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Staggered orbital currents in the half-filled two-leg ladder

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Using Abelian bosonization with a careful treatment of the Klein factors, we show that a certain phase of the half-filled two-leg ladder, previously identified as having spin-Peierls order, instead exhibits staggered orbital currents with no dimerization.

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I. INTRODUCTION

One of the most intriguing phases of strongly correlated electrons is known variously as the "orbital antiferromagnet,"[1–3] the "staggered flux phase,"[4–9] or the "*d*-density wave."[10,11] It is characterized by circulating currents which produce local magnetic moments aligned in an antiferromagnetic (staggered) way. As a consequence, time-reversal symmetry as well as translational and rotational symmetries are spontaneously broken. Another phase, the "circulating current phase,"[12] is somewhat similar, but does not break translational symmetry. These phases have received considerable attention lately, due to their possible relevance to the pseudogap region in the phase diagram of the cuprates.[12,11] A recent neutron scattering experiment[13] on underdoped $YBa_2Cu_3O_{6.6}$ has been interpreted[14] as evidence for these staggered orbital currents.

In this paper we focus on the half-filled two-leg ladder, which is the simplest system that can support the staggered flux (SF) phase (see Fig. 1). As the order parameter of this phase breaks a discrete (Z_2) symmetry, the possibility of true long-range order (LRO) of the currents is not *a priori* excluded at zero temperature in this one-dimensional system, in contrast to the situation for order parameters that break continuous symmetries, which causes their correlations to exhibit at most quasi-LRO with power-law decay.

For weak interactions, the ladder can be treated using bosonization and the perturbative renormalization group (RG). For this case, the SF phase has been found in the phase diagram for spinless electrons at half-filling.[15] Furthermore, away from half-filling, regions with dominant tendencies toward SF ordering have been found both for spinless[16] and spinful[17] electrons. Note that for general (i.e., incommensurate) fillings, true LRO of the currents is not possible due to the absence of Umklapp interactions (see Sec. V for a more detailed discussion). The results for the doped ladder with and without spin were summarized in Ref. 18, which also investigated the effects of disorder.

Here we are concerned with the SF phase for spinful electrons in a weakly interacting half-filled two-leg ladder. In contrast to the other weak-interaction studies mentioned so far, in the approach used here the nearest-neighbor hopping parameters $t_\perp$ and t along the rungs and legs, respectively, can be of the same order. We reanalyze the nature of a specific phase found in Ref. 19, and demonstrate that this phase, previously identified to be of spin-Peierls (SP) type, actually exhibits staggered orbital currents with no dimerization, and therefore in fact is the SF phase. In contrast to the infinite-U half-filled case, where the constraint of no double occupancies makes the currents an unobservable gauge artifact, the LRO currents found here are observable. Furthermore, since all excitations are gapped, the SF phase at half-filling is insulating.

We perform our calculations using Abelian bosonization, paying careful attention to the Klein factors in this formalism.[20,21] As a check of our treatment, we also reproduce the identification of the CDW phase found in Ref. 19. Furthermore, we show that our results are consistent with those found for the doped ladder.[18]

The paper is organized as follows. In Secs. II and III we discuss the ladder model and its continuum limit and bosonized form, closely following the approach of Ref. 19. In Sec. IV we define various local order parameters, derive their bosonized expressions, and calculate their expectation values in the SF phase. The results are discussed further in Sec. V. Some of the technical details have been placed in two appendices.

II. THE HALF-FILLED TWO-LEG LADDER AND ITS CONTINUUM LIMIT

A. Kinetic energy

We consider a two-leg ladder where the electrons can hop only between nearest-neighbor sites along the rungs and legs. The kinetic energy then reads

$$H_0 = -t\sum_{lms} c_{ls}^\dagger(m+1)c_{ls}(m) - t_\perp \sum_{ms} c_{1s}^\dagger(m)c_{2s}(m) + \text{H.c.} \tag{2.1}$$

The operators $c_{ls}(m)$, and $c_{ls}^\dagger(m)$, respectively, annihilate and create an electron on site $m=1,\ldots,N$ on leg $l=1,2$ with spin $s=\uparrow,\downarrow$, and obey $\{c_{ls}(m),c_{l's'}^\dagger(m')\} = \delta_{ll'}\delta_{mm'}\delta_{ss'}$, with all other anticommutators vanishing. Periodic (open) boundary conditions are used along (perpendicular to) the leg direction. Introducing even and odd combinations

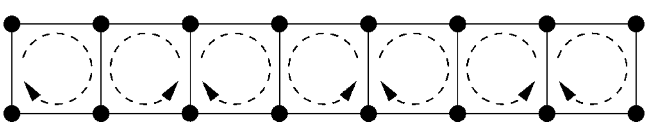

FIG. 1. Current flow in the staggered flux phase of the half-filled two-leg ladder. Reversing the currents gives the time-reversed state.

$$c_{e/o,s}=\frac{1}{\sqrt{2}}(c_{1s}\pm c_{2s}) \tag{2.2}$$

and Fourier transforming along the leg direction, the kinetic energy becomes diagonal in momentum space, describing two uncoupled bands with dispersion $\varepsilon_{e/o}(k)=-2t\cos ka \mp t_\perp$, where a is the lattice constant. Taking $t_\perp$ positive, the even (odd) combination gives a bonding (antibonding) band. We consider a half-filled system and $t_\perp<2t$, in which case the Fermi level is at zero energy and crosses both bands, thus giving rise to four Fermi points $\pm k_{Fe/o}$ which satisfy $k_{Fe}+k_{Fo}=\pi/a$.

We will assume weak interactions and focus on the low-energy, long-wavelength properties of the model, so that we may linearize H_0 around the Fermi points. It will be most convenient to work with a coordinate-space representation of the Hamiltonian. For this purpose we decompose the band operator $c_{\lambda s}(m)$ $(\lambda=e,o)$ into a sum of left- and right-moving slowly varying (on the scale of the lattice constant) continuum fields,

$$c_{\lambda s}(m)=\sqrt{a}[e^{-ik_{F\lambda}x}\psi_{L\lambda s}(x)+e^{ik_{F\lambda}x}\psi_{R\lambda s}(x)], \tag{2.3}$$

where $x\equiv ma$. The linearized kinetic energy can then be written $H_0=\int dx\mathcal{H}_0$, where

$$\mathcal{H}_0=-iv_F\sum_{\lambda s}[\psi^\dagger_{R\lambda s}\partial_x\psi_{R\lambda s}-\psi^\dagger_{L\lambda s}\partial_x\psi_{L\lambda s}]. \tag{2.4}$$

In this expression and throughout the paper it is understood that products of fermionic (and bosonic) operators that may be evaluated at the same point are to be normal-ordered. The bare Fermi velocity v_F is the same for both bands and is given by $v_F=a\sqrt{(2t)^2-t_\perp^2}$.

