Langevin equations for interacting fermions and Cooper-like pairing in trapped one-dimensional fermions

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Momentum correlations in a one-dimensional equilibrium ensemble of trapped fermions, with a point inter-action between particles of opposite spin have been studied. In the degenerate regime correlations were observed between fermions with opposite spins and momenta, similar to Cooper pairing. These correlations appear as soon as the temperature is below the Fermi energy, which is a much less stringent condition than that of the BCS transition proper. Calculations are carried out in both perturbative and non-perturbative regimes. To achieve the latter, it is shown that interacting fermionic dynamics may be solved as a stochastic linear trans-formation of Grassmann algebra generators, much in the way random *c*-number paths are introduced in the conventional quantum stochastics of bosons. Importantly, the method thus emerging is inherently free of the sign problem.

I. INTRODUCTION

A recent surge of interest in trapped ultracold atomic gases has mainly been concerned with the phenomenon of Bose-Einstein condensation [1] of bosonic atoms. More recently, interest has also arisen in properties of trapped fermionic gases [2–11]. Successful cooling of ^{40}K atoms [3] and trapping of ^{6}Li [4] and ^{40}K [5] atoms have been reported. Theoretical interest has mainly been concentrated in three areas: mean-field properties [8,9], excitations [10], and Cooper pairing and its possible manifestations [6,7,11] in trapped Fermi gases.

There are a number of generic concepts inherent to a discussion of degenerate Fermi-gases. First and foremost are the concept of the Dirac sea and the associated concepts of the Fermi surface, Fermi energy, and Fermi momentum. For interacting fermions, there is also the concept of Cooper pairing. It is not at all clear how these concepts will manifest themselves in small trapped fermionic samples. In this paper, we seek an *ab initio* answer to this question, by directly calculating two-particle momentum correlations in a one-dimensional (1D) sample of trapped interacting fermions. We consider an attractive point interaction between "spin-up" and "spin-down" fermions, well known in the theory of su-perconductivity. This kind of interaction may be realized in a sample of ^{6}Li atoms trapped in two different hyperfine states [6,7] (so that the spin up/down terminology should not be taken literally). We find that two-particle momentum corre-lations are peaked at momenta equalling $\pm p_{\mathrm{F}}$, where p_{F} corresponds to a classical excursion in momentum space of a particle with energy equal to the Fermi energy [8].

Correlations of particles with equal momenta may be ex-plained in a semiclassical way. Assume that only those atoms with energies close to the the Fermi energy contribute, so that all atoms oscillate with the same amplitude. The attrac-tive point interaction will then favor pairs of spin-up/spin-down atoms, oscillating in phase (co-oscillating pairs). These

pairs will contribute to the correlations of particles with equal momenta. However, both perturbative calculations and nonperturbative Monte Carlo simulations reveal correlations of particles with opposite momenta: in fact, these always exceed correlations of particles with equal momenta. The difference grows with interaction strength; moreover, the correlations actually extend from the Fermi surface into the Dirac sea. There does not appear to exist any semiclassical mechanism capable of producing correlations between par-ticles with opposite momenta. Observation of Cooper-like momentum correlations would, therefore, be a direct mani-festation of both the quantum and the fermionic nature of the trapped atoms. Experimentally this may be achieved by first trapping ^{6}Li atoms in a cigar-shaped trap and then turning the trap off, thus allowing the atoms to propagate freely. The Cooper-like momentum correlations should then manifest themselves as correlations in arrival times, assuming that atom detectors are placed at opposite sides of the trap along the trap axis. It is an important observation that these corre-lations appear as soon as the temperature is below the Fermi energy, which is a much less stringent condition than that of the BCS transition proper [2].

It should, however, be noted that there are serious physi-cal limitations to a 1D approximation for trapped fermions. Assuming a cigar trap, transverse motion can only be ne-glected if the transverse frequencies are large compared to both the temperature and to the chemical potential. This lim-its the number of fermions in the trap to roughly twice the ratio transverse to longitudinal frequency. Experimental ob-servations of the 1D pseudo-Cooper-pairing thus requires a very tight ("needle") trap. In 3D, this phenomenon is subject to further investigation.

The paper is organized as follows. We start from a grand-canonical formulation of trapped interacting fermions in Sec. II, then proceed with introducing first perturbative (Sec. III), and then nonperturbative (Sec. IV) approaches to this sys-tem. In Sec. V, we present numerical results; feasibility of observation of the pseudo-Cooper correlations and limita-

tions to a 1D approach to trapped fermions are also discussed.

Technically, our perturbative approach is the first nonvanishing approximation of the well-known Matsubara diagram techniques [12] (although we do not use diagrams as such). We derive perturbative corrections to a number of quantities such as statistical sum and momentum distribution, and an expression for momentum correlations between spin-up and spin-down particles. Based on a WKB approximation [13] to eigenfunctions of highly excited trap states, for the latter we also derive a simplified formula valid in the case of a highly populated trap.

The nonperturbative approach developed in this paper is an extension of the phase-space methods used for bosons [14], enabling us to treat interacting fermionic systems. Following the techniques of coherent states for Grassmann algebras [15,16], one may introduce fermionic pseudodistributions [16], in much the same way as is done for bosons [14]. Equations for the pseudodistributions can also easily be derived. The catch is that, being an element of a Grassmann algebra, a pseudodistribution is in fact a collection of 2^{2n} c numbers, where n is the number of fermionic states taken into account. Although the number of nonzero components is actually smaller, it remains of the order of the dimension of the Hilbert space (2^n) making equations for pseudodistributions impracticable for numerical methods.

For an equilibrium ensemble of trapped fermions such as we consider in this paper, the dynamics is a Matsubara-style [12] dynamics in imaginary time (extension of our results to real-time dynamics is straightforward). We show that, for a particular class of pseudodistributions, which we term B distributions, their evolution can be solved as a stochastic transformation of the set of the Grassmann algebra generators. Formally, it is determined by an $n \times n$ random matrix, obeying a linear Itô stochastic differential equation (SDE) [17] with multiplicative noise. The transformation matrix (random path) is a c number, and may hence be the subject of a Monte Carlo simulation. The dimension of the fermionic path is n^2 (and $2n^2$ for a real-time problem), which, except for very small n, is exponentially small compared to that of the Hilbert space or of the pseudodistribution itself. It is, however, large compared to $2n$ for the bosonic paths [14], so that stochastic simulations of fermions are more involved than those of bosons. On the other hand, the SDE we find for the paths is linear, and we found it more stable in simulations than the nonlinear SDEs typical for interacting bosons that are notorious for stability problems [14,18].

II. THE PHYSICAL PROBLEM

We consider a grand-canonical ensemble of trapped interacting fermions. The normalized many-body ρ matrix is given by

$$\hat{\rho}(\tau) = Z^{-1} \hat{R}(\tau), \quad (1)$$

$$\hat{R}(\tau) = e^{-\tau(\hat{H} - \mu \hat{N})}, \quad Z = \operatorname{Tr} \hat{R}(\tau), \quad (2)$$

where $\tau = 1/k_B T$, T is the temperature, μ is the chemical potential, $\hat{N}$ is the fermion number operator, and $\hat{H} = \hat{H}_1 + \hat{H}_I$ is the system Hamiltonian. The linear part, H_1, which physically contains the kinetic energy and the trap potential, will be written in a general matrix form as

$$\hat{H}_1 - \mu \hat{N} = \int dx dx' h(x,x') [\hat{a}_\uparrow^\dagger(x) \hat{a}_\uparrow(x') + \hat{a}_\downarrow^\dagger(x) \hat{a}_\downarrow(x')]$$

$$= \hat{\boldsymbol{a}}_\uparrow^\dagger h \hat{\boldsymbol{a}}_\uparrow + \hat{\boldsymbol{a}}_\downarrow^\dagger h \hat{\boldsymbol{a}}_\downarrow, \quad (3)$$

where the field operators $\hat{a}_\uparrow(x)$ and $\hat{a}_\downarrow(x)$ correspond to the spin-up and spin-down states, respectively (i.e., the trap potential does not depend on spin). As an example of a nonlinear interaction we consider a point interaction of fermions with opposite spins (well known in the theory of superconductivity [20])

$$\hat{H}_I = \kappa \int dx \hat{a}_\uparrow^\dagger(x) \hat{a}_\downarrow^\dagger(x) \hat{a}_\downarrow(x) \hat{a}_\uparrow(x)$$

$$= \kappa \int dx \hat{a}_\uparrow^\dagger(x) \hat{a}_\uparrow(x) \hat{a}_\downarrow^\dagger(x) \hat{a}_\downarrow(x). \quad (4)$$

Physically, such a Hamiltonian applies to, e.g., ^{6}Li atoms trapped in two different hyperfine states [6,7].

Although many of our formal results hold for arbitrary trap potentials, all physical calculations will be done for fermions in a 1D harmonic trap. In this case, it is convenient to introduce operators creating and annihilating spin-up (say) fermions in a kth oscillator state, $\hat{a}_{k\uparrow}^\dagger$ and $\hat{a}_{k\uparrow}$, so that

$$\hat{a}_\uparrow^\dagger(x) = \sum_k \hat{a}_{k\uparrow}^\dagger \varphi_k^*(x), \quad \hat{a}_\uparrow(x) = \sum_k \hat{a}_{k\uparrow} \varphi_k(x), \quad (5)$$

where $\varphi_k(x)$ are coordinate-space oscillator eigenfunctions, $\int dx \varphi_k(x) \varphi_{k'}(x) = \delta_{kk'}$. For the field operators in momentum representation

$$\hat{a}_\uparrow^\dagger(p) = \sum_k \hat{a}_{k\uparrow}^\dagger \varphi_{kp}^*, \quad \hat{a}_\uparrow(p) = \sum_k \hat{a}_{k\uparrow} \varphi_{kp}, \quad (6)$$

where $\varphi_{kp} = i^k \sqrt{2\pi l_0/p_0} \, \varphi_k(pl_0/p_0)$ are momentum-space oscillator eigenfunctions, $\int (dp/2\pi) \varphi_{kp} \varphi_{k'p}^* = \delta_{kk'}$, $l_0 = \sqrt{\hbar/m\omega}$, and $p_0 = \sqrt{\hbar m \omega}$. Identical definitions hold for the spin-down particles.

III. IMAGINARY-TIME PERTURBATION TECHNIQUES

A. Matsubara perturbation approach

Throughout this paper, we make use of the fact that the nonnormalized many-body ρ matrix $\hat{R}(\tau)$ may be found as a solution to the equation,

$$\frac{\partial \hat{R}(\tau)}{\partial \tau} = -(\hat{H} - \mu \hat{N}) \hat{R}(\tau), \quad \hat{R}(0) = \mathbb{1}. \quad (7)$$

As first shown by Matsubara [12], Eq. (7) can be used as a starting point for perturbative calculations. Technically it is a

Schrödinger equation for an evolution matrix in imaginary time $t = -i\tau$. Thus all one needs to do is to take the conventional nonstationary perturbation approach, based on the Schrödinger equation for a real time evolution operator, and rewrite it formally for $t = -i\tau$. In particular, one introduces the Matsubara interaction picture by splitting $\hat{R}(\tau)$ into a product

$$\hat{R}(\tau) = \hat{R}_1(\tau)\hat{R}_I(\tau), \tag{8}$$

where $\hat{R}_1(\tau) = e^{-(\hat{H}_1 - \mu\hat{N})\tau}$. Operators in the Matsubara interaction picture are defined as (with $\hat{A}$ being an operator in the Schrödinger picture)

$$\hat{A}(\tau) = \hat{R}_1^{-1}(\tau)\hat{A}\hat{R}_1(\tau), \quad \hat{A}^{\dagger}(\tau) = \hat{R}_1^{-1}(\tau)\hat{A}^{\dagger}\hat{R}_1(\tau). \tag{9}$$

For example,

$$\hat{a}_{\uparrow k}(\tau) = e^{-\varepsilon_k\tau}a_{\uparrow k}, \quad \hat{a}_{\uparrow k}^{\dagger}(\tau) = e^{\varepsilon_k\tau}a_{\uparrow k}^{\dagger}, \tag{10}$$

where $\varepsilon_k = E_k - \omega/2 - \mu$ and $E_k = (k + 1/2)\omega$ (i.e., the zero energy is eliminated from ε_k). Note that $\hat{A}(\tau)$ and $\hat{A}^{\dagger}(\tau)$ defined via Eq. (9) are no longer Hermitian conjugate (except at $\tau = 0$). Note also that a τ-dependent operator always means an operator in the interaction picture, with the exception of $\hat{R}(\tau)$, $\hat{R}_1(\tau)$, and $\hat{R}_I(\tau)$.

