Application of AAK Theory for Sparse Approximation

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Notation

$\mathbb{N}$	natural numbers (including 0)
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{D}$	open unit disc (including 0)
$\overline{\mathbb{D}}$	closed unit disc
$\partial\mathbb{D}$	unit circle (the boundary of $\mathbb{D}$)
$\ell^p(\mathbb{N})$	p-summable sequences of the form $(a_j)_{j=0}^{\infty}$
$\ell^p(\mathbb{Z})$	p-summable sequences of the form $(a_j)_{j=\infty}^{\infty}$
L^2	quadratic integrable functions on $\partial\mathbb{D}$
H^2	space of analytic functions in $\mathbb{D}$ having only positive indexed Fourier coefficients, the so-called Hardy space
$\mathcal{F}$	Fourier transform from L^2 to $\ell^2(\mathbb{Z})$
$\hat{\vartheta}$, $\mathcal{F}(\vartheta)$	vector of Fourier coefficients of a function $\vartheta \in L^2$
$\hat{\vartheta}_k$	k-th Fourier coefficient of a function $\vartheta \in L^2$
$\mathbf{P}_-, \ \mathbf{P}_+$	discrete projection operators
$\mathrm{P}_-, \ \mathrm{P}_+$	continuous projection operators
S	shift operator on $\ell^p(\mathbb{N}), \ p = 1, 2$
S^*	backward shift operator on $\ell^p(\mathbb{N}), \ p = 1, 2$
$\mathcal{S}_{\mathbf{v}}$	shift invariant subspace of $\ell^p(\mathbb{N})$ generated by a sequence $\mathbf{v} \in \ell^p(\mathbb{N})$ for $p = 1, 2$
M_z	multiplication operator on L^2 with respect to a free variable $z \in \mathbb{C}$
M_φ	multiplication operator on L^2 with respect to $\varphi \in L^2$
$\mathcal{M}_\vartheta$	multiplication invariant subspace of H^2 generated by $\vartheta \in L^2$
$\mathbf{F}_N$	Fourier matrix of size $N \times N$
$\mathbf{I}_N$	Identity matrix of size $N \times N$
$\mathbf{\Gamma_f}$	infinite Hankel matrix with respect to $\mathbf{f} \in \ell^1(\mathbb{N})$
$\mathbf{H}_{\mathbf{f}}^{(N+1)}$	the leading principal minor of $\mathbf{\Gamma_f}$ of size $(N+1) \times (N+1)$
$\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}$	periodic Hankel matrix with respect to a vector $\mathbf{f}$
$\mathbf{T_g}$	infinite triangular Toeplitz matrix with respect to $\mathbf{g} \in \ell^p(\mathbb{N}), \ p = 1, 2$
H_φ	Hankel operator with respect to $\varphi \in L^2$
$\mathcal{K}_\alpha$	reproducing kernel on H^2

$\mathbf{C}$	circulant matrix
$\mathbf{U}$	flip matrix
$\mathbf{C_p}$	companion matrix with respect to a vector $\mathbf{p}$
J	continuous flip operator
$\mathbf{J}$	discrete flip operator

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1. Introduction

Sparse approximation of structured signals is a common problem in signal processing and system theory. In particular, approximation by exponential sums often arises in natural sciences for the analysis of decay processes. In many applications it can be assumed that the signal is either exactly or approximately a finite linear combination of non-increasing exponentials with complex exponents, i.e., $\mathbf{f} := (f_k)_{k=0}^{\infty}$ satisfies

$$f_k := f(k) = \sum_{j=1}^{N} a_j z_j^k , \tag{1.1}$$

where $a_j \in \mathbb{C}\backslash\{0\}$ and $z_j \in \{z \in \mathbb{C} : 0 < |z| < 1\}$ are pairwise different. Thus, the nonlinear inverse problem of recovering the parameters z_j and a_j from a suitable number of samples f_k becomes relevant. Note that due to Kronecker's Theorem, see e.g. Theorem 8.19 in [24], the length of the above sum corresponds to the rank of the infinite Hankel matrix $\mathbf{\Gamma_f} = (f_{j+k})_{j,k=0}^{\infty}$ (and the finite Hankel matrix $\mathbf{H}_{\mathbf{f}}^{(N+1)} := (f_{j+k})_{j,k=0}^{M}$ for $M \geqslant 2N$ likewise). Hence the above parameter estimation problem is closely related to the structured low rank approximation problem of Hankel matrices. Unfortunately, the singular value decomposition cannot be applied in this case, since it does not preserve the Hankel structure of the matrix.

There have been different attempts to solve the low rank approximation problem for Hankel matrices. One possible approach is to apply a regularization method, see, e.g., [31]. The main idea in [31] is to formulate the problem as a minimization problem with the main functional enforcing the lower rank of a Hankel matrix and the regularization term enforcing the Hankel structure. Although in [31] the global optimal solution cannot be guaranteed. In [13] an alternating projection method is used to obtain a low rank approximation. In this approach a truncated singular value decomposition is applied to the Hankel matrix first and afterwards a projection to the Hankel structure is used. This procedure is repeated until a certain stopping criterion is satisfied. Also this algorithm does not converge to an optimal solution in general. Other attempts to solve the above parameter estimation problem have been studied in non-deterministic methods from Compressed Sensing [14, 21, 49]. These methods only obtain the correct parameters z_j and a_j with certain probability.

However, the most classical deterministic way to solve the above problem is the so-called Prony's method due to [48]. Note that this method is also known as annihilating filter method [22, 54] in signal processing. As pointed out in the survey by Plonka and Tasche [45], Prony's method has gained a great popularity in the last decades. Since the original method is known to have numerical stability issues, stabilized Prony-like methods such as ESPRIT [50], Matrix Pencil method [30] and Approximate Prony's Method (APM) [46] have been developed. In [32, 52, 51] Prony's method was extended to the multivariate case. A generalized Prony's method has been developed in [42], where instead of exponential functions general eigenfunctions of certain linear operators are considered.

Prony-like methods can also be applied to approximate special functions by exponential sums. These approximations are used in order to solve high-dimensional integrals by products of one-dimensional integrals in [12], in cluster analysis in quantum chemistry, see [33, 26] or for solving Schrödinger's equation [6, 7]. Many different examples for function approximation by exponential sums have been presented in [8, 9]. We remark that for special functions, where all function values are available, the Remez algorithm is an alternative to Prony-like methods. In particular, this method has been used to approximate the function $f(x) = 1/x$ by short exponential sums in [11, 25, 12]. In this thesis we will consider these approximations obtained by the Remez algorithm as benchmarks in our numerical tests.

Note that in most applications one is interested in obtaining the shortest possible exponential sum which satisfies a presumed approximation error in order to reduce further computational costs. This leads to the following problem, which we study in this thesis.

For a given $\mathbf{f}$ of the form (1.1) we want to find a new signal $\tilde{\mathbf{f}} := (\tilde{f}_k)_{k=0}^{\infty}$ given by

$$\tilde{f}_k := \tilde{f}(k) = \sum_{j=1}^{K} \tilde{a}_j \, \tilde{z}_j^k \tag{1.2}$$

with $\tilde{a}_j \in \mathbb{C}$ and $\tilde{z}_j \in \mathbb{D}$ such that $K < N$ and $\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \varepsilon$. Considering this problem two questions arise, namely:

1. Let the accuracy level $\varepsilon > 0$ be given. What is the smallest $K \in \mathbb{N}$ such that $\tilde{\mathbf{f}}$ of the form (1.2) satisfies $\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \varepsilon$ and how to compute $\tilde{z}_j \in \mathbb{D}$ and $\tilde{a}_j \in \mathbb{C}$, $j = 1, \ldots, K$?

