b \equiv \Pi_b/4$ is the PPT bound entangled state introduced in Ref. [12] in the context of the so-called nonextendible product basis, and used in Ref. [6] to prove irreversibility of asymptotic manipulations, and

$$|\psi\rangle \equiv \frac{1}{\sqrt{6}}(|00\rangle - |01\rangle - 2|11\rangle) \qquad (12)$$

is an entangled pure state that is orthogonal to all product states of Eq. (10), that is, $|\psi\rangle \in V$. For $p=0$, we recover ρ_b, for which we know that $E_C(\rho_b) > -\log_2 0.99 > D(\rho_b) = 0$. In what follows we will use facts (i)–(iv) and perturbation theory to show that for $p>0$ we encounter states $\sigma(p)$ that can be distilled, and with $E_C(\sigma_p) > E_D(\sigma_p)$.

The family of states $\sigma(p)$ in Eq. (11) has been carefully chosen to fulfill two important properties. First, $\sigma(p)$ is supported on V, since V is the support of ρ_b and also $|\psi\rangle$ is supported in V. Using (iii) and (iv b) this means that for any $p \in [0,1]$ we have a constant lower bound for E_C.

Property 1. The entanglement cost of $\sigma(p)$, $p \in [0,1]$, is bounded below by

$$E_C[\sigma(p)] > -\log_2 0.99 = 0.015 \text{ ebits.} \qquad (13)$$

Let $\rho_b^{T_A}$ denote the partial transposition of ρ_b, and P the rank four, product projector $(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$. Notice that, by construction, $\rho_b^{T_A} = \rho_b \geqslant 0$, $(P\rho_b P^\dagger)^{T_A} = P\rho_b P^\dagger$, and $P|\psi\rangle = |\psi\rangle$. The second important feature of $\sigma(p)$ is that, for any $p>0$ the partial transposition of the projection $P\sigma(p)P^\dagger$,

$$[P\sigma(p)P^\dagger]^{T_A} = (1-p)P\rho_b P^\dagger + p|\psi\rangle\langle\psi|^{T_A}, \qquad (14)$$

has a negative eigenvalue n. Therefore, because of fact (i), the corresponding state $\sigma(p)$ can be distilled.

Property 2. For $p \in (0,1]$, the state $\sigma(\rho)$ can be distilled, that is,

$$E_D[\sigma(p)] > 0 \text{ ebits.} \qquad (15)$$

For instance, for $p=0.015$, $|n|=2.7\times 10^{-4}$ (see also Fig. 1).

Property 2 has been achieved by selecting a projector P such that $(P\rho_b P^\dagger)^{T_A}$ has only rank three, and thus one vanishing eigenvalue, whereas $|\psi\rangle \in V$ has been chosen so that $(P|\psi\rangle\langle\psi|P^\dagger)^{T_A} = |\psi\rangle\langle\psi|^{T_A}$, that is, so that the negative eigenvalue of $|\psi\rangle\langle\psi|^{T_A}$ entirely contributes to Eq. (14). We can use perturbation theory to check what the effect of such choices is.

Let M and N be finite-dimensional Hermitian operators, $\Sigma_{i=0}^{l} m_i |m_i\rangle\langle m_i|$ the spectral decomposition of M, with m_i its increasingly ordered eigenvalues and $m_0 \neq m_1$, and let ϵ be a small parameter. Then the lowest eigenvalue of $M + \epsilon N$ is, as given by perturbation theory [13],

$$m_0 + \epsilon\langle m_0|N|m_0\rangle + \epsilon^2 \sum_{i=1}^{l} \frac{|\langle m_0|M|m_i\rangle|^2}{m_0 - m_i} + O(\epsilon^3). \qquad (16)$$

