Characterization of nonlocal gates

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(Received 9 July 2002; published 31 December 2002)

A nonlocal unitary transformation of two-qubits occurs when some Hamiltonian interaction couples them. Here we characterize the amount, as measured by time, of interaction required to perform two-qubit gates, when also arbitrarily fast, local unitary transformations can be applied on each qubit. The minimal required time of interaction, or *interaction cost*, defines an operational notion of the degree of nonlocality of gates. We characterize a partial order structure based on this notion. We also investigate the interaction cost of several communication tasks, and determine which gates are able to accomplish them. This classifies two-qubit gates into four categories, differing in their capability to transmit classical, as well as quantum, bits of information.

I. INTRODUCTION

An essential ingredient in quantum information processing is the ability to make two two-level systems or qubits undergo a joint unitary evolution. Accordingly, most current proposals for the implementation of a quantum computer rely on some ingenious method to realize two-qubit gates.

Irrespective of the physical substrate of the qubits, a joint unitary evolution can only be achieved through some form of interaction. This quite often couples the two qubits directly, though a third system may alternatively mediate in the transformation. The starting goal of this paper is, given any fixed two-qubit Hamiltonian, to describe how it can be used to accomplish any desired gate on the two systems.

Of course, some form of external control on the two qubits is required to conveniently modify their evolution, which would otherwise be dictated only by the coupling interaction. Inspired by the possibilities presently demonstrated in several quantum optical setups, where each qubit can be independently addressed [1], we assume here the ability to perform arbitrary local unitary operations (LU) on each of the systems. More specifically, we shall analyze the fast control limit, in which these control operations can be performed *instantaneously*. Physically, such a limit amounts to assuming a neat separation between the time scale of the interaction (which is comparatively slow) and that of the external manipulations.

The setting we consider corresponds, thus, to the so-called gate simulation under LU of Ref. [2]. This setting has been previously considered in Ref. [3], where powerful mathematical techniques were developed to study time-optimal strategies; that is, strategies that perform the desired gate by using the available interaction for the shortest time. In Ref. [4], and by elaborating on the results of Ref. [3] and of Refs. [2,5,6], time-optimal strategies have been analytically characterized for any interaction and gate of two qubits.

The main result of Ref. [4] permits therefore to assess explicitly the minimum time an interaction is required to simulate a given gate, a measure that has been called the *interaction cost* of the gate. The merit of such a measure is twofold: On one hand, time is by itself a crucial parameter in present experiments. In order to successfully process quan-tum information, unitary evolutions must in practice be enforced in a sufficiently small time as compared to the decoherence time of the quantum systems. In several settings, the time scale of gates is essentially determined by the interaction between qubits, for one-qubit unitary transformations can be performed much faster. Then, an efficient use of the interaction becomes a priority. On the other hand, the minimal realization time or interaction cost of a gate can be naturally used to compare gates, thereby endowing the set of nonlocal transformations with a partial order structure that refers to the amount of inherent interaction. This, in turn, provides us with a meaningful notion of the degree of non-locality of a gate, built upon the observation that local gates can be performed without any interaction.

In the present paper we first reproduce and extend the results of Ref. [4] concerning the time optimal use of interactions, and put these into work by characterizing the information exchange associated to a two-qubit gate. In Ref. [4], the derivation of the interaction cost rested on a previous proof of Ref. [3] which requires familiarity with several facts of differential geometry. Here we present an alternative, self–contained proof, which in addition employs ideas and a formalism that we believe to be more common to quantum information community. This new proof is complemented with an expanded analysis of the interaction cost of two-qubit gates, including several relevant examples. The overall result is an operational characterization of two-qubit gates in terms of the interaction resources needed to perform them.

For any specific information processing task, there may be several gates that can accomplish it. It is then reasonable to investigate the most efficient way to accomplish the desired task with a given interaction, that is, to search for the gate with lowest interaction cost compatible with that task. In particular, a joint gate can be used to transmit information between the qubits, and one can study the interaction cost of certain communication tasks, such as the transmission of classical and quantum bits from one system to the other.

A second main goal of this paper is precisely to characterize the minimal interaction time required to send classical, as well as quantum, information. As a byproduct, and very much in the spirit of Refs. [7] and [8], where information exchange has been used to characterize the nonlocal content

of certain gates, we obtain a complete classification of two-qubit gates with respect to their transmission capabilities, thereby supplementing the original characterization of nonlocal gates.

The results we present can be summarized as follows:

(a) Analytical characterization of the interaction cost of any two-qubit gate by any two-qubit interaction Hamiltonian, through a new, self-contained proof (Sec. III).

(b) Analytical characterization, in part of the space of two-qubit gates, of the partial order structure based on the interaction cost (Sec. III).

(c) Analytical characterization, for any two-qubit interaction, of the interaction cost of the following communication processes between two qubits (Sec. IV):

(1) Transmission of one classical bit: $\text{cbit}_{A \to B}$.

(2) Simultaneous, bidirectional transmission of two classical bits: $\text{cbit}_{A \to B}$ and $\text{cbit}_{B \to A}$.

(3) Transmission of one quantum bit: $\text{qubit}_{A \to B}$

(4) Simultaneous, bidirectional transmission of one classical bit and one quantum bit: $\text{cbit}_{A \to B}$ and $\text{qubit}_{B \to A}$.

(5) Simultaneous, bidirectional transmission of two quantum bits: $\text{qubit}_{A \to B}$ and $\text{qubit}_{B \to A}$.

(d) Analytical characterization of two-qubit gates according to their capability to perform any of the above tasks (Sec. IV).

II. DEFINITIONS AND BASIC FACTS

This section is a prelude providing the definitions and notations that will be used throughout the whole paper and reviews some facts concerning two-qubit gates which will build the basis for our further results. We shall also define the notion of majorization and collect some lemmas linked to it.

A. Two-qubit gates

Consider a system consisting of two two-dimensional subsystems (qubits), A and B. The corresponding Hilbert spaces are $\mathcal{H}_A \approx \mathbb{C}^2$ and $\mathcal{H}_B \approx \mathbb{C}^2$. The compound Hilbert space is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \approx \mathbb{C}^2 \otimes \mathbb{C}^2$.

By a two-qubit gate $\mathcal{U}$ we understand a unitary operator acting on $\mathcal{H}_{AB}$. By choosing the global phase appropriately we can always consider such a unitary to be an element of the group $SU(4,\mathbb{C})$. We speak of a *local* two-qubit gate whenever we can write $\mathcal{U} = U_A \otimes V_B$, where U_A and V_B are unitary operators acting only on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Again we can restrict ourselves to local unitaries being elements of $SU(2,\mathbb{C}) \otimes SU(2,\mathbb{C})$. Nonlocal gates are then trivially two-qubit gates which cannot be written as $U_A \otimes V_B$.

With just the help of these two definitions we can already divide the set of nonlocal gates into equivalence classes. Two two-qubit gates $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are said to be *locally equivalent* if there exist local unitaries $U_A \otimes V_B$ and $\tilde{U}_A \otimes \tilde{V}_B$ such that $\mathcal{U} = U_A \otimes V_B \tilde{\mathcal{U}} \, \tilde{U}_A \otimes \tilde{V}_B$. A useful decomposition of a general two-qubit gate developed in Refs. [3] and [5] admits to further characterize these equivalence classes enabling us to easily decide whether two gates are locally equivalent.

Lemma 1. For any two-qubit gate $\mathcal{U}$ there exist local unitaries $U_A \otimes V_B$ and $\tilde{U}_A \otimes \tilde{V}_B$ and a self-adjoint operator of the form $H = \Sigma_{k=1}^3 \alpha_k \sigma_k \otimes \sigma_k$ such that $\mathcal{U} = \tilde{U}_A \otimes \tilde{V}_B e^{-iH} U_A \otimes V_B$ [3,5].

Here the σ_k's denote the usual Pauli spin matrices. Note that the real numbers α_k are not unique, as long as we do not pose further conditions on them. This is so for two reasons: First, operators of the type $\pm \sigma_k \otimes \sigma_k$ are local and commute with H so that we can always extract such a local operator from the local parts in this decomposition and include it in H. This alters the corresponding coefficient α_k by $\pm \pi/2$. Second, there are certain local transformations of H which conserve its form but permute the coefficients α_k and change the sign of two of them. The local unitaries which cause such a transformation are of the types $\pm i\sigma_k \otimes \mathbf{1}$ and $\pm i\mathbf{1} \otimes \sigma_k$. Using this it can easily be checked that it is always possible to bring H to a form, where its coefficients obey the inequalities (see also Ref. [5])

$$\pi/4 \geq \alpha_1 \geq \alpha_2 \geq |\alpha_3|. \tag{1}$$

Note that these conditions are an arbitrary choice and that it might be necessary to relax them when we are looking for optimal simulation protocols. We will come back to this point later on.

We call the decomposition of a two-qubit gate as given in lemma 1, where the coefficients α_k fulfill Eq. (1) its *canonical form*. The purely nonlocal unitary e^{-iH} in this decomposition is termed the *interaction content* of the gate.

That the nonlocal characteristics of a two-qubit gate are determined by only three real parameters is a remarkable result in view of the fact that a general element of $SU(4,\mathbb{C})$ is fixed by 15 independent parameters. It might be mentioned here that while Ref. [3] provides a profound Lie-algebraic basis for the decomposition in lemma 1, [5] gives a constructive proof which allows to determine the coefficients α_k, as well as the local unitaries for any given gate. Based on this method we show in the Appendix how to derive the α_k for a given $\mathcal{U}$ without constructing the local unitaries.

A necessary and sufficient criterion for two gates to be locally equivalent is now obviously that they have the same interaction content. By definition it is also clear that any two-qubit gate is locally equivalent to its own interaction content, a fact on which our results concerning simulation of gates heavily rely.