For an arbitrary operator $\hat{A}$,

$$\mathrm{Tr}\,\hat{R}(\tau)\hat{A} = \mathrm{Tr}\,\hat{R}_1(\tau)\hat{R}_I(\tau)\hat{A} = Z_1\langle\hat{R}_I(\tau)\hat{A}\rangle_1, \tag{11}$$

where $Z_1 = \mathrm{Tr}\,\hat{R}_1(\tau) = \mathrm{Tr}\,e^{-(\hat{H}_1 - \mu\hat{N})\tau}$, and

$$\langle[\cdots]\rangle_1 = Z_1^{-1}\mathrm{Tr}\,\hat{R}_1(\tau)[\cdots], \tag{12}$$

is the quantum averaging in the interaction picture (that is, for the corresponding linear system). With $\hat{A} = \mathbb{1}$ we have, $Z = Z_1\langle\hat{R}_I(\tau)\rangle_1$, and hence for the quantum averaging in the interacting system,

$$\langle[\cdots]\rangle = Z^{-1}\mathrm{Tr}\,\hat{R}(\tau)[\cdots] = \frac{\langle\hat{R}_I(\tau)[\cdots]\rangle_1}{\langle\hat{R}_I(\tau)\rangle_1}. \tag{13}$$

Equations (7) and (8) yield for $\hat{R}_I(\tau)$,

$$\frac{\partial\hat{R}_I(\tau)}{\partial\tau} = -\hat{H}_I(\tau)R_I(\tau), \quad \hat{R}_I(0) = \mathbb{1}, \tag{14}$$

where $\hat{H}_I(\tau) = R_1^{-1}(\tau)\hat{H}_I R_1(\tau)$. Equations (13) and (14) form the basis of the Matsubara diagram expansion [12]. For our purposes, however, we restrict ourselves to the first nonvanishing approximation

$$\hat{R}_I(\tau) \approx \mathbb{1} - \int_0^{\tau}d\tau'\,\hat{H}_I(\tau'). \tag{15}$$

Then,

$$\langle[\cdots]\rangle = \langle[\cdots]\rangle_1(1 - \delta Z) - \int_0^{\tau}d\tau'\langle\hat{H}_I(\tau')[\cdots]\rangle_1, \tag{16}$$

where $\delta Z = -\int_0^{\tau}d\tau'\langle\hat{H}_I(\tau')\rangle_1$ is the relative correction to the statistical sum.

B. Corrections to distribution and correlation functions

For trapped fermions, calculating even the first-order corrections is somewhat of a challenge and can only be completed numerically. Our immediate goal is to derive relations to be used in the numerics. Without the nonlinear interaction the spin-up and spin-down particles are independent, so that we have

$$\delta Z = -\kappa\int_0^{\tau}d\tau'\int dx\langle\hat{n}_{\uparrow}(x,\tau)\rangle_1\langle\hat{n}_{\downarrow}(x,\tau)\rangle_1. \tag{17}$$

Employing Eqs. (5) and (10) we find

$$\langle\hat{n}_{\uparrow}(x,\tau)\rangle_1 = \langle\hat{n}_{\downarrow}(x,\tau)\rangle_1 = \sum_k|\varphi_k(x)|^2 n_k \equiv n(x), \tag{18}$$

where $n_k = (e^{\varepsilon_k\tau} + 1)^{-1}$ is the Fermi-Dirac distribution. Finally,

$$\delta Z = -\kappa\tau\int dx[n(x)]^2. \tag{19}$$

Similarly, the correction to the Fermi-Dirac distribution is found to be

$$\delta n_k = \langle\hat{a}_{\uparrow k}^{\dagger}\hat{a}_{\uparrow k}\rangle - \langle\hat{a}_{\uparrow k}^{\dagger}\hat{a}_{\uparrow k}\rangle_1$$

$$= n_k\left\{-\delta Z - \kappa\tau\int dx\,n(x)[n(x) + (1 - n_k)|\varphi_k(x)|^2]\right\}$$

$$= -\kappa\tau n_k(1 - n_k)\int dx\,n(x)|\varphi_k(x)|^2. \tag{20}$$

The correction to the momentum distribution is more complicated,

$$\delta n(p) = \langle\hat{a}_{\uparrow}^{\dagger}(p)\hat{a}_{\uparrow}(p)\rangle - \langle\hat{a}_{\uparrow}^{\dagger}(p)\hat{a}_{\uparrow}(p)\rangle_1$$

$$= \kappa\int_0^{\tau}d\tau'\int dx\,n(x)K_1(x,p,\tau')K_2(x,p,\tau'), \tag{21}$$

where

$$K_1(x,p,\tau') = \sum_k n_k e^{\varepsilon_k\tau'}\varphi_k^*(x)\varphi_{kp}, \tag{22}$$

$$K_2(x,p,\tau') = \sum_k(1 - n_k)e^{-\varepsilon_k\tau'}\varphi_k(x)\varphi_{kp}^*$$

$$= K_1^*(x,p,\tau')|_{\varepsilon_k \to -\varepsilon_k}. \tag{23}$$

Note that, in the above, τ' is a variable, unlike τ, which is a physical parameter. K_1 and K_2 depend on τ' explicitly and on τ via n_k. Finally, the spin-up/spin-down momentum correlation function reads

$$\langle \hat{n}_\uparrow(p), \hat{n}_\downarrow(p') \rangle = \langle \hat{n}_\uparrow(p) \hat{n}_\downarrow(p') \rangle - \langle \hat{n}_\uparrow(p) \rangle \langle \hat{n}_\downarrow(p') \rangle$$

$$= -\kappa \int_0^\tau d\tau' \int dx \, K_1(x,p,\tau') K_2(x,p,\tau')$$

$$\times K_1(x,p',\tau') K_2(x,p',\tau'). \qquad (24)$$

C. Highly populated trap

In this section we use oscillator units, setting the trap frequency equal to one. To return to normal units one should simply make the replacements $p \to p/p_0$, $x \to x/l_0$, $T \to k_B T/\hbar\omega$, and $\mu \to \mu/\hbar\omega$, where l_0 and p_0 were defined shortly after Eq. (6).

Equations (19)–(24) are of little use in the experimentally important case of a highly populated trap, $\mu \gg 1$. (In our units, the number of atoms in the trap $\approx 2\mu$.) Our goal is, therefore, to derive approximate formulas valid in this limit. Assuming also that the trap is "hot," $\mu \gtrsim T \gg 1$, we can use the WKB approximation for the eigenfunctions [13] and replace summation over the levels by integration. The WKB approximation for the oscillator functions reads

$$\varphi_k(x) = \sqrt{\frac{2}{\pi p_E(x)}} \cos\left[\frac{\pi k}{2} + \Phi_E(x)\right], \qquad (25)$$

$$\varphi_{kp} = i^k \sqrt{\frac{4}{x_E(p)}} \cos\left[\frac{\pi k}{2} + \Phi_E(p)\right], \qquad (26)$$

where $E = E_k = k + 1/2$ and $p_E(\xi) = x_E(\xi) = \sqrt{2E - \xi^2}$. The phase can be expressed in two equivalent forms

$$\Phi_E(x) = \int_0^x d\xi \, p_E(\xi) = \phi_E(x) + \frac{x p_E(x)}{2}, \qquad (27)$$

$$\Phi_E(p) = \int_0^p d\xi \, x_E(\xi) = \phi_E(p) + \frac{p x_E(p)}{2}, \qquad (28)$$

where $\phi_E(\xi) = \arcsin(\xi/\sqrt{2E})$.

It is instructive to consider what would happen to Eq. (24) if it were integrated over p and p'. This equation contains summation over four independent level indices, cf. Eqs. (22) and (23). Let k_1, k_2, k_1', and k_2' be the level indices occurring in $K_1(x,p,\tau'), K_2(x,p,\tau'), K_1(x,p',\tau')$, and $K_2(x,p',\tau')$, respectively. On integrating Eq. (24) over p and using the orthogonality of the momentum eigenfunctions we find that $k_1 = k_2$. As well as collapsing one summation and thus simplifying the whole expression, this also results in cancellation of the τ'-dependent exponents in $K_1(x,p,\tau')$ and $K_2(x,p,\tau')$, $e^{\varepsilon_{k_1}\tau'} e^{-\varepsilon_{k_2}\tau'} \to 1$. The same applies to $K_1(x,p',\tau')$ and $K_2(x,p',\tau')$: on integrating over p' the

level summations inside these are collapsed to a single one; we again find a cancellation of the τ'-dependent exponents, $e^{\varepsilon_{k_1'}\tau'} e^{-\varepsilon_{k_2'}\tau'} \to 1$.

It is easy to see that the same effect of cancellation of the τ'-dependent exponents may be achieved by smearing Eq. (24) over a relatively narrow momentum interval Δp. On inspecting Eq. (26) we find that such smearing results in $|k_1 - k_2| \lesssim \Delta k$, where (with $2k = k_1 + k_2$ and $E = E_k$)

$$\Delta k \Delta p \frac{\partial^2}{\partial k \partial p} \int_0^p d\xi \, x_{k+1/2}(\xi) = \frac{\Delta k \Delta p}{x_E(p)} \sim 1. \qquad (29)$$

The momentum resolution is specified by $\Delta p^2/2 = p\Delta p \sim T$, hence, for the important case of Fermi momentum, $p \sim p_F = \sqrt{2\mu}$, $x_\mu(p) \sim \sqrt{T}$, and $\Delta k \sim \sqrt{\mu/T}$. Effective cancellation of the τ'-dependent exponents thus requires $\tau\Delta k \sim \sqrt{\mu/T^3} \ll 1$.

In what follows we will assume that

$$\mu \gtrsim T \gg \mu^{1/3}, 1. \qquad (30)$$

Under these conditions, the averaging over the momenta may be introduced without actually losing resolution. Implying this averaging, the integrand in Eq. (24) becomes independent of τ'. Rather than discarding the τ'-dependent exponents, we find it convenient to formally preserve them while setting $\tau' = \tau/2$. This makes the weight factors occurring in K_1 and K_2 universal

$$e^{\epsilon_k \tau/2} n_k = e^{-\epsilon_k \tau/2} (1 - n_k) = \frac{1}{2} \cosh^{-1}\left(\frac{E_k - \mu}{2}\right). \qquad (31)$$

Hence, $K_2(x,p,\tau/2) = K_1^*(x,p,\tau/2)$, which in turn yields

$$\langle \hat{n}_\uparrow(p), \hat{n}_\downarrow(p') \rangle = -\kappa\tau \int dx |K_1(x,p,\tau/2) K_1(x,p',\tau/2)|^2. \qquad (32)$$

Using the WKB functions, and expressing the cosines by complex exponents, K_1 can be written as

$$K_1(x,p,\tau/2) = T \int dE \cosh^{-1}\left(\frac{E - \mu}{2}\right) \sqrt{\frac{1}{8\pi p_E(x) x_E(p)}}$$

$$\times \{\exp\{i[-\pi E/2 + \pi/4 + \Phi_E(x) + \Phi_E(p)]\}$$

$$+ \exp\{i[-\pi E/2 + \pi/4 - \Phi_E(x) - \Phi_E(p)]\}$$

$$+ \exp\{i[\pi E/2 - \pi/4 + \Phi_E(x) - \Phi_E(p)]\}$$

$$+ \exp\{i[\pi E/2 - \pi/4 - \Phi_E(x) + \Phi_E(p)]\}\}, \qquad (33)$$

where we have also replaced summation over the levels by integration. This integral can be evaluated by the stationary phase method. Consider, e.g., the contribution from the first exponent. Using that $\partial \Phi_E(x)/\partial E = \phi_E(x)$, we find the stationary phase condition

$$\phi_E(x) + \phi_E(p) = \frac{\pi}{2}, \qquad (34)$$

which is equivalent to

$$x^2+p^2=2E, \quad x\geqslant 0, \quad p\geqslant 0. \tag{35}$$

One can thus replace $x_E(p)\to x$ and $p_E(x)\to p$. After some more algebra, the contribution of the first exponent is found to be $e^{ipx}[2\cosh\{(x^2+p^2)/2-\mu\}/2]^{-1}\theta(x)\theta(p)$. The contributions of the remaining three exponents differ only in the step-function factors, equalling $\theta(-x)\theta(-p)$, $\theta(-x)\theta(p)$, and $\theta(x)\theta(-p)$, respectively, so that

$$K_1(x,p,\tau/2)=e^{ipx}\left[2\cosh\frac{(x^2+p^2)/2-\mu}{2}\right]^{-1}. \tag{36}$$

