D-MODULES AND PROJECTIVE STACKS

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Abstract. We study twisted D-modules on weighted projective stacks. We determine for which values of the twist and the weight the global sections functor is an equivalence, thus, proving a version of Beilinson-Bernstein Localisation Theorem.

A key observation in the proof of Kazhdan-Lusztig Conjecture by Beilinson and Bernstein is that the (generalised) flag varieties G/P are D-affine. This is known as Beilinson-Bernstein Localisation Theorem. So far these are the only known connected smooth projective D-affine varieties. In particular, Thomsen proves that a toric smooth projective D-affine variety must be a product of projective spaces [15]. On the other hand, Van den Bergh proves that weighted projective spaces are D-affine (they are singular) [16].

The goal of this paper is to re-examine the D-affinity of weighted projective spaces. Instead of looking at them as singular varieties, we consider them as stacks. We give a necessary and sufficient criterion for a weighted projective stack to be D-affine. Our method of proof is also different: Van den Bergh uses Hodges-Smith Criterion for D-affinity [11], while we do a direct calculation.

In section 1 we make general observations about D-affinity on varieties. In section 2 we establish a technical framework for working with twisted D-modules on a smooth projective stack. In section 3 we use this framework to study D-modules on weighted projective stacks.

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1. D-modules on varieties

We work with a connected algebraic variety X over an algebraically closed field $\mathbb{K}$ of characteristics zero in this section. Let $\mathcal{O}_X$ be its sheaf of functions, $\mathcal{D}_X$ its sheaf of differential operators, $D(X) = \mathcal{D}_X(X)$ its global sections. We consider the category of quasicoherent $\mathcal{D}_X$-modules $\mathcal{D}_X-$Qcoh and the category of modules over the globally defined differential operators $D(X)-$Mod. They are connected by the global sections functor

$$\Gamma : \mathcal{D}_X-\text{Qcoh} \to D(X)-\text{Mod}.$$

X is called *D-affine* if Γ is an equivalence. Affine varieties are D-affine but the converse statement is not true: the generalised flag variety G/P is a smooth projective D-affine variety [4]. In the light of this result, it is interesting to pose the following question.

Question: *Classify connected smooth projective D-affine varieties.*

It would be interesting to find other examples of such varieties besides G/P. Notice that any such example X must have zero Hodge numbers $h^{0,m}(X)$ for $m > 0$ because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, hence, has no higher cohomology. A glimmering hope for settling this question is the result of Thomsen who classified smooth toric D-affine varieties [15]. Hereby we will explain that some other classes of varieties will not give new examples.

Recall that a variety X is homogeneous if a connected algebraic (not necessarily linear) group G acts transitively on X. For a complete variety X it is equivalent to asking that the automorphism group of X acts transitively on X [13]. Such X is necessarily smooth.

Theorem 1. *Suppose X is a homogeneous complete D-affine variety. Then X is isomorphic to a generalised flag variety.*

Proof. By Borel-Remmert Theorem [13] X is a product of a partial flag variety and an abelian variety A. It remains to notice that A is not D-affine because $R^{dimA}\Gamma(A, \mathcal{O}_A) \neq 0$ by Serre's duality, unless A is a point. This would imply that $R^{dimA}\Gamma(X, \mathcal{O}_X) \neq 0$ that is impossible because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module. Thus, A is a point and X is a generalised flag variety. □

If $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, this result can be slightly improved.

Theorem 2. *Suppose X is a complex complete D-affine variety and the tangent sheaf $\mathcal{T}_X$ is generated by global sections. Then X is isomorphic to a generalised flag variety.*

Proof. Since X is a complete algebraic variety, the global (algebraic) vector fields $\mathfrak{g} = \Gamma(\mathcal{T}_X)$ form a finite dimensional Lie algebra [14, p. 95]. Let G be an analytic connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. The group G locally acts on X by the second Lie Theorem [1, p. 23]. Since X is compact, each element $a \in \mathfrak{g}$ defines a one-parameter group $\gamma_a(t)$ of (global) diffeomorphisms of X [1, p. 20]. Choosing a real basis $a_1, \ldots a_k$ of $\mathfrak{g}$, we can extend the assignment

$$\mathrm{Exp}_G(t_1 a_1) \cdot \mathrm{Exp}_G(t_2 a_2) \cdot \ldots \mathrm{Exp}_G(t_k a_k) \mapsto \gamma_{a_1}(t_1)\gamma_{a_2}(t_2) \ldots \gamma_{a_k}(t_k)$$

to a global (real) analytic action of G on X [1, p. 29].

Since $\mathcal{T}_X$ is generated by global sections, each point $x \in X$ lies in the interior of its orbit $G \cdot x$. Hence each point belongs to an open set, entirely within this point's orbit. By connectedness there is only one orbit, hence, $X \cong G/H$ as analytic manifolds.

By Borel-Remmert Theorem [1, p. 101], there exists an abelian variety A such that X is an A-fibration over a generalised flag variety Y. If A is a point, we are done. If A is not a point, $R^{dim A}\Gamma(A, \mathcal{O}_A) \neq 0$ by Serre's duality. Thus, the derived push-forward $R(X \to Y)_*(\mathcal{O}_X)$ has higher cohomology and so does $\mathcal{O}_X$. This is a contradiction. □

Observe that $\mathcal{T}_X$ is not usually a $\mathcal{D}_X$-module. This would require a flat connection on $\mathcal{T}_X$ which is quite rare. For instance, abelian varieties admit a flat connection on $\mathcal{T}_X$ as well as any other variety with a trivial tangent sheaf. On the other hand, the only generalised flag variety with a flat connection on $\mathcal{T}_X$ is a point.

Corollary 3. *If X is complex complete D-affine variety and $\mathcal{T}_X$ is a $\mathcal{D}_X$-module, then X is the point.*

It would be interesting to extend Theorem 2 and Corollary 3 to varieties over an arbitrary algebraically closed field $\mathbb{K}$. Our proof does not work because we use analytic methods.

2. D-modules on smooth projective stacks

The theory of D-modules on stacks is known [5, 7]. Let Y be a smooth algebraic variety with an action of an algebraic group G. The quotient stack $[X] = [Y/G]$ admits the standard smooth atlas $G \times Y \underset{a}{\overset{p}{\rightrightarrows}} Y$ with the action and projection maps. This atlas extends to a simplicial variety $\mathcal{X}$ where $\mathcal{X}_n = G^n \times Y$, connected by the maps

$$\mathcal{X}(\varphi) : \mathcal{X}_n \to \mathcal{X}_m, \quad \mathcal{X}(\varphi)(g_1, \ldots g_n, y) = (h_1, \ldots h_m, h_{m+1} \cdot y)$$

where (with empty products equal to 1_G)

$$h_i = \prod_{j=\varphi(i-1)+1}^{\varphi(i)} g_j,\ h_{m+1} = \prod_{j=\varphi(m)+1}^{n} g_j$$

for any non-decreasing function $\varphi : [m] \to [n] = \{0, 1, \dots, n\}$. For instance, these are the maps for the low dimensional faces (recall that $\partial_i^n : [n-1] \to [n]$ is the increasing map without i in the image):

$$\mathcal{X}(\partial_2^2)(g_1, g_2, y) = (g_1, g_2 \cdot y),\ \mathcal{X}(\partial_1^2)(g_1, g_2, y) = (g_1 g_2, y),$$
$$\mathcal{X}(\partial_0^2)(g_1, g_2, y) = (g_2, y),\ \mathcal{X}(\partial_1^1)(g, y) = g \cdot y,\ \mathcal{X}(\partial_0^1)(g, y) = y.$$

The category of quasicoherent D-modules on $[X]$ is equivalent to the category of cosimplicial D-modules on $\mathcal{X}$ [7, 6.2.2]. Recall that a cosimplicial D-module $\mathcal{V}$ consists of a quasicoherent D-module $\mathcal{V}_n$ on each $\mathcal{X}_n$ together with an isomorphism of D-modules $\mathcal{V}(\varphi) : \mathcal{X}(\varphi)^*\mathcal{V}_m \to \mathcal{V}_n$ for any non-decreasing function $\varphi : [m] \to [n]$ such that the simplicial identities hold.

A cosimplicial D-module $\mathcal{V}$ can be recovered (up to an isomorphism) from the D-module $\mathcal{V}_0$ and the D-module isomorphism

$$\gamma : p^*\mathcal{V}_0 = \mathcal{X}(\partial_0^1)^*\mathcal{V}_0 \xrightarrow{\mathcal{V}(\partial_0^1)} \mathcal{V}_1 \xrightarrow{\mathcal{V}(\partial_0^1)^{-1}} \mathcal{X}(\partial_1^1)^*\mathcal{V}_0 = a^*\mathcal{V}_0.$$

The simplicial identities in dimension two force *the cocycle condition* on the isomorphism γ, coercing $(\mathcal{V}_0, \gamma)$ into *a strongly equivariant D-module* on Y. Vice versa, a strongly equivariant D-module on Y can be extended to a cosimplicial D-module on $\mathcal{X}$. This shows that the category of quasicoherent D-modules on $[X]$ is equivalent to the category of strongly equivariant quasicoherent D-modules on Y.