For later use we mention here that self-adjoint operators of the form considered in lemma 1 are diagonal in the so-called magic basis [9] defined as

$$|\mathbf{1}\rangle = -\frac{i}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\mathbf{2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|\mathbf{3}\rangle = -\frac{i}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\mathbf{4}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{2}$$

such that we have

$$H = \sum_{k=1}^{3} \alpha_k \sigma_k \otimes \sigma_k = \sum_{j=1}^{4} \lambda_j |\mathbf{j}\rangle\langle\mathbf{j}|, \tag{3}$$

where the eigenvalues λ_j follow from the α_k by

$$\lambda_1 = \alpha_1 + \alpha_2 - \alpha_3, \quad \lambda_2 = \alpha_1 - \alpha_2 + \alpha_3,$$

$$\lambda_3 = -\alpha_1 + \alpha_2 + \alpha_3, \quad \lambda_4 = -\alpha_1 - \alpha_2 - \alpha_3. \quad (4)$$

In terms of the λ_j conditions (1) read $3\pi/4 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq -3\pi/4$. Note also that the λ_js sum up to zero (i.e., H is traceless) such that the corresponding unitary $\mathcal{U} = \exp(-iH_{\vec{\lambda}})$ is an element of the special unitary group as we have required. In the following we will characterize the interaction content of nonlocal gates either by the three-vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ or by the four-vector $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ freely switching between the representations. For operators like in Eq. (3) we write $H_{\vec{\alpha}}$ or $H_{\vec{\lambda}}$ and for the corresponding unitary $\mathcal{U}_{\vec{\alpha}}$ or $\mathcal{U}_{\vec{\lambda}}$.

B. Majorization

The relation of majorization emerged as a powerful tool in the issue of simulation, as well as in other fields of quantum information theory. From an intuitive perspective it simply makes a precise statement out of a vague notion that the components of a vector $\vec{x}$ are "less spread out" or "more equal" than are the components of a vector $\vec{y}$.

Definition 2. Let $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ be real vectors whose components are ordered nonincreasingly. Then we say that "$\vec{x}$ majorizes $\vec{y}$" and write $\vec{x} > \vec{y}$ if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \quad k = 1, \ldots, n-1,$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

A central result in the theory of majorization is the following.

Lemma 3 (Ref. [10]). Let x and y be defined as before. Then $\vec{x} > \vec{y}$ iff there exists a doubly stochastic (Ref. [15]) $n \times n$ matrix Q such that $\vec{y} = Q\vec{x}$.

We will use two facts related to doubly stochastic matrices:

(a) The first one is called Birkhoff's theorem and states that the set of doubly stochastic matrices is the convex hull of the permutation matrices. Therefore, we can write $Q = \Sigma p_i P_i$ (the $p_i \geq 0$ summing up to one and P_i being permutation matrices) for any doubly stochastic matrix Q.

(b) If we take the so-called Hadamard product of a real orthogonal matrix O with itself i.e., square it componentwise (written symbolically as $O \bigcirc O$) then we get a special type of doubly stochastic matrix called orthostochastic matrix.

Later on we will use this relation to compare four-vectors $(\vec{\lambda}, \vec{\mu}, \vec{\nu}, \ldots)$ of the kind introduced in the following section. In related works (Refs. [2,11]) it has already turned out to be convenient to have at hand an equivalent relation for the corresponding three-vectors $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \ldots)$ called the

s(pecial)-majorization relation. Let $\vec{\alpha}$ and $\vec{\beta}$ be two real and nonincreasingly ordered three vectors. Then $\vec{\alpha}$ s-majorizes $\vec{\beta}$ $(\vec{\alpha} >_s \vec{\beta})$ if

$$\alpha_1 \geq \beta_1,$$

$$\alpha_1 + \alpha_2 - \alpha_3 \geq \beta_1 + \beta_2 - \beta_3,$$

$$\alpha_1 + \alpha_2 + \alpha_3 \geq \beta_1 + \beta_2 + \beta_3. \quad (5)$$

Now let $\vec{\lambda}$ and $\vec{\mu}$ be the four-vectors related to $\vec{\alpha}$ and $\vec{\beta}$, respectively via Eq. (4). Then it is easily verified that $\vec{\lambda} > \vec{\mu}$ iff $\vec{\alpha} >_s \vec{\beta}$.

The s-majorization relation can be extended to nonordered vectors as follows. Given a vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, we construct a new "$s$-ordered" vector $\vec{\alpha}^s = (\alpha_1^s, \alpha_2^s, \alpha_3^s)$, $\alpha_1^s \geq \alpha_2^s \geq |\alpha_3^s|$ by first nonincreasingly reordering the modulus of the components α_i, and by then giving α_3^s the sign of the product $\alpha_1 \alpha_2 \alpha_3$. Then for any pair of vectors $\vec{\alpha}$ and $\vec{\beta}$, $\vec{\alpha} >_s \vec{\beta}$ denotes the set of inequalities (5) applied to $\vec{\alpha}^s$ and $\vec{\beta}^s$. We note also that according to the above discussion a gate $\mathcal{U}_{\vec{\alpha}}$ ($\vec{\alpha}$ being an *arbitrary* three-vector) is locally equivalent to the gate $\mathcal{U}_{\vec{\alpha}^s}$ corresponding to the s-ordered form of $\vec{\alpha}$.

III. INTERACTION COSTS OF GATE SIMULATION AND PARTIAL ORDER OF GATES

The main result (Theorem 1) in Ref. [4] permits to assess the interaction cost (as defined in Ref. [4]) for simulating a two-qubit gate using any given interaction Hamiltonian and fast local unitaries analytically after performing a simple optimization. The proof in Ref. [4] is based on results developed in the areas of quantum control [3] and quantum information (Refs. [5,2,6]). Here we give an alternative proof relying only on the tools introduced so far. We do this by giving a necessary and sufficient condition for the existence of a simulation protocol. Before we state and prove this result we will introduce the problem of simulating a gate (see Refs. [2,12] for a more general discussions) and describe some simplifications that can be assumed in this context.

A. Setting of gate simulation and basic assumptions

Simulating a desired two-qubit gate $\mathcal{U}$ using a given interaction described by a Hamiltonian H [16] and arbitrary local unitary transformations means to specify a series of local unitaries $\{U_1 \otimes V_1, \ldots, U_n \otimes V_n\}$ and of time intervals $\{t_1, \ldots, t_n\}$ such that

$$\mathcal{U} = (U_n \otimes V_n) e^{-iHt_n} (U_{n-1} \otimes V_{n-1}) e^{-iHt_{n-1}} \cdots e^{-iHt_2}$$

$$\times (U_1 \otimes V_1) e^{-iHt_1} (U_0 \otimes V_0). \quad (6)$$

Such a partition of a gate $\mathcal{U}$ equals a list of instructions like: "Perform transformation U_0 and V_0 on qubit A and B, respectively. Then let them interact according to H for a time t_1. Perform U_1 and V_1. Let them interact for t_2. Finally

perform U_n and V_n.'' Following this protocol one would then effectively perform the gate $\mathcal{U}$ on the two qubits no matter what their initial state was.

Posing the problem of finding such a simulation protocol naturally evokes other questions: Is there always a solution? How much time will it take to perform a possible simulation protocol? What is the minimal time of simulation? Do we have to allow for infinitesimal time steps? In case we can restrict on taking finite time steps, how many of them will suffice? In the following we will give an answer to all of them.

To do so we adopt two simplifications. At first we employ a physical idealization namely the *fast control limit* which is well justified in most of the proposed settings for quantum information processing. It states that the control operations—in our case the local unitary transformations—can be executed in times, where the natural evolution—here the interaction of the qubits—has no considerable effect on the system's state. In other words local manipulations and interactions have to take place on significantly different time scales. That is what we assume and what allows us to define the simulation time simply as $t_S = \sum_{i=1}^{n} t_i$, implying that the local transformations in Eq. (6) take effectively *no* time. We term the minimal time t_S such that we can find a simulation protocol its "interaction cost" $[\mathcal{C}_H(\mathcal{U})]$ because it actually measures the *time of interaction* required to perform the gate.

The second simplification is of pure mathematical nature and concerns the system's Hamiltonian H. Based on results of Refs. [2,6] we use that although a general two-qubit Hamiltonian has the form $H = c_0 \mathbf{1} \otimes \mathbf{1} + \sum_{i=1}^{3} a_i \sigma_i \otimes \mathbf{1} + \sum_{j=1}^{3} b_j \mathbf{1} \otimes \sigma_j + \sum_{i,j=1}^{3} c_{ij} \sigma_i \otimes \sigma_j$ we can restrict ourselves to much simpler Hamiltonians $H_{\vec{\lambda}}$ (or equivalently $H_{\vec{\alpha}}$) as given in Eq. (3). This is due to the fact that for any general Hamiltonian there exists a Hamiltonian $H_{\vec{\lambda}}$, called its *canonical form*, and efficient protocols for simulating the evolution according to the latter in terms of the first. By an efficient simulation protocol we mean that we can obtain the evolution $e^{-iH_{\vec{\lambda}}t}$ for any time t by using H for the same period of time t. (Note that such a simulation involves infinitesimal time steps, see Ref. [2].) For the purpose of simulation these Hamiltonians are equivalent in the sense that both are equally effective in simulating other Hamiltonians or gates.

B. Necessary and sufficient condition for gate simulation

We are now ready to give a necessary and sufficient condition for the existence of a simulation protocol.

Result 1. Given a two-qubit gate $\mathcal{U}$ having an interaction content $\mathcal{U}_{\vec{\beta}}$ and a Hamiltonian H having a canonical form $H_{\vec{\alpha}}$ there exists a simulation protocol of type (6) consuming a total time $t_S \geq 0$ iff a vector $\vec{n} = (n_1, n_2, n_3)$ of integers exists such that $\vec{\beta}_{\vec{n}} = \vec{\beta} + \pi/2\vec{n}$ satisfies

$$\vec{\beta}_{\vec{n}} \prec_s \vec{\alpha} t_S. \tag{7}$$

Proof. We first show that this is a necessary condition. According to the above discussion a simulation protocol for $\mathcal{U}$ using H for a time t is equivalent to a protocol for $\mathcal{U}_{\vec{\beta}}$ using

$H_{\vec{\alpha}}$ for the same time t. Moreover, we can assume that the protocol we have consists entirely of infinitesimal time steps δt since any finite time step can be decomposed into infinitesimal ones. Then Eq. (6) reads as

$$\mathcal{U}_{\vec{\beta}} = (U_n \otimes V_n) e^{-iH_{\vec{\alpha}}\delta t} (U_{n-1} \otimes V_{n-1}) \cdots e^{-iH_{\vec{\alpha}}\delta t}$$
$$\times (U_i \otimes V_i) e^{-iH_{\vec{\alpha}}\delta t} \cdots (U_1 \otimes V_1) e^{-iH_{\vec{\alpha}}\delta t} (U_0 \otimes V_0).$$

Let us assume that at a time $0 \leq t \leq t_S$ we perform the *i*th intermediate local transformation having then attained an effective transformation $\mathcal{U}_t = (U_i \otimes V_i) e^{-iH_{\vec{\alpha}}\delta t} \cdots (U_1 \otimes V_1) e^{-iH_{\vec{\alpha}}\delta t} (U_0 \otimes V_0)$. Since $\mathcal{U}_t$ is itself a gate, we can decompose it as $\mathcal{U}_t = U_t \otimes V_t \mathcal{U}_{\vec{\gamma}_t} \tilde{U}_t \otimes \tilde{V}_t$, where $\mathcal{U}_{\vec{\gamma}_t} = e^{-iH_{\vec{\gamma}_t}}$ is the interaction content of $\mathcal{U}_t$. The index t indicates the time dependence of all these unitaries.