Finally,

$$\langle\hat{n}_\uparrow(p),\hat{n}_\downarrow(p')\rangle=-\frac{\kappa}{T}\int_{-\infty}^{+\infty}dx\,\chi\left(\frac{x^2+p^2-2\mu}{2T}\right)$$
$$\times\chi\left(\frac{x^2+p'^2-2\mu}{2T}\right), \tag{37}$$

where

$$\chi(\xi)=\frac{1}{2(1+\cosh\xi)}, \quad \int_{-\infty}^{+\infty}d\xi\,\chi(\xi)=1. \tag{38}$$

In momentum, the Fermi surface is naturally defined by assuming a spread of kinetic energies $\Delta(p^2/2)\sim T$ around the Fermi-energy μ. This corresponds to the layer of the order of

$$\Delta p_{\rm F}=\frac{T}{\sqrt{\mu}}, \tag{39}$$

around the Fermi momentum

$$p_{\rm F}=\sqrt{2\mu}. \tag{40}$$

(In normal units, $\Delta p_{\rm F}=k_{\rm B}T\sqrt{m/\mu}$, and $p_{\rm F}=\sqrt{2m\mu}$; interestingly, neither formula contains Planck's constant. The latter, however, enters implicitly via the expression relating the chemical potential to the total number of trapped atoms, $2\mu=\hbar\omega N$.) Equation (39) introduces a natural physical momentum scale. We shall denote momenta measured in units $\Delta p_{\rm F}$ by capital P; in particular, $P_{\rm F}=p_{\rm F}\sqrt{\mu}/T=\sqrt{2}\,\mu/T$. New momentum units imply a rescaling of the correlation function

$$\langle\hat{n}_\uparrow(p),\hat{n}_\downarrow(p')\rangle dp\,dp'=\langle\hat{n}_\uparrow(P),\hat{n}_\downarrow(P')\rangle dP\,dP'. \tag{41}$$

Equation (37) then gets rewritten as

$$\langle\hat{n}_\uparrow(P),\hat{n}_\downarrow(P')\rangle=-\frac{\kappa T^{3/2}}{\mu}\int_{-\infty}^{+\infty}d\xi\,\chi\left(\frac{\xi^2}{2}+\frac{P^2-P_{\rm F}^2}{\sqrt{2}\,P_{\rm F}}\right)$$
$$\times\chi\left(\frac{\xi^2}{2}+\frac{P'^2-P_{\rm F}^2}{\sqrt{2}\,P_{\rm F}}\right). \tag{42}$$

Thus for a "hot" highly populated trap the overall shape of the momentum correlation function depends only on the ratio of μ to T. Furthermore, for momenta close to the Fermi surface

$$\frac{P^2-P_{\rm F}^2}{\sqrt{2}\,P_{\rm F}}\approx\sqrt{2}(|P|-P_{\rm F}), \tag{43}$$

so that for $|P|,|P'|\approx P_{\rm F}$ the shape of the correlation function is universal (i.e., independent of any physical parameters).

Using the same approximations, one can also easily derive an expression for the (unperturbed) momentum density

$$\langle\hat{n}_\uparrow(P)\rangle=\langle\hat{n}_\downarrow(P)\rangle$$
$$=\frac{T^{3/2}}{\mu^{1/2}}\int_{-\infty}^{+\infty}d\xi\left[1+\exp\left(\frac{\xi^2}{2}+\frac{P^2-P_{\rm F}^2}{\sqrt{2}\,P_{\rm F}}\right)\right]^{-1}. \tag{44}$$

Except for the overall coefficient, this expression is also universal in the vicinity of the Fermi-surface.

IV. LANGEVIN EQUATIONS FOR FERMIONS

In this section we develop a nonperturbative approach to the ensemble of trapped interacting fermions, using the techniques of coherent states for Grassmann algebras. An introduction to Grassmann algebras may be found in Refs. [15,19] (see also Ref. [16]). Note that our definition of a coherent state is different from that used by Cahill and Glauber [16].

A. An overview of Grassmann algebras

A Grassmann algebra corresponding to a single fermionic state is a set of polynomials (with complex coefficients) of the *generators*, $g,\bar{g}$, which are the "classical" counterparts of the mode creation and annihilation operators, $\hat{a},\hat{a}^\dagger$. They are "classical" in the sense that they simply anticommute, $g\bar{g}+\bar{g}g=0$, unlike $\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}=1$ for the operators. For brevity, we shall call elements of a Grassmann algebra g *numbers*; an arbitrary g number is $c_0+c_1g+c_2\bar{g}+c_3\bar{g}g$, where c_0,c_1,c_2, and c_3 are c numbers. The standard association and distribution laws apply and g numbers commute with c numbers. This allows one to work out a product of any two g numbers. One may also consider products between Hilbert-space objects (such as state vectors and operators) and g numbers. The generators are postulated to anticommute with $\hat{a}$ and $\hat{a}^\dagger$, and to commute with the vacuum state

of the fermions, $|0\rangle$. Commutation relations between more complex objects may be worked out using these definitions, e.g., (with $|1\rangle = \hat{a}^{\dagger}|0\rangle$)

$$g(|1\rangle - |0\rangle) = g\hat{a}^{\dagger}|0\rangle - g|0\rangle = -\hat{a}^{\dagger}g|0\rangle - |0\rangle g = -\hat{a}^{\dagger}|0\rangle g$$
$$-|0\rangle g = -(|1\rangle + |0\rangle)g. \quad (45)$$

For the operators, one should make use of the fact that any operator may be represented as a linear combination of $\mathbb{1}/\sqrt{2}, \hat{a}, \hat{a}^{\dagger}$ and $[\hat{a}, \hat{a}^{\dagger}]/\sqrt{2}$. It is then easy to see that one has to assume that g and $\bar{g}$ commute with diagonal operators and anticommute with purely nondiagonal ones (in the basis $|0\rangle, |1\rangle$). Importantly, these conventions are consistent with the fact that generators must commute with matrix elements (which are c numbers), e.g.,

$$g\langle 1|a^{\dagger}|0\rangle = -\langle 1|ga^{\dagger}|0\rangle = \langle 1|a^{\dagger}g|0\rangle = \langle 1|a^{\dagger}|0\rangle g. \quad (46)$$

Hermitian conjugation is extended to g numbers postulating that $g^{\dagger} = \bar{g}$ and $\bar{g}^{\dagger} = g$, and that $(XY)^{\dagger} = Y^{\dagger}X^{\dagger}$ applies if X or Y or both are g numbers. Functions involving g numbers are understood as power series, which as a rule reduce to polynomials, e.g.,

$$e^{g\hat{a}^{\dagger}} = 1 + g\hat{a}^{\dagger} + \frac{1}{2}(g\hat{a}^{\dagger})^2 + \cdots = 1 + g\hat{a}^{\dagger}, \quad (47)$$

$$e^{g\hat{a}^{\dagger} + \bar{g}\hat{a}} = 1 + g\hat{a}^{\dagger} + \bar{g}\hat{a} + \frac{1}{2}(g\hat{a}^{\dagger} + \bar{g}\hat{a})^2 + \cdots$$
$$= 1 + g\hat{a}^{\dagger} + \bar{g}\hat{a} + \bar{g}\hat{a}g\hat{a}^{\dagger} + g\hat{a}^{\dagger}\bar{g}\hat{a}$$
$$= 1 + g\hat{a}^{\dagger} + \bar{g}\hat{a} + \bar{g}g(\hat{a}^{\dagger}\hat{a} - \hat{a}\hat{a}^{\dagger}). \quad (48)$$

Left and right differentiations by the g numbers are introduced as, respectively, $(\vec{\partial}/\partial g)1 = 0$, $(\vec{\partial}/\partial g)g = 1$, and $1(\overleftarrow{\partial}/\partial g) = 0$, $g(\overleftarrow{\partial}/\partial g) = 1$; apart from this, the derivatives anticommute and commute as if they were generators: $(\vec{\partial}/\partial g)2\hat{a}\bar{g} = 2(\vec{\partial}/\partial g)\hat{a}\bar{g} = -2\hat{a}(\vec{\partial}/\partial g)\bar{g} = -2\hat{a}$, etc. The (left) integration is defined so as to coincide with the left differentiation: $\int dg\, 1 = 0$, $\int dg\, g = 1$; apart from that, the differentials also behave as generators.

B. Coherent states

For a given basis of n single-particle states, determined by the annihilation and creation operators, $\hat{a}_k, \hat{a}_k^{\dagger}$, one introduces $2n$ generators $g_k, \bar{g}_k$ ($k = 1, \ldots, n$). Any two generators anticommute; an element of the Grassmann algebra is then determined by 2^{2n} c-number coefficients. Any generator commutes with the vacuum and anticommutes with any creation or annihilation operator. Differentiations and integrations are introduced component wise.

A coherent state is defined as

$$|\boldsymbol{g}\rangle = e^{-\boldsymbol{g}\hat{\boldsymbol{a}}^{\dagger}}|0\rangle, \quad (49)$$

where $\boldsymbol{g} = \{g_1, \ldots, g_n\}$ and $\boldsymbol{g}\hat{\boldsymbol{a}}^{\dagger} = g_1\hat{a}_1^{\dagger} + \cdots + g_n\hat{a}_n^{\dagger}$. The coherent state is an even object (i.e., it contains only even products of the anticommuting quantities) and hence commutes with all g numbers. It readily follows that

$$\hat{a}_k|\boldsymbol{g}\rangle = g_k|\boldsymbol{g}\rangle = |\boldsymbol{g}\rangle g_k, \quad \hat{a}_k^{\dagger}|\boldsymbol{g}\rangle = -\frac{\vec{\partial}}{\partial g_k}|\boldsymbol{g}\rangle = |\boldsymbol{g}\rangle\frac{\overleftarrow{\partial}}{\partial g_k}. \quad (50)$$

Conjugating, we have,

$$\langle\bar{\boldsymbol{g}}|\hat{a}_k^{\dagger} = \langle\bar{\boldsymbol{g}}|\bar{g}_k = \bar{g}_k\langle\bar{\boldsymbol{g}}|, \quad \langle\bar{\boldsymbol{g}}|\hat{a}_k = -\langle\bar{\boldsymbol{g}}|\frac{\overleftarrow{\partial}}{\partial\bar{g}_k} = \frac{\vec{\partial}}{\partial\bar{g}_k}\langle\bar{\boldsymbol{g}}|, \quad (51)$$

where $\langle\bar{\boldsymbol{g}}| = (|\boldsymbol{g}\rangle)^{\dagger} = \langle 0|e^{-\hat{\boldsymbol{a}}\bar{\boldsymbol{g}}}$.

We stress that our definition of a coherent state differs from a normalized coherent state as introduced by Cahill and Glauber [16]. The coherent states (49) are unnormalized.

$$\langle\bar{\boldsymbol{g}}|\boldsymbol{g}\rangle = e^{\bar{\boldsymbol{g}}\boldsymbol{g}}, \quad (52)$$

and thus correspond more closely to the Bargmann states [15], rather than to the normalized coherent states as used in quantum optics. Indeed, with α being a c number, a Bargmann state is introduced as, $|(\alpha)\rangle = e^{\alpha\hat{a}^{\dagger}}|0\rangle = e^{|\alpha|^2/2}|\alpha\rangle$, and $\langle(\alpha)|(\alpha)\rangle = e^{|\alpha|^2}$. The latter is a c number analogue of Eq. (52). Whereas for a coherent state, $|\alpha\rangle = e^{\alpha\hat{a}^{\dagger} - \alpha^*\hat{a}}|0\rangle$ and $\langle\alpha|\alpha\rangle = 1$.