Further significant clarification is possible. Consider a $\mathcal{D}_Y$-module M with a compatible G-action, i.e., ${}^g(dm) = {}^g d\, {}^g m$ for all $g \in G$, $d \in D$, $m \in M$. This is sometimes called *a weakly equivariant D-module.* Such a G-action yields an isomorphism of $\mathcal{O}_G \otimes \mathcal{D}_Y$-modules $\gamma : p^*M \to a^*M$ [10].

The Lie algebra $\mathfrak{g}$ of G acts on M in two ways: via the differential of the action $\mathfrak{g} \to \mathcal{D}_Y$ and via the differential of the G-action. These two actions coincide if and only if $\gamma : p^*M \to a^*M$ is an isomorphism of $\mathcal{D}_G \otimes \mathcal{D}_Y$-modules (note that $\mathcal{D}_G \otimes \mathcal{D}_Y \cong \mathcal{D}_{G\times Y}$) [10]. This gives an alternative definition of a strongly equivariant D-module.

The preceding discussion enables us (modulo equivalences of categories) to define a quasicoherent $\mathcal{D}_{[X]}$-module as a quasicoherent strongly G-equivariant $\mathcal{D}_Y$-module.

There are different notions of a projective stack, for instance, a stack whose coarse moduli space is a projective variety. Here we use a more restrictive notion: a projective stack is a smooth closed substack of a weighted projective stack [17]. Let us spell it out. Let $V = \bigoplus V_k$ be a positively graded $n+1$-dimensional $\mathbb{K}$-vector space. Naturally we treat it as a $\mathbb{G}_m$-module with positive weights by $\lambda \bullet \mathbf{v}_k = \lambda^k \mathbf{v}_k$ where $\mathbf{v}_k \in V_k$. Let Y be a smooth closed $\mathbb{G}_m$-invariant subvariety of $V \setminus \{0\}$. We define *a projective stack* as the stack $[X] = [Y/\mathbb{G}_m]$. The G.I.T.-quotient $X = Y//\mathbb{G}_m$ is the coarse moduli space of $[X]$.

Let us describe the category $\mathcal{O}_{[X]}-\mathrm{Qcoh}$ of quasicoherent sheaves on $[X]$. Choose a homogeneous basis $\mathbf{e}_i$ on V with $\mathbf{e}_i \in V_{d_i}$, $i = 0, 1, \ldots, n$. Let $\mathbf{x}_i \in V^*$ be the dual basis. Then $\mathbb{K}[V] = \mathbb{K}[\mathbf{x}_0, ..., \mathbf{x}_n]$ possesses a natural grading with $\deg(\mathbf{x}_i) = d_i$. Let I be the defining ideal of $\overline{Y}$. Since Y is $\mathbb{G}_m$-invariant, the ideal I and the ring

$$\mathbb{A} \coloneqq \mathbb{K}[\overline{Y}] = \mathbb{K}[\mathbf{x}_0, ..., \mathbf{x}_n]/I$$

are graded. Both X and $[X]$ can be thought of as the projective spectrum of $\mathbb{A}$. The scheme X is naturally isomorphic to the scheme theoretic $\operatorname{Proj} \mathbb{A}$. The stack $[X]$ is the Artin-Zhang projective spectrum $\operatorname{Proj}_{AZ}\mathbb{A}$ [3], i.e. its category of quasicoherent sheaves $\mathcal{O}_{[X]}-\mathrm{Qcoh}$ is equivalent to the quotient category $\mathbb{A}-\mathrm{Grmod}/\mathbb{A}-\mathrm{Tors}$ where $\mathbb{A}-\mathrm{Grmod}$ is the category of $\mathbb{Z}$-graded $\mathbb{A}$-modules, $\mathbb{A}-\mathrm{Tors}$ is its full subcategory of torsion modules.

Recall that

$$\tau_{\mathbb{A}}(M) = \{m \in M \mid \exists N\ \forall k > N\ \mathbb{A}_k m = 0\}$$

is *the torsion submodule of* M. M is said to be *torsion* if $\tau_{\mathbb{A}}(M) = M$. It can be seen as well that the torsion submodule of M is the sum of all the finite dimensional submodules of M since $\mathbb{A}$ is connected.

Denote by

$$\pi_{\mathbb{A}} : \mathbb{A}-\mathrm{Grmod} \to \mathbb{A}-\mathrm{Grmod}/\mathbb{A}-\mathrm{Tors}$$

the quotient functor. Since $\mathbb{A}-\mathrm{Grmod}$ has enough injectives and $\mathbb{A}-\mathrm{Tors}$ is dense then there exists a section functor

$$\omega_{\mathbb{A}} : \mathbb{A}-\mathrm{Grmod}/\mathbb{A}-\mathrm{Tors} \to \mathbb{A}-\mathrm{Grmod}$$

which is right adjoint to $\pi_{\mathbb{A}}$ in the sense that

$$\mathrm{Hom}_{\mathbb{A}-\mathrm{Grmod}}(N, \omega_{\mathbb{A}}(\mathcal{M})) \cong \mathrm{Hom}_{\mathbb{A}-\mathrm{Grmod}/\mathbb{A}-\mathrm{Tors}}(\pi_{\mathbb{A}}(N), \mathcal{M}).$$

Recall that $\pi_{\mathbb{A}}$ is exact, $\omega_{\mathbb{A}}$ is left exact and $\pi_{\mathbb{A}}\omega_{\mathbb{A}} \cong Id_{\mathbb{A}-\mathrm{Grmod}/\mathbb{A}-\mathrm{Tors}}$. We call $\omega_{\mathbb{A}}\pi_{\mathbb{A}}(M)$ the $\mathbb{A}$-*saturation* of M. We say that a module is $\mathbb{A}$-*saturated* if it is isomorphic to the saturation of a module. It can be seen from the adjunction that an $\mathbb{A}$-saturated module is torsion-free and is isomorphic to its own saturation. If M and N are $\mathbb{A}$-saturated, then being isomorphic in $\mathbb{A}-\text{Grmod}/\mathbb{A}-\text{Tors}$ is equivalent to being isomorphic in $\mathbb{A}-\text{Grmod}$.

We need a description of the global sections functor on $[X]$ in these terms:

$$\Gamma : \mathcal{O}_{[X]}-\text{Qcoh} \to \text{VS}_{\mathbb{K}}, \quad \Gamma(\mathcal{M}) = \omega_{\mathbb{A}}(\mathcal{M})_0.$$

In particular, if M is an $\mathbb{A}$-saturated module then

$$\Gamma(\pi_{\mathbb{A}}(M)) = M_0.$$

The sheaf $\mathcal{O}_{[X]}(k)$ is defined as $\pi_{\mathbb{A}}(\mathbb{A}[k])$ where $\mathbb{A}[k]$ is the shifted regular module and the grading is given by $\mathbb{A}[k]_m = \mathbb{A}_{k+m}$.

In particular, $\Gamma(\mathcal{O}_{[X]}(k)) = \mathbb{A}_k$ if $\mathbb{A}[k]$ is $\mathbb{A}$-saturated which is the case for polynomial rings of more than two variables [2]. A well-known example of a ring, not $\mathbb{A}$-saturated (as an $\mathbb{A}$-module), is the polynomial ring in one variable $\mathbb{A} = \mathbb{K}[x]$. Its $\mathbb{A}$-saturation is the Laurent polynomial ring $\mathbb{K}[x, x^{-1}]$ seen as an $\mathbb{A}$-module. Finally we will need the push-forward functor

$$\pi_* : \mathcal{O}_{[X]}-\text{Qcoh} \to \mathcal{O}_X-\text{Qcoh},$$

given by associating a sheaf on X to a graded $\mathbb{A}$-module. In general, it is not an equivalence. For instance, $\mathcal{O}_{[X]}(k)$ is an invertible sheaf but $\mathcal{O}_X(1) \cong \pi_*(\mathcal{O}_{[X]}(1))$ is not invertible, in general [6].