To determine how $\vec{\gamma}_t$ varies with t we take the next infinitesimal time step $e^{-iH_{\vec{\alpha}}\delta t}$ in the protocol and get

$$e^{-iH_{\vec{\alpha}}\delta t}\mathcal{U}_t = e^{-iH_{\vec{\alpha}}\delta t} U_t \otimes V_t \mathcal{U}_{\vec{\gamma}_t} \tilde{U}_t \otimes \tilde{V}_t$$
$$= U_{t+\delta t} \otimes V_{t+\delta t} \mathcal{U}_{\vec{\gamma}_{t+\delta t}} \tilde{U}_{t+\delta t} \otimes \tilde{V}_{t+\delta t}.$$

For convenience we change here to the four-vector representation [as defined in Eq. (4)]. Denote by $\vec{\lambda}$, $\vec{\nu}$, and $\vec{\xi}$ the vectors corresponding to $\vec{\alpha}$, $\vec{\gamma}_t$, and $\vec{\gamma}_{t+\delta t}$, respectively. After local transformations the last identity can be written as

$$e^{-iH_{\vec{\lambda}}\delta t} U \otimes V \mathcal{U}_{\vec{\nu}} = W \otimes X \mathcal{U}_{\vec{\xi}} Y \otimes Z, \tag{8}$$

where $W \otimes X$ and $Y \otimes Z$ are appropriately defined local unitaries and all time indices are omitted. The right-hand side of Eq. (8) is a decomposition of the left-hand side, but we do not require this to be the canonical form as defined in Sec. II A. We therefore, have the possibility to put further conditions on the unitaries in this decomposition.

If we multiply from the left by $U^{\dagger} \otimes V^{\dagger}$ and sandwich this equation between $|\mathbf{k}\rangle$, one of the magic states, we find

$$\langle \psi_k | e^{-iH_{\vec{\lambda}}\delta t} | \psi_k \rangle e^{-i\nu_k} = \langle \psi_k | W \otimes X \mathcal{U}_{\vec{\xi}} Y \otimes Z | \mathbf{k} \rangle, \tag{9}$$

where $|\psi_k\rangle := U \otimes V|\mathbf{k}\rangle$. In order to have equality for $\delta t = 0$ we make use of the above mentioned freedom and require for this case $W \otimes X = U \otimes V$, $Y \otimes Z = \mathbf{1} \otimes \mathbf{1}$ and $\vec{\xi} = \vec{\nu}$.

For infinitesimal δt we can thus expand

$$\langle \psi_k | W \otimes X = \langle \mathbf{k}| + \langle \delta \mathbf{k}^{\perp}|,$$

$$Y \otimes Z |\mathbf{k}\rangle = |\mathbf{k}\rangle + |\delta \bar{\mathbf{k}}^{\perp}\rangle,$$

$$\vec{\xi} = \vec{\nu} + \delta \vec{\nu}.$$

where we may assume $\langle \delta \mathbf{k}^{\perp} | \mathbf{k} \rangle = \langle k | \delta \bar{\mathbf{k}}^{\perp} \rangle = 0$. Combining everything in Eq. (9) and collecting terms up to first order we find

$$\langle \psi_k | H_{\vec{\lambda}} | \psi_k \rangle \delta t = \delta \nu_k$$

which has to hold for all k.

Let us now take a closer look at the diagonal elements $\langle\psi_k|H_{\vec{\lambda}}|\psi_k\rangle$. With regard to the definition $|\psi_k\rangle$ and now again including the time dependence of $U_t\otimes V_t$ we have $\langle\psi_k|H_{\vec{\lambda}}|\psi_k\rangle=\langle\mathbf{k}|(U_t\otimes V_t)^{\dagger}H_{\vec{\lambda}}(U_t\otimes V_t)|\mathbf{k}\rangle$. In the magic basis local unitaries take on the form of real orthogonal matrices $[(U_t\otimes V_t)^{\dagger}\rightarrow O(t)]$ and the Hamiltonian gets diagonal $[H_{\vec{\lambda}}\rightarrow D_{\vec{\lambda}}:=\mathrm{diag}(\vec{\lambda})]$. Therefore, $\delta\nu_k=\delta t(OD_{\vec{\lambda}}O^T)_{kk}$ $=\delta t[(O\bigcirc O)\vec{\lambda}]_k$, where $(O\bigcirc O)$ denotes the Hadamard product of the real orthogonal matrix $O(t)$ with itself. Defining $Q(t):=O(t)\bigcirc O(t)$ we can write compactly

$$\frac{\delta\vec{\nu}}{\delta t}=Q(t)\vec{\lambda}. \tag{10}$$

Recall that $\delta\vec{\nu}$ is the variation of the interaction content at some intermediate time $0\leq t\leq t_S$ in our simulation protocol. The overall interaction content $\vec{\nu}(t_S)$ is found by integrating Eq. (10) from 0 to t_S. As initial condition we have $\vec{\nu}(0)=\vec{0}$ since our simulation protocol starts from the identity having no interaction content. We then find

$$\vec{\nu}(t_S)=\int_0^{t_S}Q(t)dt\vec{\lambda}=S\vec{\lambda}t_S,$$

where $S:=1/t_S\int_0^{t_S}Q(t)dt$ is again a doubly stochastic matrix. To see this observe $\Sigma_{j=1}^4 S_{jk}=(1/t_S)\int_0^{t_S}\Sigma_{j=1}^4 Q(t)_{jk}dt=(1/t_S)\int_0^{t_S}1dt=1$. The same holds for summation over k.

With Lemma 3 we can state that $\vec{\nu}(t_S)\prec\vec{\lambda}t_S$ or switching again to the three-vector representation $\vec{\gamma}(t_S)\prec_s\vec{\alpha}t_S$ [see the definitions preceding Eq. (8)]. Remember that our basic assumption was that we have a simulation protocol for a gate $\mathcal{U}=U\otimes V\mathcal{U}_{\vec{\beta}}\widetilde{U}\otimes\widetilde{V}$. However, by means of $\vec{\gamma}(t_S)$ we can find a—possibly different—decomposition since $\mathcal{U}=\mathcal{U}_{t_S}=U_{t_S}\otimes V_{t_S}\mathcal{U}_{\vec{\gamma}_{t_S}}\widetilde{U}_{t_S}\otimes\widetilde{V}_{t_S}$. From the discussion in Sec. II A we know that the vectors $\vec{\beta}$ and $\vec{\gamma}_{t_S}$ have to be related via the local operations specified there. There are two operations that can be done to alter $\vec{\beta}$: (i) add multiples of $\pi/2$ to its components, i.e., build $\vec{\beta}_{\vec{n}}=\vec{\beta}+\pi/2\vec{n}$ for a vector $\vec{n}=(n_1,n_2,n_3)$, and (ii) permute and simultaneously change the sign of two components, which can be expressed easily by multiplication with an appropriate matrix P. Therefore, we must have $\vec{\gamma}(t_S)=P\vec{\beta}_{\vec{n}}\prec_s\vec{\alpha}t_S$ for some P and $\vec{n}$. Recalling the definition of s ordering of vectors [see Eq. (5) and the remarks there] we find $(P\vec{\beta}_{\vec{n}})_s=(\vec{\beta}_{\vec{n}})_s$ and therefore $\vec{\beta}_{\vec{n}}\prec_s\vec{\alpha}t_S$.

We now turn to the second part of our proof and show sufficiency. Since this has already been proven in Ref. [11] we will just sketch this proof. Let $\vec{\mu}$ and $\vec{\lambda}$ be the four-vectors corresponding to $\vec{\beta}_{\vec{n}}$ and $\vec{\alpha}$. Then Eq. (7) reads as $\vec{\mu}\prec\vec{\lambda}t_S$ and it follows by Brikhoff's theorem (see Sec. II B) that we can write $\vec{\mu}=\Sigma_{i=1}^n p_iP_i\vec{\lambda}t_S=\Sigma_{i=1}^n P_i\vec{\lambda}t_i$ where we defined $t_i=p_it_S$. Using that each of the $4!=24$ permutations

P_i of the magic states $\{|\mathbf{j}\rangle\}$ can be performed through an appropriate local unitaries $U_i\otimes V_i$ we have

$$\mathcal{U}_{\vec{\beta}_{\vec{n}}}=e^{-iH_{\vec{\mu}}}=\exp\left(-i\sum_{i=1}^n H_{P_i\vec{\lambda}t_i}\right)$$

$$=\exp\left(-i\sum_{i=1}^n U_i\otimes V_iH_{\vec{\lambda}}U_i^{\dagger}\otimes V_i^{\dagger}t_i\right)$$

$$=\prod_{i=1}^n U_i\otimes V_ie^{-iH_{\vec{\lambda}}t_i}U_i^{\dagger}\otimes V_i^{\dagger}.$$

For the last line we took into account that $[U_i\otimes V_iH_{\vec{\lambda}}U_i^{\dagger}\otimes V_i^{\dagger},U_j\otimes V_jH_{\vec{\lambda}}U_j^{\dagger}\otimes V_j^{\dagger}]=0\forall i,j$ since the local transformations involved only permute the eigen vectors of $H_{\vec{\lambda}}$. The last line provides clearly a proper simulation protocol for $\mathcal{U}_{\vec{\beta}_{\vec{n}}}$ and—by applying appropriate local unitaries at the beginning and at the end—for all locally equivalent gates (including $\mathcal{U}_{\vec{\beta}}$).