Making the proper identifications, we realize that the negative eigenvalue $n(p)$ of the operator in Eq. (14) is
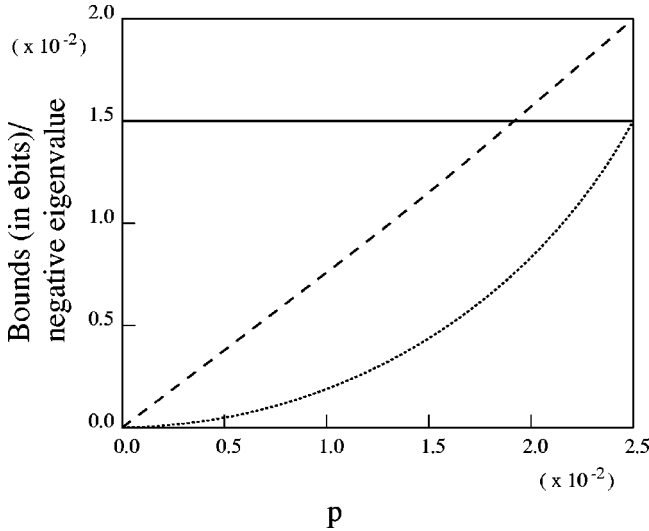
$$n = -|k|p^2 + O(p^3), \qquad (17)$$

FIG. 1. Finite gap between entanglement cost E_C and distillable entanglement E_D for distillable states. We obtain, as a function of p, an upper bound $E_N[\sigma(p)]$ (diagonal, dashed line) for $E_D[\sigma(p)]$, which in some regime is smaller than the lower bound (horizontal line) for the entanglement cost $E_C[\sigma(p)]$. Both bounds are expressed in ebits. The lower, dotted curve corresponds to $20|n|$, where $|n|$ is the modulus of the negative eigenvalue of the operator $[P\sigma(p)P^\dagger]^{T_A}$, and indicates that the distillable entanglement $E_D[\sigma(p)]$ is finite.

where $|k|>0$ is of the order of 1 and the zero- and first-order contributions vanish due, respectively, to the fact that the smallest eigenvalue of Eq. (14) vanishes for $p=0$, and to the fact that the corresponding eigenvector, $|\tau\rangle \equiv |0\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2}$, fulfills

$$\langle\tau|(|\psi\rangle\langle\psi|^{T_A})|\tau\rangle = \langle\tau|\psi\rangle\langle\psi|\tau\rangle = 0. \quad (18)$$

Finally, for $p \lesssim 1$ such that contributions $O(p^3)$ may become important, numerical calculations show that $|n|$ grows monotonically with p [see Fig. (1)].

Summarizing, so far we have learned that $\sigma(p)$ can be distilled for any $p>0$, while the entanglement cost is bounded below by Eq. (13). In order to complete the result we need to prove that the distillable entanglement of $\sigma(p)$ is, in some regime of $p \in (0,1]$, smaller than the lower bound (13). This would already follow from the above if $E_D[\sigma(p)]$ were a continuous function of p. For $p=0$ we have the bound entangled state ρ_b, that is, $E_D[\sigma(0)]=0$ ebits, whereas at the other extreme, $p=1$, we have the pure-entangled state $|\psi\rangle$, whose distillable entanglement E_D (and entanglement cost E_C) can be easily computed and reads $E_D[\sigma(1)]=0.55$ ebits. But, unfortunately, we cannot base our argument in the continuity of $E_D[\sigma(p)]$ as a function of p, to conclude that an intermediate p must exist such that the distillable entanglement is nonzero and still below the bound (13). Whereas it may well be that $E_D(\rho)$ is a continuous function of ρ, this has not been proved. Notice that a plausible objection to continuity relies on the fact that $E_D(\rho)$ is actually a function of $\rho^{\otimes N}$ in the large N limit. Therefore, a

small perturbation of ρ, which produces a large deviation in $\rho^{\otimes N}$, may imply a discontinuous change in $E_D(\rho)$.