Let us now describe the (twisted) $\mathcal{D}_{[X]}$-modules. Let $\partial_i = \partial/\partial\mathbf{x}_i$, $i = 0, 1, \dots, n$. The Weyl algebra $D(V) = \mathbb{K}\langle \mathbf{x}_0, \dots, \mathbf{x}_n, \partial_0, \dots, \partial_n\rangle$ gets a grading from the $\mathbb{G}_m$-action on V: $\deg(\mathbf{x}_i) = d_i$, $\deg(\partial_i) = -d_i$. We define *the reduced Weyl algebra* as

$$\mathbb{D} := \text{End}_{D(V)}(D(V)/ID(V)) \cong \mathbb{I}(ID(V))/ID(V)$$

where

$$\mathbb{I}(ID(V)) = \{\mathbf{w} \in D(V) \mid \mathbf{w}ID(V) \subseteq ID(V)\}$$

is the idealiser of $ID(V)$ in $D(V)$. Notice that $\mathbb{D}$ is graded: I is graded, then $ID(V)$ is graded, then $\mathbb{I}(ID(V))$ is graded, and finally $\mathbb{D}$ is graded. Observe that $\mathbb{A}$ is a graded subalgebra of $\mathbb{D}$ since $\mathbb{K}[\mathbf{x}_i] \subseteq \mathbb{I}(ID(V))$. It is known that for $\mathbf{w} \in D(V)$ [12, 15.5.9]

$$\mathbf{w} \in ID(V) \Leftrightarrow \mathbf{w}(\mathbb{K}[\mathbf{x}_i]) \subseteq I \quad \text{and} \quad \mathbf{w} \in \mathbb{I}(ID(V)) \Leftrightarrow \mathbf{w}(I) \subseteq I$$

where $\mathbf{w}$ acts naturally on polynomials in I. This defines an algebra embedding $\mathbb{D} \hookrightarrow \text{End}_{\mathbb{K}}(\mathbb{A})$ whose image lies in $D(\overline{Y})$, the ring of differential operators on $\mathbb{A}$.

Proposition 4. [12, 15.5.13] *The map* $\phi : \mathbb{D} \to D(\overline{Y})$ *is an isomorphism.*

The element $\sum_i d_i \mathbf{x}_i \partial_i$ belongs to the idealiser $\mathbb{I}(ID(V))$. We call its image in $\mathbb{D}$ *the Euler field*

$$\mathbf{E} = \sum_i d_i \mathbf{x}_i \partial_i + ID(V).$$

It belongs to $\mathbb{D}_0$ and defines the grading of $\mathbb{D}$ and its subalgebra $\mathbb{A}$.

Lemma 5. *Let* $\mathbf{x} \in \mathbb{D}$*. Then* $\mathbf{x} \in \mathbb{D}_k$ *if and only if* $\mathbf{E}\mathbf{x} - \mathbf{x}\mathbf{E} = k\mathbf{x}$*.*

Proof. It suffices to check it on the generators:

$$\mathbf{E}\mathbf{x}_i = \sum_j d_j \mathbf{x}_j \partial_j \mathbf{x}_i = \mathbf{x}_i \mathbf{E} + d_i \mathbf{x}_i.$$

Similarly,

$$\mathbf{E}\partial_i = \partial_i \mathbf{E} - d_i \partial_i.$$

□

The Euler field can be used to define gradings on $\mathbb{D}$-modules.

Lemma 6. *Let* M *be a* $\mathbb{D}$*-module. The span* M' *of all eigenvectors of the Euler field* $\mathbf{E}$ *is a* $\mathbb{K}$*-graded* $\mathbb{D}$*-submodule of* M*.*

Proof. Let $m \in M^\lambda$, the λ-eigenspace of $\mathbf{E}$. Using Lemma 5,

$$\mathbf{E}\mathbf{x}_i m = \mathbf{x}_i \mathbf{E} m + d_i \mathbf{x}_i m = (\lambda + d_i)\mathbf{x}_i m,$$

so

$$\mathbf{x}_i m \in M^{\lambda + d_i}.$$

Similarly,

$$\mathbf{E}\partial_i m = \partial_i \mathbf{E} m - d_i \partial_i m = (\lambda - d_i)\partial_i m$$

and

$$\partial_i m \in M^{\lambda - d_i}.$$

□

Let us fix $\lambda \in \mathbb{K}$. In general,

$$M \geq M' = \oplus_{\mu \in \mathbb{K}} M^\mu \geq M^{(\lambda)} := \oplus_{n \in \mathbb{Z}} M^{\lambda + n}.$$

A $\mathbb{D}$-module M is called λ-*Euler* if $M = M^{(\lambda)}$. A λ-Euler $\mathbb{D}$-module M admits a canonical $\mathbb{Z}$-grading given by $M_k = M^{k+\lambda}$. *The category of* λ*-Euler* $\mathbb{D}$*-modules* $\mathbb{D}\text{–}\mathrm{Grmod}^\lambda$ is a full subcategory of the category of graded $\mathbb{D}$-modules $\mathbb{D}\text{–}\mathrm{Grmod}$. The full subcategory of the torsion (as $\mathbb{A}$-modules) modules is denoted $\mathbb{D}\text{–}\mathrm{Tors}^\lambda$. Notice as well that the torsion submodule of a graded $\mathbb{D}$-module is a graded $\mathbb{D}$-module and that if, moreover, it is λ-Euler, then the torsion submodule is λ-Euler too.

$\mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}$ is a locally small category. $\mathbb{D}\text{–}\,\mathrm{Tors}^{\lambda}$ is a Serre subcategory of $\mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}$ which is closed under taking arbitrary direct sums. Therefore, $\mathbb{D}\text{–}\,\mathrm{Tors}^{\lambda}$ is a localising subcategory of $\mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}$ [9] and the quotient functor

$$\pi^{\lambda}_{\mathbb{D}} : \mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda} \to \mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}/\mathbb{D}\text{–}\,\mathrm{Tors}^{\lambda}$$

is exact and has a right adjoint section functor

$$\omega^{\lambda}_{\mathbb{D}} : \mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}/\mathbb{D}\text{–}\,\mathrm{Tors}^{\lambda} \to \mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}.$$

It follows that we have

$$\mathrm{Hom}_{\mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}}(N, \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})) \cong \mathrm{Hom}_{\mathbb{D}\text{–}\,\mathrm{Grmod}^{\lambda}/\mathbb{D}\text{–}\,\mathrm{Tors}^{\lambda}}(\pi^{\lambda}_{\mathbb{D}}(N), \mathcal{M}).$$

Theorem 7. *The category* $\mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh}$ *of quasicoherent D-modules on the stack* $[X]$ *is equivalent to the quotient category* $\mathbb{D}\text{–}\,\mathrm{Grmod}^{0}/\mathbb{D}\text{–}\,\mathrm{Tors}^{0}$.

Proof. The category of D-modules on $\overline{Y}$ is just the category of $D(\overline{Y})$-modules since $\overline{Y}$ is affine. The category of weakly $\mathbb{G}_m$-equivariant D-modules on $\overline{Y}$ is $D(\overline{Y})\text{–}\,\mathrm{Grmod}$. The two actions of the Lie algebra of the multiplicative group $\mathbb{G}_m$ are given by the Euler element $\mathbf{E}$ and by the grading. Thus, the category of strongly $\mathbb{G}_m$-equivariant D-modules on $\overline{Y}$ is the category of 0-Euler D-modules $D(\overline{Y})\text{–}\,\mathrm{Grmod}^{0}$.

By definition, the category $\mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh}$ is the category of strongly $\mathbb{G}_m$-equivariant D-modules on Y. Thus, taking sections on the open set Y induces an exact functor

$$\Gamma(Y, \_\_\,) : \mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh} \to D(Y)\text{–}\,\mathrm{Grmod}$$

where $D(Y)$ is the ring of global differential operators on Y. Proposition 4 makes the global sections $\Gamma(Y, \mathcal{M})$ into a graded $\mathbb{D}$-module via the restriction map $\mathbb{D} \cong D(\overline{Y}) \to D(Y)$. This module is 0-Euler, because $\mathcal{M}$ is strongly equivariant. Thus, we obtain exact functors

$$\Gamma(Y, \_\_\,) : \mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh} \to \mathbb{D}\text{–}\,\mathrm{Grmod}^{0} \quad \text{and}$$

$$\pi^{0}_{\mathbb{D}} \circ \Gamma(Y, \_\_\,) : \mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh} \to \mathbb{D}\text{–}\,\mathrm{Grmod}^{0}/\mathbb{D}\text{–}\,\mathrm{Tors}^{0}.$$

Let us examine the sheafification functor $\mathbb{D}\text{–}\,\mathrm{Grmod}^{0} \to \mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh}$. The sheafification of an object in $\mathbb{D}\text{–}\,\mathrm{Tors}^{0}$ is supported at 0. Hence objects in $\mathbb{D}\text{–}\,\mathrm{Tors}^{0}$ give the zero sheaf on Y. So it induces a functor on the quotient

$$\widetilde{\ } : \mathbb{D}\text{–}\,\mathrm{Grmod}^{0}/\mathbb{D}\text{–}\,\mathrm{Tors}^{0} \to \mathcal{D}_{[X]}\text{–}\,\mathrm{Qcoh}$$

which is quasiinverse to $\pi^{0}_{\mathbb{D}} \circ \Gamma(Y, \_\_\,)$. □

An inquisitive reader may observe that we have defined the category $\mathcal{D}_{[X]}-$Qcoh without defining the object $\mathcal{D}_{[X]}$. Later on we remedy this partially by constructing an object $D^{\lambda}_{[X]}$ for each $\lambda \in \mathbb{K}$ so that $\mathcal{D}_{[X]} = \pi^0_{\mathbb{D}}(D^0_{[X]})$. Let us define *the category* $\mathcal{D}^{\lambda}_{[X]}-$Qcoh *of twisted D-modules on* $[X]$ as the quotient $\mathbb{D}-\mathrm{Grmod}^{\lambda}/\mathbb{D}-\mathrm{Tors}^{\lambda}$. It is possible to define the category internally and then prove a version of Theorem 7 but we see no value in doing it here.