We remark here that Ref. [11] shows how to find explicitly the probability distribution $\{p_i\}$ and permutations $\{P_i\}$ which determine the time steps $\{t_i\}$ and the local unitaries $\{U_i\otimes V_i\}$. There also the maximal number n of evolution steps sufficient in any simulation protocol was determined. It turned out to be three for time optimal protocols.

This condition for the simulation of gates is an analogue to that established in Ref. [2] for efficient Hamiltonian simulation. Such a correspondence was, in principle, only expected for infinitesimal gates. It is remarkable that it extends in such a tight analogy to finite gates. The main difference is that here we have to include all different decompositions of the gate under consideration by allowing for variations $\vec{\beta}_{\vec{n}}=\vec{\beta}+(\pi/2\vec{n})$. There is no analog to this in the case of Hamiltonian simulation. The reason for this is that here we have to accommodate the periodicity properties of unitary operators while in the setting of Hamiltonian simulation we deal with a linear space of Hermitian operators.

C. Interaction costs

To finally assess the interaction cost $\mathcal{C}_H(\mathcal{U})$—i.e., the minimal time to simulate $\mathcal{U}$ using H and local unitaries as defined in Ref. [4]—we just have to optimize condition (7) with respect to both t_S and $\vec{n}$. Doing so we reproduce the main result of Ref. [4]:

Result 2. The interaction cost $\mathcal{C}_H(\mathcal{U})$ is the minimal value of $t_S\geq 0$ such that either $\vec{\beta}_{(0,0,0)}\prec_s\vec{\alpha}t_S$ or $\vec{\beta}_{(-1,0,0)}\prec_s\vec{\alpha}t_S$ holds.

Proof. This is equivalent to Result 1 under the restriction that it suffices to look at $\vec{n}$ being (0,0,0) or (−1,0,0) to find the smallest t_S. This is because in case $\vec{n}$ is not one of these two vectors we can show that either $\vec{\beta}_{(0,0,0)}\prec_s\vec{\beta}_{\vec{n}}$ or $\vec{\beta}_{(-1,0,0)}\prec_s\vec{\beta}_{\vec{n}}$. For the minimal time t_S such that $\vec{\beta}_{\vec{n}}\prec_s\vec{\alpha}t_S$ for a given $\vec{n}$ we therefore essentially have either $\vec{\beta}_{(0,0,0)}\prec_s\vec{\alpha}t_S$ or $\vec{\beta}_{(-1,0,0)}\prec_s\vec{\alpha}t_S$ for the same time t_S. Obviously letting $\vec{n}$ be (0,0,0) or (−1,0,0) will make for *at least* the

same minimal time. The optimization for $(0,0,0)$ or $(-1,0,0)$ cannot be avoided since in general $\vec{\beta}_{(0,0,0)}$ and $\vec{\beta}_{(-1,0,0)}$ are incomparable according to the s-majorization relation. To show that either $\vec{\beta}_{(0,0,0)} <_s \vec{\beta}_{\vec{n}}$ or $\vec{\beta}_{(-1,0,0)} <_s \vec{\beta}_{\vec{n}}$ for all $\vec{n}$ different than $(0,0,0)$ or $(-1,0,0)$ we distinguish two cases. (i) First we look at vectors $\vec{n}$ having at least one component $|n_j| > 1$. Since the components of $\vec{\beta}$ have to fulfill Eq. (1) the maximal component of the reordered form of $\vec{\beta}_{\vec{n}}$ (see Sec. II A) is at least $3\pi/4$. We then have $\vec{\beta}_{\vec{n}} >_s (3\pi/4,0,0)$ and this last vector clearly s majorizes both $\vec{\beta}_{(0,0,0)}$ and $\vec{\beta}_{(-1,0,0)}$. (ii) The vectors $\vec{n}$ satisfying $|n_j| \leqslant 1 \forall j$ have to be checked case by case. We find $\vec{\beta}_{(-1,0,0)} <_s \vec{\beta}_{\vec{n}}$ for $\vec{n} \in \{(-1,-1,-1),(0,1,0),(0,0,-1),(0,0,1)\}$ and $\vec{\beta}_{(0,0,0)} <_s \vec{\beta}_{\vec{n}}$ for the remaining $\vec{n}$.

Let us formulate result 2 as a kind of recipe. In order to time optimally perform a gate $\mathcal{U}$ using an interaction described by a Hamiltonian H together with arbitrary local unitaries proceed as follows:

(1) Determine $\vec{\beta}$ characterizing the interaction content of $\mathcal{U}$ following Ref. [5] (see, also the Appendix). Using Refs. [2,6] compute the canonical form of H to get $\vec{\alpha}$.

(2) Test whether $\vec{\beta}$ or $\vec{\beta}_{(-1,0,0)}$ is s majorized by $\vec{\alpha} t_S$ for a smaller time t_S.

(3) For the vector yielding the better result, as well as for $\vec{\alpha}$ compute the corresponding four-vectors $\vec{\mu}$ and $\vec{\lambda}$, respectively. Following Ref. [11] find the permutations P_i and probabilities p_i $(i=1,2,3)$ such that $\vec{\mu} = \Sigma_{i=1}^3 p_i P_i \vec{\lambda} t_S$.

(4) The p_i determine the time steps t_i and the P_i give the local unitaries to be applied in between. This provides a simulation protocol for $\mathcal{U}_{\vec{\mu}}$ using $H_{\vec{\lambda}}$ for at most three finite time steps.

(5) Simulate the evolutions according to $H_{\vec{\lambda}}$ by using the Hamiltonian H for the same period of time following Ref. [2]. Apply appropriate local unitaries (determined using Ref. [5]) in the beginning and at the end of the overall simulation to effectively perform $\mathcal{U}$.

We now discuss certain special cases for which some of the above points can be dropped or get simpler.

(a) In case the Hamiltonian we use describes solely pure interaction, that is to say is of the form $H = \Sigma_{i,j=1}^3 c_{ij} \sigma_i \otimes \sigma_j$ without any local parts, we can attain its canonical form by a local transformation $H_{\vec{\alpha}} = U \otimes V H U^{\dagger} \otimes V^{\dagger}$ (see Ref. [2]). Since $e^{-iH_{\vec{\alpha}} t_i} = U \otimes V e^{-iH t_i} U^{\dagger} \otimes V^{\dagger}$ we do not have to employ infinitesimal simulations [as required in step (5)] and the simulation protocol will only contain three finite time steps.

(b) In case the interaction content of the desired gate is characterized by a vector $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ satisfying $\vec{\beta} <_s \vec{\beta}_{(-1,0,0)}$ we can skip the optimization [step (2)] and state directly: The interaction cost $\mathcal{C}_H(\mathcal{U})$ is the minimal value of t_S such that $\vec{\beta} <_s \vec{\alpha} t_S$. The condition on $\vec{\beta}$ for $\vec{\beta} <_s \vec{\beta}_{(-1,0,0)}$ to be true is $\beta_1 + |\beta_3| \leqslant \pi/4$. To see this we have to apply the inequalities (5) defining the s majorization to $\vec{\beta}$ and $\vec{\beta}_{(-1,0,0)}^s = (\pi/2 - \beta_1, \beta_2, -\beta_3)$, the s-ordered version of

$\vec{\beta}_{(-1,0,0)}$ (see Sec. II A). We find

$$\beta_1 \leqslant \pi/2 - \beta_1,$$

$$\beta_1 + \beta_2 \mp \beta_3 \leqslant \pi/2 - \beta_1 + \beta_2 \pm \beta_3.$$

The first inequality is fulfilled trivially since $\beta_1 \leqslant \pi/4$ in any case. The last two inequalities are equivalent to $\beta_1 + |\beta_3| \leqslant \pi/4$ and this is what we claimed. The reverse $\vec{\beta}_{(-1,0,0)} <_s \vec{\beta}$ is never true because the first inequality is violated for any β_1. In all the other cases where $\vec{\beta}$ and $\vec{\beta}_{(-1,0,0)}$ are incomparable it will depend on the Hamiltonian which of the two vectors yields the optimal time.

D. Interaction costs of basic gates

As an illustration we shall give here explicitly the interaction costs for three specific gates [controlled NOT (CNOT), double CNOT (DCNOT), SWAP] and for the whole class of controlled-U gates. We choose those not only because they play a prominent role in quantum information but also due to their role as "landmarks" in the set of two-qubit gates as we will show in the following section. Let us list them here by first giving their definition in terms of their action on the computational basis $\{|i,j\rangle\}_{i,j=0}^1$, then characterizing their interaction content by the corresponding vector $\vec{\beta}$ and finally assessing the interaction costs pursuant to a general Hamiltonian H with canonical form $H_{\vec{\alpha}}$.

1. CNOT gate and controlled-U gates

The CNOT gate is the prototypical two-qubit quantum logic gate. Its action is defined compactly as $|i\rangle_A \otimes |j\rangle_B \rightarrow |i\rangle_A \otimes |i \oplus j\rangle_B$, where $\oplus$ denotes addition modulo 2. That is, it flips the second (target) qubit iff the first (control) qubit is in state $|1\rangle$. Let us denote the CNOT gate by $\mathcal{U}_{CNOT}^{AB}$, where the first superscript indicates the control and the second the target qubit. In the Appendix we show that the interaction content of this gate is given by $\vec{\beta} = (\pi/4)(1,0,0)$. Therefore, the CNOT belongs to the special class of gates where we can skip the optimization in result 2 and go straight ahead to majorization in order to determine the interaction cost. Requiring $\vec{\beta} <_s \vec{\alpha} t_S$ is equivalent to

$$\pi/4 \leqslant \alpha_1 t_S,$$

$$\pi/4 \leqslant (\alpha_1 + \alpha_2 \pm \alpha_3) t_S.$$

Clearly the first inequality yields the tighter bound. The interaction cost for simulating a CNOT is $\mathcal{C}_H(CNOT) = (\pi/4) 1/\alpha_1$.