2. Vice versa, let the "storage budget" $K \in \mathbb{N}$ be given. How do we have to choose the parameters $\tilde{a}_j \in \mathbb{C}$ and $\tilde{z}_j \in \mathbb{D}$ in order to achieve the smallest possible error $\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})}$?

For solving the above problem we employ the theory of Adamjan, Arov and Krein (AAK theory) [1] in this work. The main theorem from [1] can be seen as a structured low rank approximation approach for infinite matrices, see Section 2.5 in [34], and is widely used by engineers for model reduction. It states that an infinite Hankel matrix $\mathbf{\Gamma_f} = (f_{j+k})_{j,k=0}^{\infty}$ bounded on $\ell^p(\mathbb{N})$ for $p \in \{1,2\}$ and generated by $\mathbf{f} = (f_k)_{k=0}^{\infty}$ can be approximated by an infinite Hankel matrix $\mathbf{\Gamma_{\tilde{f}}}$ of finite rank K such that

$$\|\mathbf{\Gamma_f} - \mathbf{\Gamma_{\tilde{f}}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \min_{\mathrm{Rank}\mathbf{\Gamma_g}\leqslant K} \|\mathbf{\Gamma_f} - \mathbf{\Gamma_g}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \sigma_K,$$

where $\sigma_0 \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots$ denote singular values of $\mathbf{\Gamma_f}$ ordered by size and repeated according to multiplicities. This result is non-trivial, since the usual approximation approach using the truncated spectral decomposition of $\mathbf{\Gamma_f}$ does not preserve the Hankel structure. There exist several equivalent formulations of the AAK result. For instance in the original paper [1] the above theorem is presented for an arbitrary bounded function $f \in L^{\infty}([0,2\pi))$ and provides the existence of the best approximation by a function $\tilde{f}$ from the subspace $H^{\infty,[K]}$ $(K \in \mathbb{N})$ of the Hardy space H^{∞}. It states that the best approximation $\tilde{f}$ satisfies

$$\|f - \tilde{f}\|_{\infty} = \min_{g\in H^{\infty,[K]}} \|f - g\|_{\infty} = \sigma_K(\mathbf{\Gamma_f}),$$

where $\mathbf{\Gamma_f}$ is the infinite Hankel matrix generated by the negative indexed Fourier coefficients $\mathbf{f} := (\hat{f}(-k-1))_{k=0}^{\infty}$ of f. Here $H^{\infty,[K]}$ denotes the space of functions in H^{∞} which possess at most K poles in $\mathbb{D}$. Further formulations of the AAK Theorem can be found in [41, 15, 35, 37, 39, 55]. Note that the proofs of this theorem which can be found in the literature involve some deep results from complex analysis. They are based on the analysis of bounded functions in Hardy spaces and operator theory, using the fundamental theorems such as Nehari's Theorem [38] and Beurling Theorem [5].

For earlier attempts to apply the AAK theory in order to solve sparse approximation problems by exponential sums we refer to [3] and [8]. In [3] the connection between the AAK theory for discrete and continuous settings on $\mathbb{R}^+$ and on an interval has been studied and some asymptotical results considering truncated Hankel operators are provided. In [8] a finite-dimensional approximation problem using finite number of samples of a continuous function on an interval is considered. On the first glance the main result in [8] seems very similar to a finite version of the AAK Theorem. It can be summarized as follows. Using $2N+1$ samples $\mathbf{f} = (f_k)_{k=0}^{2N}$ the authors consider a finite Hankel matrix $\mathbf{H_f}^{(N+1)}$ of size $(N+1)\times(N+1)$ and it's singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_N$ with the corresponding singular vectors $\mathbf{v}_0, \mathbf{v}_1 \ldots, \mathbf{v}_N$. Then they propose to compute the N roots $z_1^{(n)}, \ldots, z_N^{(n)}$ of the finite polynomial $P_{\mathbf{v}_n}(z) = \sum_{k=0}^{N} v_k z^k$ corresponding to the n-th singular vector $\mathbf{v}_n$ of $\mathbf{H_f}^{(N+1)}$, which can be used as the nodes of the approximating sum. Finally, the authors prove that if the zeros of $P_{\mathbf{v}_n}(z)$ have multiplicity 1, then there exist coefficients $c_1, \ldots, c_N$ such that

$$\|\mathbf{f} - (\tilde{f}_k)_{k=0}^{2N}\|_2 \leqslant \sigma_n, \qquad \tilde{f}_k := \sum_{j=1}^{N} c_j (z_j^{(n)})^k.$$

The coefficients can be found by solving a certain Vandermonde system. Note that the approximating sum in the above equation has length N and not $n < N$. However, in numerical examples the authors notice that roughly $N - n$ terms in the above sum are small enough to be discarded, although the theoretical foundation for this fact is not provided in [8]. Eventually the authors propose a procedure which can be used for reduction of number of terms.

Contribution of this work. The main contribution of this thesis is a new algorithm for solving the K-term approximation problem (1.2), such that

$$\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \sigma_K,$$

where σ_K denotes the K-th singular value of the infinite Hankel matrix $\mathbf{\Gamma_f}$. We give an explicit procedure for the computation of the nodes $\tilde{z}_j$ and coefficients $\tilde{a}_j$ for $j = 1, \ldots, K$, based on the AAK results. Our algorithm also includes a technique for the computation of all singular values of $\mathbf{\Gamma_f}$, which is not trivial due to the fact that $\mathbf{\Gamma_f}$ is an infinite matrix. For this purpose we consider $\mathbf{\Gamma_f}$ with finite rank N and investigate the structure of the singular vectors corresponding to the non-zero singular values of $\mathbf{\Gamma_f}$. This structure allows us to reduce the singular value problem for the infinite matrix $\mathbf{\Gamma_f}$ to an equivalent singular value problem for an $N \times N$ kernel matrix. Further, we provide a new proof of the AAK Theorem for Hankel matrices with finite rank in the discrete context. To our knowledge this is the first proof, which employs only tools from linear algebra and Fourier analysis and completely avoids the fundamental theorems from operator theory. For this purpose we characterize all mathematical objects used in our proof in the framework of linear algebra and establish the connection to the continuous setting using the Fourier transform. Also the connection between the AAK theory and Prony's method becomes clear in this work. Thus, this thesis can be seen as a solid groundwork for further investigations and approaching this theory from the field of linear algebra.

This dissertation is organized as follows. In Chapter 2 we compile the main definitions of the mathematical objects used in this work and their basic properties. We distinguish between the so-called discrete setting in Section 2.1 and the continuous setting in Section 2.2. In the discrete setting we consider absolute or square summable sequences, finite and infinite matrices with certain structure and their interaction with each other. The continuous setting handles analytic functions on the unit disc, L^2 and it's subspaces and linear operators mapping between them. Further we establish a connection between the objects from Sections 2.1 and 2.2 via the Fourier transform. This connection is summarized in the table at the end of Chapter 2.