C. *B* representation

In a similar manner to that used in the pseudoprobabilistic phase-space representations of quantum optics, we introduce a *B representation* for an arbitrary Schrödinger operator $\hat{A}$,

$$\hat{A} = \int d\bar{\boldsymbol{g}}d\boldsymbol{g}|\boldsymbol{g}\rangle B_{\hat{A}}(\bar{\boldsymbol{g}}, \boldsymbol{g})\langle\bar{\boldsymbol{g}}|, \quad (53)$$

where $d\bar{\boldsymbol{g}}d\boldsymbol{g} = d\bar{g}_1 dg_1 \cdots d\bar{g}_n dg_n$ (note the order) and $B_{\hat{A}}(\bar{\boldsymbol{g}}, \boldsymbol{g})$ is a g number. Note that $\langle\bar{\boldsymbol{g}}'|\boldsymbol{g}\rangle = e^{\bar{\boldsymbol{g}}'\boldsymbol{g}}$, and that the double Fourier transformation is a unity operation, $\int dg' e^{g'g''}\int dg e^{gg'}f(g) = f(g'')$. Then, taking the matrix element

$$\langle\bar{\boldsymbol{g}}'|\hat{A}|\boldsymbol{g}'\rangle = \int d\bar{\boldsymbol{g}}d\boldsymbol{g}\langle\bar{\boldsymbol{g}}'|\boldsymbol{g}\rangle B_{\hat{A}}(\bar{\boldsymbol{g}}, \boldsymbol{g})\langle\bar{\boldsymbol{g}}|\boldsymbol{g}'\rangle$$
$$= \int d\bar{\boldsymbol{g}}d\boldsymbol{g}B_{\hat{A}}(\bar{\boldsymbol{g}}, \boldsymbol{g})e^{\bar{\boldsymbol{g}}'\boldsymbol{g} + \bar{\boldsymbol{g}}\boldsymbol{g}'}, \quad (54)$$

and inverting the Fourier transformations we have

$$B_{\hat{A}}(\bar{\boldsymbol{g}}, \boldsymbol{g}) = \int d\boldsymbol{g}' d\bar{\boldsymbol{g}}' e^{-\bar{\boldsymbol{g}}\boldsymbol{g}' - \bar{\boldsymbol{g}}'\boldsymbol{g}}\langle\bar{\boldsymbol{g}}'|\hat{A}|\boldsymbol{g}'\rangle, \quad (55)$$

where $d\boldsymbol{g}'d\bar{\boldsymbol{g}}' = dg_1'd\bar{g}_1'\cdots dg_n'd\bar{g}_n'$ (once more, note the order). This proves the existence of a *B* representation for an

arbitrary operator: unlike bosons, all integrals over a Grass-mann algebra converge trivially. For example, for the unity operator

$$B_1 = e^{-\bar{g}g}. \tag{56}$$

Relations (50) and (51) result in

$$B_{\hat{a}_k\hat{A}} = g_k B_{\hat{A}}, \quad B_{\hat{A}\hat{a}_k^\dagger} = B_{\hat{A}} \bar{g}_k, \quad B_{\hat{a}_k^\dagger\hat{A}} = (\vec{\partial}/\partial g_k) B_{\hat{A}},$$

$$B_{\hat{A}\hat{a}_k} = B_{\hat{A}}(\vec{\partial}/\partial\bar{g}_k). \tag{57}$$

These formulas allow one readily to transform operator equations into equations for the corresponding B representations.

D. Dynamics of the B function

The B representation of a many-body ρ matrix of a fermionic system will be called a B *function*. We consider the unnormalized equilibrium B function for the system of trapped interacting fermions

$$B(\bar{g},g,\tau) = B_{\hat{R}(\tau)}, \tag{58}$$

where $\hat{R}(\tau)$ is defined by Eqs. (2), (3), and (4). The Matsubara-style equation (7) for $\hat{R}(\tau)$ is readily transformed into the equation for the B function

$$\frac{\partial B(\bar{g},g,\tau)}{\partial\tau} = -\left[\sum_{\sigma=\uparrow,\downarrow} \frac{\vec{\partial}}{\partial g_\sigma} h g_\sigma \right.$$

$$\left. + \kappa \int dx \frac{\vec{\partial}}{\partial g_\uparrow(x)} g_\uparrow(x) \frac{\vec{\partial}}{\partial g_\downarrow(x)} g_\downarrow(x) \right] B(\bar{g},g,\tau), \tag{59}$$

with the initial condition given by Eq. (56). For brevity, we use a "vector/matrix" notation, $g = \{g_\uparrow, g_\downarrow\}$, $g_\sigma = \{g_\sigma(x)\}$ $(\sigma = \uparrow, \downarrow)$, $(\vec{\partial}/\partial g_\sigma) h g_\sigma = \int dx dx' (\vec{\partial}/\partial g_\sigma(x)) h(x,x') g_\sigma(x')$, etc. [cf. also Eq. (3)]. Note that we do not assume summation over repeated indices: unless a matrix notation is used all sums are shown explicitly.

1. Linear interaction

We start by considering the simplest possible case—a linear interaction of two fermionic modes. Equation (59) then reads

$$\frac{\partial B(\bar{g},g,\tau)}{\partial\tau} = -\sum_{k,l=1,2} \frac{\vec{\partial}}{\partial g_k} h_{kl} g_l B(\bar{g},g,\tau)$$

$$= -\left(\frac{\vec{\partial}}{\partial g} h g\right) B(\bar{g},g,\tau). \tag{60}$$

We note that

$$\frac{\vec{\partial}}{\partial g} h g = \mathrm{Tr}\, h - g h^{\mathrm{T}} \frac{\vec{\partial}}{\partial g}, \tag{61}$$

where superscript T means "transposed," and consider the action of a particular term in the sum, e.g., $g_2(\vec{\partial}/\partial g_1)$. Regarded as a function of the generator g_1, the B function can be written as $B(g_1) = G_0 + g_1 G_1$, where neither G_0 nor G_1 contains g_1. Then $g_2(\vec{\partial}/\partial g_1)B(g_1) = g_2 G_1$ and

$$\left(1 + h_{12} d\tau g_2 \frac{\vec{\partial}}{\partial g_1}\right) B(g_1) = G_0 + g_1 G_1 + h_{12} d\tau g_2 G_1 = B(g_1$$

$$+ h_{12} d\tau g_2). \tag{62}$$

That is, recalling that the B function is a polynomial of the Grassmann algebra generators, the infinitesimal operation $1 + d\tau g h^{\mathrm{T}}(\vec{\partial}/\partial g)$ induces a linear transformation of the set of the generators, leaving the coefficients of the polynomial intact. Only the set $g = \{g_1, g_2\}$ is transformed, while $\bar{g} = \{\bar{g}_1, \bar{g}_2\}$ stay untouched. This means that the transformed generators are no longer Hermitian conjugate to $\bar{g} = \{\bar{g}_1, \bar{g}_2\}$.

These considerations suggest that equation (60) may be solved by the ansatz

$$B(\bar{g},g,\tau) = f(\beta) B(\bar{g},g^\tau,0), \quad g^0 = g. \tag{63}$$

The τ-dependent generators g^τ are found as a linear transformation of the basic set,

$$g^\tau = \beta^{-1} g, \tag{64}$$

where $\beta = \beta(\tau)$ is a c-number matrix. It is fixed by the requirement [cf Eq. (61)]

$$\frac{\partial g^\tau}{\partial\tau} = \left(g h^{\mathrm{T}} \frac{\vec{\partial}}{\partial g}\right) g, \tag{65}$$

resulting in the equation for $\beta(\tau)$,

$$\frac{\partial\beta(\tau)}{\partial\tau} = -h\beta(\tau), \quad \beta(0) = \mathbb{1}. \tag{66}$$

(The standard rule of product differentiation holds for $g h^{\mathrm{T}}(\vec{\partial}/\partial g)$, which is an even quantity, so that it suffices to consider its action on a single generator.) For the coefficient we have $\partial f(\beta)/\partial\tau = -\mathrm{Tr}\, h\, f(\beta)$, satisfied by $f(\beta) = \mathrm{Det}\,\beta$. This numerical factor results in an important property—"conservation of the phase volume"

$$\int d\bar{g}dg B(\bar{g},g,\tau) F(\bar{g},g^\tau) = \int d\bar{g}dg^\tau B(\bar{g},g^\tau,0) F(\bar{g},g^\tau). \tag{67}$$

(On changing variables in Grassmann algebras see Ref. [15]). These considerations are readily extended to an arbitrary linear interaction.

Note that ansatz (63) reduces computational complexity even in the simplest case of $n = 2$ fermions. Indeed, in this

case, the number of independent c number coefficients determining $B(\bar{g}, g, \tau)$ equals $2^{2n} = 16$. Equation (60) is in fact a coupled system of 16 c-number equations. Whereas β is an $n \times n = 2 \times 2$ matrix, and Eq. (66) is a coupled system of $n^2 = 4$ c-number equations.

It is worth stressing that success of our approach is rooted in properties of Bargmann states leading to Eqs. (57). Consider what happens if one works with normalized coherent states [16], $e^{-\bar{g}g/2}|g\rangle$ and $e^{-\bar{g}g/2}\langle\bar{g}|$. Note that we do not introduce any notation for these states: the notation $|g\rangle$ and $\langle\bar{g}|$ is retained for the unnormalized (Bargmann type) coherent states. Equation (53) is replaced by a definition of a P representation,

$$\hat{A} = \int d\bar{g} dg |g\rangle e^{-\bar{g}g} P_{\hat{A}}(\bar{g}, g) \langle\bar{g}|. \tag{68}$$

Similarly to Eq. (57) we have

$$P_{\hat{a}_k \hat{A}} = g_k P_{\hat{A}}, \quad P_{\hat{a}_k^\dagger \hat{A}} = \left(\frac{\vec{\partial}}{\partial g_k} + \frac{\bar{g}_k}{2}\right) P_{\hat{A}}, \tag{69}$$

so that the equation for the B function (60) is replaced by

$$\frac{\partial P(\bar{g}, g, \tau)}{\partial \tau} = -\left[\left(\frac{\vec{\partial}}{\partial g} + \frac{\bar{g}}{2}\right) h g\right] P(\bar{g}, g, \tau). \tag{70}$$

We see that normalizing the coherent states results in additional "product" terms in the equation for the pseudodistribution. For bosons, these terms simply make the corresponding Langevin equations nonlinear [14]. For fermions, the presence of product terms spells disaster: Equation (70) mixes different powers of the algebra generators and hence cannot, in principle, be solved by a linear transformation of the generators.

The technique of algebra transformations is readily extended to the real-time evolution. In this case, the ρ matrix obeys the von-Neumann equation, $i\partial\hat{\rho}/\partial t = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}$, resulting in the equation for the B function

$$i\frac{\partial B(\bar{g}, g, \tau)}{\partial \tau} = \left(\frac{\vec{\partial}}{\partial g} h g\right) B(\bar{g}, g, \tau) - B(\bar{g}, g, \tau) \left(\bar{g} h \frac{\overleftarrow{\partial}}{\partial \bar{g}}\right), \tag{71}$$

where we have assumed that the (linear) system Hamiltonian reads $H = a^\dagger h a$. The first term on the RHS here is taken care of by transformation (64), the equation for β being $i\partial\beta/\partial t = h\beta$. The second term on the RHS is taken care of by a transformation of the conjugated set of the generators, $\bar{g}^\tau = \bar{\beta}^{-1}\bar{g}$, the equation for $\bar{\beta}$ being, $i\partial\bar{\beta}/\partial t = -\bar{\beta}h$. That is, in the case of time evolution of a linear system, the g and $\bar{g}$ sets undergo conjugate transformations and hence stay conjugate. This last property, however, does not generalize to the nonlinear case.