Given a module M in $\mathbb{D}-\mathrm{Grmod}^{\lambda}$, we call $\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$ the $\mathbb{D}^{\lambda}$*-saturation* of M. We say that a module is $\mathbb{D}^{\lambda}$*-saturated* is it is isomorphic to the $\mathbb{D}^{\lambda}$-saturation of a module. It can be seen from the adjunction that a $\mathbb{D}^{\lambda}$-saturated module is torsion-free and is isomorphic to its own saturation.

We shall prove now that an $\mathbb{A}$-saturated λ-Euler $\mathbb{D}$-module is automatically $\mathbb{D}^{\lambda}$-saturated. This will make our forthcoming calculations easier.

Lemma 8. *Let M be a λ-Euler $\mathbb{D}$-module. Then the $\mathbb{D}^{\lambda}$-saturation of M is an $\mathbb{A}$-submodule of its $\mathbb{A}$-saturation.*

Proof. We have a map
$$M \to \omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$$
in $\mathbb{D}-\mathrm{Grmod}^{\lambda}$ [2]. The kernel and cokernel of this map are torsion which implies that
$$\pi_{\mathbb{A}}(\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)) \cong \pi_{\mathbb{A}}(M).$$
From adjunction, this isomorphism is the image of a map in $\mathbb{A}-\mathrm{Grmod}$,
$$\phi : \omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M) \to \omega_{\mathbb{A}}\pi_{\mathbb{A}}(M).$$
We claim that this map is injective. Since $\pi_{\mathbb{A}}(\phi)$ is an isomorphism then $\mathrm{Ker}\phi$ is a torsion $\mathbb{A}$-module. Consider $\mathbb{D}\mathrm{Ker}\phi$ (which contains $\mathrm{Ker}\phi$), it is a left $\mathbb{D}$-submodule of $\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$. Take $m \in \mathrm{Ker}\phi$ then there exists an integer N such that
$$\mathbb{A}_{\geqslant N} m = 0.$$
For any $d \in \mathbb{D}$ of order k we have
$$\mathbb{A}_{\geqslant N+k}(dm) \leqslant \mathbb{D}\mathbb{A}_{\geqslant N} m = 0.$$
It follows that it is a torsion submodule of $\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$ but $\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$ is torsion-free. Hence $\mathrm{Ker}\phi = 0$ □

An immediate corollary is the following:

Corollary 9. *Any $\mathbb{A}$-saturated λ-Euler $\mathbb{D}$-module is $\mathbb{D}^{\lambda}$-saturated.*

Let us give examples of objects in $\mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh}$. The sheaf $\mathcal{O}_{[X]}(k)$ is an object in $\mathcal{D}^k_{[X]}-\mathrm{Qcoh}$. We introduce

$$D^\lambda_{[X]} := \mathbb{D}/\mathbb{D}(\mathbf{E}-\lambda).$$

Another interesting object in $\mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh}$ is

$$\mathcal{D}^\lambda_{[X]} := \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]}).$$

It plays the role of the sheaf of twisted differential operators, although $D^\lambda_{[X]}$ is not an algebra because $\mathbb{D}(\mathbf{E}-\lambda)$ is not a two-sided ideal, in general. However, $\mathbf{E}$ is a central element of $\mathbb{D}_0$, so

$$D^\lambda_{[X]_0} = \mathbb{D}_0/\mathbb{D}_0(\mathbf{E}-\lambda)$$

is an algebra. It plays the role of the algebra of global sections of the twisted differential operators on $[X]$. $D^\lambda_{[X]}$ is a $\mathbb{D}-D^\lambda_{[X]_0}$-bimodule.

In the next section the adjoint functors of global sections and localisation will play a role. This adjoint pair $(\Gamma_\lambda, L_\lambda)$ is defined as:

$$\Gamma_\lambda : \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh} \to D^\lambda_{[X]_0}-\mathrm{Mod},\ \Gamma_\lambda(\mathcal{M}) := \omega^\lambda_{\mathbb{D}}(\mathcal{M})_0 = \omega^\lambda_{\mathbb{D}}(\mathcal{M})^\lambda,$$

$$L_\lambda : D^\lambda_{[X]_0}-\mathrm{Mod} \to \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh},\quad L_\lambda(N) := \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} N).$$

The ways we defined our global sections functors for $\mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh}$ and $\mathcal{O}_{[X]}-\mathrm{Qcoh}$ are not necessarily equivalent. Yet we know that

$$\Gamma_\lambda(\pi^\lambda_{\mathbb{D}}(M)) \leqslant \Gamma(\pi_{\mathbb{A}}(M))$$

as $\mathbb{A}$-modules for any λ-Euler $\mathbb{D}$-module M.

The exposition would be greatly simplified if restricting the section functor $\omega_{\mathbb{A}}$ to $\mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh}$ were equivalent to $\omega^\lambda_{\mathbb{D}}$. This explains why we have different global sections functor for different λ although geometrically only one is needed. However, to ensure that we obtain λ-Euler $\mathbb{D}$-modules and not just $\mathbb{A}$-modules we use $\omega^\lambda_{\mathbb{D}}$.

3. D-modules on weighted projective space

In this section we consider $Y = V \setminus \{0\}$, the punctured vector space of dimension at least 2 and $[X] = [Y/\mathbb{G}_m] = [\mathbb{P}(V)]$, the weighted projective stack. In this case $I = \{0\}$, $\mathbb{A} = \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ where the degree of $\mathbf{x}_i$ is $d_i > 0$ and $\mathbb{D} = \mathbb{K}\langle \mathbf{x}_0, \ldots, \mathbf{x}_n, \partial_0, \ldots, \partial_n\rangle$ is the Weyl algebra. Without loss of generality, we assume that $0 < d_0 \leq d_1 \leq \ldots \leq d_n$.

Let us look at the $\mathbb{D}$-module Δ generated by the delta-function at zero $\delta = \delta_0(\mathbf{x}_0, \ldots, \mathbf{x}_n)$

$$\Delta = \mathbb{D}\delta \cong \mathbb{D}/(\mathbb{D}\mathbf{x}_0 + \mathbb{D}\mathbf{x}_1 + \ldots + \mathbb{D}\mathbf{x}_n)\,.$$

The linear map
$$\mathbb{K}[\partial_0, \dots, \partial_n] \to \Delta, \quad f(\partial_0, \dots, \partial_n) \mapsto f(\partial_0, \dots, \partial_n) \cdot \delta$$
is an isomorphism of vector spaces. If we identify $\mathbb{K}[\partial_0, \dots, \partial_n]$ with Δ using this linear map, then ∂_i acts by multiplication and $\mathbf{x}_i$ acts by derivation $\partial_j \mapsto -\delta_{i,j}$. In particular,
$$\mathbf{E} \cdot \delta = \mathbf{E} \cdot 1 = \sum_j d_j \mathbf{x}_j \cdot \partial_j = \sum_j -d_j = -(\sum_j d_j)\delta.$$
Hence, Δ is k-Euler for each integer k. Its canonical k-Euler grading is given by
$$\delta \in \Delta^{-\sum_j d_j} = \Delta_{-k-\sum_j d_j}, \quad \partial_i \cdot \delta \in \Delta_{-k-d_i-\sum_j d_j}.$$

Let $J = (\mathbf{x}_0, \dots, \mathbf{x}_n) \lhd \mathbb{A}$. If M is a $\mathbb{D}$-module, $\tau_{\mathbb{A}}(M) = \{m \in M \mid \exists k \; J^k m = 0\}$ is its torsion $\mathbb{D}$-submodule (a reader can easily verify that if $J^k m = 0$, then $J^{k+1}\partial_i m = 0$). The torsion $\mathbb{D}$-modules are those, supported set theoretically on the zero $0 \in V$. By Kashiwara's theorem, any $\mathbb{D}$-module supported at 0 is a direct sum of copies of Δ.

Let us introduce some notations. Suppose that M and N are two $\mathbb{Z}$-graded $\mathbb{A}$-modules. We say that an $\mathbb{A}$-module homomorphism $f : M \to N$ has *degree* l if $f(M_i) \subset N_{i+l}$ for all i. Denote by $\mathrm{Hom}(M,N)_l$ the set of all degree l $\mathbb{A}$-module homomorphisms and write
$$\underline{\mathrm{Hom}}_{\mathbb{A}}(M,N) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}(M,N)_l.$$
Now let $\mathrm{Ext}^q(M,N)_l$ be the derived functor of $\mathrm{Hom}(M,N)_l$ and write
$$\underline{\mathrm{Ext}}^q_{\mathbb{A}}(M,N) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Ext}^q(M,N)_l.$$
Artin and Zhang prove [2] that for any graded $\mathbb{A}$-module M,
$$\tau_{\mathbb{A}}(M) \cong \varinjlim \underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, M),$$
$$R^1\tau_{\mathbb{A}}(M) \cong \varinjlim \underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, M)$$
and that there exists a long exact sequence of $\mathbb{A}$-modules
$$0 \to \tau_{\mathbb{A}}(M) \to M \to \omega_{\mathbb{A}}\pi_{\mathbb{A}}(M) \to R^1\tau_{\mathbb{A}}(M) \to 0$$
where $\tau_{\mathbb{A}}(M)$ and $R^1\tau_{\mathbb{A}}(M)$ are torsion modules. This implies the following proposition.