B. Interactions

The continuum description of general, but weak, finite-ranged, spin-independent interactions, to leading order in the interaction strengths, was carefully discussed in Refs. 19, 22, and 23. One can restrict attention to terms which are both marginal (i.e., consisting of four-fermion interactions with no spatial derivatives) and nonchiral (i.e., containing two right-moving and two left-moving fermions). These terms can be classified according to whether they conserve momentum or not. The Hamiltonian density for momentum-conserving terms reads

$$\mathcal{H}_I^{(1)}=\sum_{\lambda\mu}\{b^\rho_{\lambda\mu}J_{R\lambda\mu}J_{L\lambda\mu}-b^\sigma_{\lambda\mu}\boldsymbol{J}_{R\lambda\mu}\cdot\boldsymbol{J}_{L\lambda\mu}+f^\rho_{\lambda\mu}J_{R\lambda\lambda}J_{L\mu\mu}-f^\sigma_{\lambda\mu}\boldsymbol{J}_{R\lambda\lambda}\cdot\boldsymbol{J}_{L\mu\mu}\}. \tag{2.5}$$

Here f and b refer to forward and backward scattering processes, respectively, and

$$J_{P\lambda\mu}=\sum_s\psi^\dagger_{P\lambda s}\psi_{P\mu s}, \tag{2.6}$$

$$\boldsymbol{J}_{P\lambda\mu}=\frac{1}{2}\sum_{ss'}\psi^\dagger_{P\lambda s}\boldsymbol{\sigma}_{ss'}\psi_{P\mu s'}, \tag{2.7}$$

where σ^x, σ^y, and σ^z are the Pauli matrices. The following general symmetries hold: $b^\nu_{eo}=b^\nu_{oe}$ and $f^\nu_{eo}=f^\nu_{oe}$, where $\nu=\rho,\sigma$. To avoid double-counting, we set $f^\nu_{\lambda\lambda}=0$. At half-filling the model also has particle-hole symmetry, which implies $b^\nu_{oo}=b^\nu_{ee}$, leaving six independent couplings of this type.

Half-filling also allows for non-momentum-conserving (i.e., Umklapp) terms. The Hamiltonian density for these interactions reads

$$\mathcal{H}_I^{(2)}=\sum_{\lambda\mu}\{u^\rho_{\lambda\mu}I^\dagger_{R\lambda\mu}I_{L\overline{\lambda}\overline{\mu}}-u^\sigma_{\lambda\mu}\boldsymbol{I}^\dagger_{R\lambda\mu}\cdot\boldsymbol{I}_{L\overline{\lambda}\overline{\mu}}+\text{H.c.}\}, \tag{2.8}$$

where $\overline{e}=o$ and $\overline{o}=e$. Here we have defined

$$I_{P\lambda\mu}=\sum_{ss'}\psi_{P\lambda s}\epsilon_{ss'}\psi_{P\mu s'}, \tag{2.9}$$

$$\boldsymbol{I}_{P\lambda\mu}=\frac{1}{2}\sum_{ss'}\psi_{P\lambda s}(\epsilon\boldsymbol{\sigma})_{ss'}\psi_{P\mu s'}, \tag{2.10}$$

where $\epsilon=-i\sigma^y$. We may take $u^\nu_{eo}=u^\nu_{oe}$ since $I_{P\lambda\mu}=I_{P\mu\lambda}$ and $\boldsymbol{I}_{P\lambda\mu}=-\boldsymbol{I}_{P\mu\lambda}$. The latter result also implies $\boldsymbol{I}_{P\lambda\lambda}=0$, so that we can take $u^\sigma_{\lambda\lambda}=0$. In addition, particle-hole symmetry gives $u^\rho_{ee}=u^\rho_{oo}$, leaving three independent Umklapp couplings. Thus a total of nine independent coupling constants must be taken into account in this model of the half-filled two-leg ladder.

III. BOSONIZATION

In the Abelian bosonization formalism,[20,21,24] the fermionic field operators $\psi_{P\lambda s}$ can be expressed in terms of dual Hermitian bosonic fields $\phi_{\lambda s}$ and $\theta_{\lambda s}$ as[25]

$$\psi_{P\lambda s}=\frac{1}{\sqrt{2\pi\epsilon}}\kappa_{\lambda s}\exp[i(P\phi_{\lambda s}+\theta_{\lambda s})], \tag{3.1}$$

where ϵ is a short-distance cutoff, and $P=R/L=\pm 1$. The bosonic fields satisfy the commutation relations

$$[\phi_{\lambda s}(x),\phi_{\lambda' s'}(x')]=[\theta_{\lambda s}(x),\theta_{\lambda' s'}(x')]=0, \tag{3.2a}$$

$$[\phi_{\lambda s}(x),\theta_{\lambda' s'}(x')]=i\pi\delta_{\lambda\lambda'}\delta_{ss'}\Theta(x-x'), \tag{3.2b}$$

the latter result written for $\epsilon\to 0$. Here $\Theta(x)$ is the Heaviside function. The long-wavelength normal-ordered fermionic densities can be expressed in terms of the bosonic fields as $\psi^\dagger_{P\lambda s}\psi_{P\lambda s}=\partial_x(\phi_{\lambda s}+P\theta_{\lambda s})/2\pi$.

The Klein factors $\kappa_{\lambda s}$ commute with the bosonic fields, and satisfy

$$\{\kappa_{\lambda s},\kappa_{\lambda' s'}\}=2\delta_{\lambda\lambda'}\delta_{ss'}. \tag{3.3}$$

Note that the Klein factors used here are Hermitian (instead of unitary), since we follow the common procedure of neglecting the number-changing property of the Klein factors in the thermodynamic limit.[20]

Charge and spin operators are now defined as

$$\phi_{\lambda\rho}=\frac{1}{\sqrt{2}}(\phi_{\lambda\uparrow}+\phi_{\lambda\downarrow}), \tag{3.4a}$$

$$\phi_{\lambda\sigma}=\frac{1}{\sqrt{2}}(\phi_{\lambda\uparrow}-\phi_{\lambda\downarrow}), \tag{3.4b}$$

with similar definitions of the θ operators. We also define

$$\phi_{r\nu}=\frac{1}{\sqrt{2}}(\phi_{e\nu}+r\phi_{o\nu}), \tag{3.5}$$

where $r=\pm$ and $\nu=\rho,\sigma$. Again, similar definitions apply to the θ operators. Both Eqs. (3.4) and (3.5) are unitary transformations, which implies that the commutation relations for the new sets of operators are of the same type as those in Eq. (3.2).