In Chapter 3 we present a special case of the AAK theorem in two versions, namely for infinite Hankel matrices with finite rank as well as for Hankel operators with finite rank. In Section 3.1 we provide a new proof of the discrete version of the theorem using only tools from linear algebra. The proof is "translated" to the continuous setting in Section 3.2, where the structure of the "discrete" proof is being carried over.

In Chapter 4 we give a brief overview over the Prony-like methods for parameter estimation of exponential sums, which are used in our algorithm in

Chapter 5. Section 4.1 deals with the classical Prony's method, which recovers the exponents and the corresponding coefficients from the exact sequence of samples $\mathbf{f}$ in (1.1). Since the classical Prony's method is numerically instable for non-exact data, we introduce the approximative Prony's method APM from [46] in Section 4.2. The APM algorithm approximates the original samples $\mathbf{f}$ by the sequence $\tilde{\mathbf{f}}$ with exponential sum structure. This method is proven to be numerically stable for noisy data.

In Chapter 5 we present one of the main results of this dissertation, namely a new algorithm for sparse approximation of exponential sums. The algorithm itself is derived in Section 5.1. Additionally, in Section 5.2 we discuss stability issues of the presented algorithm and provide some solutions. Furthermore, in Section 5.3 we outline an algorithm for the low rank approximation for the special case of periodic sequences $\mathbf{f}$. In this case one can show that all nodes z_j lie on the unit circle and the truncated SVD preserve the Hankel structure of the matrix.

In Chapter 6 we provide numerical experiments demonstrating the performance of Algorithms 5.1 and 5.4. Our algorithm is tested for approximation of exponential sums in Section 6.1. Further, in Sections 6.2 and 6.3 the approximation of other decaying functions is considered, where in particular a new non-equidistant sampling approach is presented. Finally, in Section 6.5 we briefly show the performance of Algorithm 5.4.

2. Preliminaries

In the following we denote by $\ell^p(\mathbb{N})$ for $p \in \{1,2\}$ the space of $p-$summable sequences

$$\ell^p(\mathbb{N}) := \left\{ \mathbf{v} = (v_j)_{j=0}^{\infty} \ : \ \sum_{j=0}^{\infty} |v_j|^p < \infty \right\}, \quad p = 1,2$$

and the norm

$$\|\mathbf{v}\|_{\ell^p(\mathbb{N})} = \left(\sum_{j=0}^{\infty} |v_j|^p \right)^{1/p}$$

and by $\mathbb{D}$ the open unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

Clearly, we can also consider the space $\ell^p(\mathbb{Z})$ defined analogously, containing doubly infinite sequences of the form $\mathbf{v} = (v_j)_{j=-\infty}^{\infty}$. It holds $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$. Note that $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ are Hilbert spaces with inner products given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\ell^2(\mathbb{N})} = \sum_{j=0}^{\infty} u_j \overline{v}_j \qquad \text{and} \qquad \langle \mathbf{u}, \mathbf{v} \rangle_{\ell^2(\mathbb{Z})} = \sum_{j=-\infty}^{\infty} u_j \overline{v}_j, \tag{2.1}$$

respectively. For every sequence $\mathbf{v} \in \ell^p(\mathbb{N})$ and $z \in \mathbb{D}$ we denote the corresponding Laurent polynomial by

$$P_{\mathbf{v}}(z) := \sum_{j=0}^{\infty} v_j z^j \tag{2.2}$$

and its Fourier series by $P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}\omega})$, where $\omega \in [0, 2\pi)$.

2.1. Discrete Setting

Let $\mathbf{u} := (u_k)_{k=-\infty}^{\infty}$ and $\mathbf{w} := (w_k)_{k=-\infty}^{\infty}$ be two sequences in $\ell^2(\mathbb{Z})$. We define the *discrete convolution* as follows,

$$(\mathbf{u} * \mathbf{w})_k = \sum_{j=-\infty}^{\infty} u_j w_{k-j}, \qquad k \in \mathbb{Z}.$$

The convolution of sequences can be written in terms of infinite matrix-vector multiplication

$$\left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \ddots & u_0 & u_{-1} & u_{-2} & u_{-3} & \ddots & \ddots & \\ \ddots & u_1 & u_0 & u_{-1} & u_{-2} & u_{-3} & \ddots & \ddots \\ \ddots & u_2 & u_1 & u_0 & u_{-1} & u_{-2} & u_{-3} & \ddots \\ \hline \ddots & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} & \ddots \\ \ddots & \ddots & u_3 & u_2 & u_1 & u_0 & u_{-1} & \ddots \\ & \ddots & \ddots & u_3 & u_2 & u_1 & u_0 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \left(\begin{array}{c} \vdots \\ w_{-3} \\ w_{-2} \\ w_{-1} \\ \hline w_0 \\ w_1 \\ w_2 \\ \vdots \end{array}\right) = \left(\begin{array}{c} \vdots \\ (\mathbf{u} * \mathbf{w})_{-3} \\ (\mathbf{u} * \mathbf{w})_{-2} \\ (\mathbf{u} * \mathbf{w})_{-1} \\ \hline (\mathbf{u} * \mathbf{w})_0 \\ (\mathbf{u} * \mathbf{w})_1 \\ (\mathbf{u} * \mathbf{w})_2 \\ \vdots \end{array}\right) = \mathbf{u} * \mathbf{w}.$$

The above matrix is usually called *infinite Toeplitz matrix* with respect to $\mathbf{u}$ or a *convolution matrix*. This representation gives rise to the definition of infinite Hankel matrices and triangular Toeplitz matrices below. For $\mathbf{f} \in \ell^1(\mathbb{N})$ we define the infinite *Hankel matrix*

$$\mathbf{\Gamma}_{\mathbf{f}} := \begin{pmatrix} f_0 & f_1 & f_2 & \cdots \\ f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(f_{k+j}\right)_{k,j=0}^{\infty}.$$

Note that by extending the sequence spaces to $\ell^p(\mathbb{Z})$, multiplication of $\mathbf{\Gamma}_{\mathbf{f}}$ with a vector $\mathbf{v} \in \ell^p(\mathbb{N})$ can be seen as the part with negative indices of the convolution vector $\mathbf{u} * \mathbf{w}$, where

$$\mathbf{u} = (u_j)_{j=-\infty}^{\infty} := \begin{cases} f_{-j-1}, & j < 0 \\ 0, & j \geqslant 0 \end{cases} \qquad \text{and} \qquad \mathbf{w} = (w_j)_{j=-\infty}^{\infty} := \begin{cases} 0, & j < 0 \\ v_j, & j \geqslant 0 \end{cases},$$

since

$$\left(\begin{array}{ccc|cccc} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & f_0 & f_1 & f_2 & \ddots & \ddots & \\ & & f_0 & f_1 & f_2 & \ddots & \ddots \\ & & & f_0 & f_1 & f_2 & \ddots \\ \hline & & & & f_0 & f_1 & \ddots \\ & & & & & f_0 & \ddots \\ & & & & & & \ddots \end{array}\right) \left(\begin{array}{c} \vdots \\ 0 \\ \hline v_0 \\ v_1 \\ v_2 \\ \vdots \end{array}\right) = \left(\begin{array}{c} \vdots \\ (\mathbf{u} * \mathbf{w})_{-3} \\ (\mathbf{u} * \mathbf{w})_{-2} \\ (\mathbf{u} * \mathbf{w})_{-1} \\ \hline (\mathbf{u} * \mathbf{w})_0 \\ (\mathbf{u} * \mathbf{w})_1 \\ (\mathbf{u} * \mathbf{w})_2 \\ \vdots \end{array}\right) = \left(\begin{array}{c} \vdots \\ (\mathbf{\Gamma}_{\mathbf{f}}\, \mathbf{v})_2 \\ (\mathbf{\Gamma}_{\mathbf{f}}\, \mathbf{v})_1 \\ (\mathbf{\Gamma}_{\mathbf{f}}\, \mathbf{v})_0 \\ \hline (\mathbf{u} * \mathbf{w})_0 \\ (\mathbf{u} * \mathbf{w})_1 \\ (\mathbf{u} * \mathbf{w})_2 \\ \vdots \end{array}\right).$$

