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Separating transformation in a problem on extremal decomposition of the complex plane

Dedicated to Prof. Yu. B. Zelinskii on the occasion of his 70^{th} *birthday*

The paper is devoted to investigation of the problems of geometric function theory of a complex variable. A general problem of the description of extremal configurations maximizing the product of the inner radii of mutually non-overlapping domains is studied.

Робота присвячена дослідженню відкритих проблем геометричної теорії функцій комплексної змінної. Зокрема, вивчається загальна проблема з описання екстремальних конфігурацій, що мінімізують добуток внутрішніх радіусів попарно неперетинних областей.

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively; $\mathbb{C}$ be the Complex plane and let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0,\infty)$. Let $r(B,a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ [1, 2]. An inner radius is a generalization of a conformal radius for multiply connected domains.

Definition 1. *A finite collection of arbitrary fixed domains* $\{B_k\}_{k=1}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, *such as* $B_k \subset \overline{\mathbb{C}}$, $B_k \cap B_m = \emptyset$, $k \neq m$, $k,m = \overline{1,n}$ *is called a system of non-overlapping domains.*

Definition 2. *Let* $n \in \mathbb{N}$, $n \geqslant 2$. *A set of points* $A_n :=$ $:= \left\{a_k \in \mathbb{C} : k = \overline{1,n}\right\}$ *is called* n*-radial system if:* $|a_k| \in \mathbb{R}^+$, $k = \overline{1,n}$, $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Denote $P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\}$, $\theta_k := \arg a_k, a_{n+1} := a_1, \theta_{n+1} := 2\pi$, $\alpha_k := \frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k=\overline{1,n}$. Let $\chi(t)=\frac{1}{2}(t+t^{-1})$. For an arbitrary n-radial system of points $A_n=\{a_k\}$, $k=\overline{1,n}$, we assume

$$\mathcal{L}(A_n) := \prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|.$$

The class of n-radial systems of points for which $\mathcal{L}(A_n)=1$ automatically includes all systems of n distinct points that are located on the unit circle.

The goal of the present work is the construction of sharp upper bounds for a functional of the form

$$J_n(\gamma)=\left[r\left(B_0,0\right)r\left(B_\infty,\infty\right)\right]^{\gamma}\prod_{k=1}^{n}r\left(B_k,a_k\right),$$

where $\gamma\in\mathbb{R}^+$, $A_n=\{a_k\}_{k=1}^{n}$ is n-radial system of points, B_0, B_∞, $\{B_k\}_{k=1}^{n}$ is system of pairwise disjoint domains, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{0,n}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$.

Note that to describe the extremal configurations of the domains we use notion of quadratic differential (see, for example, [1, P. 63–70]).

The following statement holds

Theorem 1. *Let $n\in\mathbb{N}$, $n\geqslant 7$, $0<\gamma\leqslant\gamma_n$, $\gamma_n=0{,}08\,n^2$. Then for any n-radial system of points $A_n=\{a_k\}_{k=1}^{n}$ such that $\mathcal{L}(A_n)=1$ and any set of mutually non-overlapping domains B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the following inequality holds*

$$\left[r\left(B_0,0\right)r\left(B_\infty,\infty\right)\right]^{\gamma}\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant\left[r\left(\Lambda_0,0\right)r\left(\Lambda_\infty,\infty\right)\right]^{\gamma}\prod_{k=1}^{n}r\left(\Lambda_k,\lambda_k\right),$$

where domains Λ_0, Λ_∞, Λ_k, and points 0, ∞, λ_k, $k=\overline{1,n}$, are, respectively, circular domains and poles of the quadratic differential

$$Q(w)dw^2=-\frac{\gamma w^{2n}+(n^2-2\gamma)w^n+\gamma}{w^2(w^n-1)^2}\,dw^2. \qquad (1)$$

Proof. Consider the system of functions

$$\zeta = \pi_k(w) = -i\left(e^{-i\theta_k}w\right)^{\frac{1}{\alpha_k}}, \quad k=\overline{1,n}.$$