Next we consider the bosonized form of the Hamiltonian density $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I^{(1)}+\mathcal{H}_I^{(2)}$, which is most succinctly expressed in terms of the variables $\phi_{r\nu}$ and $\theta_{r\nu}$. The kinetic-energy density reads

$$\mathcal{H}_0=\frac{v_F}{2\pi}\sum_{r\nu}\left[(\partial_x\phi_{r\nu})^2+(\partial_x\theta_{r\nu})^2\right]. \tag{3.6}$$

The momentum-conserving part of the interactions can be written $\mathcal{H}_I^{(1)}=\mathcal{H}_I^{(1a)}+\mathcal{H}_I^{(1b)}$, where

$$\mathcal{H}_I^{(1a)}=\frac{1}{2\pi^2}\sum_{r\nu}A_{r\nu}[(\partial_x\phi_{r\nu})^2-(\partial_x\theta_{r\nu})^2]. \tag{3.7}$$

Here $A_{r\nu}=h_\nu[b_{ee}^\nu+rf_{eo}^\nu]$ with $h_\rho=1$, $h_\sigma=-1/4$. Furthermore,

$$\begin{aligned}\mathcal{H}_I^{(1b)}=-\frac{1}{(2\pi\epsilon)^2}&[2\hat{\Gamma}b_{eo}^\sigma\cos 2\theta_{-\rho}\cos 2\phi_{+\sigma}\\&-\cos 2\phi_{+\sigma}(2b_{ee}^\sigma\cos 2\phi_{-\sigma}+2\hat{\Gamma}f_{eo}^\sigma\cos 2\theta_{-\sigma})\\&+\cos 2\theta_{-\rho}(\hat{\Gamma}b_{eo}^+\cos 2\phi_{-\sigma}+b_{eo}^-\cos 2\theta_{-\sigma})],\end{aligned} \tag{3.8}$$

with $b_{eo}^\pm=b_{eo}^\sigma\pm 4b_{eo}^\rho$ and $\hat{\Gamma}=\kappa_{e\uparrow}\kappa_{e\downarrow}\kappa_{o\uparrow}\kappa_{o\downarrow}$. Finally, the bosonized form of the Umklapp interaction density reads

$$\begin{aligned}\mathcal{H}_I^{(2)}=-\frac{2}{(2\pi\epsilon)^2}\cos 2\phi_{+\rho}&[8\hat{\Gamma}u_{ee}^\rho\cos 2\theta_{-\rho}+2u_{eo}^\sigma\cos 2\phi_{+\sigma}\\&+u_{eo}^+\cos 2\phi_{-\sigma}+\hat{\Gamma}u_{eo}^-\cos 2\theta_{-\sigma}],\end{aligned} \tag{3.9}$$

with $u_{eo}^\pm=u_{eo}^\sigma\pm 4u_{eo}^\rho$.

Since the Hermitian operator $\hat{\Gamma}$ obeys $\hat{\Gamma}^2=I$, $\hat{\Gamma}$ has eigenvalues $\Gamma=\pm 1$. Furthermore, since $[H,\hat{\Gamma}]=0$, H and $\hat{\Gamma}$ can be simultaneously diagonalized.

IV. THE STAGGERED FLUX PHASE

A. Pinned fields

Numerical integration of the one-loop RG equations for the couplings shows[19,22,23] that some of the couplings remain small, while the others grow (sometimes after a sign change) and eventually diverge. The weak-coupling RG flow must be cut off before it leaves the regime of its perturbative validity. The ratios of the diverging couplings at the cutoff scale are found to approach fixed constants in the limit of asymptotically small bare couplings, with different sets of ratios corresponding to different phases of the ladder. In the SF phase b_{ee}^ρ and b_{ee}^σ are negligible, while the diverging couplings are given by[19]

$$f_{eo}^\rho=-\frac{1}{4}f_{eo}^\sigma=-b_{eo}^\rho=\frac{1}{4}b_{eo}^\sigma=\frac{1}{2}u_{eo}^\sigma=-2u_{eo}^\rho=2u_{ee}^\rho\equiv g>0. \tag{4.1}$$

The resulting low-energy effective Hamiltonian can be mapped onto an SO(8) Gross-Neveu model, whose integrability may be exploited to extract the exact energies, degeneracies and quantum numbers of all the low-energy excited states.[19] However, for our discussion, a semiclassical reasoning will suffice. Since the single coupling constant g flows toward large values, in the semiclassical ground state the bosonic fields in the Hamiltonian will be pinned to values which minimize the cosine interaction $\mathcal{H}_I^{(1b)}+\mathcal{H}_I^{(2)}$. Note that this argument would not be valid if the cosine interactions were to contain both the dual fields $\phi_{-\sigma}$ and $\theta_{-\sigma}$, since then the uncertainty principle would forbid both fields to be pinned. However, $\phi_{-\sigma}$ disappears from the cosine interaction because b_{ee}^σ is negligible and $b_{eo}^+=u_{eo}^+=0$. The pinned fields are then $\phi_{+\rho}$, $\phi_{+\sigma}$, $\theta_{-\rho}$, and $\theta_{-\sigma}$. Since all four bosonic modes $r\nu$ are pinned, the SF phase has no gapless excitations at half-filling.

The possible solutions for the pinned fields are found by minimizing $\langle\Gamma|H|\Gamma\rangle$, where $|\Gamma\rangle$ is the eigenstate of $\hat{\Gamma}$ with eigenvalue Γ (these solutions will depend on Γ, but the physics will of course not, as will be seen explicitly in Sec. III B). There are infinitely many solutions for the pinned fields that minimize $\langle\Gamma|H|\Gamma\rangle$. However, this multitude of solutions is only apparent; taking into account the fact that the bosonic fields are not gauge-invariant, it can be shown that there are only two physically distinct ground states.[19] The pinned-field configurations that we will use to specify these ground states are given in Table I.