Therefore the infinite Hankel matrix $\mathbf{\Gamma_f}$ determines an operator $\mathbf{\Gamma_f} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ for $p \in \{1, 2\}$ given by

$$\mathbf{\Gamma_f}\,\mathbf{v} = \left(\sum_{j=0}^{\infty} f_{k+j} v_j \right)_{k=0}^{\infty} \qquad \text{for } \mathbf{v} := (v_k)_{k=0}^{\infty} \in \ell^p(\mathbb{N}).$$

This is a direct consequence of Young's inequality and the convolution representation above, since with $\mathbf{u}, \mathbf{w}$ as before it holds

$$\|\mathbf{\Gamma_f}\,\mathbf{v}\|_{\ell^p(\mathbb{N})} \leqslant \|\mathbf{u} * \mathbf{w}\|_{\ell^p(\mathbb{Z})} \leqslant \|\mathbf{u}\|_{\ell^1(\mathbb{Z})} \|\mathbf{w}\|_{\ell^p(\mathbb{Z})} = \|\mathbf{f}\|_{\ell^1(\mathbb{N})} \|\mathbf{v}\|_{\ell^p(\mathbb{N})} < \infty.$$

For $\mathbf{g} = (g_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$, we define the infinite *triangular Toeplitz matrix* $\mathbf{T_g}$ by

$$\mathbf{T_g} := \begin{pmatrix} g_0 & & & \\ g_1 & g_0 & & \\ g_2 & g_1 & g_0 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $\mathbf{g} \in \ell^1(\mathbb{N})$ this matrix determines a bounded operator $\mathbf{T_g} : \ell^\nu(\mathbb{N}) \to \ell^\nu(\mathbb{N})$ for $\nu \geqslant 1$ given by

$$\mathbf{T_g v} := \left(\sum_{j=0}^{k} g_{k-j} v_j \right)_{k=0}^{\infty}, \qquad \mathbf{v} \in \ell^\nu(\mathbb{N}),$$

since for

$$\tilde{\mathbf{g}} = (\tilde{g}_j)_{j=-\infty}^{\infty} := \begin{cases} 0, & j < 0 \\ g_j, & j \geqslant 0 \end{cases} \qquad \text{and} \qquad \tilde{\mathbf{v}} = (\tilde{v}_j)_{j=-\infty}^{\infty} := \begin{cases} 0, & j < 0 \\ v_j, & j \geqslant 0 \end{cases},$$

it holds

$$\|\mathbf{T_g v}\|_{\ell^\nu(\mathbb{N})} = \|\tilde{\mathbf{g}} * \tilde{\mathbf{v}}\|_{\ell^\nu(\mathbb{Z})} \leqslant \|\tilde{\mathbf{g}}\|_{\ell^1(\mathbb{Z})} \|\tilde{\mathbf{v}}\|_{\ell^\nu(\mathbb{Z})} = \|\mathbf{g}\|_{\ell^1(\mathbb{N})} \|\mathbf{v}\|_{\ell^\nu(\mathbb{N})}$$

by Young's inequality.

2.1.1. Basic properties of infinite Hankel and Toeplitz matrices

Let $\mathbf{v} := (v_k)_{k=0}^{\infty}$ be a sequence in $\ell^p(\mathbb{N})$ and $p \in \{1, 2\}$. We define the *(forward) shift operator* $\mathrm{S} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ by

$$\mathrm{S}\mathbf{v} := (0, v_0, v_1, v_2, \ldots)$$

and the *backward shift operator* $\mathrm{S}^* : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ by

$$\mathrm{S}^*\mathbf{v} := (v_1, v_2, v_3, \ldots).$$

The shift operator is closely related to the infinite Hankel and Toeplitz matrices. It is obvious that the Toeplitz matrix $\mathbf{T_g}$ contains the forward shifted vectors

$\mathrm{S}^k\mathbf{g}$ as its k-th column. Furthermore the k-th column (and row) of the Hankel matrix $\mathbf{\Gamma_f}$ is the backward shift $(\mathrm{S}^*)^k\mathbf{f}$ of the vector $\mathbf{f}$. Thus, for $\mathbf{v} \in \ell^2(\mathbb{N})$ we have

$$\mathbf{\Gamma_f}\mathrm{S}\mathbf{v} = \left(\sum_{j=1}^{\infty} f_{k+j}\, v_{j-1}\right)_{k=0}^{\infty} = \left(\sum_{j=0}^{\infty} f_{k+1+j} v_j\right)_{k=0}^{\infty} = \mathrm{S}^*\mathbf{\Gamma_f}\mathbf{v}.$$

This commutator relation determines the structure of a Hankel operator and can be even used as a formal definition of $\mathbf{\Gamma_f}$, see [29]. Beyond that, some less obvious connections can be observed, for investigation of which we need the following definition.

The *shift invariant subspace* of $\ell^2(\mathbb{N})$ generated by the sequence $\mathbf{v} \in \ell^2(\mathbb{N})$ (resp. $\mathbf{v} \in \ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$) is denoted by

$$\mathcal{S}_\mathbf{v} := \mathrm{clos}_{\ell^2(\mathbb{N})}\mathrm{span}\,\{\mathrm{S}^k\mathbf{v} : k \in \mathbb{N}\}.$$

With the help of the concept of shift invariant subspaces we obtain a useful characterization of the kernel of $\mathbf{\Gamma_f}$ presented in the following lemma.

Lemma 2.1. *Let* $\mathbf{f} := (f_k)_{k=0}^{\infty}$ *be a sequence in* $\ell^1(\mathbb{N})$ *and* $\mathbf{\Gamma_f}$ *the corresponding infinite Hankel matrix as above. Then the following assertions hold.*

(1) *The kernel space* $\mathrm{Ker}\,(\mathbf{\Gamma_f}) := \{\mathbf{v} \in \ell^2(\mathbb{N}) : \mathbf{\Gamma_f}\mathbf{v} = \mathbf{0}\}$ *is* S*-invariant, i.e., for* $\mathbf{v} \in \mathrm{Ker}(\mathbf{\Gamma_f})$ *we have* $\mathcal{S}_\mathbf{v} \subset \mathrm{Ker}(\mathbf{\Gamma_f})$.

(2) *A vector* $\mathbf{v} \in \ell^2(\mathbb{N})$ *is in* $\mathrm{Ker}\,(\mathbf{\Gamma_f})$ *if and only if* $\mathbf{f} \in (\mathcal{S}_{\overline{\mathbf{v}}})^{\perp}$.

Proof.