Nevertheless, following fact (ii), the logarithmic negativity $E_N[\sigma(p)]$ is a continuous upper bound for $E_D[\sigma(p)]$ [see Fig. (1)]. A direct calculation of $E_N[\sigma(p)]$ finally proves the irreversibility of the preparation-distillation cycle for distillable bipartite states. In particular, for $p=0.0015$ we have

$$E_D[\sigma(0.0015)] < E_N[\sigma(0.0015)] = 0.012 \text{ ebits.} \quad (19)$$

Thus, $\sigma(0.0015)$ is an example of distillable state with a finite gap $E_C-E_D>0.003$ ebits.

We can now further use the properties of the logarithmic negativity E_N to show that such a gap remains even when pure-state entanglement is loaned [7] to assist in the transformations, as it was done with ρ_b in Ref. [6]. This is achieved by considering a distillation process starting from N copies of $\sigma(p)$ together with L copies of $|\Phi\rangle$,

$$\sigma^{\otimes N} \otimes |\Phi\rangle\langle\Phi|^{\otimes L} \Rightarrow |\Phi\rangle^{\otimes L + NE_D^c(\sigma)}, \quad (20)$$

where $E_D^c(\sigma)$ denotes the distillable entanglement of σ in the catalytic setting. For any N and L, we have

$$E_N(\sigma^{\otimes N} \otimes |\Phi\rangle\langle\Phi|^{\otimes L}) = E_N(\sigma^{\otimes N}) + E_N(\Phi^{\otimes L}) \quad (21)$$

$$= NE_N(\sigma) + L, \quad (22)$$

where we have used additivity of E_N and the fact that $E_N(\Phi)=1$. This means that even in the large N limit, and once the L loaned states $|\Phi\rangle$ have been discounted from the distillation outcome, at most $NE_N(\sigma)$ ebits of entanglement has been distilled, so that even in the catalytic scenario the bound $E_D^c(\sigma) < E_N(\sigma)$ holds. Therefore, the irreversibility result of the paper also applies to this case.

We have shown that the irreversibility in the asymptotic manipulation of bipartite mixed states is not a phenomena restricted to bound entangled states, by providing a specific example of distillable state with a finite gap between its entanglement cost E_C and its distillable entanglement E_D. Notice that these results legitimize the use of different measures of entanglement, such as E_C and E_D, to quantify, in the asymptotic limit, the resources of entangled mixed states. The search for an intrinsic irreversibility in the asymptotic manipulation of bipartite systems has motivated, through several contributions—see, for instance, [6,11,14–17], the development of many techniques for the study of entanglement and has certainly implied an important gain in insight. Paradoxically, a remaining open question is now whether a nontrivial example [18] of a bipartite mixed state exists for which the processes of preparation and distillation can be performed in a fully reversible fashion.

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$$\rho = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|, \tag{23}$$

where each $|\Phi_i\rangle \equiv (|\phi_i\psi_i\rangle + |\phi_i^\perp\psi_i^\perp\rangle)/\sqrt{2}$ satisfies that $\mathrm{tr}(\rho_{A_i}\rho_{A_j})=0$ when $i \neq j$, for $\rho_{A_i} \equiv \mathrm{tr}_B|\Phi_i\rangle\langle\Phi_i|$. Then from one copy of ρ, we can distill one ebit of entanglement by using a local measurement in part A of the system, given by projectors E_i onto the range of ρ_{A_i}, since $E_i \otimes I_B \rho E_i^\dagger \otimes I_B = p_i|\Phi_i\rangle\langle\Phi_i|$. At the same time, one ebit of entanglement is manifestly sufficient to create a copy of ρ. Thus, asymptotically we have $E_C(\rho) \leqslant 1$ ebit and $E_D(\rho) \geqslant 1$ ebit, but since entanglement cannot be created using LOCC, we have $E_C \geqslant E_D$, which implies $E_D(\rho) = E_C(\rho) = 1$ ebit.

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