B. Order parameters

In this subsection we explicitly show that the phase characterized by the couplings in Eq. (4.1) is not of spin-Peierls type with a (π,π) modulation in the kinetic energy,[19] but instead is the SF phase. We first define the relevant order

TABLE I. The Γ-dependent pinned-field configurations used for the ground states in the SF and CDW phases (we choose $\langle\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|\Gamma\rangle=i$; see Appendix A). The two configurations listed here for a given ground state are physically equivalent, as can be seen from Table II.

Ground state	Γ	$\langle\phi_{+\rho}\rangle$	$\langle\phi_{+\sigma}\rangle$	$\langle\theta_{-\rho}\rangle$	$\langle\theta_{-\sigma}\rangle$
SF 1	1	0	0	0	0
SF 2	1	π	0	0	0
SF 1	-1	π	0	$\pi/2$	$\pi/2$
SF 2	-1	0	0	$\pi/2$	$\pi/2$
CDW 1	1	0	0	$\pi/2$	0
CDW 2	1	π	0	$\pi/2$	0
CDW 1	-1	0	0	0	$\pi/2$
CDW 2	-1	π	0	0	$\pi/2$

parameters. The fundamental definition of the current operator comes from interpreting the Heisenberg equation of motion for the number operator

$$n_l(m)=\sum_s c_{ls}^\dagger(m)c_{ls}(m) \tag{4.2}$$

as a discretized continuity equation. We will assume that the SF phase is a low-energy phase of a lattice Hamiltonian whose interactions commute with $n_l(m)$. This is, e.g., the case for density-density and spin-exchange interactions. Then the components of the current operator take their conventional forms (see Fig. 2)

$$j_\perp(m)=it_\perp a\sum_s [c_{2s}^\dagger(m)c_{1s}(m)-\text{H.c.}], \tag{4.3}$$

$$j_l(m)=ita\sum_s [c_{ls}^\dagger(m+1)c_{ls}(m)-\text{H.c.}]. \tag{4.4}$$

Furthermore, the local kinetic-energy operator is

$$k_l(m)=-t\sum_s [c_{ls}^\dagger(m+1)c_{ls}(m)+\text{H.c.}]. \tag{4.5}$$

For completeness, in our discussion we also include the number operator $n_l(m)$ itself, since we will later check that our calculations reproduce the results for the CDW phase found in Ref. 19.

Next we outline the derivation of the bosonized expressions for these order parameters. It will be convenient to define an auxiliary operator,

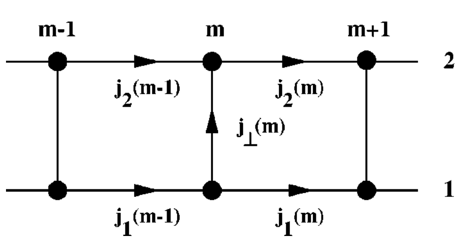

FIG. 2. Currents as defined in Eqs. (4.3) and (4.4). Current conservation is expressed by Eq. (4.13).

$$G_l(m,u,v)=\sum_s [c_{ls}^\dagger(m+u)c_{ls}(m)+v\cdot\text{H.c.}]. \tag{4.6}$$

Then $j_l(m)=itaG_l(m,1,-1)$, $k_l(m)=-tG_l(m,1,1)$, and $n_l(m)=G_l(m,0,1)$. The continuum version of $G_l(m,u,v)$ will contain products of type $:\psi_{P\lambda s}^\dagger(x+ua)\psi_{P'\lambda's'}(x):$, i.e., with the argument of the field operators differing by a lattice constant when $u=1$ (here we have temporarily included the normal-ordering symbol explicitly). One can safely Taylor-expand within the normal-ordering symbol to obtain $:\psi_{P\lambda s}^\dagger(x)\psi_{P'\lambda's'}(x):+ua:\partial_x\psi_{P\lambda s}^\dagger(x)\psi_{P'\lambda's'}(x):$ (note that due to the normal-ordering, all order parameters will be measured with respect to their values in the noninteracting ground state). For now, we only keep the zeroth-order term in the Taylor expansion, and comment briefly on higher-order terms later. We find

$$\begin{aligned}G_l(m,u,v)=\frac{a}{2}\sum_{P\lambda s}\{&\psi_{P\lambda s}^\dagger\psi_{P\lambda s}(e^{-iPk_{F\lambda}ua}+v\,e^{iPk_{F\lambda}ua})\\&+\psi_{P\lambda s}^\dagger\psi_{-P\lambda s}e^{-2iPk_{F\lambda}x}e^{-iPk_{F\lambda}ua}(1+v)\\&-(-1)^l[\psi_{P\lambda s}^\dagger\psi_{P\bar{\lambda}s}e^{-iP(k_{F\lambda}-k_{F\bar{\lambda}})x}\\&\times(e^{-iPk_{F\lambda}ua}+v\,e^{iPk_{F\bar{\lambda}}ua})\\&+\psi_{P\lambda s}^\dagger\psi_{-P\bar{\lambda}s}e^{-iP(k_{F\lambda}+k_{F\bar{\lambda}})x}\\&\times(e^{-iPk_{F\lambda}ua}+v\,e^{-iPk_{F\bar{\lambda}}ua})]\}.\end{aligned} \tag{4.7}$$

The expectation value of the normal-ordered long-wavelength density $\psi_{P\lambda s}^\dagger\psi_{P\lambda s}$ is zero. Bosonizing $\psi_{P\lambda s}^\dagger\psi_{-P\lambda s}$ produces exponentials containing the fields $\phi_{+\rho}$, $\phi_{+\sigma}$, $\phi_{-\rho}$ and $\phi_{-\sigma}$. Bosonizing $\psi_{P\lambda s}^\dagger\psi_{P\bar{\lambda}s}$ produces exponentials containing the fields $\theta_{-\rho}$, $\theta_{-\sigma}$, $\phi_{-\rho}$ and $\phi_{-\sigma}$. Since $\phi_{-\rho}$ and $\phi_{-\sigma}$ are dual to the pinned fields $\theta_{-\rho}$ and $\theta_{-\sigma}$, respectively, they will fluctuate strongly due to the uncertainty principle, and the expectation value of exponentials of these fields will vanish. We are therefore left with a term which contains products of type $\psi_{P\lambda s}^\dagger\psi_{-P\bar{\lambda}s}$. Bosonizing this produces exponentials containing the four pinned fields, so this term will have a nonzero expectation value.