1. Let $\mathbf{v} \in \mathrm{Ker}(\mathbf{\Gamma_f})$. Then the commutator relation implies

$$\mathbf{\Gamma_f}\mathrm{S}\mathbf{v} = \mathrm{S}^*\mathbf{\Gamma_f}\mathbf{v} = \mathrm{S}^*\mathbf{0} = \mathbf{0},$$

and thus $\mathrm{S}\mathbf{v}$ is also in $\mathrm{Ker}(\mathbf{\Gamma_f})$.

2. Using the definition of $\mathcal{S}_\mathbf{v}$ we obtain

$$\begin{aligned}\mathbf{\Gamma_f}\mathbf{v} = 0 \;&\Leftrightarrow\; \sum_{k=0}^{\infty} f_{k+j}\, v_k = 0 \quad \forall\, j \in \mathbb{N}\\ &\Leftrightarrow\; \sum_{k=0}^{\infty} (\mathrm{S}^j\mathbf{v})_k f_k = 0 \quad \forall\, j \in \mathbb{N}\\ &\Leftrightarrow\; \langle \mathbf{f}, \mathrm{S}^j\bar{\mathbf{v}}\rangle_{\ell^2(\mathbb{N})} = 0 \quad \forall\, j \in \mathbb{N}\\ &\Leftrightarrow\; \mathbf{f} \perp \mathcal{S}_{\overline{\mathbf{v}}}\end{aligned}$$

for every $\mathbf{v} \in \ell^2(\mathbb{N})$.

□

Lemma 2.2. *For two sequences* $\mathbf{f} \in \ell^1(\mathbb{N})$ *and* $\mathbf{g} \in \ell^1(\mathbb{N})$, *we have the following.*

(1) *The convolution* $\mathbf{f} * \mathbf{g}$ *is a sequence in* $\ell^1(\mathbb{N})$ *and*

$$\mathbf{T}_{\mathbf{f}*\mathbf{g}} = \mathbf{T}_{\mathbf{f}} \cdot \mathbf{T}_{\mathbf{g}} = \mathbf{T}_{\mathbf{g}} \cdot \mathbf{T}_{\mathbf{f}}.$$

(2) *For* $\mathbf{g} \in \ell^1(\mathbb{N})$ $\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}}$ *is a bounded Hankel operator on* $\ell^\nu(\mathbb{N})$ *for* $\nu \geqslant 1$.

(3) *It holds* $\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}} = \mathbf{T}_{\mathbf{g}}^\top \mathbf{\Gamma}_{\mathbf{f}}$.

(4) *Further we have* $\mathbf{T}_{\mathbf{g}}^* \mathbf{f} = \mathbf{\Gamma}_{\mathbf{f}}\mathbf{g}$.

Proof.

1. We observe that for $l \geqslant k$

$$(\mathbf{T}_{\mathbf{g}}\mathbf{T}_{\mathbf{f}})_{l,k} = \sum_{r=0}^{l-k} f_{(l-k)-r} g_r = \sum_{r=0}^{l-k} f_r g_{(l-k)-r} = (\mathbf{T}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}})_{l,k} = (\mathbf{f} * \mathbf{g})_{l-k},$$

   while $(\mathbf{T}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}})_{l,k} = 0$ for $l < k$. Young's inequality ensures that $(\mathbf{f} * \mathbf{g}) \in \ell^1(\mathbb{N})$ and thus the product of Toeplitz operators is a bounded operator on $\ell^\nu(\mathbb{N})$ for $\nu \geqslant 1$.

2. Since the j-th row of $\mathbf{\Gamma}_{\mathbf{f}}$ is $(\mathrm{S}^*)^j \mathbf{f}$ and the k-th column of $\mathbf{T}_{\mathbf{g}}$ is $\mathrm{S}^k \mathbf{g}$, it follows that

$$(\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}})_{j,k} = ((\mathrm{S}^*)^j \mathbf{f})^\top (\mathrm{S}^k \mathbf{g}) = \mathbf{f}^\top (\mathrm{S}^{j+k}\mathbf{g}),$$

   thus the entries of $\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}}$ only depend on the sum of their indices. Therefore, $\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}}$ has again Hankel structure. The obtained Hankel matrix is generated by $\mathbf{\Gamma}_{\mathbf{f}}\mathbf{g} \in \ell^1(\mathbb{N})$.

3. Similarly, since the j-th row of $\mathbf{T}_{\mathbf{g}}$ is $\mathrm{S}^j \mathbf{g}$ and the k-th column of $\mathbf{\Gamma}_{\mathbf{f}}$ is $(\mathrm{S}^*)^k \mathbf{f}$ we obtain

$$(\mathbf{T}_{\mathbf{g}}^\top \mathbf{\Gamma}_{\mathbf{f}})_{j,k} = (\mathrm{S}^j \mathbf{g})^\top ((\mathrm{S}^*)^k \mathbf{f}) = (\mathrm{S}^{j+k}\mathbf{g})^\top \mathbf{f} = \mathbf{f}^\top (\mathrm{S}^{j+k}\mathbf{g}) = (\mathbf{\Gamma}_{\mathbf{f}}\mathbf{T}_{\mathbf{g}})_{j,k}.$$

4. Again, using the shift structure of the matrices $\mathbf{T}_{\mathbf{g}}$ and $\mathbf{\Gamma}_{\mathbf{f}}$ we have

$$\mathbf{T}_{\mathbf{g}}^* \mathbf{f} = (\overline{\langle \mathrm{S}^j \mathbf{g}, \mathbf{f} \rangle})_{j=0}^\infty = (\overline{\langle \mathbf{g}, (\mathrm{S}^*)^j \mathbf{f} \rangle})_{j=0}^\infty = (\langle (\mathrm{S}^*)^j \mathbf{f}, \mathbf{g} \rangle)_{j=0}^\infty = \mathbf{\Gamma}_{\mathbf{f}}\mathbf{g}.$$

□

2.1.2. Infinite Hankel matrices with finite rank

Let $\mathbf{f} = (f_k)_{k=0}^\infty$ be sequence of the form (1.1) with $N \in \mathbb{N}$, $a_j \in \mathbb{C}\backslash\{0\}$ and pairwise different nodes $z_j \in \mathbb{D}\backslash\{0\}, j \in \{1, \ldots, N\}$. Then $\mathbf{f} = (f_k)_{k=0}^\infty \in \ell^1(\mathbb{N})$, since

$$\|\mathbf{f}\|_1 = \sum_{k=0}^{\infty} |f_k| = \sum_{k=0}^{\infty} \left| \sum_{j=1}^{N} a_j z_j^k \right| \leqslant \sum_{j=1}^{N} \left(\sum_{k=0}^{\infty} \left| a_j z_j^k \right| \right) = \sum_{j=1}^{N} \frac{|a_j|}{1 - |z_j|} < \infty. \qquad (2.3)$$

First, we recall the following property of the corresponding infinite Hankel matrix $\mathbf{\Gamma_f}$, the so-called Kronecker's Theorem. Note that there exist different formulations of the following theorem, for instance Theorem 16.13 in [55] or Theorem 8.19 in [24], which contain more or less extended statements. We present here a shortened version, which fits best in our setting.

Theorem 2.3. (Kronecker's Theorem). *The Hankel operator* $\mathbf{\Gamma_f} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ *generated by* $\mathbf{f} = (f_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$ *of the form* (1.1) *has finite rank* N.