The family of functions $\{\pi_k(w)\}_{k=1}^n$ is called admissible for separating transformation of domains B_0, B_∞, B_k, $k=\overline{1,n}$, with respect to the angles $\{P_k\}_{k=1}^n$. Let $\Omega_k^{(1)}$, $k=\overline{1,n}$, denote a domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_k\bigcap \overline{P}_k)$ containing the point $\pi_k(a_k)$ with the own symmetric reflection relative to the imaginary axis. In turn, by $\Omega_k^{(2)}$, $k=\overline{1,n}$, we denote the domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_{k+1}\bigcap \overline{P}_k)$ containing the point $\pi_k(a_{k+1})$ with the own symmetric reflection relative to the imaginary axis, $B_{n+1}:=B_1$, $\pi_n(a_{n+1}):=\pi_n(a_1)$. In addition, by $\Omega_k^{(0)}$ we denote the domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_0\bigcap \overline{P}_k)$ containing the point $\zeta=0$ with the own symmetric reflection relative to the imaginary axis. Accordingly, by $\Omega_k^{(\infty)}$ we denote the domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_\infty\bigcap \overline{P}_k)$ containing the point $\zeta=\infty$ with the own symmetric reflection relative to the imaginary axis. Denote $\pi_k(a_k):=\omega_k^{(1)}$, $\pi_k(a_{k+1}):=\omega_k^{(2)}$, $k=\overline{1,n}$. It follows from the definition of the function $\pi_k(w)$ that

$$|\pi_k(w)-\omega_k^{(1)}| \sim \frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1}\cdot|w-a_k|, \quad w\to a_k, \quad w\in\overline{P_k},$$

$$|\pi_k(w)-\omega_k^{(2)}| \sim \frac{1}{\alpha_k}|a_{k+1}|^{\frac{1}{\alpha_k}-1}\cdot|w-a_{k+1}|, \quad w\to a_{k+1}, \quad w\in\overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w\to 0, \quad w\in\overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w\to \infty, \quad w\in\overline{P_k}.$$

Taking into account corresponding results of papers [2, 3], we have the inequalities

$$r\left(B_k,a_k\right) \leqslant \left[\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1}\cdot\frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right]^{\frac{1}{2}}, \quad k=\overline{1,n}, \qquad (2)$$

$$r\left(B_0,0\right)\leqslant\left[\prod_{k=1}^{n}r^{\alpha_k^2}\left(\Omega_k^{(0)},0\right)\right]^{\frac{1}{2}},$$
$$r\left(B_\infty,\infty\right)\leqslant\left[\prod_{k=1}^{n}r^{\alpha_k^2}\left(\Omega_k^{(\infty)},\infty\right)\right]^{\frac{1}{2}}. \qquad (3)$$

An equality in (2)–(3) is fully investigated in [2, Theorem 1.9]. By using this inequalities we have the expansion

$$J_n(\gamma)\leqslant\prod_{k=1}^{n}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\frac{\gamma\alpha_k^2}{2}}\times$$
$$\times\left(\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1}\cdot\frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right)^{\frac{1}{2}}.$$

Elementary calculations show that

$$J_n(\gamma)\leqslant\left(\prod_{k=1}^{n}\alpha_k\right)\prod_{k=1}^{n}\frac{|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}}{\left(|a_k||a_{k+1}|\right)^{\frac{1}{2\alpha_k}}}\cdot|a_k|\times$$

$$\times\left\{\prod_{k=1}^{n}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\right\}^{\frac{1}{2}}.$$

Hence

$$J_n(\gamma)\leqslant\left(\prod_{k=1}^{n}\alpha_k\right)\prod_{k=1}^{n}\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}+\left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|\times$$

$$\times\left\{\prod_{k=1}^{n}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\right\}^{\frac{1}{2}}.$$

Thus

$$\prod_{k=1}^{n}\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}+\left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|=\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|=\mathcal{L}(A_n),$$

and we obtain

$$J_n(\gamma)\leqslant 2^n\cdot\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\mathcal{L}(A_n)\times$$

$$\times\prod_{k=1}^{n}\left\{\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\right\}^{\frac{1}{2}}.$$