Next consider $j_\perp(m)$. Its continuum expression only contains products of type $\psi_{P\lambda s}^\dagger\psi_{P\bar{\lambda}s}$ and $\psi_{P\lambda s}^\dagger\psi_{-P\bar{\lambda}s}$. Thus only the latter product will contribute to the expectation value of this operator. Note that in order to calculate $j_\perp(m)$ no Taylor expansion is necessary, since both fermion operators are taken at the same value of m from the outset.

Using $k_{Fe}+k_{Fo}=\pi/a$ and $2t\cos k_{Fo}a=t_\perp$ to simplify expressions, we find

$$\langle j_\perp(m)\rangle=it_\perp a^2(-1)^m\langle F_{-1}(x)\rangle+\text{c.c.}, \tag{4.8}$$

$$\langle j_l(m)\rangle=\tfrac{1}{2}\,it_\perp a^2(-1)^{l+m}\langle F_{-1}(x)\rangle+\text{c.c.}, \tag{4.9}$$

$$\langle k_l(m)\rangle=ia\sqrt{t^2-(t_\perp/2)^2}(-1)^{l+m}\langle F_1(x)\rangle+\text{c.c.}, \tag{4.10}$$

$$\langle n_l(m)\rangle=-a(-1)^{l+m}\langle F_1(x)\rangle+\text{c.c.} \tag{4.11}$$

Here we have defined the operator

$$F_p(x)=\sum_s [\psi_{Les}^\dagger\psi_{Ros}+p\psi_{Los}^\dagger\psi_{Res}]. \tag{4.12}$$

The expectation value of $F_p(x)$ is independent of x. It then follows from Eqs. (4.8) and (4.9) that if currents exist, they will flow around the plaquettes in a staggered pattern as shown in Fig. 1. Current conservation is expressed by

$$\langle j_l(m)\rangle=\langle j_l(m-1)\rangle+(-1)^l\langle j_\perp(m)\rangle \tag{4.13}$$

(see Fig. 2). Since expression (4.9) for $\langle j_l(m)\rangle$ only contains the zeroth-order term in the Taylor series expansion of the field operators, we conclude from Eq. (4.13) that the higher-order terms do not contribute to the plaquette currents.

The bosonized expression for $F_p(x)$ can be written

$$F_p(x)=\frac{1}{2\pi\epsilon}\sum_{\lambda s} d_{\lambda\bar\lambda}^{\alpha(p)}\kappa_{\lambda s}\kappa_{\bar\lambda s}\times\exp[i(\phi_{+\rho}+s\phi_{+\sigma}-d_{\lambda\bar\lambda}\theta_{-\rho}-sd_{\lambda\bar\lambda}\theta_{-\sigma})]. \tag{4.14}$$

Here we have defined $d_{eo}=-d_{oe}=1$, $s=\uparrow\downarrow=\pm1$, and

$$\alpha(p)=\begin{cases}1, & p=-1\\ 2, & p=1.\end{cases} \tag{4.15}$$

Let $|n;\Gamma\rangle\equiv|n(\Gamma)\rangle\otimes|\Gamma\rangle$ be a simultaneous eigenstate of H and $\hat\Gamma$. The eigenstate $|n(\Gamma)\rangle$ lives in the Hilbert space where the bosonic operators act, while $|\Gamma\rangle$ was introduced in Sec. IV A. We now consider a particular ground state, denoted by $|0;\Gamma\rangle$, and calculate the expectation value of $F_p(x)$ in this state. First we insert the completeness relation (A8) between the rightmost Klein factor and the exponential in Eq. (4.14), and use Eq. (A9). Upon introducing $\tilde\phi_{+\rho}=\phi_{+\rho}-\langle\phi_{+\rho}\rangle$ etc., we encounter the expression

$$\langle\exp[i(\tilde\phi_{+\rho}+s\tilde\phi_{+\sigma}-d_{\lambda\bar\lambda}\tilde\theta_{-\rho}-sd_{\lambda\bar\lambda}\tilde\theta_{-\sigma})]\rangle. \tag{4.16}$$

By construction, the pinned tilde-fields have zero expectation values. We also define $\tilde\theta_{+\rho}=\theta_{+\rho}$ etc., for the fields dual to the pinned fields. As the Hamiltonian is invariant under a sign change of any of these tilde-fields, and their commutation relations are invariant under a combined sign change of any two dual fields, this expectation value is independent of s and $d_{\lambda\bar\lambda}$, as these variables can only be ±1. A qualitative estimate for this expectation value is calculated in Appendix B. Denoting the expectation value by B, we obtain

$$\langle F_p(x)\rangle=\frac{B}{2\pi\epsilon}\sum_{\lambda s} d_{\lambda\bar\lambda}^{\alpha(p)}\langle\kappa_{\lambda s}\kappa_{\bar\lambda s}\rangle\times\exp[i(\langle\phi_{+\rho}\rangle+s\langle\phi_{+\sigma}\rangle-d_{\lambda\bar\lambda}\langle\theta_{-\rho}\rangle-sd_{\lambda\bar\lambda}\langle\theta_{-\sigma}\rangle)]. \tag{4.17}$$

Here we have suppressed the Γ dependence of the expectation values appearing after the summation sign. This expression can now be evaluated for the ground-state configurations for the SF phase in Table I by inserting values for the pinned fields and using Eqs. (3.3) and (A3). The results are listed in Table II. In the SF phase $\langle F_{-1}(x)\rangle$ is nonzero and *imaginary*, which implies that the currents are nonzero. Explicitly, we find

TABLE II. The quantity $\tilde F_p\equiv2\pi\epsilon\langle F_p(x)\rangle/B$, as calculated from Eq. (4.17), for the ground states in Table I. The physical properties of these states are seen to be independent of the "gauge" Γ, as they should be.