2. Stochastic transformation of the generators

It is natural to expect that, as in the way it is done for bosons [14], nonlinear fermionic interactions may be accounted for by making the transformation (trajectory) β stochastic. With this in mind, consider an ansatz

$$B(\bar{g}, g, \tau) = \overline{B^\beta}, \tag{72}$$

$$B^\beta = \text{Det } \beta \, B(\bar{g}, g^\tau, 0). \tag{73}$$

Here, the set g of the Grassmann algebra generators undergoes a random linear transformation, $g^\tau = \beta^{-1}g$, and the upper bar in Eq. (72) denotes averaging over the statistics of the transformations (not to be confused with the upper bar denoting the conjugate set of generators). These statistics are specified by postulating an Itô stochastic differential equation (SDE) [17]

$$d\beta = -m\beta d\tau + dQ\beta, \tag{74}$$

for the trajectory $\beta = \beta(\tau)$. It is assumed that m is a certain time-independent matrix, and dQ is a matrix of which the elements are linear combinations of Itô increments with constant coefficients. To determine m and dQ, one should find the corresponding equation for the B function and then match it to Eq. (59).

Our immediate goal is, therefore, to derive equations for B^β and $B = \overline{B^\beta}$ from the SDE (74). To simplify this task, we assume the following strategy. The SDE for B^β is derived in Stratonovich calculus [17]. This greatly simplifies the derivation because the rules of normal calculus may be used; in essence we just repeat the above considerations in the linear case. We then find the equivalent Itô SDE for B^β. In turn, this greatly simplifies the averaging in Eq. (72).

The equivalent Stratonovich equation for β reads [17]

$$d\beta = -\left(m + \frac{X}{2}\right)\beta d\tau + dQ\beta, \tag{75}$$

where the constant matrix X is defined (in Itô calculus, strictly speaking) by $dQ^2 = X d\tau$. It is then straightforward to show that (remember that Stratonovich calculus applies)

$$d \text{ Det } \beta = \text{Det } \beta \text{ Tr } \delta\beta, \tag{76}$$

$$dg^\tau = -\left(g \delta\beta^{\text{T}} \frac{\vec{\partial}}{\partial g}\right) g^\tau, \tag{77}$$

where $\delta\beta = d\beta \, \beta^{-1}$ and superscript T stands for transposed. Relation (77) may be extended to an arbitrary function of the transformed generators, because the standard rule of product differentiation applies to both the differential d and the differential operator on the Grassmann algebra $g\delta\beta^{\text{T}}(\vec{\partial}/\partial g)$. Noting that

$$\text{Tr } \delta\beta - g \delta\beta^{\text{T}} \frac{\vec{\partial}}{\partial g} = \frac{\vec{\partial}}{\partial g} \delta\beta g, \tag{78}$$

we find the Stratonovich SDE for B^β,

$$dB^\beta = \left(\frac{\vec{\partial}}{\partial \boldsymbol{g}} \delta\beta \boldsymbol{g} \right) B^\beta = \left\{ \frac{\vec{\partial}}{\partial \boldsymbol{g}} \left[-\left(m + \frac{X}{2} \right) d\tau + dQ \right] \boldsymbol{g} \right\} B^\beta.$$

(79)

To find the equivalent Itô SDE, note that B^β is a polynomial of the Grassmann algebra monomials with c-number coefficients. We can regard these coefficients as a c-number vector $\mathcal{B}$, and equation (79) as an equation for this vector,

$$d\mathcal{B} = \mathcal{M}(\delta\beta)\mathcal{B}.$$

(80)

where $\mathcal{M}(\delta\beta)$ is a matrix. The important point is that this matrix depends linearly on its argument $\delta\beta$. Using this, we easily find the equivalent Itô SDE for $\mathcal{B}$,

$$\mathcal{B} = \left\{ \mathcal{M}(\delta\beta) + \frac{1}{2} [\mathcal{M}(dQ)]^2 \right\} \mathcal{B}.$$

(81)

This can equally well be written as the Itô SDE sought for B^β,

$$dB^\beta = \left\{ \frac{\vec{\partial}}{\partial \boldsymbol{g}} \left[-\left(m + \frac{X}{2} \right) d\tau + dQ \right] \boldsymbol{g} + \frac{1}{2} \left(\frac{\vec{\partial}}{\partial \boldsymbol{g}} dQ \boldsymbol{g} \right)^2 \right\} B^\beta.$$

(82)

Since Itô increments are uncorrelated with β at the same "time" τ, averaging of this equation is straightforward and yields the equation sought for B,

$$dB = \left[-\frac{\vec{\partial}}{\partial \boldsymbol{g}} \left(m d\tau + \frac{\overline{dQ^2}}{2} \right) \boldsymbol{g} + \frac{1}{2} \overline{\left(\frac{\vec{\partial}}{\partial \boldsymbol{g}} dQ \boldsymbol{g} \right)^2} \right] B.$$

(83)

The averagings here are understood in Itô calculus (and can actually be omitted since, by the rules of Itô calculus, quantities quadratic in Itô increments are nonrandom).

3. Nonlinear interaction

To simplify our considerations, we treat the fermions as being on a spatial grid. The interaction (4) is then that within a single grid point. Specifically, x is assumed to be an equidistant grid of points separated by Δx. Then, the integral, $\int dx$, and the delta function, $\delta(x-x')$, are understood as $\Delta x \Sigma_x$ and $\Delta x^{-1} \delta_{xx'}$, respectively, where $\delta_{xx'}$ is the Kronecker symbol. This assumption is in itself a regularization, thanks to which certain formulas start making mathematical sense [and which otherwise would remain merely symbolic, like Eq. (87) below]. An alternative regularization would be to consider the limit of a potential interaction of fermions with opposite spins, like

$$H_{\mathrm{nl}}^{\mathrm{pot}} = \int dx dx' V(x-x') \hat{a}_\uparrow^\dagger(x) \hat{a}_\uparrow(x) \hat{a}_\downarrow^\dagger(x') \hat{a}_\downarrow(x').$$

(84)

These two viewpoints should agree for momenta $p \ll \hbar/\Delta x$, where Δx is either the grid increment or the interaction

range. At the same time, on-the-grid fermions are simpler mathematically, so we will stick with them.

Comparing Eq. (59) with Eq. (83), we see that the nonlinear interaction can indeed be modeled via stochastically. Namely, we introduce a standardized Gaussian real white-noise variable $\eta(x,\tau)$, such that $\overline{\eta(x,\tau)} = 0$ and $\overline{\eta(x,\tau)\eta(x',\tau')} = \delta(\tau-\tau')\delta(x-x')$, and write $(\sigma,\sigma' = \uparrow,\downarrow)$,

$$dQ_{\sigma\sigma'}(x,x') = \delta_{\sigma\sigma'} \delta(x-x') \sqrt{-\kappa} \, \eta(x,\tau) d\tau.$$

(85)

Then

$$\frac{1}{2} \overline{\left(\frac{\vec{\partial}}{\partial \boldsymbol{g}} dQ \boldsymbol{g} \right)^2} = \frac{1}{2} \overline{\left(\frac{\vec{\partial}}{\partial \boldsymbol{g}_\uparrow} dQ_{\uparrow\uparrow} \boldsymbol{g}_\uparrow \right)^2} + \frac{1}{2} \overline{\left(\frac{\vec{\partial}}{\partial \boldsymbol{g}_\downarrow} dQ_{\downarrow\downarrow} \boldsymbol{g}_\downarrow \right)^2}$$

$$+ \overline{\left(\frac{\vec{\partial}}{\partial \boldsymbol{g}_\uparrow} dQ_{\uparrow\uparrow} \boldsymbol{g}_\uparrow \right) \left(\frac{\vec{\partial}}{\partial \boldsymbol{g}_\downarrow} dQ_{\downarrow\downarrow} \boldsymbol{g}_\downarrow \right)}$$

$$= -\kappa d\tau \int dx \left[\frac{1}{2} \frac{\vec{\partial}}{\partial g_\uparrow(x)} g_\uparrow(x) \frac{\vec{\partial}}{\partial g_\uparrow(x)} g_\uparrow(x) \right.$$

$$+ \frac{1}{2} \frac{\vec{\partial}}{\partial g_\downarrow(x)} g_\downarrow(x) \frac{\vec{\partial}}{\partial g_\downarrow(x)} g_\downarrow(x)$$

$$\left. + \frac{\vec{\partial}}{\partial g_\uparrow(x)} g_\uparrow(x) \frac{\vec{\partial}}{\partial g_\downarrow(x)} g_\downarrow(x) \right].$$

(86)

The third term here exactly reproduces the nonlinear contribution to Eq. (59). The first two are transformed into effectively linear contributions

$$-\frac{\kappa d\tau}{2} \sum_{\sigma=\uparrow,\downarrow} \int dx \frac{\vec{\partial}}{\partial g_\sigma(x)} g_\sigma(x) \frac{\vec{\partial}}{\partial g_\sigma(x)} g_\sigma(x)$$

$$= -\frac{\kappa d\tau}{2} \sum_{\sigma=\uparrow,\downarrow} \int dx \frac{\vec{\partial}}{\partial g_\sigma(x)} \left[\delta(0) \right.$$

$$\left. - \frac{\vec{\partial}}{\partial g_\sigma(x)} g_\sigma(x) \right] g_\sigma(x)$$

$$= -\frac{\kappa d\tau}{2\Delta x} \sum_{\sigma=\uparrow,\downarrow} \int dx \frac{\vec{\partial}}{\partial g_\sigma(x)} g_\sigma(x),$$

(87)

where we have used the fact that, under the regularization, $\delta(0) = 1/\Delta x$. On the other hand,

$$-\frac{1}{2} \frac{\vec{\partial}}{\partial \boldsymbol{g}} \overline{dQ^2} \boldsymbol{g} = \frac{\kappa d\tau}{2\Delta x} \sum_{\sigma=\uparrow,\downarrow} \int dx \frac{\vec{\partial}}{\partial g_\sigma(x)} g_\sigma(x),$$

(88)

so that contributions (87) and (88) cancel each other. (This emphasizes that the choice of stochastic calculus may also be regarded as a regularization [21].) Thus the matrix m is found to be

$$m_{\uparrow\uparrow}(x,x') = m_{\downarrow\downarrow}(x,x') = h(x',x),$$

(89)

$$m_{\uparrow\downarrow}(x,x')=m_{\downarrow\uparrow}(x,x')=0. \tag{90}$$

In other words, for the system in question the transformation β factorizes into two identical transformations of the spin-up and spin-down sets, $\boldsymbol{g}_\uparrow$ and $\boldsymbol{g}_\downarrow$,

$$\boldsymbol{g}_\sigma^\tau = \alpha^{-1}\boldsymbol{g}_\sigma, \tag{91}$$

where $\alpha(x,x')=\beta_{\uparrow\uparrow}(x,x')=\beta_{\downarrow\downarrow}(x,x')$, and $\beta_{\downarrow\uparrow}(x,x') =\beta_{\uparrow\downarrow}(x,x')=0$. The stochastic trajectory α obeys the Itô SDE

$$d\alpha(x,x')=-d\tau\int dx''h(x,x'')\alpha(x'',x')$$
$$+\sqrt{-\kappa}\,dW(x)\alpha(x,x'), \tag{92}$$

where the Itô increment $dW(x)$ is normalized so that $[dW(x)]^2=d\tau/\Delta x$.