Proposition 10. *A λ-Euler $\mathbb{D}$-module M is $\mathbb{D}^\lambda$-saturated if it is torsion-free and $\varinjlim \underline{\mathrm{Ext}}^1(\mathbb{A}/\mathbb{A}_{\geqslant k}, M) = 0$.*

The next lemma will prove primordial in the proof that $\Gamma_\lambda L_\lambda \cong Id_{D^\lambda_{[X]_0}-\mathrm{Mod}}$ for any λ and $n \geqslant 2$.

Lemma 11. *For $n \geqslant 2$, $D^{\lambda}_{[X]}$ is $\mathbb{D}^{\lambda}$-saturated.*

Proof. Recall that $D^{\lambda}_{[X]} = \mathbb{D}/\mathbb{D}(\mathbf{E} - \lambda)$. It is easier to compute Ext groups by taking a projective resolution of the left argument than an injective one of the right argument. Since $\mathbb{A}/\mathbb{A}_{\geqslant 1} \cong \mathbb{K}$, the first three terms of the Koszul resolution are given by

$$\ldots \to \bigoplus_{i_0<i_1} \mathbb{A}(-d_{i_0} - d_{i_1}) \to \bigoplus_{i=0}^{n} \mathbb{A}(-d_i) \to \mathbb{A} \to \mathbb{A}/\mathbb{A}_{\geqslant 1} \to 0.$$

Take away $\mathbb{A}/\mathbb{A}_{\geqslant 1}$ and apply $\underline{\mathrm{Hom}}_{\mathbb{A}}(\_, D^{\lambda}_{[X]})$ to the above exact sequence to get

$$0 \to D^{\lambda}_{[X]} \xrightarrow{\phi_1} \bigoplus_{i=0}^{n} D^{\lambda}_{[X]}(d_i) \xrightarrow{\phi_2} \bigoplus_{i_0<i_1} D^{\lambda}_{[X]}(d_{i_0} + d_{i_1}) \to \ldots$$

where

$$\phi_1 \colon \overline{m} \mapsto (\mathbf{x}_i \overline{m})_{i=0}^{n}$$

and

$$\phi_2 \colon (\overline{m}_i)_{i=0}^{n} \mapsto (\mathbf{x}_{i_0}\overline{m}_{i_1} - \mathbf{x}_{i_1}\overline{m}_{i_0})_{i_0<i_1}.$$

It follows that

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) \cong \mathrm{Ker}(\phi_1),$$
$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) \cong \frac{\mathrm{Ker}(\phi_2)}{\mathrm{Im}(\phi_1)}.$$

Both $\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$ and $\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$ vanish. Let us first compute $\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$. Pick $\overline{m} \in \mathrm{Ker}(\phi_1)$, then $\mathbf{x}_i\overline{m} = 0$ for each i, where

$$\overline{m} = m + \mathbb{D}(\mathbf{E} - \lambda).$$

We can assume m to be homogeneous, so

$$\mathbf{x}_i m = p_i(\mathbf{E} - \lambda)$$

for some homogeneous $p_i \in \mathbb{D}$. We want to show that $p_i \in \mathbf{x}_i\mathbb{D}$. Suppose, for a contradiction, that it is not. Then we can write

$$p_i = \mathbf{x}_i m' + \mathbf{f}\partial^{\underline{\beta}} + LT$$

where $m' \in \mathbb{D}$, $\mathbf{f} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term which is non-zero by assumption, free of $\mathbf{x}_i$, $\underline{\beta}$ the biggest power and LT are the lower terms using **DegLex** for the ordering of the monomials in ∂. Without loss of generality, we can assume that $i \neq 0$. It follows that

$$\mathbf{x}_i m = \mathbf{x}_i m'' + d_0\mathbf{f}\mathbf{x}_0\partial^{\underline{\beta}+\underline{e_0}} + LT$$

since $\mathbf{f}\partial^{\underline{\beta}}\mathbf{x}_0\partial_0 = \mathbf{f}\mathbf{x}_0\partial^{\underline{\beta}+\underline{e_0}} + LT$. But $\mathbf{f}\mathbf{x}_0$ is not divisible by $\mathbf{x}_i$ and we obtain a contradiction. Thus,

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) = 0.$$

Similarly, let us show that $\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, \mathbb{D}^{\lambda}_{[X]})$ vanishes. To proceed, choose $(\overline{m}_i)_{i=0}^n \in \mathrm{Ker}(\phi_2)$. Then for all i, j, there exists a $\theta_{ij} \in \mathbb{D}$ such that

$$\mathbf{x}_i m_j = \mathbf{x}_j m_i + \theta_{ij}(\mathbf{E} - \lambda).$$

Write

$$m_j = \mathbf{x}_j m'_j + \mathbf{f}\partial^{\underline{\beta}} + LT$$

where $m'_j \in \mathbb{D}$, $\mathbf{f} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term, free of $\mathbf{x}_j$, $\underline{\beta}$ is the highest power and LT are the lower terms using **DegLex** for the ordering of the monomials in ∂. Let us suppose, for the sake of a contradiction, that $|\underline{\beta}| \neq 0$. Then without loss of generality, we can assume that $\underline{\beta}$ is the lowest among all the possible representatives of $\overline{m}_j$. Write

$$\theta_{ij} = \mathbf{x}_j\theta' + \mathbf{g}\partial^{\underline{\gamma}} + LT$$

where $\mathbf{g} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term, free of $\mathbf{x}_j$. If $\mathbf{g} = 0$ then we are done. Suppose that $\mathbf{g} \neq 0$ so that

$$\mathbf{x}_i\mathbf{x}_j m'_j + \mathbf{x}_i\mathbf{f}\partial^{\underline{\beta}} + LT = \mathbf{x}_j(m_i + \theta'(\mathbf{E} - \lambda)) + \mathbf{g}\partial^{\underline{\gamma}}(\mathbf{E} - \lambda) + LT.$$

Again without loss of generality, suppose that $i, j \neq 0$ as $n \geqslant 2$. By comparing the highest terms, free of $\mathbf{x}_j$, we get

$$\mathbf{x}_i\mathbf{f}\partial^{\underline{\beta}} = d_0\mathbf{g}\mathbf{x}_0\partial^{\underline{\gamma}+\underline{e_0}}$$

with $|\underline{\gamma}| < |\underline{\beta}|$. Therefore,

$$\mathbf{f}\partial^{\underline{\beta}} = d_0\frac{\mathbf{g}}{\mathbf{x}_i}\mathbf{x}_0\partial^{\underline{\gamma}+\underline{e_0}} = \frac{\mathbf{g}}{\mathbf{x}_i}\partial^{\underline{\gamma}}(\mathbf{E} - \lambda) + LT.$$

So $m_j - \frac{\mathbf{g}}{\mathbf{x}_i}\partial^{\underline{\gamma}}(\mathbf{E} - \lambda)$ is another representative of $\overline{m}_j$ which has an index $\underline{\gamma}$ lower than $\underline{\beta}$, contrary to our hypothesis. Thus $\mathbf{g} = 0$ and

$$m_j = \mathbf{x}_j m'_j$$

For all i, j, we have

$$\mathbf{x}_i\mathbf{x}_j m'_j = \mathbf{x}_i\mathbf{x}_j m'_i + \theta_{ij}(\mathbf{E} - \lambda)$$

which implies that

$$\mathbf{x}_i\mathbf{x}_j(m'_j - m'_i) \in \mathbb{D}(\mathbf{E} - \lambda).$$

By using the first argument twice, we obtain that for all i, j

$$m'_j - m'_i \in \mathbb{D}(\mathbf{E} - \lambda).$$

Write

$$\overline{m'} := \overline{m'_j} = \overline{m'_i}$$

for the residues of m'_j and m'_i. Then for all i,

$$\overline{m_i} = \mathbf{x}_i \overline{m'}.$$

Hence,

$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^\lambda_{[X]}) = 0.$$

To finish our proof, for each k we have a short exact sequence of graded $\mathbb{A}$-modules:

$$0 \to \mathbb{A}_{\geqslant k}/\mathbb{A}_{\geqslant k+1} \to \mathbb{A}/\mathbb{A}_{\geqslant k+1} \to \mathbb{A}/\mathbb{A}_{\geqslant k} \to 0$$

and $\mathbb{A}_{\geqslant k}/\mathbb{A}_{\geqslant k+1}$ is isomorphic to a finite direct sum of copies of $\mathbb{A}/\mathbb{A}_{\geqslant 1}$. By applying $\underline{\mathrm{Hom}}_{\mathbb{A}}(\_, D^\lambda_{[X]})$ to this short exact sequence and by induction on k, we conclude that for all k:

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^\lambda_{[X]}) = 0,$$
$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^\lambda_{[X]}) = 0.$$

Taking direct limit [2] it follows that

$$\tau_{\mathbb{A}}(D^\lambda_{[X]}) = 0, \quad \text{and} \quad \varinjlim \underline{\mathrm{Ext}}^1(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^\lambda_{[X]}) = 0.$$

Hence $D^\lambda_{[X]}$ is $\mathbb{D}^\lambda$-saturated by Proposition 10. □

The condition on n in the last proof is necessary. We can prove that $D^\lambda_{[X]}$ is not $\mathbb{D}^\lambda$-saturated for all λ when $n = 1$. For this, it suffices to notice that for $\lambda = 0$,

$$(-d_1\partial_1, d_0\partial_0) \in \mathrm{Ker}(\phi_2)$$

but

$$(-d_1\partial_1, d_0\partial_0) \notin \mathrm{Im}(\phi_1)$$

since $d_0\mathbf{x}_0\partial_0 = -d_1\mathbf{x}_1\partial_1 + \mathbf{E}$.

Lemma 12. *Let $n \geqslant 2$. If Γ_λ is exact then $\Gamma_\lambda L_\lambda \cong Id_{D^\lambda_{[X]_0}-\mathrm{Mod}}$*

Proof. Let N be a $D^\lambda_{[X]_0}$-module. Take the first two terms of a free resolution of N

$$P_1 \to P_0 \to N \to 0$$

where $P_i = \bigoplus_{j\in I_i} D^\lambda_{[X]_0}$ and I_i is an index set. Since both $D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} -$ and $\pi^\lambda_{\mathbb{D}}$ are right exact functors, it follows that

$$\Gamma_\lambda L_\lambda(P_1) \to \Gamma_\lambda L_\lambda(P_0) \to \Gamma_\lambda L_\lambda(N) \to 0$$

is exact. We can compute the first two terms explicitly:

$$\begin{aligned}\Gamma_\lambda L_\lambda(P_i) &= (\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} P_i))_0 \\ &= (\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} \bigoplus_{j\in I_i} D^\lambda_{[X]_0}))_0 \\ &\cong (\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(\bigoplus_{j\in I_i} D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} D^\lambda_{[X]_0}))_0 \\ &\cong (\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(\bigoplus_{j\in I_i} D^\lambda_{[X]}))_0\end{aligned}$$

since the tensor product commutes with arbitrary direct sums and that $D^\lambda_{[X]}\otimes_{D^\lambda_{[X]_0}} D^\lambda_{[X]_0} \cong D^\lambda_{[X]}$. The category $\mathbb{D}-\mathrm{Grmod}^\lambda$ is locally noetherian [8, Prop. 4.18]. By a result of Gabriel, the section functor $\omega^\lambda_{\mathbb{D}}$ commutes with inductive limits and, in particular, with arbitrary direct sums [9, p. 379]. Moreover, $\pi^\lambda_{\mathbb{D}}$ is left adjoint to $\omega^\lambda_{\mathbb{D}}$, so $\pi^\lambda_{\mathbb{D}}$ commutes as well with arbitrary direct sums. This yields the following sequence of natural isomorphisms:

$$\begin{aligned}\Gamma_\lambda L_\lambda(P_i) &\cong (\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(\bigoplus_{j\in I_i} D^\lambda_{[X]}))_0 \\ &\cong (\bigoplus_{j\in I_i} \omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]}))_0 \\ &\cong (\bigoplus_{j\in I_i} D^\lambda_{[X]})_0 \\ &\cong \bigoplus_{j\in I_i} D^\lambda_{[X]_0} \\ &\cong P_i\end{aligned}$$

since $D^\lambda_{[X]}$ is $\mathbb{D}^\lambda$-saturated and that $(\_)_0$ commutes with arbitrary direct sums. Thus, we constructed a commutative diagram with exact rows:

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & \Gamma_\lambda L_\lambda(N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ P_1 & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where α and β are isomorphisms, so $\Gamma_\lambda L_\lambda(N) \cong N$ is a natural isomorphism by the four lemma. □

Theorem 13. *Let $\mathcal{A}$ be the $\mathbb{Z}_{\geq 0}$-span of all d_i-s. If $\lambda \in \mathbb{K} \setminus (-\sum_i d_i - \mathcal{A})$, then the global sections functor $\Gamma_\lambda : \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh} \to D^\lambda_{[X]_0}-\mathrm{Mod}$*

is exact. In this case, Γ_λ *defines an equivalence between the quotient category* $\mathcal{D}^\lambda_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_\lambda$ *and* $D^\lambda_{[X]_0}-\text{Mod}$.

Proof. The category $\mathcal{D}^\lambda_{[X]}-\text{Qcoh}$ is the quotient category of the category of λ-Euler modules by the category of torsion modules. The canonical grading on a λ-Euler module M is given by $M_k = M^{k+\lambda}$. The torsion modules are direct sums of Δ. The global sections functor Γ_λ is

$$\Gamma_\lambda : \mathcal{M} \mapsto \omega^\lambda_{\mathbb{D}}(\mathcal{M})_0 = \omega^\lambda_{\mathbb{D}}(\mathcal{M})^\lambda.$$

We know that $\omega^\lambda_{\mathbb{D}}$ is a left exact functor. Taking λ-eigenspaces is an exact functor, so we are left to prove that Γ_λ is right exact. An epimorphism $f : \mathcal{M} \to \mathcal{N}$ induces the exact sequence

$$\omega^\lambda_{\mathbb{D}}(\mathcal{M}) \to \omega^\lambda_{\mathbb{D}}(\mathcal{N}) \to \text{coker}(\omega^\lambda_{\mathbb{D}}(f)) \to 0$$

where $\text{coker}(\omega^\lambda_{\mathbb{D}}(f))$ is a torsion $\mathbb{D}$-module. Taking the zeroeth graded part, we get the exact sequence

$$\Gamma_\lambda(\mathcal{M}) \to \Gamma_\lambda(\mathcal{N}) \to \text{coker}(\omega^\lambda_{\mathbb{D}}(f))_0 \to 0.$$

Our restriction on λ provides that $\text{coker}(\omega^\lambda_{\mathbb{D}}(f))_0 = 0$. Indeed, if $\lambda \notin \mathbb{Z}$, then $\text{coker}(\omega^\lambda_{\mathbb{D}}(f)) = 0$. If $\lambda \in \mathbb{Z}$, then $\text{coker}(\omega^\lambda_{\mathbb{D}}(f)) = \oplus\Delta$ and $\text{coker}(\omega^\lambda_{\mathbb{D}}(f))_0 = \oplus\Delta^\lambda$. Since the $\mathbf{E}$-weights of Δ are $-\sum_i d_i - \mathcal{A}$, $\text{coker}(\omega^\lambda_{\mathbb{D}}(f))_0 = 0$. Hence Γ_λ is exact.

The kernel $\text{Ker}\Gamma_\lambda$ is the full subcategory of $\mathcal{D}^\lambda_{[X]}-\text{Qcoh}$ whose objects are those $\mathcal{M}$ without non-trivial global sections, i.e., with $\Gamma_\lambda(\mathcal{M}) = 0$. Since Γ_λ is exact, it is a Serre subcategory, and Γ_λ descends to a functor

$$\widetilde{\Gamma}_\lambda : \mathcal{D}^\lambda_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_\lambda \to D^\lambda_{[X]_0}-\text{Mod}.$$

and let

$$Q : \mathcal{D}^\lambda_{[X]}-\text{Qcoh} \to \mathcal{D}^\lambda_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_\lambda$$

be the quotient functor. We claim that QL_λ is a quasiinverse of $\widetilde{\Gamma}_\lambda$. Now in one direction,

$$\begin{aligned}\widetilde{\Gamma}_\lambda(QL_\lambda)(N) &= (\widetilde{\Gamma}_\lambda Q)L_\lambda(N)\\ &= \Gamma_\lambda L_\lambda(N)\\ &\cong N\end{aligned}$$

since Γ_λ is exact. Thus,

$$\widetilde{\Gamma}_\lambda QL_\lambda \cong Id_{D^\lambda_{[X]_0}-\text{Mod}}.$$

In the opposite direction, we have a natural transformation

$$QL_\lambda\widetilde{\Gamma}_\lambda \to Id_{\mathcal{D}^\lambda_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_\lambda}.$$