Proof. If $\mathbf{f}$ can be written in the form (1.1), we define the characteristic polynomial (Prony polynomial)

$$P(z) := \prod_{j=1}^{N}(z - z_j) = \sum_{k=0}^{N} b_k z^k, \tag{2.4}$$

where $b_N = 1$. Then

$$\sum_{l=0}^{N} b_l f_{k+l} = \sum_{l=0}^{N} b_l \sum_{j=1}^{N} a_j z_j^{k+l} = \sum_{j=1}^{N} a_j z_j^k \left(\sum_{l=0}^{N} b_l z_j^l \right) = 0 \tag{2.5}$$

for all $k \in \mathbb{N}$, i.e., the $(N+k)$-th column of $\mathbf{\Gamma_f}$ is a linear combination of the N preceding columns. Thus Rank $\mathbf{\Gamma_f} \leqslant N$. Now we want to show that Rank $\mathbf{\Gamma_f} = N$. Due to the structure (1.1) of $\mathbf{f}$ for the truncated Hankel matrix we have the factorization

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_{N-1} \\ f_1 & f_2 & \cdots & f_N \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1} & f_N & \cdots & f_{2N-2} \end{pmatrix} = \mathbf{V}\mathbf{C}\mathbf{V}^{\top},$$

where $\mathbf{V}$ is a Vandermonde matrix given by

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}$$

and $\mathbf{C} = \operatorname{diag}(a_1, \ldots, a_N)$ is the diagonal matrix containing the coefficients a_j. Since all z_j are in $\mathbb{D} \backslash \{0\}$ and both, z_j and a_j are pairwise different, the matrices $\mathbf{V}$ and $\mathbf{C}$ have full rank. Thus the first N columns (and rows) of $\mathbf{\Gamma_f}$ are linearly independent and therefore Rank $\mathbf{\Gamma_f} = N$. □

In order to examine the converse of the above theorem we need some insights into the theory of difference equations. An equation of the form

$$f_{j+N} + c_{N-1} f_{j+N-1} + \cdots + c_1 f_{j+1} + c_0 f_j = 0, \quad j = 0, 1, 2, \ldots, \tag{2.6}$$

where the coefficients c_k are real or complex numbers is called a *homogeneous difference equation of order* N. The equivalent sum notation of (2.6) is given by

$$f_{j+N} = -\sum_{k=0}^{N-1} c_k f_{j+k}, \quad j = 0, 1, 2, \dots . \tag{2.7}$$

The polynomial with the same coefficients as in (2.7),

$$P_{\mathbf{c}}(x) := \sum_{k=0}^{N} c_k x^k \quad \text{for } c_N = 1, \tag{2.8}$$

is called the *characteristic polynomial* of the difference equation (2.7). The above difference equation has obviously the trivial solution $f_j \equiv 0$ for all $j \in \mathbb{N}$. To find further solutions of the homogeneous equation we choose the ansatz $f_j = z^j$ for some $z \in \mathbb{C}$, $z \neq 0$. Then with $c_N = -1$ we obtain the identity

$$\sum_{k=0}^{N} c_k f_{j+k} = \sum_{k=0}^{N} c_k z^{j+k} = z^j \sum_{k=0}^{N} c_k z^k = z^k P_{\mathbf{c}}(z),$$

which is zero if and only if $P_{\mathbf{c}}(z)$ is zero. Thus $f_j = z^j$ is a non-trivial solution of the homogeneous equation (2.7) if and only if z is a root of the characteristic polynomial $P_{\mathbf{c}}(x)$. In the following theorem, which can be found as Theorem 1.1 in [4], the generalization of this statement is presented.

Theorem 2.4. *Let $z_1, \dots, z_N$ be distinct roots of $P_{\mathbf{c}}(x)$. Then*

$$f_k = \sum_{j=0}^{N} a_j z_j^k \tag{2.9}$$

is the unique solution of the homogeneous equation (2.7)*, where a_j are some real or complex coefficients.*

Now we assume that the infinite Hankel matrix $\mathbf{\Gamma_f}$ has rank N. This means that the $(N+1)$-th column (or row) of $\mathbf{\Gamma_f}$ is a linear combination of the N previous columns (or rows). Due to the structure of $\mathbf{\Gamma_f}$ we easily see that in this case $\mathbf{f}$ satisfies a difference equation of order N of the above form. Assuming that the zeros z_j, $j = 1, \dots, N$ of the characteristic polynomial $P_{\mathbf{c}}(x)$ are pairwise different, the above theorem yields that $\mathbf{f}$ can be written in the form (1.1). Note that the zeros have modulus smaller than 1, since $\mathbf{f}$ has been assumed to be in $\ell^1(\mathbb{N})$, as has been shown in (2.3).

2.1.3. Con-diagonalization of matrices

The idea of the con-diagonalization arises in analogy to the unitary diagonalization of hermitian matrices and resp. compact self-adjoint operators. In the following we will apply the concept of con-similarity and con-diagonalization

for finite matrices, see e.g. [8]. We begin with the following definition.
For a matrix $\mathbf{A} \in \mathbb{C}^{N\times N}$ we call $\lambda \in \mathbb{C}$ a *con-eigenvalue* with the corresponding *con-eigenvector* $\mathbf{v} \in \mathbb{C}^N$ if it satisfies

$$\mathbf{A}\overline{\mathbf{v}} = \lambda\,\mathbf{v}. \tag{2.10}$$

Observe that for a pair $(\lambda, \mathbf{v})$ satisfying (2.10) it also holds

$$\mathbf{A}(\overline{\mathrm{e}^{\mathrm{i}\alpha}\mathbf{v}}) = \mathrm{e}^{-\mathrm{i}\alpha}\mathbf{A}\overline{\mathbf{v}} = (\mathrm{e}^{-\mathrm{i}\alpha}\lambda)\mathbf{v} = (\mathrm{e}^{-2\mathrm{i}\alpha}\lambda)(\mathrm{e}^{\mathrm{i}\alpha}\mathbf{v})$$

for all $\alpha \in \mathbb{R}$. Thus, for each con-eigenvalue λ of $\mathbf{A}$ we can find a corresponding real non-negative con-eigenvalue $\sigma = |\lambda|$ by this rotation trick. We will always restrict the con-eigenvalues to their unique non-negative representatives. Note that this concept can also be found in the literature as Takagi factorization due to [53] and can be applied to infinite matrices defining a bounded operator on $\ell^2(\mathbb{N})$, such as $\mathbf{\Gamma_f}$, see, e.g. Section 4.2 in [15].
In the following we explore the connection between con-diagonalization and the singular value decomposition (SVD). For that purpose we distinguish between symmetric matrices (such as Hankel matrices) and non-symmetric matrices.