For convenience, right side of the last inequality we multiply and divide by the value $\sqrt{\gamma}$ and get the inequality

$$J_n(\gamma)\leqslant\left(\frac{2}{\sqrt{\gamma}}\right)^n\cdot\left(\prod_{k=1}^{n}\alpha_k\sqrt{\gamma}\right)\cdot\mathcal{L}(A_n)\times$$

$$\times\prod_{k=1}^{n}\left\{\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\right\}^{\frac{1}{2}},$$

where $|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}=|\omega_k^{(2)}-\omega_k^{(1)}|$, $k=\overline{1,n}$. Equality in the last inequality is attained if and only if the equality is attained in inequalities (2) – (3) for all $k=\overline{1,n}$. Each expression contained in braces in the last inequality is value of the functional

$$K_\tau=\left[r\left(B_0,0\right)r\left(B_\infty,\infty\right)\right]^{\tau^2}\cdot\frac{r\left(B_1,a_1\right)r\left(B_2,a_2\right)}{|a_1-a_2|^2}\qquad(4)$$

on the system of non-overlapping domains $\{\Omega_k^{(0)},\Omega_k^{(1)},\Omega_k^{(2)},\Omega_k^{(\infty)}\}$ and corresponding points system $\{0,\omega_k^{(1)},\omega_k^{(2)},\infty\}$, $k=\overline{1,n}$. Functional evaluation (4) in the case of fixed poles was first obtained by V. N. Dubinin, then by G. V. Kuzmina, E. Emelyanov, A. L. Tarhonskii. Note that in our case the points a_1 and a_2 of the functional (4) are not fixed and not symmetrical. So, fractional-linear transformations can not reduce them to -1 and 1. V. N. Dubinin and G. V. Kuzmina considered the case when the points a_1 and a_2 are symmetrical and equal to 1 and -1, respectively. Therefore decisive role is played corollary of Theorem 4.1.1 in [1, p. 169]. From here, based on invariance of the functional (4) as in the proof of Theorem 4.1.1 in [1], we have that $K_\tau \leqslant \Phi(\tau)$, $\tau \geqslant 0$, where $\Phi(\tau) = \tau^{2\tau^2} \cdot |1-\tau|^{-(1-\tau)^2} \cdot (1+\tau)^{-(1+\tau)^2}$. Then

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \left(\prod_{k=1}^{n} \alpha_k \sqrt{\gamma}\right) \left[\prod_{k=1}^{n} \Phi(\tau_k)\right]^{1/2} =$$

$$= \left(\frac{2}{\sqrt{\gamma}}\right)^n \left[\prod_{k=1}^{n} \left(\tau_k^{2\tau_k^2+2} \cdot |1-\tau_k|^{-(1-\tau_k)^2} \cdot (1+\tau_k)^{-(1+\tau_k)^2}\right)\right]^{\frac{1}{2}},$$

where $\tau_k = \sqrt{\gamma} \cdot \alpha_k$, $k = \overline{1,n}$. Let

$$S(x) = x^{2x^2+2} \cdot |1-x|^{-(1-x)^2} \cdot (1+x)^{-(1+x)^2} \quad \text{and} \quad \Psi(x) = \ln(S(x)).$$

$S(x)$ is logarithmically convex function on the interval $[0, x_0]$, $x_0 \approx 0{,}88441$. Further similarly [3, 4] we consider the following extremal problem

$$\prod_{k=1}^{n} S(x_k) \longrightarrow \max, \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma}, \quad x_k = \alpha_k \sqrt{\gamma}.$$

Let $X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$ be any extremal set of points in above mentioned problem. In a similar way as in [4], we obtain the following result: *if* $0 < x_k^{(0)} < x_j^{(0)}$, *then*

$$\Psi'(x_k^{(0)}) = \Psi'(x_j^{(0)}), \tag{5}$$

where $k,j = \overline{1,n}$, $k \neq j$,

$$\Psi'(x) = 4x\ln(x) - 2(x-1)\ln|x-1| - 2(x+1)\ln(x+1) + \frac{2}{x}$$

(*see Fig.* 1).