E. Quantum averaging

1. Normally ordered characteristic function

In order to learn how to calculate quantum averages, we consider the normally-ordered characteristic function

$$F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)=\mathrm{Tr}[R(\tau)e^{\hat{a}^\dagger}\boldsymbol{g}'e^{\overline{\boldsymbol{g}}'\hat{a}}]. \tag{93}$$

Employing Eq. (53), this can be expressed as

$$F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)=\int d\overline{\boldsymbol{g}}d\boldsymbol{g}B(\overline{\boldsymbol{g}},\boldsymbol{g},\tau)e^{\overline{\boldsymbol{g}}\boldsymbol{g}'+\overline{\boldsymbol{g}}'\boldsymbol{g}-\overline{\boldsymbol{g}}\boldsymbol{g}}. \tag{94}$$

In obtaining this result we used the fact that $\mathrm{Tr}|\boldsymbol{g}\rangle\langle\overline{\boldsymbol{g}}|=e^{\overline{\boldsymbol{g}}\boldsymbol{g}}$ (thus $\mathrm{Tr}|\boldsymbol{g}\rangle\langle\overline{\boldsymbol{g}}|\neq\langle\overline{\boldsymbol{g}}|\boldsymbol{g}\rangle=e^{\overline{\boldsymbol{g}}\boldsymbol{g}}$). Then, (i) using Eq. (73), (ii) changing variables $\overline{\boldsymbol{g}},\boldsymbol{g}\rightarrow\overline{\boldsymbol{g}},\boldsymbol{g}^\tau$ following Eq. (67), (iii) using Eq. (56), and (iv) expressing $\boldsymbol{g}=\beta\boldsymbol{g}^\tau$, we arrive at

$$F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)=\overline{\int d\overline{\boldsymbol{g}}d\boldsymbol{g}^\tau\exp[-\overline{\boldsymbol{g}}(\beta+1)\boldsymbol{g}^\tau+\overline{\boldsymbol{g}}'\beta\boldsymbol{g}^\tau-\boldsymbol{g}'\overline{\boldsymbol{g}}]}. \tag{95}$$

Under averaging we have a Gaussian integral on the Grassmann algebra that is calculated directly [15] resulting in

$$F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)=\overline{\mathrm{Det}(1+\beta)e^{\overline{\boldsymbol{g}}'(1+\beta^{-1})^{-1}\boldsymbol{g}'}}. \tag{96}$$

Care needs to be exercised with the order of operators and Grassmann-algebra objects if calculating averages using the characteristic function. Consider, for example, an average of the density operators, $\langle:\hat{n}_{k_1}\hat{n}_{k_2}\cdots\hat{n}_{k_m}:\rangle$, where $\hat{n}_k=\hat{a}_k^\dagger\hat{a}_k$, and $k_l,l=1,\ldots,m$ are certain fermionic states (which may or may not be all different, so that normal ordering is essential). If $x_{k_1},x_{k_2},\ldots,x_{k_m}$ and $y_{k_1},y_{k_2},\ldots,y_{k_m}$ are two groups of pairwise anticommuting objects, it is easy to prove that

$$x_{k_1}y_{k_1}x_{k_2}y_{k_2}\cdots x_{k_m}y_{k_m}=x_{k_1}x_{k_2}\cdots x_{k_m}y_{k_m}y_{k_{m-1}}\cdots y_{k_1}. \tag{97}$$

(First move y_{k_1} to the right, then move y_{k_2} to the left of y_{k_1}, etc.) Using this property we find [with $\rho(\tau)$ being the normalized ρ matrix]

$$\langle:\hat{n}_{k_1}\hat{n}_{k_2}\cdots\hat{n}_{k_m}:\rangle=\mathrm{Tr}[\rho(\tau):\hat{n}_{k_1}\hat{n}_{k_2}\cdots\hat{n}_{k_m}:]$$

$$=\mathrm{Tr}[\rho(\tau)\hat{a}_{k_1}^\dagger\hat{a}_{k_2}^\dagger\cdots\hat{a}_{k_m}^\dagger\hat{a}_{k_m}\hat{a}_{k_{m-1}}\cdots\hat{a}_{k_1}]$$

$$=\mathrm{Tr}[\hat{a}_{k_m}\hat{a}_{k_{m-1}}\cdots\hat{a}_{k_1}\rho(\tau)\hat{a}_{k_1}^\dagger\hat{a}_{k_2}^\dagger\cdots\hat{a}_{k_m}^\dagger]$$

$$=Z^{-1}\frac{\vec{\partial}}{\partial\overline{g}'_{k_1}}\frac{\vec{\partial}}{\partial\overline{g}'_{k_2}}\cdots\frac{\vec{\partial}}{\partial\overline{g}'_{k_m}}F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)$$

$$\times\frac{\overleftarrow{\partial}}{\partial g'_{k_m}}\frac{\overleftarrow{\partial}}{\partial g'_{k_{m-1}}}\cdots\frac{\overleftarrow{\partial}}{\partial g'_{k_1}}\Big|_{\overline{g}',g'\to 0}, \tag{98}$$

where $Z=\mathrm{Tr}\,R=F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)|_{\overline{g}',g'\to 0}$ This relation is quite general and holds even if $\rho(\tau)$ contains fermionic coherence. For the characteristic function given by Eq. (96) a simpler relation holds,

$$\langle:\hat{n}_{k_1}\hat{n}_{k_2}\cdots\hat{n}_{k_m}:\rangle=Z^{-1}\frac{\vec{\partial}}{\partial g'_{k_1}}\frac{\vec{\partial}}{\partial\overline{g}'_{k_1}}\frac{\vec{\partial}}{\partial g'_{k_2}}\frac{\vec{\partial}}{\partial\overline{g}'_{k_2}}\cdots$$

$$\times\frac{\vec{\partial}}{\partial g'_{k_m}}\frac{\vec{\partial}}{\partial\overline{g}'_{k_m}}F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)|_{\overline{g}',g'\to 0}. \tag{99}$$

If the k_l are all different, the normal ordering may be omitted, and

$$\langle\hat{n}_{k_1}\hat{n}_{k_2}\cdots\hat{n}_{k_m}\rangle=Z^{-1}\overline{\mathrm{Det}(1+\beta)[(1+\beta^{-1})^{-1}]_{k_1k_1}[(1+\beta^{-1})^{-1}]_{k_2k_2}\cdots[(1+\beta^{-1})^{-1}]_{k_mk_m}}, \tag{100}$$

while for the partition function we have, $Z=\overline{\mathrm{Det}(1+\beta)}$. If any two indices coincide, Eq. (99) gives zero.

2. Example: free thermal fermions

Consider a grand-canonical ensemble of noninteracting fermions with chemical potential μ, occupying n states with energies e_k, $k=1,\ldots,n$. In this case, $h_{kk'}=\delta_{kk'}\epsilon_k$, where $\epsilon_k=e_k-\mu$. Equation (66) is readily solved, yielding $\beta_{kk'}=\delta_{kk'}e^{-\epsilon_k\tau}$ and the characteristic function is found to be

$$F(\overline{\boldsymbol{g}}',\boldsymbol{g}',\tau)=\prod_k\frac{1+e^{\epsilon_k\tau}+\overline{g}'_k g'_k}{e^{\epsilon_k\tau}}. \tag{101}$$
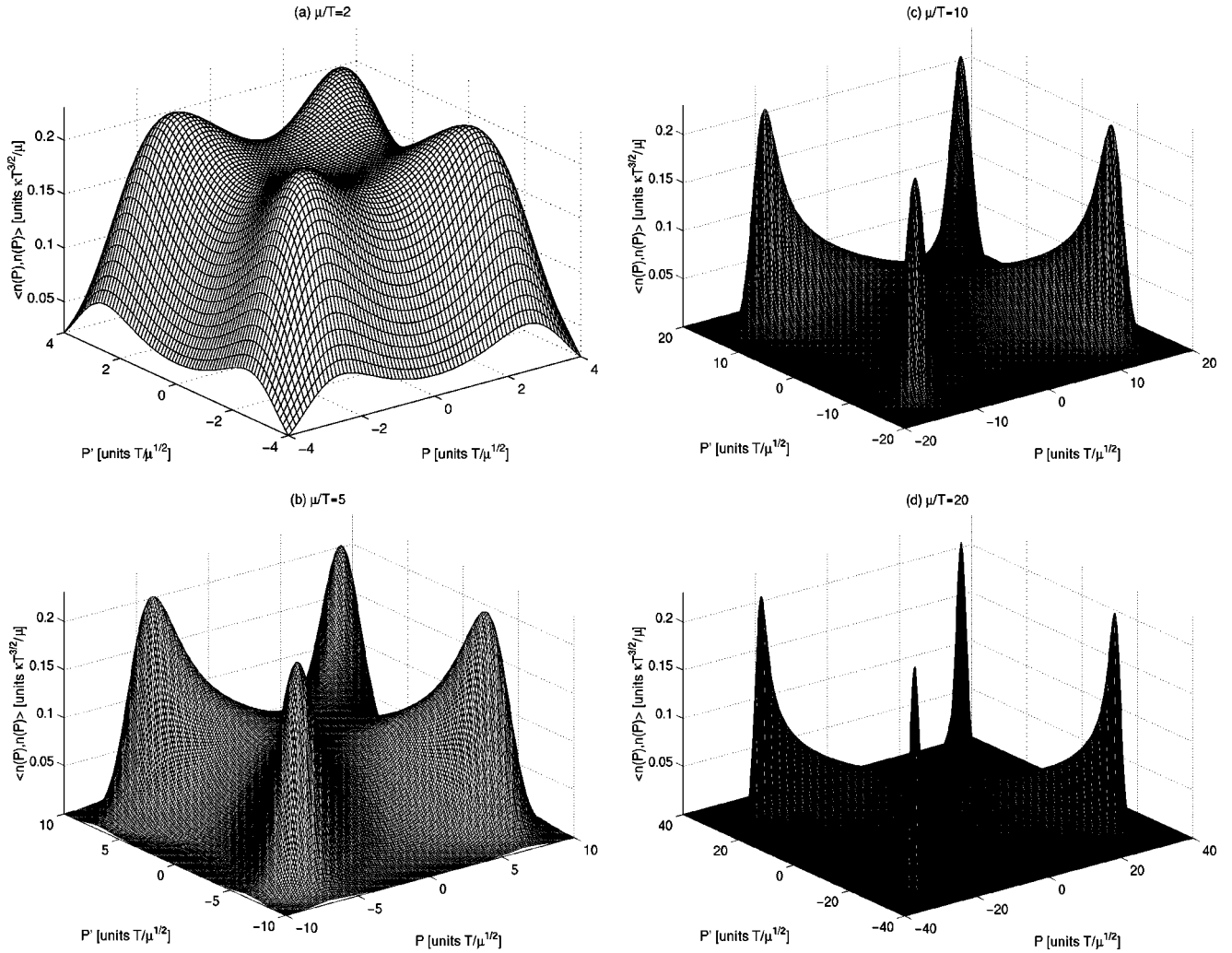
FIG. 1. Momentum correlation function $\langle \hat{n}_\uparrow(P), \hat{n}_\downarrow(P') \rangle$ for fermions in a 1D harmonic trap calculated via the approximate perturbative formula (42); (a) $\mu/T=2$, (b) $\mu/T=5$, (c) $\mu/T=10$, and (d) $\mu/T=20$.

Thus the partition function is

$$Z(\tau) = \text{Tr}\, R(\tau) = F(\bar{\boldsymbol{g}}',\boldsymbol{g}',\tau)\big|_{\bar{\boldsymbol{g}}',\boldsymbol{g}'\to 0} = \prod_k \frac{1+e^{\epsilon_k \tau}}{e^{\epsilon_k \tau}}, \tag{102}$$

which is indeed a correct partition function for a grand-canonical ensemble of free fermions. Then, for the normalized characteristic function

$$Z^{-1} F(\bar{\boldsymbol{g}}',\boldsymbol{g}',\tau) = \prod_k \left(1 + \frac{\bar{g}'_k g'_k}{1+e^{\epsilon_k \tau}} \right) = \prod_k (1+\bar{g}'_k g'_k n_k), \tag{103}$$

where $n_k = (1+e^{\epsilon_k \tau})^{-1}$ is the Fermi-Dirac distribution. Using Eq. (99) we have (with all indices different)

$$\langle \hat{n}_{k_1} \hat{n}_{k_2} \cdots \hat{n}_{k_m} \rangle = n_{k_1} n_{k_2} \cdots n_{k_m}, \tag{104}$$

as expected for noninteracting particles.