Take an object $\widetilde{\mathcal{M}}$ in $\mathcal{D}^{\lambda}_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_{\lambda}$. Then there exists an object $\mathcal{M}$ in $\mathcal{D}^{\lambda}_{[X]}-\text{Qcoh}$ such that $\widetilde{\mathcal{M}} = Q(\mathcal{M})$. Hence,

$$\begin{aligned} QL_{\lambda}\widetilde{\Gamma}_{\lambda}(\widetilde{\mathcal{M}}) &= QL_{\lambda}\Gamma_{\lambda}(\mathcal{M}) \\ &= Q\pi^{\lambda}_{\mathbb{D}}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} (\omega^{\lambda}_{\mathbb{D}}(\mathcal{M}))_0). \end{aligned}$$

On a level of a λ-Euler module M (with its canonical grading), the natural map

$$D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} M_0 \to M$$

gives rise to the long exact sequence

$$0 \to K \to D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} M_0 \to M \to N \to 0$$

where K is its kernel and N is its cokernel. Since $\pi^{\lambda}_{\mathbb{D}}$ is exact,

$$0 \to \pi^{\lambda}_{\mathbb{D}}(K) \to \pi^{\lambda}_{\mathbb{D}}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} M_0) \to \pi^{\lambda}_{\mathbb{D}}(M) \to \pi^{\lambda}_{\mathbb{D}}(N) \to 0$$

is a long exact sequence as well. If $M = \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})$, applying Γ_{λ} yields

$$0 \to \Gamma_{\lambda}\pi^{\lambda}_{\mathbb{D}}(K) \to \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0 \to \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0 \to \Gamma_{\lambda}\pi^{\lambda}_{\mathbb{D}}(N) \to 0$$

since $\Gamma_{\lambda}\pi^{\lambda}_{\mathbb{D}}(\omega^{\lambda}_{\mathbb{D}}(\mathcal{M})) \cong \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0$ and $\Gamma_{\lambda}L_{\lambda} \cong Id_{D^{\lambda}_{[X]_0}-\text{Mod}}$ when Γ_{λ} is exact. The middle map

$$\omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0 \to \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0$$

is the identity map and hence an isomorphism. It follows that $\pi^{\lambda}_{\mathbb{D}}(K)$ and $\pi^{\lambda}_{\mathbb{D}}(N)$ are objects in $\text{Ker}(\Gamma_{\lambda})$. Therefore,

$$\pi^{\lambda}_{\mathbb{D}}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0) \to \pi^{\lambda}_{\mathbb{D}}(\omega^{\lambda}_{\mathbb{D}}(\mathcal{M}))$$

is an isomorphism in $\mathcal{D}^{\lambda}_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_{\lambda}$ and

$$\begin{aligned} QL_{\lambda}\widetilde{\Gamma}_{\lambda}(\widetilde{\mathcal{M}}) &\cong Q\pi^{\lambda}_{\mathbb{D}}(\omega^{\lambda}_{\mathbb{D}}(\mathcal{M})) \\ &\cong Q(\mathcal{M}) \\ &\cong \widetilde{\mathcal{M}}. \end{aligned}$$

It follows that $QL_{\lambda}\widetilde{\Gamma}_{\lambda} \cong I_{\mathcal{D}^{\lambda}_{[X]}-\text{Qcoh}/\text{Ker}\Gamma_{\lambda}}$. □

We are left to study when $\text{Ker}\Gamma_{\lambda}$ is a zero category so that Γ_{λ} defines an equivalence between the quotient category $\mathcal{D}^{\lambda}_{[X]}-\text{Qcoh}$ and $D^{\lambda}_{[X]_0}-\text{Mod}$.

Lemma 14. *Suppose that* $\lambda \in \mathbb{Z} \setminus \mathcal{A}$ *or that the greatest common divisor* $\gcd_i(d_i) \neq 1$. *Then* $\text{Ker}\Gamma_{\lambda} \neq 0$.

Proof. If $k \in \mathbb{Z}$, then $\mathcal{O}_{[X]}(k) = \pi_{\mathbb{D}}^{\lambda}(\mathbb{A}[k])$ is a non-zero $\mathbb{D}^k$-saturated (since it is $\mathbb{A}$-saturated [2]) object of $\mathcal{D}_{[X]}^k-\mathrm{Qcoh}$ because $1 \in \mathbb{A}_0 = \mathbb{A}[k]_{-k}$ and

$$\mathbf{E} \cdot 1 = 0 = (-k+k)1.$$

The global sections

$$\Gamma_k(\mathcal{O}_{[X]}(k)) = \mathbb{A}[-k]_0 = \mathbb{A}_k$$

are non-zero if and only if $k \in \mathcal{A}$. Thus, if $\lambda \in \mathbb{Z} \setminus \mathcal{A}$, then $\mathcal{O}_{[X]}(\lambda)$ is a non-zero object of $\mathrm{Ker}\Gamma_\lambda$.

Now let us assume that the greatest common divisor d of $d_0, ..., d_n$ is greater than 1. It easily follows that

$$\mathbb{D}_1 = \mathbb{D}_2 = \ldots = \mathbb{D}_{d-1} = 0.$$

Let M be the $\mathbb{K}$-vector space with a basis of all formal monomials $\mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n}$, $a_i \in \mathbb{K}$. It is a $\mathbb{D}$-module under the following operations, defined on the monomials by

$$\mathbf{x}_i \cdot \mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n} = \mathbf{x}_0^{a_0} \ldots \mathbf{x}_i^{1+a_i} \mathbf{x}_{i+1}^{a_{i+1}} \ldots \mathbf{x}_n^{a_n},$$
$$\partial_i \cdot \mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n} = a_i \mathbf{x}_0^{a_0} \ldots \mathbf{x}_i^{-1+a_i} \mathbf{x}_{i+1}^{a_{i+1}} \ldots \mathbf{x}_n^{a_n}.$$

Given $\lambda \in \mathbb{K}$, we consider the $\mathbb{D}$-submodule $N = \mathbb{D}\mathbf{x}_0^{(\lambda-1)/d_0}$. Since

$$\mathbf{E} \cdot \mathbf{x}_0^{(\lambda-1)/d_0} = d_0 \mathbf{x}_0 \partial_0 \cdot \mathbf{x}_0^{(\lambda-1)/d_0} = (\lambda-1)\mathbf{x}_0^{(\lambda-1)/d_0},$$

the module N is λ-Euler and $\mathbf{x}_0^{(\lambda-1)/d_0} \in N^{\lambda-1} = N_{-1}$ in the canonical λ-Euler grading. Put $\mathcal{N} = \pi_{\mathbb{D}}^{\lambda}(N)$. By definition, N is torsion-free. Denote by $\tau_{\mathbb{D}}^{\lambda}$ the restriction of $\tau_{\mathbb{A}}$ to $\mathbb{D}-\mathrm{Grmod}^{\lambda}$. The long exact sequence [2]

$$0 \to \tau_{\mathbb{D}}^{\lambda}(N) \to N \to \omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \to R^1\tau_{\mathbb{D}}^{\lambda}(N) \to 0$$

reduces to the short exact sequence

$$0 \to N \to \omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \to R^1\tau_{\mathbb{D}}^{\lambda}(N) \to 0.$$

But $R^1\tau_{\mathbb{D}}^{\lambda}(N)$ is a torsion $\mathbb{D}$-module, hence it is a direct sum of copies of Δ. The $\mathbf{E}$-weights of N are congruent to -1 modulo d and the $\mathbf{E}$-weights of the module Δ are congruent to 0 modulo d. It follows that the short exact sequence splits and

$$\omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \cong N \oplus R^1\tau_{\mathbb{D}}^{\lambda}(N).$$

Since $\omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N)$ is torsion free, $\omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \cong N$ and $R^1\tau_{\mathbb{D}}^{\lambda}(N) = 0$. This means that N is $\mathbb{D}^{\lambda}$-saturated and

$$\Gamma_\lambda(\mathcal{N}) = N_0 = 0.$$

Hence, $\mathcal{N}$ is a non-zero object in $\mathrm{Ker}\Gamma_\lambda$. □

In all the other cases the kernel is trivial.

Lemma 15. *Let us assume that the greatest common divisor* $\gcd_i(d_i)$ *is equal to* 1*. If* $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$*, then* $\mathrm{Ker}\Gamma_\lambda$ *is a zero category.*

Proof. Let m be the least common multiple of $d_0, \ldots, d_n$. Suppose that $\mathcal{M}$ is a non-zero object in $\mathcal{D}^\lambda_{[X]} - \mathrm{Qcoh}$. Then $M := \omega^\lambda_{\mathbb{D}}(\mathcal{M})$ is a non-zero λ-Euler torsion-free $\mathbb{D}$-module. We need to show that $M_0 \neq 0$. Let us suppose that the contrary is true, i.e., $M_0 = 0$. We proceed to arrive at a contradiction via a sequence of claims.