Con-diagonalization vs. SVD for symmetric matrices

First, it can be simply observed that a symmetric infinite Hankel matrix $\mathbf{\Gamma_f}$ with finite rank is a compact operator and unitarily con-diagonalizable, see [29]. Since $\mathbf{\Gamma_f}\overline{\mathbf{v}} = \lambda\,\mathbf{v}$ implies

$$(\mathbf{\Gamma_f}\mathbf{\Gamma_f^*})\mathbf{v} = \mathbf{\Gamma_f}\overline{\mathbf{\Gamma_f}\overline{\mathbf{v}}} = \lambda\mathbf{\Gamma_f}\overline{\mathbf{v}} = |\lambda|^2\mathbf{v},$$

we directly conclude that the non-negative con-eigenvalues and con-eigenvectors of $\mathbf{\Gamma_f}$ are also singular values and singular vectors of $\mathbf{\Gamma_f}$, respectively.
Conversely, let σ be a singular value of $\mathbf{\Gamma_f}$ with geometric multiplicity 1 and $\mathbf{0} \neq \mathbf{v} \in \ell^2(\mathbb{N})$ the corresponding left singular vector, i.e.

$$\mathbf{\Gamma_f}\mathbf{\Gamma_f^*}\mathbf{v} = \mathbf{\Gamma_f}\overline{\mathbf{\Gamma}}_\mathbf{f}\mathbf{v} = \sigma^2\mathbf{v}.$$

Then $\mathbf{\Gamma_f}\overline{\mathbf{v}}$ is also a singular vector of $\mathbf{\Gamma_f}$ corresponding to the same singular value σ, since

$$\mathbf{\Gamma_f}\mathbf{\Gamma_f^*}(\mathbf{\Gamma_f}\overline{\mathbf{v}}) = \mathbf{\Gamma_f}\overline{\mathbf{\Gamma}}_\mathbf{f}(\mathbf{\Gamma_f}\overline{\mathbf{v}}) = \mathbf{\Gamma_f}(\overline{\mathbf{\Gamma_f}\overline{\mathbf{\Gamma}}_\mathbf{f}\mathbf{v}}) = \sigma^2\,\mathbf{\Gamma_f}\overline{\mathbf{v}}$$

and therefore it holds $\mathbf{\Gamma_f}\overline{\mathbf{v}} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}$. On the other hand we have

$$\sigma^2\mathbf{v} = \mathbf{\Gamma_f}\overline{\mathbf{\Gamma}}_\mathbf{f}\mathbf{v} = \bar{\lambda}\mathbf{\Gamma_f}\overline{\mathbf{v}} = |\lambda|^2\mathbf{v}$$

and hence $|\lambda| = \sigma$.

Remark 2.5. Note that if we start with a right singular vector of $\mathbf{\Gamma_f}$, we use the singular value equation with the matrix $\mathbf{\Gamma_f^*\Gamma_f}$. In this case we can show analogously that $\overline{\mathbf{\Gamma_f v}}$ satisfies the singular value equation

$$\mathbf{\Gamma_f^*\Gamma_f}(\overline{\mathbf{\Gamma_f v}}) = \sigma^2\,\overline{\mathbf{\Gamma_f v}}$$

and end up with the con-eigenvalue equation of the form

$$\mathbf{\Gamma_f v} = \lambda\overline{\mathbf{v}},$$

which can be used as an alternative definition for (2.10).

△

Con-diagonalization vs. SVD for non-symmetric matrices

In the case of non-symmetric matrices the sufficient condition for the existence of con-eigenvalues and con-eigenvectors is given in Theorem 4.6.6 in [29], namely the existence of non-negative eigenvalues of $\overline{\mathbf{A}}\mathbf{A}$ with the corresponding eigenvectors. The theorem also states how to determine the con-eigenvalues and vectors of $\mathbf{A}$ from the non-zero eigenvalues and eigenvectors of $\overline{\mathbf{A}}\mathbf{A}$. Let $(\sigma, \mathbf{w})$ be an eigenpair of $\overline{\mathbf{A}}\mathbf{A}$ with $\sigma \neq 0$, then $(\lambda, \mathbf{v})$ is a con-eigenpair of $\mathbf{A}$, where $\lambda = \sqrt{\sigma}$ and $\mathbf{v} := \mathbf{A}\mathbf{w} + \lambda\overline{\mathbf{w}}$, since

$$\begin{aligned}\mathbf{A}\overline{\mathbf{v}} &= \mathbf{A}(\overline{\mathbf{A}\mathbf{w}} + \lambda\mathbf{w})\\ &= \overline{\overline{\mathbf{A}}\mathbf{A}\mathbf{w}} + \lambda\mathbf{A}\mathbf{w}\\ &= \lambda^2\overline{\mathbf{w}} + \lambda\mathbf{A}\mathbf{w} = \lambda\mathbf{v}.\end{aligned}$$

2.1.4. Circulant matrices

In this section we consider a special class of finite matrices, the so-called circulant matrices. These matrices are strongly related to finite Hankel- and Toeplitz matrices, as we will see below.
Let $L \in \mathbb{N}$ and $\mathbf{c} = (c_j)_{j=0}^{L-1}$ be a vector in $\mathbb{C}^L$. We call a matrix $\mathbf{C} \in \mathbb{C}^{L\times L}$ circulant if it is of the form

$$\mathbf{C} = \mathrm{circ}(\mathbf{c}) := \begin{pmatrix} c_0 & c_1 & & \cdots & c_{L-1} \\ c_{L-1} & c_0 & c_1 & & \vdots \\ & c_{L-1} & c_0 & \ddots & \\ \vdots & & \ddots & \ddots & c_1 \\ c_1 & \cdots & & c_{L-1} & c_0 \end{pmatrix} = \left(c_{(k-j)\,\mathrm{mod}\,L}\right)_{j,k=0}^{L-1}.$$

Note that $\mathrm{circ}(\mathbf{c})$ is a special case of a finite Toeplitz matrix, since it has identical entries on its diagonals. Let

$$\mathbf{U} := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & 1 \\ \vdots & & \iddots & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix} \tag{2.11}$$

be the flip matrix, which gives us a certain permutation of the columns or rows by a right or left multiplication. Then we observe that

$$\mathbf{UC} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{L-1} \\ c_1 & c_2 & c_3 & \cdots & c_0 \\ c_2 & c_3 & c_4 & \cdots & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_0 & c_1 & \cdots & c_{L-2} \end{pmatrix}$$

has Hankel structure as well as

$$\mathbf{CU} = \begin{pmatrix} c_0 & c_{L-1} & c_{L-2} & \cdots & c_1 \\ c_{L-1} & c_{L-2} & c_{L-3} & \cdots & c_0 \\ c_{L-2} & c_{L-3} & c_{L-4} & \cdots & c_{L-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_0 & c_{L-1} & \cdots & c_2 \end{pmatrix}.$$

Note that $\mathbf{UU} = \mathbf{I}$ and thus $\mathbf{UUC}$ is circulant again. The properties of circulant matrices are well-studied and can be found in [17]. However, one of the most interesting and useful properties, in particular for our algorithm presented in Chapter 5, can be summarized as follows.