The CNOT is a representative of the general class of controlled-U gates. These gates apply a unitary operation on the target qubit iff the control qubit is in state $|1\rangle$. Thus they have the form

$$\mathcal{U}_{ctrl-U} = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes U.$$

In the Appendix we show that the interaction content of a controlled-U gate is always described by $\vec{\beta} = (\beta,0,0)$, where

β is fixed by the eigenvalues of U. The interaction cost to simulate such a gate is $\mathcal{C}_H(\mathcal{U}_{ctrl-U}) = \beta/\alpha_1$.

2. DCNOT gate

The DCNOT gate is the concatenation of two CNOT gates in the following way $\mathcal{U}_{DCNOT}^{AB} = U_{CNOT}^{BA} \mathcal{U}_{CNOT}^{AB}$ and its action on the computational basis can be described as $|i\rangle_A \otimes |j\rangle_B \to |j\rangle_A \otimes |i \oplus j\rangle_B$. This gate was introduced in Ref. [7] as an intermediate gate between the CNOT and the SWAP gates. In the following we will emphasize the special role of the DC-NOT gate. Its interaction content is described by $\vec{\beta} = (\pi/4) \times (1,1,0)$ such that the DCNOT falls as well under the class of gates where we do not have to care about the optimization. For the interaction cost we find $\mathcal{C}_H(DCNOT) = (\pi/4)2/(\alpha_1 + \alpha_2 - |\alpha_3|)$.

3. SWAP gate

The SWAP gate is the unique gate having the effect to exchange the states of two qubits i.e., transforming $|i\rangle_A \otimes |j\rangle_B \to |j\rangle_A \otimes |i\rangle_B$. It is well known that $\mathcal{U}_{SWAP} = U_{CNOT}^{AB} \mathcal{U}_{CNOT}^{BA} \mathcal{U}_{CNOT}^{AB}$ and regarding the two other gates not very surprising that its interaction content is $\vec{\beta} = \pi/4(1,1,1)$. Once more recalling conditions (1) we can say that this is maximal. Now the optimization cannot be avoided. We find $\vec{\beta}_{(-1,0,0)}^s = \pi/4(1,1,-1)$ and it turns to be optimal to simulate $\vec{\beta}$ ($\vec{\beta}_{(-1,0,0)}^s$) if $\alpha_3 > 0$ ($\alpha_3 < 0$). In case $\alpha_3 = 0$ the interaction costs are equal for both alternatives. In any case we find the interaction costs $\mathcal{C}_H(SWAP) = (\pi/4)3/(\alpha_1 + \alpha_2 + |\alpha_3|)$.

E. Order of gates

What we see by these examples and what was to be expected is that the interaction costs depend strongly on the interaction resource—i.e., the Hamiltonian—we have at our disposal. But once the interaction is fixed the notion of interaction cost induces an order in the set of gates allowing us to compare the ''nonlocality'' of two gates in terms of the resources needed to perform them. Of course this order is always relative to the Hamiltonian and may change when we choose another one. For example if we use the Ising interaction $\sigma_1 \otimes \sigma_1$ we find the CNOT to be less nonlocal than the DCNOT and this one in turn to be less nonlocal than the SWAP. On the contrary with the exchange interaction $\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3$ at hand the SWAP is less time consuming than the DCNOT and in this sense less nonlocal. However, in a restricted region of the set of two-qubit gates this order is absolute, in that it does not depend on the interaction Hamiltonian. We will first define this order properly and then state and prove this result.

We say gate $\mathcal{U}$ is *more nonlocal* than gate $\mathcal{V}$, and write $\mathcal{V} \leq \mathcal{U}$, when for all interactions H the interaction cost of $\mathcal{U}$ is never smaller than that of $\mathcal{V}$,

$$\mathcal{V} \leq \mathcal{U} \equiv \mathcal{C}_H(\mathcal{V}) \leq \mathcal{C}_H(\mathcal{U}) \quad \forall H.$$

Result 3. Let $\mathcal{U}$ and $\mathcal{V}$ be two two-qubit gates with corresponding ordered vectors $\vec{\beta}_{\mathcal{U}}$ and $\vec{\beta}_{\mathcal{V}}$ such that in both cases

the restriction $\beta_1 + |\beta_3| \leq \pi/4$ holds. Then gate $\mathcal{U}$ is more nonlocal than gate $\mathcal{V}$ if and only if $\vec{\beta}_{\mathcal{V}} <_s \vec{\beta}_{\mathcal{U}}$.

Proof. Since both vectors $\vec{\beta}$ satisfy $\beta_1 + |\beta_3| \leq \pi/4$ the interaction costs $\mathcal{C}_H(\mathcal{V})$ and $\mathcal{C}_H(\mathcal{U})$ are given, respectively, by the smallest $t_{\mathcal{U}}, t_{\mathcal{V}} \geq 0$ such that

$$\vec{\beta}_{\mathcal{V}} <_s \vec{\alpha} t_{\mathcal{V}}$$

$$\vec{\beta}_{\mathcal{U}} <_s \vec{\alpha} t_{\mathcal{U}}.$$

Suppose first $\mathcal{V} \leq \mathcal{U}$, that is, for any Hamiltonian H we have $\mathcal{C}_H(\mathcal{V}) \leq \mathcal{C}_H(\mathcal{U})$ and in particular $\vec{\beta}_{\mathcal{V}} <_s \vec{\alpha} \mathcal{C}_H(\mathcal{U})$. If we rewrite this relation for the particular Hamiltonian where $\vec{\alpha} = \vec{\beta}_{\mathcal{U}}$ and use that in this case $\mathcal{C}_H(\mathcal{U}) = 1$ we find $\vec{\beta}_{\mathcal{V}} <_s \vec{\beta}_{\mathcal{U}}$. This proves the direct implication. The inverse follows right away by using the partial order property of majorization. $\vec{\alpha} \mathcal{C}_H(\mathcal{U}) >_s \vec{\beta}_{\mathcal{U}} >_s \vec{\beta}_{\mathcal{V}}$ directly implies $\mathcal{C}_H(\mathcal{U}) \geq \mathcal{C}_H(\mathcal{V})$ (see the proof of result 2).

Once more coming back to the problem of Hamiltonian simulation we mention that the corresponding partial order there has been solved completely [2]. The reason why the partial order established in result 3 only holds in the region of gates, where $\beta_1 + |\beta_3| \leq \pi/4$ is again that we have to deal here with the rather involved periodic structure of SU(4). It is exactly this restricted region where we can evade this difficulty by suppressing the otherwise essential optimization between $\vec{\beta}_{(0,0,0)}$ and $\vec{\beta}_{(-1,0,0)}$ (step 2 in the recipe given in Sec. III C).

IV. TRANSMISSION OF INFORMATION AND CLASSES OF GATES

By now we analyzed two-qubit gates in terms of the time expense they cause in the context of simulation. There the main objective is to perform a given gate on two qubits using a minimum time of interaction seen as a valuable resource. The notion of interaction cost thereby obtained gave a measure for how nonlocal (either relative to a specific interaction or absolute as in result 3) a gate is. In this section we change the perspective. We now want to prescribe the tasks a gate has to accomplish and ask how nonlocal it, therefore, has to be. In this setting we consider the gate and its inherent nonlocality to be the valuable resource. The task we have in mind here is the transmission of information in form of classical as well as quantum bits.

This section is organized as follows: First we motivate why the capability of gates to transmit bits is a proper measure for their nonlocality. After having given some basic definitions, we collect a number of known results for certain gates. Then we treat the problems of transmitting a cbit or a qubit in one direction, as well as all possible combinations of them in both directions by using a two-qubit gate and determine the interaction content necessary to do so. The subsequent discussion of the results will allow us to distinguish various classes of gates differing in their capability for quantum-communicational tasks which will give a characterization of the nonlocality of a gate as well.

A. Transmission capability and nonlocal content of gates

Nonlocal gates result physically from an interaction taken place between the qubits by some means. Interaction between two physical systems conditions on the other hand the transmission of information between them since after having interacted (at least one of) the subsystem's states will have changed depending on the states of *both* subsystems as they were before the interaction. Hence there must have been some kind of information exchange in the process of interaction. It is, therefore, natural to ask whether we can utilize a nonlocal gate to send (classical or quantum) information. The amount of information we can transmit using a gate will give us then a characterization of its degree of nonlocality. A similar point of view was captured in Refs. [7,8] where the amount of classical and quantum information necessary to implement a gate was adapted as a measure for its nonlocal content.

What do we mean by the transmission of classical or quantum information? Consider two parties Alice and Bob holding a qubit A and B, respectively. Assume further that somehow they manage to perform a gate $\mathcal{U}$ on their qubits. Then we say that $\mathcal{U}$ allows for the *transmission of a classical bit* from Alice to Bob (denoted by $\mathrm{cbit}_{A \to B}$) if after the application of $\mathcal{U}$ Bob can distinguish with probability 1 whether Alice's qubit was in $|0\rangle$ or $|1\rangle$. We speak of the *transmission of a quantum bit* from Alice to Bob ($\mathrm{qubit}_{A \to B}$) if under the action of $\mathcal{U}$ Bob's qubit takes on the state of Alice's qubit.

Let us make some remarks here. (i) The essential difference between these two effects of a gate is, that in the case of $\mathrm{cbit}_{A \to B}$ we do not require superpositions of $|0\rangle$ and $|1\rangle$ to be transmitted faithfully whereas in the case $\mathrm{qubit}_{A \to B}$ we do. The possibility to send a qubit trivially includes the one to transmit a cbit resembling the fact that quantum information incorporates classical information. (ii) Without further specifying $\mathcal{U}$ we can state directly that in case $\mathrm{qubit}_{A \to B}$ Alice loses her state after sending it due to the no-cloning-theorem. (iii) If Alice's qubit is maximally entangled to some ancilla qubit on her side then the transmission $\mathrm{qubit}_{A \to B}$ swaps the entanglement thus establishing a maximally entangled pair of qubits (ebit) between Alice and Bob. That is why the authors of Refs. [7,8] identified the capabilities of a gate to send a qubit and to create an ebit. Here we want to distinguish between the actual *creation of entanglement* without ancilla systems as treated in Ref. [5] and *entanglement swapping* by the transmission of a qubit. This differentiation is essential for example in the case of a CNOT gate which can be used to create an ebit $[\mathcal{U}_{CNOT}^{AB}(1/\sqrt{2})(|0\rangle + |1\rangle) \otimes |0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)]$ but not to transmit a qubit as we will show in the following.