V. RESULTS AND DISCUSSION

In this section, we apply the techniques outlined above to an ensemble of fermions in a harmonic trap. We assume a "needle" trap (cf. Sec. I) with the ratio of radial to longitudinal frequencies

$$\bar{\mu} = \frac{\omega_r}{\omega_z} \gg 1. \tag{105}$$

As in Sec. III C we use here oscillator units assuming also that the trap frequency $\omega = \omega_z = 1$. The 1D approximation is valid if

$$\mu, T \ll \bar{\mu}. \tag{106}$$

This sets the limit $2\bar{\mu}$ for the number of particles in the trap. For the effective 1D interacton we then have [18]

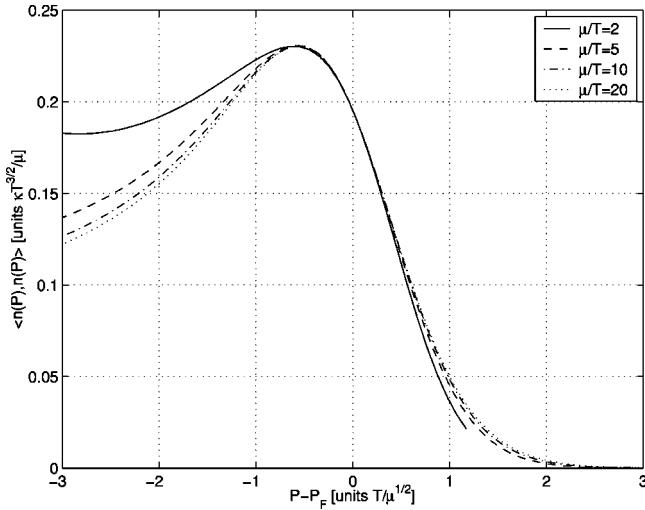$$\kappa = \frac{2a\bar{\mu}}{l_0} \equiv \kappa_0 \bar{\mu}, \tag{107}$$

FIG. 2. Cross section for $P=P'$ of the momentum correlation function calculated via the approximate perturbative formula (42), for values of μ/T as in Fig. 1.

where a is the scattering length. Assuming (in normal units) $a=-1140$ Å and $\omega=144$ Hz [7], for trapped ^{6}Li atoms we find $\kappa_0=0.067$.

A. Perturbation approximation

1. Pseudo-Cooper pairing

Figure 1 shows the correlation function, $\langle\hat{n}_\uparrow(P),\hat{n}_\downarrow(P')\rangle$, calculated via Eq. (42) for $\mu/T=2,5,10,20$, as a 3D graph. In all four cases, the correlations are peaked at $P,P'\sim P_F$, resulting in a characteristic four-peak shape. There are also noticeable correlations away from the Fermi surface, for parallel and antiparallel momenta, $P\sim\pm P'$. There are no correlations for $|P|\neq|P'|$ (with the exception of $\mu/T=2$, but this is an artifact of a badly defined Fermi surface for these parameters). In accordance with the aforementioned universality of Eq. (42) for $|P|,|P'|\sim P_F$, the height of the peaks
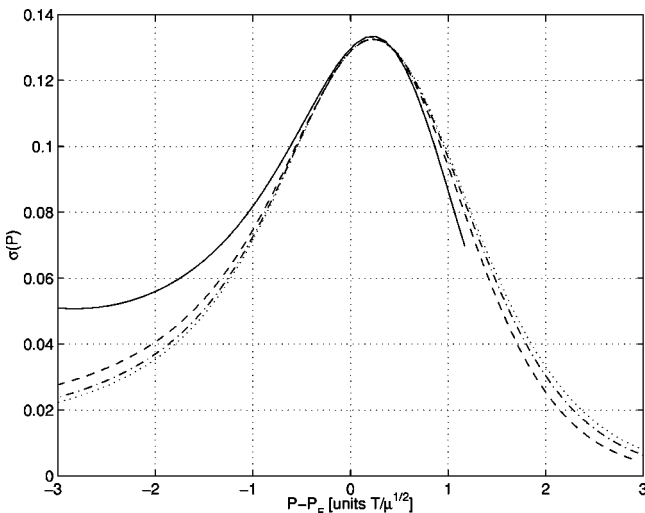
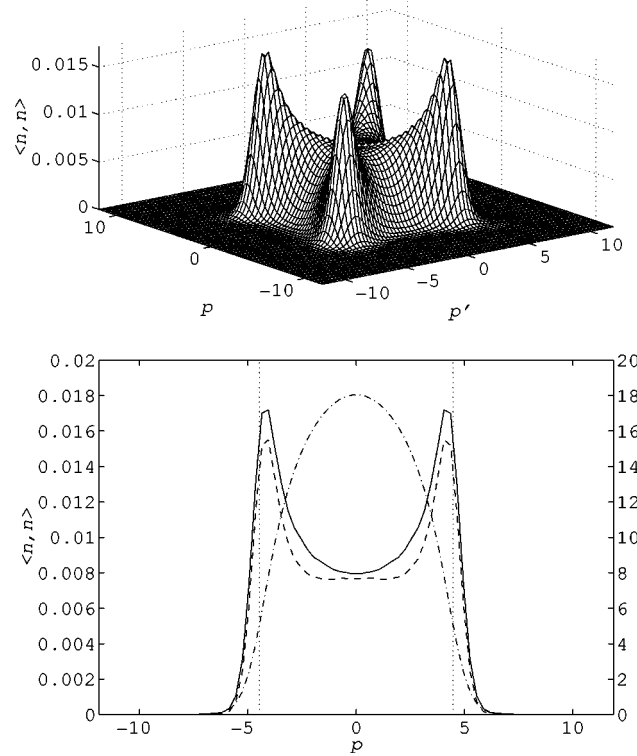FIG. 3. Function $\sigma(P)$ determining the signal-to-noise ratio, cf. Eq. (108).

FIG. 4. Top: momentum correlation function $\langle\hat{n}_\uparrow(p),\hat{n}_\downarrow(p')\rangle$ for fermions in a 1D harmonic trap calculated via the technique of stochastic transformations. Bottom: cross sections of the correlation function for $p=-p'$ (solid line) and $p=p'$ (dashed line), and the momentum distribution $\langle\hat{n}(p)\rangle$ (dash-dotted line). Vertical dotted lines mark the boundaries of the classically allowed region for a particle on the Fermi surface. $T=2$, $\mu=10$, $\kappa=-0.1$ (all oscillator units).

does not depend on the parameter μ/T. [To make this feature evident, we plot the correlation function in units of $|\kappa|T^{3/2}/\mu$, cf. Eq. (42)]. This universality is even more pronounced in Fig. 2, where the cross section of the correlation function for $P=P'$ is plotted as a function of $\Delta P=P-P_F$. We see that there is very little change to the shape of the cross section for $\mu/T\geqslant5$. The case of $\mu/T=2$ is different, which again reflects a badly defined Fermi surface; even in this case the universality is well pronounced for $P\geqslant P_F-\Delta p_F$.

Formally, the four-peak structure of the correlation function simply reflects the fact that, for larger k, the momentum probability density for the kth oscillator state is predominantly concentrated at the boundaries of the classically allowed region. The particles close to the Fermi surface hence spend most of their time at the boundaries of the classically allowed region in momentum space. This is exactly the property reflected by the momentum correlation function. However, physically, the correlations at opposite momenta (pseudo-Cooper correlations) remain difficult to explain. Naively, one might expect that the following simple picture should hold for temperatures that are not too low. Highly excited oscillator states correspond to a motion, which is in essence classical. This suggests a picture of trapped fermions
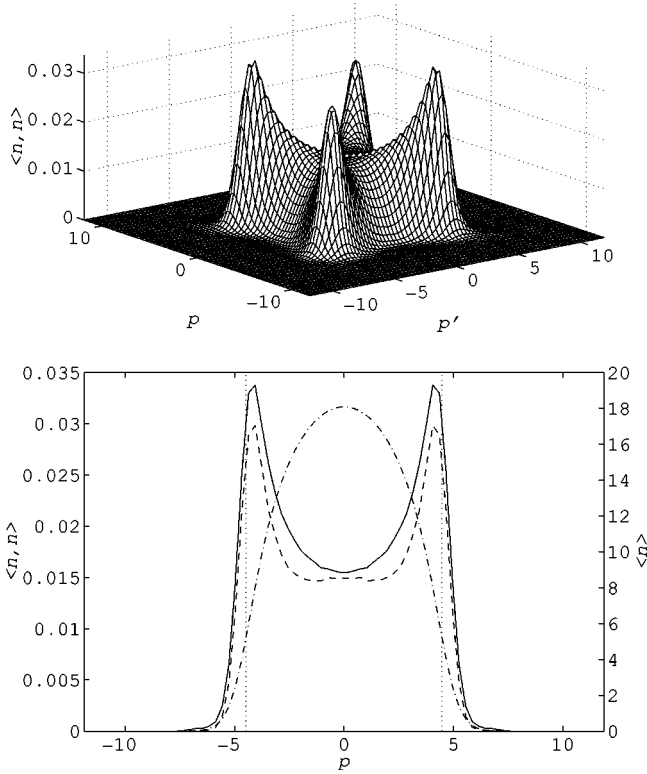
FIG. 5. Same as Fig. 4, for $T=2$, $\mu=10$, $\kappa=-0.2$ (all oscillator units).

as a microcanonical ensemble of trapped classical particles, where the temperature plays the role of the energy spread. Classical statistics should then favor co-oscillating pairs of fermions, where a spin-up particle and a spin-down particle oscillate with close amplitude and phase. However, in such a pair momenta are always equal, and, therefore, can only contribute to the equal-momentum correlations. Since things should grow more classical with temperature, only correlations for equal momenta should survive as temperature increases.

It is instructive to estimate the parameters for which this picture should be valid. The energy width of the Fermi-surface is assessed as T. For the motion of the Fermi-surface particle to become classical, this width should accommodate enough oscillator states to form a coherent state. This results in the condition, $\sqrt{\mu}\lesssim T$, for the Cooper-like correlation to vanish. It does not contradict $1,\mu^{1/3}\ll T\lesssim \mu$ for which Eq. (42) is valid (one example is $T=100$, $\mu=1000$). Hence, if this semiclassical picture held, no pseudo-Cooper correlations would appear in our results, whereas according to Eq. (42) the correlations for $P=P'$ and $P=-P'$ are equally strong. Thus the very presence of these correlations, and especially the fact that they persist at higher temperatures, is a manifestation of a quantum and fermionic nature of trapped particles.

Can the pseudo-Cooper correlations be observed? In principle, this is possible by placing two particle detectors at the opposite sides of the "needle." Then, the momentum correlations will manifest themselves as time-of-flight correlations after the trap is switched off and the fermionic sample is

allowed to expand freely. Assume on each side we have counted atoms that have momenta within an interval of the order of Δp_F in the vicinity of p_F. In the "large-P" units we use, this corresponds to an interval of unity, so that the power of correlations we wish to observe equals the correlation function itself. The number of particles in this sample is $\langle \hat{n}(p)\rangle$. In the ideal case of 100% detection efficiency and the only noise source in this measurement being the shot noise, the single-run signal-to-noise ratio squared is found to be, (accounting for the two species of fermions)

$$\left[\frac{S}{N}\right]^2 = \frac{2\langle \hat{n}_\uparrow(P),\hat{n}_\downarrow(P)\rangle}{2[\langle \hat{n}_\uparrow(P)\rangle+\langle \hat{n}_\downarrow(P)\rangle]} = \frac{|\kappa|}{2\mu^{1/2}}\sigma(P)=\frac{|\kappa_0|\bar{\mu}}{2\mu^{1/2}}\sigma(P),$$
(108)

where $\sigma(P)$ is the ratio of the integrals of the RHS of Eqs. (42) and (44) for $P=P'$.