Claim 1. $M_{-mt} = 0$ *for any* $t \in \mathbb{Z}_{>0}$.
Proof of Claim: If $a \in M_{-mt}$, then $\mathbf{x}_i^{mt/d_i} \cdot a = 0$ for all $i = 0, \ldots, n$ since it is an element of M_0. Hence, a generates a torsion $\mathbb{D}$-submodule of M but M is torsion-free. Hence $a = 0$. □

Claim 2. $M_{-mt+kd_i} = 0$ *for all* i *and* $0 \leqslant k \leqslant \frac{mt}{d_i}$*. In particular,* $M_{-kd_i} = 0$ *for all* $k \geqslant 0$.
Proof of Claim: We proceed by induction. The case $k = 0$ is Claim 1. Assume that this is true for k, and let us prove it for $k+1$. If $-mt + (k+1)\,d_i = 0$, then we are done. Otherwise, let us pick a non-zero element $a \in M_{-mt+(k+1)d_i}$. It follows that

$$\partial_i \cdot a \in M_{-mt+kd_i}$$

which is zero by induction. Moreover, $\mathbf{x}_i^{-(k+1)+mt/d_i} \cdot a \in M_0$ which is zero again. Since

$$\left[\partial_i, \mathbf{x}_i^{-(k+1)+mt/d_i}\right] = \left(\frac{mt}{d_i} - (k+1)\right) \mathbf{x}_i^{-(k+2)+mt/d_i},$$

we conclude that $\mathbf{x}_i^{-(k+2)+mt/d_i} \cdot a = 0$. We can repeat this argument to conclude that $\mathbf{x}_i^{-(k+l)+mt/d_i} \cdot a = 0$ for all positive l with $\frac{mt}{d_i} - (k+l) \geq 0$. In particular, $a = \mathbf{x}_i^0 \cdot a = 0$. □

Claim 3. *If* $c_0, ..., c_k$ *are positive integers and* g *is their greatest common divisor, then there exist integers* $r_0 \leqslant 0$*, and* $r_1, \ldots, r_k \geqslant 0$ *such that* $r_0 c_0 + \ldots + r_k c_k = g$.
Proof of Claim: Let l be the least common multiple of $c_0, ..., c_k$. By the Euclidean algorithm there exist integers $s_0, \ldots, s_k$ such that

$$s_0 c_0 + \ldots + s_k c_k = 1.$$

Now we can add $-\frac{l}{c_0} c_0 + \frac{l}{c_i} c_i = 0$ for various i to this relations to get integers $r_0, ..., r_k$ such that

$$r_0 c_0 + \ldots + r_k c_k = 1$$

and $r_1, \ldots, r_k \geqslant 0$. Inevitably, $r_0 \leqslant 0$. □

Claim 4. *For all integer* $b_0, \ldots, b_l \geqslant 0$*,* $M_{-(b_0 d_0 + \ldots + b_l d_l)} = 0$.

Proof of Claim: We proceed by induction on l. The base case $l = 0$ is Claim 2. Assume this is true for $l-1$. In particular, it is true if $b_i = 0$ for some i.

Let $g_l = \gcd(d_0, \ldots, d_l)$ and fix a positive integer k. Consider a non-zero element $a \in M_{-kg_l}$. There exist positive integers $c_0, c_1, \ldots, c_l$ such that

$$\partial_0^{c_0} \cdot a = \partial_1^{c_1} \cdot a = \ldots = \partial_l^{c_l} \cdot a = 0.$$

Indeed, by Claim 3, there exist $r_i \leqslant 0$ and $r_0, \ldots, r_{i-1}, r_{i+1}, \ldots r_l \geqslant 0$ such that

$$r_0 d_0 + \ldots + r_l d_l = g_l$$

Now if $c_i = -kr_i \geqslant 0$, then

$$\partial_i^{c_i} \cdot a \in M_{-c_i d_i - kg_l} = M_{-k(r_0 d_0 + \ldots + r_{i-1} d_{i-1} + r_{i+1} d_{i+1} + \ldots + r_l d_l)} = 0,$$

by induction. Let us consider the Weyl algebra

$$\widetilde{\mathbb{D}} = \mathbb{K}\langle \mathbf{x}_0, \ldots, \mathbf{x}_l, \partial_0, \ldots, \partial_l \rangle$$

and its polynomial subalgebra $\widetilde{\mathbb{A}} = \mathbb{K}[\partial_0, \ldots, \partial_l]$. The $\widetilde{\mathbb{A}}$-module $\widetilde{\mathbb{D}}a$ is supported at zero, hence, it must be a direct sum of copies of $\widetilde{\Delta} = \widetilde{\mathbb{D}}\delta(\partial_0, \ldots, \partial_l) \cong \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_l]$. It follows that

$$\mathbf{x}_0^{b_0} \ldots \mathbf{x}_l^{b_l} \cdot a \neq 0 \quad \text{for all} \quad b_0, \ldots, b_l \geqslant 0.$$

We want to determine for which k, we can find $b_0, \ldots, b_l \geqslant 0$ such that $\mathbf{x}_0^{b_0} \ldots \mathbf{x}_l^{b_l} \cdot a \in M_0 = 0$. We get a contradiction and hence $M_{-kg_l} = 0$ for such k. The condition is that

$$b_0 d_0 + \ldots + b_l d_l = kg_l,$$

i.e. $kg_l \in \mathbb{Z}_{\geqslant 0} d_0 + \mathbb{Z}_{\geqslant 0} d_1 + \ldots + \mathbb{Z}_{\geqslant 0} d_l$. □

In particular, it is true for $l = n$, i.e., $M_{-k} = 0$ for all $k \in \mathcal{A}$. Now let us finish the proof of the theorem. By Schur's Theorem there exists[1] $K \geqslant 0$ such that $k \in \mathcal{A}$ for all $k > K$, in particular, $M_{-k} = 0$ for all $k > K$. Thus, M is supported at zero as a $\mathbb{K}[\partial_0, \ldots \partial_n]$-module. By Kashiwara's Theorem M is a direct sum of copies of $\mathbb{A} = \mathbb{K}[\mathbf{x}_0, \ldots \mathbf{x}_n]$. If $\lambda \in \mathbb{K} \setminus \mathbb{Z}$ then $\mathbb{A}$ is not λ-Euler. Thus, $M = 0$. Finally, if $\lambda \in \mathbb{Z}$ then $\mathbb{A}$ is λ-Euler. Moreover, as a graded module M is a direct sum of copies of $\mathbb{A}[\lambda]$. Observe that $\mathbb{A}[\lambda]_0 = \mathbb{A}_\lambda \neq 0$ if and only if $\lambda \in \mathcal{A}$. Thus, if $\lambda \in \mathcal{A}$, then $M = 0$ as well. □

[1] The smallest such K is called the Frobenius number. It is a NP-hard problem to find such K. There is no known closed formula that gives K as a function of $d_0, ..., d_n$ for $n \geqslant 2$.

Combining the last two claims, we obtain a characterisation of the kernel of the global sections functor.

Theorem 16. *The greatest common divisor* $\gcd_i(d_i)$ *is equal to* 1 *and* $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$ *if and only if* $\mathrm{Ker}\Gamma_\lambda$ *is a zero category.*

Together with Theorem 13 this gives the following corollaries.

Corollary 17. *Let us suppose that* $\lambda \in (\mathbb{K}\setminus\mathbb{Z})\cup\mathcal{A}$ *and* $\gcd(d_0, ..., d_n) = 1$*. Then* $\Gamma_\lambda : \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh} \to D^\lambda_{[X]_0}-\mathrm{Mod}$ *is an equivalence of categories.*

In particular, we obtain a necessary and sufficient condition for a weighted projective stack to be D-affine.

Corollary 18. *The weighted projective stack* $[X] = [\mathbb{P}(V)]$ *is D-affine if and only if* $\gcd_i(d_i)$ *is equal to* 1*.*

Proof. D-affinity deals with the case of $\lambda = 0$. Γ_0 is exact, and its kernel is zero if and only if $\gcd_i(d_i)$ is equal to 1. □

A similar functor for varieties

$$\Gamma'_\lambda : \mathcal{D}^\lambda_X-\mathrm{Qcoh} \to D^\lambda_{[X]_0}-\mathrm{Mod}$$

is studied by Van den Bergh [16]. It is instructive to compare it with the push-forward functor

$$\pi_* : \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh} \to \mathcal{D}^\lambda_X-\mathrm{Qcoh}.$$

The functors $\Gamma'_\lambda\pi_*$ and Γ_λ are naturally equivalent, so we can conclude the final corollary.

Corollary 19. *Let us suppose that* $\lambda \in \mathbb{K} \setminus \mathbb{Z} \cup \mathcal{A}$ *and* $\gcd_{i\neq j}(d_i) = 1$ *for every* j *(the well-formedness condition). Then the push-forward functor* $\pi_* : \mathcal{D}^\lambda_{[X]}-\mathrm{Qcoh} \to \mathcal{D}^\lambda_X-\mathrm{Qcoh}$ *is an equivalence of categories.*

It can be noticed as well that the condition of well-formedness is not required for a weighted projective stack to be D-affine. We only need the greatest common divisor of its weights to be equal to one to guarantee it. As varieties, this condition was added to prove D-affinity of weighted projective spaces.

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