For the gates introduced in Sec. III D it is well known and easy to see how they can be used to transmit bits. Regarding the definitions given there the following is effortless verified:

(a) $\mathcal{U}_{CNOT}^{AB}|i0\rangle = |ii\rangle$, $i = 0,1$ and therefore the CNOT is sufficient to send a cbit from Alice to Bob. Since Alice's qubit does not change at all under the action of this gate it is impossible for her to send a qubit to Bob (see remark (ii) above). This is not true if Alice and Bob share entanglement as an additional resource. See the remark below.

(b) $\mathcal{U}_{DCNOT}^{AB}|\varphi 0\rangle = |0\varphi\rangle$ where $|\varphi\rangle$ is an arbitrary qubit state transmitted by the action of $\mathcal{U}_{DCNOT}^{AB}$. Moreover, we find $\mathcal{U}_{DCNOT}^{AB}|\varphi 1\rangle = |1\rangle \otimes \sigma_x|\varphi\rangle$ telling us that Bob may send at the same time a cbit to Alice under the condition that in case he sent $|1\rangle$ he flips his qubit after the transmission in order to recover the correct state $|\varphi\rangle$. Since he knows what he sent, as we can assume, this requires no additional communication.

(c) $\mathcal{U}_{SWAP}|\varphi\psi\rangle = |\psi\varphi\rangle$, where $|\varphi\rangle$ and $|\psi\rangle$ are arbitrary states both being transmitted faithfully.

We can summarize this by the implications

$$CNOT \to \mathrm{cbit}_{A \to B},$$

$$DCNOT \to \mathrm{qubit}_{A \to B} + \mathrm{cbit}_{B \to A},$$

$$SWAP \to \mathrm{qubit}_{A \to B} + \mathrm{qubit}_{B \to A}.$$

Obviously, due to the symmetry of the nonlocal content of two-qubit gates under exchange of parties, the same expressions hold if we make the substitutions $A \leftrightarrow B$. These relations hold strictly for the case where the communicating parties have no ancilla systems and no prior entanglement at hand, but have to be read as lower bounds on the capabilities of these gates to transfer information if we allow for additional resources of this kind. It is a central result in quantum information that the capacities to transmit information can be increased if the parties possess shared entanglement (ebits) [7,8].

B. Transmission of information in the context of gate simulation

Assume now Alice and Bob want to send some given amount of information (possibly in both directions) by using some fixed interaction described by a Hamiltonian H and arbitrary local transformations of their qubits. They could do so by choosing appropriately one of the above gates providing the necessary transmission capability and then simulate it according to the results we derived so far. The interaction costs thereby incurred are given in Sec. III D. But is this optimal? There might be gates which are suitable for the same task but have an interaction content different from those of CNOT, DCNOT, or SWAP yielding smaller interaction costs. In the following we want to single out which gate is both sufficient for a certain transmission task and optimal in terms of interaction costs. We do this by deriving necessary and sufficient conditions on the interaction content of a gate to be capable for the transmission of a given amount of information. All we have to do then is to find the gate which fulfills the appropriate condition and causes the minimal interaction cost.

1. $cbit_{A \to B}$

Assume Alice encodes a classical bit into her qubit by preparing it in $|0\rangle$ or $|1\rangle$ and Bob holds some arbitrary state $|\varphi\rangle$. Then the bit is by definition transmitted if after an application of a gate $\mathcal{U}$ Bob's qubit takes on a state $|\psi\rangle$ or $|\psi^\perp\rangle$ (some state orthogonal to $|\psi\rangle$) depending on whether Alice

sent "0" or "1." At the same time Alice's qubit may change arbitrarily. The action of $\mathcal{U}$ we have to require is described by

$$|0\varphi\rangle \rightarrow |\chi\psi\rangle,$$

$$|1\varphi\rangle \rightarrow |\tilde{\chi}\psi^\perp\rangle. \quad (11)$$

More precisely we can state: A necessary condition for a gate $\mathcal{U}$ to be capable of transmitting a cbit is, that there exist states $|\varphi\rangle$, $|\chi\rangle$, $|\tilde{\chi}\rangle$, $|\psi\rangle$, and $|\psi^\perp\rangle$ such that relations (11) hold. Assume now that this is indeed the case. What can we say about the interaction content of $\mathcal{U}$? Since independent local transformations before and after the application of $\mathcal{U}$ do not affect its interaction content, we can look for unitaries fulfilling $Z|\varphi\rangle = |0\rangle$, $Y|\chi\rangle = |0\rangle$, $X|\psi\rangle = |0\rangle$ and $X|\psi^\perp\rangle = |1\rangle$ and define $\mathcal{U}' = (X_A \otimes Y_B)\mathcal{U}(\mathbf{1}_A \otimes Z_B)$ having a simpler action given by

$$|00\rangle \rightarrow |00\rangle,$$

$$|10\rangle \rightarrow |\alpha 1\rangle, \quad (12)$$

where $|\alpha\rangle = Y|\tilde{\chi}\rangle$. $\mathcal{U}'$ and $\mathcal{U}$ are locally equivalent and therefore have the same interaction content. To derive conditions on this interaction content we apply $\mathcal{U}'$ to the state $\varrho := \frac{1}{2}\mathbf{1}_A \otimes |0\rangle_B\langle 0|$—transforming under the terms of Eq. (12)—and take the partial trace with respect to system A:

$$\mathrm{tr}_A\{\mathcal{U}'\varrho\mathcal{U}'^\dagger\} = \frac{1}{2}\mathrm{tr}\{|00\rangle\langle 00| + |\alpha 1\rangle\langle \alpha 1|\} = \frac{1}{2}\mathbf{1}_B. \quad (13)$$

When we, on the other hand, assume a decomposition $\mathcal{U}' = (\tilde{U} \otimes \tilde{V})\mathcal{U}_{\vec{\beta}}(U \otimes V)$ we find

$$\mathrm{tr}_A\{\mathcal{U}'\varrho\mathcal{U}'^\dagger\} = \frac{1}{2}\mathrm{tr}_A\{[\tilde{V}_B\mathcal{U}_{\vec{\beta}}^{AB}V_B]\mathbf{1}_A \otimes |0\rangle_B\langle 0|[\tilde{V}_B\mathcal{U}_{\vec{\beta}}^{AB}V_B]^\dagger\}. \quad (14)$$

Equating the right-hand sides of Eqs. (13) and (14) and multiplying from the left by $\tilde{V}_B^\dagger$ and from the right by $\tilde{V}_B$ yields

$$\mathbf{1}_B = \mathrm{tr}_A\{\mathcal{U}_{\vec{\beta}}^{AB}\mathbf{1}_A \otimes |\omega\rangle_B\langle\omega|\mathcal{U}_{\vec{\beta}}^{AB\dagger}\},$$

where we have abbreviated $V|0\rangle = |\omega\rangle$. Expressing without loss of generality $|\omega\rangle = \cos(\omega)|0\rangle + e^{-i\theta}\sin(\omega)|1\rangle$ one can workout the trace explicitly and finds

$$\mathbf{1}_B = \begin{pmatrix} 1-a & b \\ b* & 1+a \end{pmatrix},$$

$$a = \cos(2\omega)\cos(2\beta_1)\cos(2\beta_2),$$

$$b = \sin(2\omega)\cos(2\beta_3)[\cos(\theta)\cos(2\beta_2) + i\sin(\theta)\sin(2\beta_1)]. \quad (15)$$

Let us stop here and consider what Eq. (15) tells us. The left-hand side was an immediate consequence of the necessary conditions on $\mathcal{U}$ to properly transmit a cbit while the right-hand side results from the general ansatz $\mathcal{U}' = (\tilde{U}$

$\otimes \tilde{V})\mathcal{U}_{\vec{\beta}}(U \otimes V)$, where the unitary V contains the parameters ω, θ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ characterizes the interaction content $\mathcal{U}_{\vec{\beta}}$. Equation (15) thus puts certain conditions on the parameters in the decomposition of $\mathcal{U}'$. Obviously we have to require $a = b = 0$. This in turn is fulfilled in various cases, for example whenever two of the coefficients $\beta_k = \pi/4$, the third being arbitrary. However, it is also easy to see that there are solutions, where only *one* of the coefficients $\beta_k = \pi/4$. In this case we have to choose either ω or θ appropriately. This puts conditions on the state $|\varphi\rangle$ in Eq. (11) denoting the input state Bob has to choose in order to properly receive the cbit Alice aims to send him. Three solutions of this kind are for example given by $\{\beta_1 = \pi/4, \omega = 0\}$, $\{\beta_2 = \pi/4, \omega = 0\}$, and $\{\beta_3 = \pi/4, \omega = \pi/4\}$ where in each case the remaining parameters can be chosen arbitrarily. All in all we have shown that it is a *necessary* condition for the transmission of a cbit to have at least one of the coefficients β_k equal to $\pi/4$ and without loss of generality we can always require this to be β_1.

To be systematic we should now continue and show, that any gate characterized by a vector $\vec{\beta} = (\pi/4, \beta_2, \beta_3)$ is also *sufficient* for this task. But at this point we will not do so for two reasons. First we already know that an interaction content $\vec{\beta} = (\pi/4, 0, 0)$ is sufficient to transmit a cbit because this basically fixes a CNOT or any gate locally equivalent to a CNOT. Second we find $(\pi/4, 0, 0) <_s (\pi/4, \beta_2, \beta_3)$ for all $0 \leq |\beta_3| \leq \beta_2 \leq \pi/4$ and therefore $\mathcal{C}_H(\text{CNOT}) \leq \mathcal{C}_H(\mathcal{U}_{(\pi/4, \beta_2, \beta_3)})$ for all H. Thus looking for gates other than those out of the CNOT class has no advantage in terms of interaction costs. Let us state this as

Result 4. The cheapest (time optimal) way to transmit a cbit using some given interaction is to simulate a CNOT gate. The interaction cost is $\mathcal{C}_H(cbit_{A \rightarrow B}) = (\pi/4)1/\alpha_1$.

The following results will show that the transmission capability scales up with the coefficients β_k becoming bigger. Just by continuity it follows then right away that any gate having an interaction content $\vec{\beta} = (\pi/4, \beta_2, \beta_3)$ is also sufficient to tranmit at least a cbit.