Ground state	Γ	$\tilde F_1$	$\tilde F_{-1}$
SF 1	±1	0	$4i$
SF 2	±1	0	$-4i$
CDW 1	±1	4	0
CDW 2	±1	-4	0

$$\langle j_\perp(m)\rangle=2(-1)^l\langle j_l(m)\rangle=\mp\frac{B}{2\pi\epsilon}8t_\perp a^2(-1)^m, \tag{4.18}$$

where the upper (lower) sign refers to ground state SF 1 (SF 2). Furthermore, $\langle F_1(x)\rangle$ vanishes identically, so that there is no modulation in neither $\langle k_l(m)\rangle$ nor $\langle n_l(m)\rangle$. We have also shown that the first order contribution to $\langle k_l(m)\rangle$ is zero in the SF phase.[26]

Finally, we note that the ground-state degeneracy can be broken in a formal way by adding to the Hamiltonian a term proportional to the order parameter. Thus, for the SF phase, one can let $H\to H-\eta j_\perp(1)$, where η is an infinitesimal constant. Depending on whether $\eta\gtrless0$, ground state SF 1 or SF 2 will have the lower energy. The small imaginary-valued symmetry breaking term perturbs the purely real-valued Hamiltonian, selecting a particular ground state which is intrinsically complex-valued with large imaginary components in the many-body amplitude.

V. DISCUSSION

As an additional check of our calculations, we have also reproduced the results for the CDW phase found in Ref. 19. In this phase, the signs of b_{eo}^ρ, b_{eo}^σ, and u_{ee}^ρ are opposite to the ones given in Eq. (4.1). The same bosonic fields are pinned as in the SF phase, but their expectation values are different. The CDW phase also has a twofold-degenerate ground state; the pinned-field configurations we have used are listed in Table I. The rest of the calculation is identical to the one presented in Sec. IV, including the calculation of B in Appendix B. Our results for $\langle F_p(x)\rangle$ for the CDW phase are summarized in Table II. We find that $\langle F_1(x)\rangle$ is nonzero and *real*, so that $\langle n_l(m)\rangle$ is modulated. This phase has no currents, since $\langle F_{-1}(x)\rangle=0$.

It is perhaps worth commenting more explicitly on how the Klein factors affect the calculation of the expectation values of the various order parameters considered in Sec. IV B. Two aspects are important. First, the $\langle\kappa_{\lambda s}\kappa_{\bar\lambda s}\rangle$ in Eq. (4.17) contribute relative signs to the various terms in the (λ,s) summation. These signs are crucial for determining whether $\langle F_p(x)\rangle$ is nonzero, or if it instead vanishes identically due to cancellations. Second, if $\langle F_p(x)\rangle$ is nonzero, the fact that $\langle \kappa_{\lambda s}\kappa_{\bar{\lambda} s}\rangle$ is purely imaginary affects whether $\langle F_p(x)\rangle$ is real or imaginary, which in turn determines whether the expectation value of a given order parameter that depends on $\langle F_p(x)\rangle$ will be nonzero; see Eqs. (4.8)–(4.11).

In this paper we have used the so-called "field-theoretic" bosonization.[24] We have also performed the calculations using the more rigorous "constructive" bosonization[21] (however, we still neglect the number-changing property of the Klein factors). In the latter approach, Eq. (3.2b) is replaced by $[\phi_{\lambda s}(x),\theta_{\lambda' s'}(x')]=i(\pi/2)\delta_{\lambda\lambda'}\delta_{ss'}\mathrm{sgn}(x-x')$. Consequently, the anticommutation between right- and left-moving fermions with the same band and spin indices must now be taken care of by the Klein factors, which therefore acquire an additional R/L index. As a result, 12 different products $\hat{\Gamma}_i$ of four Klein factors appear in the Hamiltonian. One must identify all relations between the $\hat{\Gamma}_i$, as these relations put restrictions on the permissible sets of eigenvalues Γ_i.[20] Thus the treatment of Klein factors is more complicated than in the field-theoretic approach, where a single operator $\hat{\Gamma}$ appears. However, the final results for the expectation values of the order parameters are found to be the same.[26]

Our results imply that the SF phase occurs in the phase diagram of a weakly interacting general SO(5) invariant model on the half-filled two-leg ladder.[19] However, the basin of attraction of the SF phase is not restricted to have SO(5) symmetry. In fact, for all bare couplings studied in Ref. 19, including attractive interactions that *break* SO(5) symmetry, it was found that the RG flow goes to the SO(5) subspace, where the SF phase is one of the "attracting directions." It would be very interesting to undertake a complete exploration of the parameter space, to see if the SF phase could possibly be reached from purely repulsive off-site density-density interactions, supplemented by various spin-exchange interactions.

Next, we discuss the possibility of SF order away from half-filling. For generic incommensurate fillings, Umklapp interactions are absent. Thus the total charge mode $\phi_{+\rho}$ will be gapless (making the system metallic), so that $\langle \exp(i\phi_{+\rho})\rangle$ will vanish. The currents will then only show quasi-LRO.[17,18,27] Strictly speaking, the system is then no longer in the SF phase, but shows a dominant tendency toward SF ordering. On the other hand, for commensurate fillings, higher-order Umklapp interactions are present,[28,29] so that if these interactions are not irrelevant,[30] $\phi_{+\rho}$ may be pinned, making true LRO possible. These conclusions are consistent with those obtained from symmetry arguments: In the absence of Umklapp interactions, the Hamiltonian is invariant under the continuous symmetry $\phi_{+\rho}\rightarrow\phi_{+\rho}+c$ (i.e., the constant c can take arbitrary values), and pinning of $\phi_{+\rho}$ is forbidden by the Mermin-Wagner theorem. This theorem no longer applies when Umklapp interactions are present, since then the symmetry is reduced to a discrete one (i.e., c can only take particular values).

Finally, we show that our results for the half-filled SF phase are consistent with the results obtained for the doped ladder for generic incommensurate fillings. In Table II of Ref. 18 the values of the three pinned fields in the phase with dominant tendency toward SF ordering are taken to be $\langle\phi_{+\sigma}\rangle=\pi/2$, $\langle\theta_{-\rho}\rangle=0$, $\langle\theta_{-\sigma}\rangle=0$. Using the SF couplings in Eq. (4.1), and taking $\Gamma=-1$, one sees that these expectation values of the pinned fields indeed minimize $H_I^{(1b)}$, and also $H_I^{(2)}$ at half-filling when $\langle\phi_{+\rho}\rangle$ is taken to be an odd multiple of $\pi/2$.