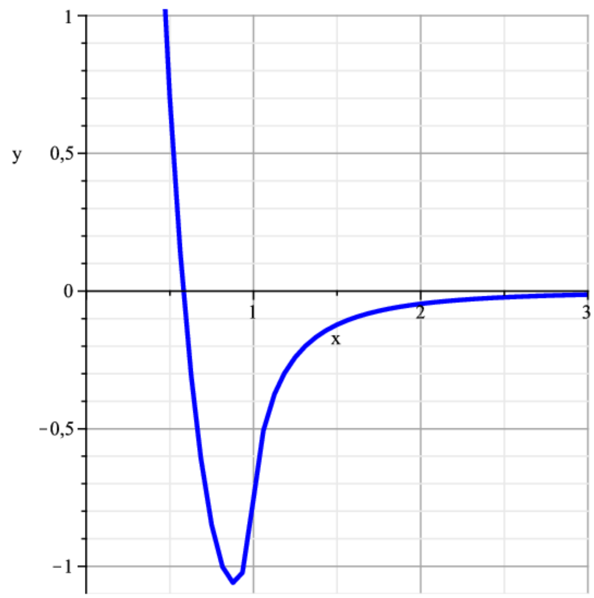

Fig. 1: Graph of the function $y = \Psi'(x)$

By using similar arguments as in the paper [4] and the relation (5) we will prove that $x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)}$. Taking into account properties of the function $\Psi'(x)$ and conditions of Theorem we obtain the following relation: $(x_1 - 0{,}56)\,n + (x_2 - x_1) > 0$ for $n \geqslant 7$. Therefore, we have $nx_1 + (x_2 - x_1) > 0{,}56\,n$. Finally we get

$$(n-1)x_1 + x_2 > 2\sqrt{\gamma_n}, \quad \gamma_n = 0{,}08\,n^2, \quad n \geqslant 7.$$

Thus, in accordance with [4], we agree to say that the set of points $\left\{x_k^{(0)}\right\}_{k=1}^{n}$, $n \geqslant 7$, can not be extremal if $x_n^{(0)} \in (x_0, 2]$. Consequently, for an extremal set $\left\{x_k^{(0)}\right\}_{k=1}^{n}$ is only possible in the case when $x_k^{(0)} \in (0, x_0]$, $k = \overline{1,n}$, and $x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}$. For $\gamma < \gamma_n$, $n \geqslant 7$, all previous arguments are valid. The equality case is straightforward to verify. Theorem 1 is proved.

Corollary 1. *Under conditions of Theorem 1 the following inequality holds*

$$[r(B_0,0)r(B_\infty,\infty)]^\gamma \prod_{k=1}^{n} r(B_k,a_k) \leqslant$$

$$\leqslant \left(\frac{4}{n}\right)^n \left[\frac{\left(\frac{4\gamma}{n^2}\right)^{\left(\frac{4\gamma}{n^2}\right)}}{\left|\frac{2\sqrt{\gamma}}{n}-1\right|^{\left(\frac{2\sqrt{\gamma}}{n}-1\right)^2}\left(\frac{2\sqrt{\gamma}}{n}+1\right)^{\left(\frac{2\sqrt{\gamma}}{n}+1\right)^2}}\right]^{\frac{n}{2}}.$$

Equality is attained if 0, ∞, a_k, *and* B_0, B_∞, B_k, $k=\overline{1,n}$, *are, respectively, poles and circular domains of the quadratic differential* (1).

Corollary 2. *Let* $n \in \mathbb{N}$, $n \geqslant 7$, $0 < \gamma \leqslant \gamma_n$, $\gamma_n = 0{,}08\,n^2$. *Then for any different points on the circle* $|a_k| = 1$, $k = \overline{1,n}$, *and any pairwise non-overlapping domains* B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, *the following inequality holds*

$$[r\,(B_0,0)\,r\,(B_\infty,\infty)]^\gamma \prod_{k=1}^{n} r\,(B_k,a_k) \leqslant [r\,(\Lambda_0,0)\,r\,(\Lambda_\infty,\infty)]^\gamma \prod_{k=1}^{n} r\,(\Lambda_k,\lambda_k)\,,$$

where domains Λ_0, Λ_∞, Λ_k, *and points* 0, ∞, λ_k, $k = \overline{1,n}$, *are, respectively, circular domains and poles of the quadratic differential* (1).

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