2. $cbit_{A \rightarrow B}$ and $cbit_{B \rightarrow A}$

Again let Alice encode a logical bit into her qubit as $|0\rangle$ or $|1\rangle$. Further assume Bob wants to send "0" and therefore prepares $|0\rangle$. To properly transmit their two messages they have to find a gate, which transforms the states like

$$|00\rangle \rightarrow |\varphi\chi\rangle,$$

$$|10\rangle \rightarrow |\psi\chi^\perp\rangle. \quad (16)$$

To detect the messages being sent to him, Bob has to measure the observable $\sigma_\chi = |\chi\rangle\langle\chi| - |\chi^\perp\rangle\langle\chi^\perp|$. Conversely Alice has to measure σ_φ or σ_ψ (defined similarly) depending on whether her message was "0" or "1." Consider now the same situation but let Bob's message be "1." The same reasoning as before yields

$$|01\rangle \rightarrow |\varphi^{\perp}\omega\rangle,$$

$$|11\rangle \rightarrow |\psi^{\perp}\omega^{\perp}\rangle. \tag{17}$$

Now Bob has to measure σ_{ω}. The transformation behavior characterized so far lacks of one essential condition: it is not unitary. Unitary transformations map an orthonormal basis into another one and this is so long not fulfilled, since f.e. $\langle \varphi\chi|\psi^{\perp}\omega^{\perp}\rangle \neq 0$. Imposing that the vectors on the right-hand side of Eqs. (16) and (17) build again a basis one finds four possible cases: (i) $\langle \varphi|\psi^{\perp}\rangle = 0$ and $\langle \psi|\varphi^{\perp}\rangle = 0$, (ii) $\langle \chi|\omega^{\perp}\rangle = 0$ and $\langle \omega|\chi^{\perp}\rangle = 0$, (iii) $\langle \varphi|\psi^{\perp}\rangle = 0$ and $\langle \omega|\chi^{\perp}\rangle = 0$, and (iv) $\langle \psi|\varphi^{\perp}\rangle = 0$ and $\langle \chi|\omega^{\perp}\rangle = 0$. The last two cases are more restrictive than (i) and (ii) since there the states of *both* qubits have to meet certain conditions. We are, however, interested to stay as less restrictive as possible so that we are going to focus on (i) in which case we have to require $|\psi\rangle = e^{-i\alpha}|\varphi\rangle$ and $|\psi^{\perp}\rangle = e^{-i\beta}|\varphi^{\perp}\rangle$. Let us summarize what we have found so far:

$$|00\rangle \rightarrow |\varphi\chi\rangle,$$

$$|10\rangle \rightarrow e^{-i\alpha}|\varphi\chi^{\perp}\rangle,$$

$$|01\rangle \rightarrow |\varphi^{\perp}\omega\rangle,$$

$$|11\rangle \rightarrow e^{-i\beta}|\varphi^{\perp}\omega^{\perp}\rangle.$$

Including the phases into $|\chi^{\perp}\rangle$ and $|\omega^{\perp}\rangle$ and again adjusting the axes by local transformations to cleanse the notation (as we did for the cbit$_{A\rightarrow B}$-problem) we can write equivalently

$$|00\rangle \rightarrow |00\rangle,$$

$$|10\rangle \rightarrow |01\rangle,$$

$$|01\rangle \rightarrow |1\,\omega\rangle,$$

$$|11\rangle \rightarrow |1\,\omega^{\perp}\rangle. \tag{18}$$

We can see that Bob has to measure a different observable depending on what he sent. For the case (ii) above we would find similar transformations but then being Alice the one who has to adapt her observable. Therefore case (i) gets identical with (ii), if we let Alice and Bob exchange their names which in turn cannot have any relevance for the interaction content of the gate they use. Or more mathematically: (i) can be transformed into (ii) by conjugating the gate with the SWAP and this does not alter the interaction content.

We can now parametrize $|\omega\rangle = \cos(\omega)|0\rangle + e^{-i\theta}\sin(\omega)|1\rangle$ and $|\omega^{\perp}\rangle = e^{-i\eta}(-\sin(\omega)|0\rangle + e^{-i\theta}\cos(\omega)|1\rangle)$ and determine the interaction content of the gate

$$\mathcal{U}(\eta,\theta,\omega) = e^{-i\pi/4}e^{i(\eta+\theta)/4}$$

$$\times \begin{pmatrix} e^{-i(\eta+\theta)}\cos(\omega) & 0 & e^{-i\theta}\sin(\omega) & 0 \\ -e^{-i\eta}\sin(\omega) & 0 & \cos(\omega) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\tag{19}$$

written in the computational basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ in this order. The global phase assures $\mathcal{U}(\eta,\theta,\omega)$ being a *special* unitary operator. Following Appendix one finds for the vector $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ characterizing the interaction content $\mathcal{U}_{\vec{\beta}}$ of $\mathcal{U}(\eta,\theta,\omega)$,

$$\beta_1 = \pi/4,$$

$$\beta_2 = \pi/4,$$

$$\beta_3 = \pi/4 - \vartheta,$$

where ϑ is a solution to $\tan^2(2\vartheta) = \sec^2[(\eta+\theta)/2]\sec^2(\omega) - 1$. ϑ therefore parametrizes a family of gates, of which each element has the desired capability to transmit cbit$_{A\leftrightarrow B}$.

Note especially that the DCNOT $[\vec{\beta} = (\pi/4, \pi/4, 0)]$ and the SWAP $[\vec{\beta} = (\pi/4, \pi/4, \pi/4)]$ belong to this family as we should expect according to the discussion in Sec. III D. These gates are attained for the choice $\vartheta = \pi/4$ and $\vartheta = 0$, respectively. In terms of (η, θ, ω) this corresponds f.e. to set $(\eta = \pi, \theta = 0, \omega = \pi/2)$ and $(\eta + \theta = 0, \omega = 0)$ for the DCNOT and the SWAP, respectively yielding the expected result when inserted in Eq. (19).

If we want to tranmit the cbits using some given interaction we can freely choose the parameter ϑ out of $[0, \pi/2]$ in order to keep down the interaction costs. Let us present the optimal choice in

Result 5. The cheapest (time optimal) way to transmit cbits in both directions using some given interaction is to simulate a gate holding an interaction content $\vec{\beta} = (\pi/4)[1, 1, 2\alpha_3/(\alpha_1 + \alpha_2)]$. The corresponding interaction cost is $\mathcal{C}_H(cbit_{A\leftrightarrow B}) = (\pi/4)2/(\alpha_1 + \alpha_2)$.

Proof. Define $b := 1/2 - 2/\pi \cdot \vartheta$ and parametrize $\vec{\beta}(\vartheta) = \vec{\beta}(b) = (\pi/4)(1, 1, 2b)$. We have to find $b \in [-1/2, 1/2]$ and $t_S \geq 0$ such that either $\vec{\beta}(b) \prec_s \vec{\alpha}t_s$ or $\vec{\beta}_{(-1,0,0)}(b) \prec_s \vec{\alpha}t_s$ holds and t_S is minimal. First note that $\vec{\beta}^s_{(-1,0,0)}(b) = \vec{\beta}(-b)$. The optimization with respect to b therefore includes that with respect to $\vec{\beta}$ and $\vec{\beta}_{(-1,0,0)}$. The minimal time such that $\vec{\beta}(b) \prec_s \vec{\alpha}t_s$ is fulfilled is given by

$$t_{\min}(b) = \max\left\{ \frac{\pi}{4}\frac{1}{\alpha_1}, \frac{\pi}{2}\frac{1-b}{\alpha_1 + \alpha_2 - \alpha_3}, \frac{\pi}{2}\frac{1+b}{\alpha_1 + \alpha_2 + \alpha_3} \right\}.$$

Optimization with respect to b yields the interaction cost

TABLE I. Interaction content, transmission capability and interaction costs are listed for various classes of gates. With respect to their transmission capability the whole set of two-qubit gates resolves into a natural hierarchy of four classes.

	Interaction content			Transmission capability			Interaction cost
	β_1	β_2	β_3	$\mathrm{cbit}_{A \to B}$	$\mathrm{qubit}_{A \to B}$ and $\mathrm{cbit}_{B \to A}$	$\mathrm{qubit}_{A \leftrightarrow B}$	$\mathcal{C}_H(\mathcal{U})$
Controlled-U	x	0	0	$\times$	$\times$	$\times$	$\dfrac{x}{\alpha_1}$
CNOT	$\pi/4$	0	0	$\surd$	$\times$	$\times$	$\dfrac{\pi}{4}\dfrac{1}{\alpha_1}$
I	$\pi/4$	y	z	$\surd$	$\times$	$\times$	$\geqslant \dfrac{\pi}{4}\dfrac{1}{\alpha_1}$
DCNOT	$\pi/4$	$\pi/4$	0	$\surd$	$\surd$	$\times$	$\dfrac{\pi}{4}\dfrac{2}{\alpha_1+\alpha_2-\lvert\alpha_3\rvert}$
II	$\pi/4$	$\pi/4$	z	$\surd$	$\surd$	$\times$	$\dfrac{\pi}{4}\dfrac{2}{\alpha_1+\alpha_2}$ for $z=\dfrac{\pi}{4}\dfrac{2\alpha_3}{\alpha_1+\alpha_2}$
SWAP	$\pi/4$	$\pi/4$	$\pi/4$	$\surd$	$\surd$	$\surd$	$\dfrac{\pi}{4}\dfrac{3}{\alpha_1+\alpha_2+\lvert\alpha_3\rvert}$

$\mathcal{C}_H(\mathrm{cbit}_{A \leftrightarrow B})$

$$= \min_{b \in [-1/2,1/2]} [t_{\min}(b)]$$

$$= \min_{b \in [-1/2,1/2]}$$

$$\times \left[\max\left\{ \frac{\pi}{4}\frac{1}{\alpha_1}, \frac{\pi}{2}\frac{1-b}{\alpha_1+\alpha_2-\alpha_3}, \frac{\pi}{2}\frac{1+b}{\alpha_1+\alpha_2+\alpha_3} \right\} \right].$$

This is an exercise in linear optimization which has to be solved under the condition $\pi/4 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \lvert\alpha_3\rvert$. An elementary calculation yields $\mathcal{C}_H(\mathrm{cbit}_{A \leftrightarrow B}) = (\pi/2)1/(\alpha_1+\alpha_2)$ for $b = \alpha_3/(\alpha_1+\alpha_2)$.