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APPENDIX A: MATRIX ELEMENTS OF PRODUCTS OF KLEIN FACTORS

The Hermitian operator $\hat{\Gamma}=\kappa_{e\uparrow}\kappa_{e\downarrow}\kappa_{o\uparrow}\kappa_{o\downarrow}$ enters into bosonic expressions for $\mathcal{H}_I^{(1b)}$ and $\mathcal{H}_I^{(2)}$ [Eqs. (3.8)–(3.9)]. Its eigenvalues are $\Gamma=\pm 1$, and the associated eigenstates $|\Gamma\rangle$ obey $\langle\Gamma|\Gamma'\rangle=\delta_{\Gamma\Gamma'}$. The completeness relation in the space spanned by these eigenstates is

$$\sum_{\Gamma=\pm 1}|\Gamma\rangle\langle\Gamma|=I. \tag{A1}$$

We want to calculate various matrix elements of $\kappa_{es}\kappa_{os}$, which appear in the expectation values of the order parameters considered in Sec. IV B. We have

$$\kappa_{e\uparrow}\kappa_{o\uparrow}\hat{\Gamma}=\kappa_{e\uparrow}\kappa_{o\uparrow}\kappa_{e\uparrow}\kappa_{e\downarrow}\kappa_{o\uparrow}\kappa_{o\downarrow}=\kappa_{e\downarrow}\kappa_{o\downarrow}. \tag{A2}$$

Using $\hat{\Gamma}|\Gamma\rangle=\Gamma|\Gamma\rangle$, one obtains

$$\langle\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|\Gamma\rangle=\Gamma\langle\Gamma|\kappa_{e\downarrow}\kappa_{o\downarrow}|\Gamma\rangle, \tag{A3}$$

$$\langle-\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|\Gamma\rangle=\Gamma\langle-\Gamma|\kappa_{e\downarrow}\kappa_{o\downarrow}|\Gamma\rangle. \tag{A4}$$

The complex conjugate of Eq. (A4) can be rewritten as $\langle\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|-\Gamma\rangle=\Gamma\langle\Gamma|\kappa_{e\downarrow}\kappa_{o\downarrow}|-\Gamma\rangle$. Letting $\Gamma\rightarrow-\Gamma$ in this relation, and comparing with Eq. (A4), shows that the off-diagonal matrix elements are zero;

$$\langle-\Gamma|\kappa_{es}\kappa_{os}|\Gamma\rangle=0. \tag{A5}$$

Next, consider the equation $\langle\Gamma|\hat{\Gamma}|\Gamma\rangle=\Gamma$. Anticommuting the two inner Klein factors and inserting Eq. (A1) gives

$$\sum_{\Gamma'=\pm 1}\langle\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|\Gamma'\rangle\langle\Gamma'|\kappa_{e\downarrow}\kappa_{o\downarrow}|\Gamma\rangle=-\Gamma. \tag{A6}$$

Using Eqs. (A5) and (A3), we obtain $\langle\Gamma|\kappa_{es}\kappa_{os}|\Gamma\rangle^2=-1$, i.e.

$$\langle\Gamma|\kappa_{es}\kappa_{os}|\Gamma\rangle=\pm i. \tag{A7}$$

This result is consistent with $\kappa_{es}\kappa_{os}$ being an anti-Hermitian operator, thus having a purely imaginary expectation value.

In order to determine the matrix elements, one can, e.g., fix the two diagonal matrix elements for one of the spin directions. The other matrix elements are then determined from Eqs. (A3) and (A5). In this paper we choose to set $\langle\Gamma|\kappa_{e\uparrow}\kappa_{o\uparrow}|\Gamma\rangle=+i$.

Finally, we consider the space spanned by the states $|n;\Gamma\rangle$ defined in Sec. IV B. The completeness relation in this space reads

$$\sum_n \sum_{\Gamma=\pm1} |n;\Gamma\rangle\langle n;\Gamma| = I. \tag{A8}$$

We will also need the matrix element

$$\begin{aligned}\langle 0;\Gamma|\kappa_{\lambda s}\kappa_{\bar{\lambda}s}^{-}|n;\Gamma'\rangle &= \langle\Gamma|\kappa_{\lambda s}\kappa_{\bar{\lambda}s}^{-}|\Gamma'\rangle\langle 0(\Gamma)|n(\Gamma')\rangle\\ &= \langle\Gamma|\kappa_{\lambda s}\kappa_{\bar{\lambda}s}^{-}|\Gamma\rangle\,\delta_{\Gamma\Gamma'}\delta_{n0}.\end{aligned} \tag{A9}$$

APPENDIX B: EXPECTATION VALUES OF EXPONENTIALS OF PINNED FIELDS

In this appendix, we calculate a qualitative estimate of the expectation value in Eq. (4.16) by employing a simplified treatment of the Hamiltonian, in which the pinned fields are expanded to quadratic order around their expectation values. This gives $\langle\Gamma|H|\Gamma\rangle=\Sigma_{r\nu}H_{r\nu}$ with $[H_{r\nu},H_{r'\nu'}]=0$, where $H_{r\nu}$ is of Klein-Gordon form,

$$\mathcal{H}_{+\nu}=\frac{1}{2\pi}\left[uK(\partial_x\tilde{\theta}_{+\nu})^2+\frac{u}{K}(\partial_x\tilde{\phi}_{+\nu})^2+w^2\tilde{\phi}_{+\nu}^2\right], \tag{B1}$$

$$\mathcal{H}_{-\nu}=\frac{1}{2\pi}\left[uK(\partial_x\tilde{\phi}_{-\nu})^2+\frac{u}{K}(\partial_x\tilde{\theta}_{-\nu})^2+w^2\tilde{\theta}_{-\nu}^2\right], \tag{B2}$$

where $w^2=24g/(\pi\epsilon^2)$, and

$$u=\sqrt{v_F^2-(g/\pi)^2},\quad K=\sqrt{\frac{v_F-g/\pi}{v_F+g/\pi}}. \tag{B3}$$

Furthermore, we have defined $\tilde{\phi}_{+\rho}=\phi_{+\rho}-\langle\phi_{+\rho}\rangle$ etc. for the pinned fields, and $\tilde{\theta}_{+\rho}=\theta_{+\rho}$ etc. for the fields dual to them. Written in terms of these fields, $H_{r\nu}$ is independent of Γ (and also of whether we consider the SF or CDW phase). Clearly, $\mathcal{H}_{-\nu}$ can be obtained from $\mathcal{H}_{+\nu}$ by letting $\tilde{\theta}_{+\nu}\to\tilde{\phi}_{-\nu}$ and $\tilde{\phi}_{+\nu}\to\tilde{\theta}_{-\nu}$. In addition,

$$[\partial_x\tilde{\theta}_{+\nu}(x),\tilde{\phi}_{+\nu}(x')]=[\partial_x\tilde{\phi}_{-\nu'}(x),\tilde{\theta}_{-\nu'}(x')]. \tag{B4}$$