The function $\sigma(P)$ is plotted in Fig. 3 for $\mu/T =2,5,10,20$. On inspecting this figure, we see that the maximal signal-to-noise ratio is achieved slightly above P_F and corresponds to $\sigma(P)=0.13$. With $\kappa_0=-0.067$, and assuming that the trap is filled to the possible limit, $\mu=\bar{\mu}$, the ideal signal-to-noise ratio is thus estimated as $0.005\bar{\mu}^{1/2}$. Thus in the ideal case no more than 200 experimental runs are needed to beat the shot noise. How other noises and experimental imperfections would affect this result is a subject for further discussion.
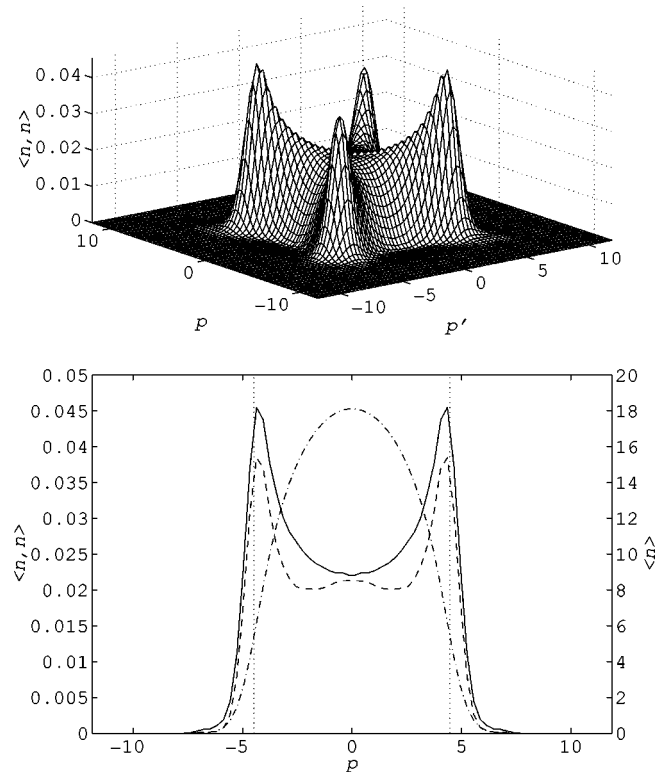

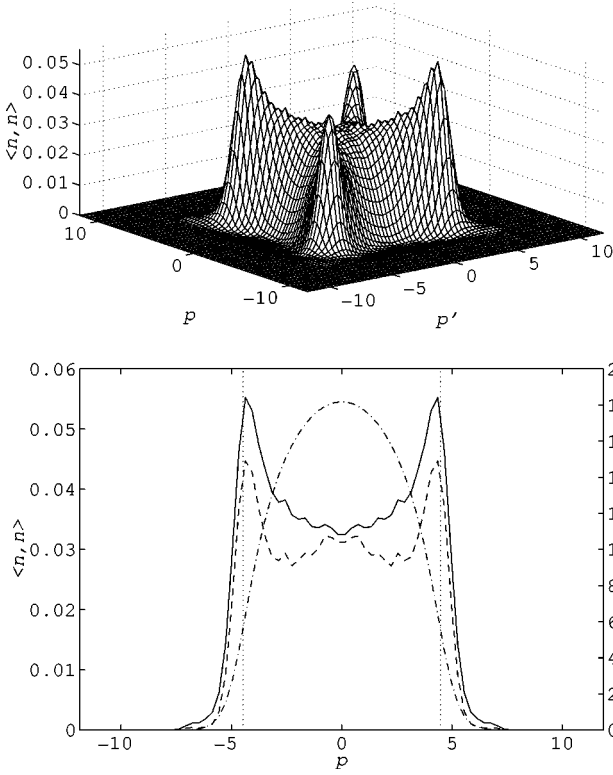FIG. 6. Same as Fig. 4, for $T=2$, $\mu=10$, $\kappa=-0.3$ (all oscillator units).

FIG. 7. Same as Fig. 4, for $T=2$, $\mu=10$, $\kappa=-0.4$ (all oscillator units).

FIG. 8. Same as Fig. 4, for $T=2$, $\mu=10$, $\kappa=-0.5$ (all oscillator units).

2. Limits to 1D perturbative treatment

We will now consider the limits of the perturbative treatment. An effective energy a fermion acquires due to interactions is particle density times κ. Since the number of particles is $\sim\mu$ and the size of the trapped sample is of the order of $\sqrt{\mu}$, the density is estimated as $\sqrt{\mu}$. The same estimate applies to particles at the Fermi surface: their number is estimated as T, and they are concentrated in the interval δx_F found from $T\sim\delta x_F^2/2=\sqrt{2\mu}\,\delta x_F$, so that their density $T/\delta x_F\sim\sqrt{\mu}$. Thus the perturbative treatment is applicable if

$$\kappa\sqrt{\mu}\ll\mu,T. \tag{109}$$

The former inequality here guarantees that the trap potential is not disturbed by the Dirac sea. The latter allows interactions within the Fermi surface to be treated perturbatively.

In a needle trap, perturbation results are applicable if

$$\kappa_0\bar{\mu}\sqrt{\mu}\ll T,\mu. \tag{110}$$

Comparing this to Eq. (108), we find that the applicability of the perturbation treatment limits the signal-to-noise ratio squared, $[S/N]^2$, by $\kappa_0 T/2\mu$. We need $\mu/T\gtrsim5$ to establish the pseudo-Cooper pairing feature, which in turn limits $[S/N]^2$ to 0.007. This does not appear to be a problem. A more serious problem is that the applicability of the 1D perturbative treatment sets a limit to the number of particles in the trap: $\kappa_0\bar{\mu}\ll\sqrt{\mu}$ and $\mu\lesssim\bar{\mu}$ yields $\bar{\mu}\ll1/k_0^2=250$.
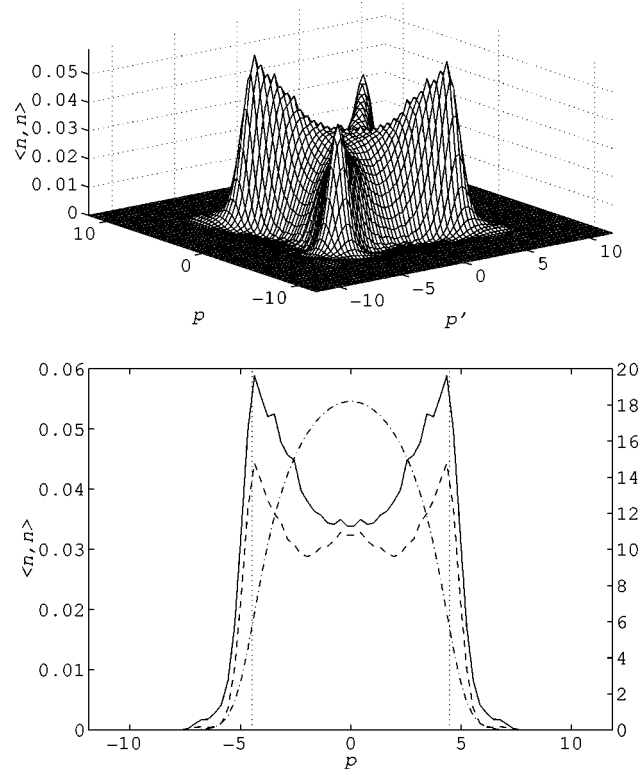
B. Nonperturbative regime

Not unexpectedly, simulations using the random transformation techniques turned out to be quite involved. Reliable results were only obtained for few atoms in a trap. At the same time, these simulations revealed certain details and tendencies that are quite likely to be present in momentum correlations in larger samples. They also give an indication of certain effects characteristic of finite fermionic samples that could be a subject to further investigation.

In Figs. 4–8, we present the results of the nonperturbative Monte Carlo simulations of the trapped fermions, for $T=2$, $\mu=10$, and κ ranging from $k=-0.1$ (Fig. 4) to $k=-0.5$ (Fig. 8). Equation (92) was simulated in a basis of 30 lower oscillator states. For each value of κ, we calculated the momentum distribution and the spin up/spin down momentum correlation function using the formulas

$$\langle\hat{n}(p)\rangle=\langle\hat{n}_\uparrow(p)\rangle+\langle\hat{n}_\downarrow(p)\rangle$$
$$=2Z^{-1}\overline{\text{Det}^2(1+\alpha)[(1+\alpha^{-1})^{-1}]}_{pp}, \tag{111}$$

$$\langle\hat{n}_\uparrow(p),\hat{n}_\downarrow(p')\rangle$$

$$=\langle\hat{n}_\uparrow(p)\hat{n}_\downarrow(p')\rangle-\langle\hat{n}_\uparrow(p)\rangle\langle\hat{n}_\downarrow(p')\rangle$$
$$=Z^{-1}\overline{\text{Det}^2(1+\alpha)[(1+\alpha^{-1})^{-1}]_{pp}[(1+\alpha^{-1})^{-1}]_{p'p'}}$$
$$-\tfrac{1}{4}\langle\hat{n}(p)\rangle^2, \tag{112}$$

where $Z=\overline{\text{Det}^2(1+\alpha)}$ (recall that $\beta=\alpha\oplus\alpha$). These quantities were found as averages over samples of 10^6 trajectories.
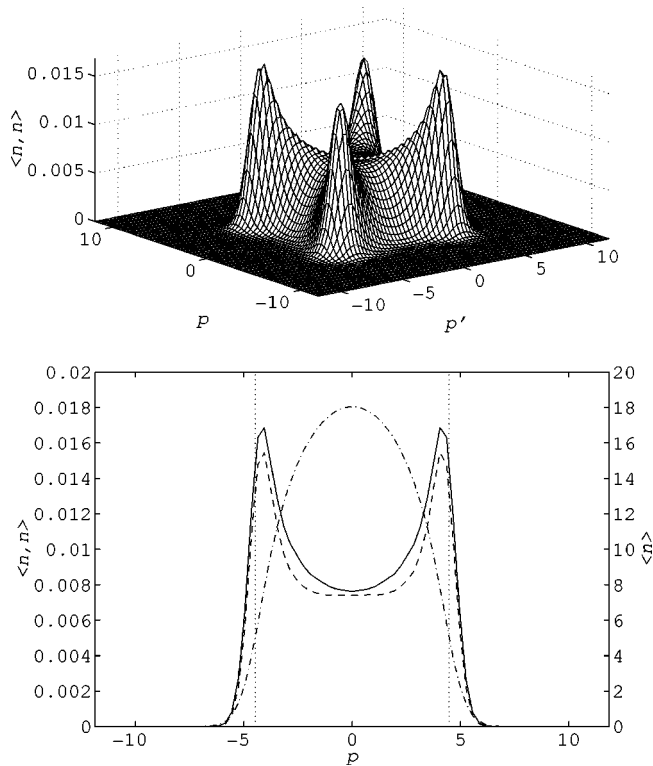
FIG. 9. Same as Fig. 4 but calculated via the exact perturbative relation (24), for $T=2$, $\mu=10$, $\kappa=-0.1$ (all oscillator units). This figure should be compared to Fig. 4.

The top panels of Figs. 4–8 show the correlation function as a 3D plot. In the bottom panels of the figures, we plot the cross sections of this function for opposite momenta, $p'=-p$, as solid lines, and for equal momenta, $p'=p$, as dashed lines. The momentum distributions, $\langle\hat{n}(p)\rangle=\langle\hat{n}_\downarrow(p)\rangle+\langle\hat{n}_\uparrow(p)\rangle$, are shown as dash-dotted lines. In Fig. 9, we also present the results of the perturbative calculation using the exact perturbative formula (24) for $\kappa=-0.1$. We see that the data presented in Figs. 4 and 9 are barely distinguishable,

providing us with verification both of the techniques of random algebra transformations and of the numerical algorithm.

As opposed to the approximate formula (42), both the exact perturbative and the nonperturbative results show that pseudo-Cooper correlations always exceed those at equal momenta. The relative increase in the Cooper correlations at lower temperatures may be regarded as a precursor to a BCS transition proper. (Which in a final system should manifest itself as correlations at opposite momenta becoming *much* larger than correlations at equal momenta). As can be further seen from the figures, stronger interactions lead first to quantitative and then to qualitative changes in the momentum correlations. Correlations grow linearly as κ grows. On the top of that, we see that pseudo-Cooper correlations start to dominate the system. Interestingly, not only the difference between the opposite and same-momentum correlations grows, but pseudo-Cooper correlations also extend from the Fermi surface into the Dirac sea. This effect, which is a consequence of the final size of the system, becomes well pronounced at $-\kappa=0.5$.

So what can we expect in larger samples? Qualitatively, it is quite probable that the pseudo-Cooper correlations will be present. The difference between these and correlations at equal momenta, which is in essence the Cooper-paring proper, is much less likely to survive. A quantitative theory of these effects in larger samples, as well as an extension of our results to 3D (so as to get rid of the restrictions of a needle trap), remain subject to further work.

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