3. qubit$_{A \to B}$ and (qubit$_{A \to B}$ and cbit$_{B \to A}$)

To reliably transmit a qubit we require

$$\lvert 00 \rangle \to \lvert \varphi\chi \rangle,$$

$$\lvert 10 \rangle \to \lvert \varphi\chi^\perp \rangle.$$

The remaining vectors $\lvert 01 \rangle$ and $\lvert 11 \rangle$ may transform arbitrarily but have to stay orthogonal to both among themselves and with respect to $\lvert \varphi\chi \rangle$ and $\lvert \varphi\chi^\perp \rangle$. The least restrictive choice yields similar to the foregoing section

$$\lvert 01 \rangle \to \lvert \varphi^\perp \omega \rangle,$$

$$\lvert 11 \rangle \to \lvert \varphi^\perp \omega^\perp \rangle.$$

Without loss of generality we can identify $\lvert \varphi \rangle = \lvert 0 \rangle, \lvert \varphi^\perp \rangle = \lvert 1 \rangle, \lvert \chi \rangle = \lvert 0 \rangle$, and $\lvert \chi^\perp \rangle = \lvert 1 \rangle$ ending up with the same gate

(18) as for $\mathrm{cbit}_{A \leftrightarrow B}$. The optimal interaction content and cost to send a $\mathrm{qubit}_{A \to B}$ is therefore the same as in result 5.

Regarding the transformations given in Eq. (18) it is obvious that this gate is also capable to send at the same time a $\mathrm{cbit}_{B \to A}$. To do so Bob encodes his bit into $\lvert 0 \rangle$ or $\lvert 1 \rangle$. Applying $\mathcal{U}$ sends the bit to Alice. The qubit Bob gets from Alice comes in faithfully if Bob sent "0." In the other case he has to recover the qubit by a local transformation obeying $V\lvert \omega \rangle = \lvert 0 \rangle$ and $V\lvert \omega^\perp \rangle = \lvert 1 \rangle$. An interaction content $\vec{\beta} = (\pi/4, \pi/4, \pi/4 - \vartheta)$ is therefore sufficient for the transmission $\mathrm{qubit}_{A \to B}$ and $\mathrm{cbit}_{B \to A}$. This is also necessary since any interaction content showing less than $\pi/4$ in the first two entries is not sufficient to send a $\mathrm{qubit}_{A \to B}$. Again we can refer to the values given in result 5 for the optimal interaction content and cost.

4. qubit$_{A \leftrightarrow B}$

This problem is trivial since the exchange of the two quantum states completely fixes the transformation of the basis states and therefore also the gate. The SWAP is the only gate providing the required action. Interaction content and cost are given in Sec. III D.

C. Classes of gates

The results of the foregoing sections are summarized in Table I.

One can see that the capability of a gate to transmit information increases when the coefficients β_k characterizing its interaction content approach their maximal values $\pi/4$. Es-

pecially when one of them takes on this maximum value, the corresponding gate acquires a new feature. The special gates CNOT, DCNOT, and SWAP (and all their local equivalents) mark these thresholds and that is why we announced them being ''landmarks'' in the set of two-qubit gates. This allows us to distinguish four classes of gates differing in their transmission capability: (i) gates with $\pi/4 > \beta_1 \geqslant \beta_2 \geqslant |\beta_3|$ (no transmission capability), (ii) CNOT and type I, (iii) DCNOT and type II, and (iv) SWAP. This classification endows the coefficients β_k with physical significance and therefore complements earlier work, where a gate's interaction content $\mathcal{U}_{\vec{\beta}}$ was associated with its capability to create entanglement [5].

V. CONCLUSIONS

In this work we addressed the problem of simulating two-qubit gates using some given interaction and local unitary transformations in the fast control limit. For this to be possible we presented a necessary and sufficient condition linking the gate, the Hamiltonian characterizing the interaction and the total time of simulation. Optimization with respect to time gave a measure $\mathcal{C}_H(\mathcal{U})$—termed interaction cost—for how costly such a simulation in terms of time of interaction is and thereby recovered a result already attained in Ref. [4]. The interaction cost has been computed for various gates and was shown to induce a partial order in a region of the set of two-qubit gates thus establishing a meaningful notion of and measure for the nonlocality of a gate.

To give an application, as well as a supplementation of these results we then turned to the problem of transmitting information between two parties using two-qubit gates. Necessary and sufficient conditions on gates were established to be capable of transferring classical and quantum bits in all combinations and directions. This allowed us to compute explicitly the interaction costs for these tasks. Beyond it the transmission capability of a gate provided a classification of two-qubit gates.

All results derived here concern two-qubit systems. All the underlying problems can naturally be extended to higher dimensional systems and therefore, it would be desirable to generalize the results. The main obstacle to do so is that in higher dimensions there is no decomposition like in Eq. (1) for a general unitary operator.

ACKNOWLEDGMENTS

APPENDIX: INTERACTION CONTENT OF NONLOCAL GATES

In Lemma 1 we presented a decomposition for two-qubit gates of the form $\mathcal{U} = \tilde{U}_A \otimes \tilde{V}_B e^{-iH} U_A \otimes V_B$, where $H = \exp(\Sigma_{k=1}^3 \alpha_k \sigma_k \otimes \sigma_k)$. Here we demonstrate a method based on Ref. [5] to determine the α_k for a general given $\mathcal{U}$.

In Sec. II A we gave an alternative representation of H in terms of its eigenvalues λ_k. The method actually admits to compute the λ_ks and relies on the following two observations: (i) Hamiltonians of the special form considered here are diagonal in the magic basis as we have already shown in Sec. II A. (ii) Local unitaries are real in the magic basis [9]. Especially they become real orthogonal matrices since of course they stay to be unitary. This fact resembles the homomorphism $SU(2,\mathbb{C}) \otimes SU(2,\mathbb{C}) \simeq SO(4,\mathbb{R})$ [14] becoming manifest in the magic basis. Using these two facts the decomposition takes on the form $\mathcal{U} = \tilde{O} D O$ when written in the magic basis where $D = \text{diag}(e^{-i\lambda_1}, e^{-i\lambda_2}, e^{-i\lambda_3}, e^{-i\lambda_4})$ and $\tilde{O}, O$ are real orthogonal matrices corresponding to $\tilde{U}_A \otimes \tilde{V}_B$ and $U_A \otimes V_B$. Therefore $\mathcal{U}^T \mathcal{U} = O^T D \tilde{O}^T \tilde{O} D O = O^T D^2 O$. Hence, if we compute the eigenvalues of $\mathcal{U}^T \mathcal{U}$ we will find them to be $\{e^{-2i\lambda_1}, e^{-2i\lambda_2}, e^{-2i\lambda_3}, e^{-2i\lambda_4}\}$. Taking the arguments of these phases and dividing by two will give us the λ_ks and via Eq. (4) the α_ks.

As an example let us determine the α_ks for the CNOT gate. In the computational basis [in the order $(|11\rangle, |10\rangle, |01\rangle, |00\rangle)$] and the magic basis [in the order given by the enumeration in Eq. (2)] we find respectively

$$\mathcal{U}_{CNOT}^{AB} = e^{-i\pi/4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{CB}$$

$$= \frac{e^{-i\pi/4}}{2} \begin{pmatrix} 1 & -i & -1 & -i \\ i & 1 & i & -1 \\ -1 & -i & 1 & -1 \\ i & -1 & i & 1 \end{pmatrix}_{MB}.$$

The overall phase included assures that $\det(\mathcal{U}_{CNOT}) = 1$ and therefore $\mathcal{U}_{CNOT} \in SU(4)$. The eigenvalues of $\mathcal{U}_{CNOT}^T \mathcal{U}_{CNOT}$ turn out to be $\{i, i, -i, -i\}$. Taking the square root and then ordering the arguments in decreasing order we find $\vec{\lambda} = \pi/4(1,1,-1,-1)$. Solving Eq. (4) we get $\vec{\alpha} = \pi/4(1,0,0)$.

However, in some cases simple algebraic considerations provide a more elegant way to find the interaction content. We shall demonstrate this on the basis of the class of controlled-U gates. These gates are of the form $\mathcal{U}_{ctrl-U}^{AB} = P_0 + P_1 \mathbf{1} \otimes U$, where $P_i = |i\rangle_A \langle i| \otimes \mathbf{1}_B$ as we mentioned in Sec. III D. If we now take the transpose $\mathcal{U}_{ctrl-U}^T$ in the magic basis and take into account that $P_0^T = P_1$ and $(\mathbf{1} \otimes U)^T = \mathbf{1} \otimes U^\dagger$ we find $\mathcal{U}_{ctrl-U}^T \mathcal{U}_{ctrl-U} = (P_1 + P_0 \mathbf{1} \otimes U^\dagger)(P_0 + P_1 \mathbf{1}$

$\otimes U) = P_0 \mathbf{1} \otimes U^\dagger + P_1 \mathbf{1} \otimes U = |0\rangle\langle 0| \otimes U^\dagger + |1\rangle\langle 1| \otimes U$. This operator is block diagonal in the computational basis and therefore has the same eigenvalues as U but with multiplicity 2, i.e., has a spectrum $\{e^{i2\beta}, e^{i2\beta}, e^{-i2\beta}, e^{-i2\beta}\}$, where $e^{\pm i2\beta}$

are the eigenvalues of U. Solving Eqs. (4) for $\lambda_1 = \lambda_2 = \beta, \lambda_3 = \lambda_4 = -\beta$ we find $\vec{\alpha} = (\beta, 0, 0) [\vec{\alpha} = (\pi/2 - \beta, 0, 0)]$ for $\beta \leq \pi/4 [\beta \geq \pi/4]$. For the CNOT we have especially $U = \sigma_x$ and thus $\beta = \pi/4$ as it shall be.

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[15] A matrix is called doubly stochatic if its entries are all nonnegative and each row and column adds up to one.

[16] Throughout the paper we put $\hbar = 1$ and consider time to be dimensionless.