Thus all Hamiltonians $H_{r\nu}$ are equivalent. It therefore suffices to consider, e.g., $H_{+\nu}$. We expand the fields as

$$\begin{aligned}\tilde{\phi}_{+\nu}(x,t)&=\sqrt{uK}\sqrt{\frac{\pi}{L}}\sum_{q\neq0}e^{-\epsilon|q|/2}\frac{1}{\sqrt{2\omega_q}}\\ &\quad\times\{a_{+\nu q}e^{i(qx-\omega_q t)}+a_{+\nu q}^{\dagger}e^{-i(qx-\omega_q t)}\},\end{aligned} \tag{B5}$$

$$\begin{aligned}\partial_x\tilde{\theta}_{+\nu}(x,t)&=\frac{i}{\sqrt{uK}}\sqrt{\frac{\pi}{L}}\sum_{q\neq0}e^{-\epsilon|q|/2}\sqrt{\frac{\omega_q}{2}}\\ &\quad\times\{a_{+\nu q}e^{i(qx-\omega_q t)}-a_{+\nu q}^{\dagger}e^{-i(qx-\omega_q t)}\},\end{aligned} \tag{B6}$$

where $\omega_q=\omega_{-q}$, and $a_{+\nu q}$ and $a_{+\nu q}^{\dagger}$ are canonical boson operators satisfying $[a_{+\nu q},a_{+\nu q'}^{\dagger}]=\delta_{qq'}$. These expansions give the correct equal-time commutation relations and equations of motion. The Hamiltonian can then be written on diagonal form,

$$H_{+\nu}=\sum_{q\neq0}e^{-\epsilon|q|}\omega_q a_{+\nu q}^{\dagger}a_{+\nu q}, \tag{B7}$$

with $\omega_q=\sqrt{u^2q^2+uKw^2}$.

Next we consider the ground-state expectation value of $\exp[ic\tilde{\phi}_{+\nu}(x)]$, where c is an arbitrary c-number. Let

$$\tilde{\phi}_{+\nu}\equiv\Phi_{+\nu}+\Phi_{+\nu}^{\dagger}, \tag{B8}$$

where $\Phi_{+\nu}$ $(\Phi_{+\nu}^{\dagger})$ contains the annihilation (creation) part of $\tilde{\phi}_{+\nu}$. The ground-state expectation value can be written

$$\langle\exp[ic\tilde{\phi}_{+\nu}(x)]\rangle=\exp\left(-\frac{c^2}{2}[\Phi_{+\nu}(x),\Phi_{+\nu}^{\dagger}(x)]\right), \tag{B9}$$

where

$$[\Phi_{+\nu}(x),\Phi_{+\nu}^{\dagger}(x)]=\frac{\pi}{L}uK\sum_{q>0}\frac{e^{-\epsilon q}}{\omega_q}. \tag{B10}$$

In actuality the coupling constant g is not constant up to arbitrarily high momenta. Rather, g is really a function $g(q)$, with $g(q\to0)=g$, and $g(q\to\infty)=0$. It follows that u, K, and w also become momentum dependent, and ω_q acquires an additional momentum dependence. Thus a more correct expression for the commutator is

$$[\Phi_{+\nu}(x),\Phi_{+\nu}^{\dagger}(x)]=\frac{\pi}{L}\sum_{q>0}u(q)K(q)\frac{e^{-\epsilon q}}{\omega_q(q)}. \tag{B11}$$

For simplicity, we will assume that there is a characteristic momentum cutoff $1/\Lambda$ such that for $q\ll1/\Lambda$, $g(q)$ is well approximated by g, and for $q\gg1/\Lambda$, $g(q)\approx0$. Multiplying the integrand by $[e^{-\Lambda q}+(1-e^{-\Lambda q})]$, we then approximate $g(q)\approx g$ in the term containing $e^{-\Lambda q}$, and $g(q)\approx0$ in the term containing $(1-e^{-\Lambda q})$. With $\Lambda\gg\epsilon$, this gives

$$\begin{aligned}[\Phi_{+\nu}(x),\Phi_{+\nu}^{\dagger}(x)]&\approx\frac{\pi}{L}\sum_{q>0}\left(uK\frac{e^{-\Lambda q}}{\omega_q}+\frac{e^{-\epsilon q}-e^{-\Lambda q}}{q}\right)\\ &=\frac{\pi K}{4}[\boldsymbol{H}_0(z)-Y_0(z)]+\frac{1}{2}\ln(\Lambda/\epsilon),\end{aligned} \tag{B12}$$

where $z\equiv\Lambda w\sqrt{K/u}$. Here $\boldsymbol{H}_0(z)$ is a Struve function, and $Y_0(z)$ is a Bessel function of the second kind.

Due to the equivalence of the Hamiltonians $H_{r\nu}$, we have

$$\langle e^{ic\tilde{\phi}_{+\rho}}\rangle=\langle e^{ic\tilde{\phi}_{+\sigma}}\rangle=\langle e^{ic\tilde{\theta}_{-\rho}}\rangle=\langle e^{ic\tilde{\theta}_{-\sigma}}\rangle. \tag{B13}$$

It then follows from Eq. (B9) that the expectation value in Eq. (4.16) is independent of s and $d_{\lambda\bar{\lambda}}$, as these variables are restricted to be ± 1. Denoting this expectation value by B, we find

$$B \approx \frac{\epsilon}{\Lambda} \exp\left\{ -\frac{\pi K}{2} [\boldsymbol{H}_0(z) - Y_0(z)] \right\}. \tag{B14}$$

The ϵ in the prefactor cancels the $1/\epsilon$ in the prefactor of Eq. (4.17). In the unpinned limit $(g \to 0)$, $B \approx \sqrt{24g/\pi v_F} \to 0$, while in the limit of maximum pinning $(g \to \pi v_F)$,

$$B \approx \frac{\epsilon}{\Lambda} \exp\left(-\frac{\epsilon}{\Lambda} \sqrt{\frac{\pi v_F - g}{24g}} \right) \to \frac{\epsilon}{\Lambda}.$$

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