Proposition 2.6. *Let $\mathbf{C}$ be a circulant matrix as above. Then it can be diagonalized by the Fourier matrix $\mathbf{F}_L$ of size $L \times L$, defined as*

$$\mathbf{F}_L := \frac{1}{\sqrt{L}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_L & \omega_L^2 & \cdots & \omega_L^{L-1} \\ 1 & \omega_L^2 & \omega_L^4 & \cdots & \omega_L^{2(L-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_L^{L-1} & \omega_L^{2(L-1)} & \cdots & \omega_L^{(L-1)(L-1)} \end{pmatrix} = \left(\omega_L^{ij}\right)_{i,j=0}^{L-1},$$

where $\omega_L := \mathrm{e}^{-2\mathrm{i}\pi/L}$ is the L-th primitive root of unity. It holds

$$\mathbf{C} = \mathbf{F}_L^* \mathbf{D} \mathbf{F}_L, \tag{2.12}$$

where $\mathbf{D}$ is a diagonal matrix containing the eigenvalues of $\mathbf{C}$, which are given by

$$\lambda_j = \sum_{l=0}^{L-1} c_l \omega_L^{-lj} \qquad \text{for } j = 0, \ldots, L-1.$$

Proof. We show (2.12) using the definition of $\mathbf{F}_L$ and the assumed form of the eigenvalues λ_j. We obtain

$$\begin{aligned}
\mathbf{F}_L^* \mathbf{D} \mathbf{F}_L &= \overline{\mathbf{F}}_L \mathbf{D} \mathbf{F}_L \\
&= \frac{1}{L} (\omega_L^{-ij})_{i,j=0}^{L-1} \cdot \mathrm{diag}(\lambda_0, \ldots, \lambda_{L-1}) \cdot (\omega_L^{jk})_{j,k=0}^{L-1} \\
&= \frac{1}{L} (\lambda_j \omega_L^{-ij})_{i,j=0}^{L-1} \cdot (\omega_L^{jk})_{j,k=0}^{L-1} \\
&= \frac{1}{L} \left(\sum_{j=0}^{L-1} \lambda_j \omega_L^{-ij} \omega_L^{jk} \right)_{i,k=0}^{L-1} \\
&= \frac{1}{L} \left(\sum_{j=0}^{L-1} \sum_{l=0}^{L-1} c_l \omega_L^{-lj} \omega_L^{j(k-i)} \right)_{i,k=0}^{L-1} \\
&= \frac{1}{L} \left(\sum_{l=0}^{L-1} c_l \sum_{j=0}^{L-1} \omega_L^{j(k-i-l)} \right)_{i,k=0}^{L-1}.
\end{aligned}$$

Due to the properties of roots of unity we have for every term in the sum

$$c_l \sum_{j=0}^{L-1} \omega_L^{j(k-i-l)} = \begin{cases} Lc_l, & k-i \equiv l \mod L, \\ 0, & \text{else} \end{cases}$$

for $l = 0, \ldots, L-1$ and thus we obtain

$$\mathbf{F}_L^* \mathbf{D} \mathbf{F}_L = (c_{(k-i) \mathrm{mod} L})_{i,k=0}^{L-1} = \mathbf{C}.$$

□

2.2. Continuous Setting

2.2.1. L^2 on the unit circle and Hardy spaces

In this work L^2 denotes the Hilbert space of square integrable functions on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| = 1\}$,

$$L^2 := \left\{ \phi : \partial\mathbb{D} \to \mathbb{C} \ : \ \int_0^{2\pi} |\phi(\mathrm{e}^{\mathrm{i}t})|^2 \mathrm{d}t < \infty \right\}$$

with the inner product

$$\langle \phi, \psi \rangle_{L^2} := \frac{1}{2\pi} \int_0^{2\pi} \phi(\mathrm{e}^{\mathrm{i}t}) \overline{\psi(\mathrm{e}^{\mathrm{i}t})} \mathrm{d}t,$$

where $\phi, \psi \in L^2$. Note that L^2 can be identified with the space $L^2[0, 2\pi)$ via $z = \mathrm{e}^{\mathrm{i}t}$. Thus, due to the standard theory of Fourier analysis, every function $\psi \in L^2$ can be written as a *Fourier series* on $\partial\mathbb{D}$

$$\psi(\mathrm{e}^{\mathrm{i}t}) = \sum_{k=-\infty}^{\infty} \hat{\psi}(k) \mathrm{e}^{\mathrm{i}tk},$$

where the Fourier coefficients $\hat{\psi}(k)$ are given by the Fourier transform

$$\mathcal{F} : L^2 \mapsto l^2(\mathbb{Z}),$$
$$(\mathcal{F}\psi)(k) := \hat{\psi}(k) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\mathrm{e}^{\mathrm{i}t}) \mathrm{e}^{-\mathrm{i}tk} \mathrm{d}t.$$

The transmission between the function domain and the Fourier domain will become crucial for the understanding of the objects of this section.
Further we introduce the space L^∞ of essentially bounded Lebesgue-measurable $\mathbb{C}$-valued functions on $\partial\mathbb{D}$,

$$L^\infty := \left\{ \phi : \partial\mathbb{D} \to \mathbb{C} \ : \ \operatorname*{ess\,sup}_{|z|=1} |\phi(z)| < \infty \right\}.$$

Essentially bounded means that it is bounded on the complement of a set of measure zero. Note that L^∞ is a Banach space equipped with the essential supremum norm

$$\|\phi\|_{L^\infty} := \operatorname*{ess\,sup}_{|z|=1} |\phi(z)|.$$

Furthermore, L^∞ is a proper subspace of L^2 and it holds

$$\|\phi\|_{L^2} \leqslant \|\phi\|_{L^\infty} \qquad \forall \ \phi \in L^\infty.$$

By H^2 we denote the so-called *Hardy space*. There exist two different possibilities to define H^2. The first definition is based on the Fourier transform above.

We define H^2 as a subspace of L^2, which contains functions having only Fourier coefficients with non-negative index, i.e.

$$H^2 := \left\{ \phi \in L^2 \ : \ \phi(\mathrm{e}^{\mathrm{i}t}) = \sum_{k=0}^{\infty} \hat{\phi}(k)\mathrm{e}^{\mathrm{i}tk} \right\}.$$

Such functions on the unit circle $\partial\mathbb{D}$ can be naturally identified with the power series $\sum_{k=0}^{\infty} \hat{\phi}(k) z^k$ on the open unit disk $\mathbb{D}$. Considering z as an independent variable, such power series defines an analytic function in $\mathbb{D}$. This leads to an alternative definition of the Hardy space, namely the space of analytic functions on $\mathbb{D}$ with

$$\|\phi\|_{H^2} := \sup_{r\in(0,1)} \frac{1}{2\pi} \int_0^{2\pi} |\phi(r\mathrm{e}^{\mathrm{i}t})|^2 \mathrm{d}t < \infty.$$

The transmission from the open unit disk to the unit circle is obtained by Fatou's theorem, which states that for $\phi \in \mathbb{D}$ the radial limit

$$\tilde{\phi}(\mathrm{e}^{\mathrm{i}t}) = \lim_{r\to 1} \phi(r\mathrm{e}^{\mathrm{i}t})$$

exists almost everywhere for $t \in (0, 2\pi]$. This result can be found e.g. in [55], Theorem 13.10. Indeed $\tilde{\phi}$ is a function in L^2. In the following we will make no distinction between the analytic function ϕ in $\mathbb{D}$ and the corresponding limit function $\tilde{\phi}$, which has its domain on the unit circle $\partial\mathbb{D}$. Note that due to the power series representation of functions in H^2 it can be easily seen that the set $\{1, z, z^2, \ldots\}$ with $z = \mathrm{e}^{\mathrm{i}t}$ defines the standard orthonormal basis in H^2.
Using the Parseval's identity, we consider the scalar product on H^2 in the Fourier domain, that is, for two functions

$$\phi(z) := \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \psi(z) := \sum_{k=0}^{\infty} b_k z^k$$

in H^2 we have

$$\langle \phi, \psi \rangle_{H^2} := \sum_{k=0}^{\infty} a_k \overline{b_k}.$$

For further properties of L^2, H^2 and L^∞ defined above we refer to the Section 13 in [55].

2.2.2. On rational functions in L^2 and their Fourier series

Let q and p be polynomials, which have no common zeros and p is not the zero polynomial. Further let p have only simple zeros $z_1, \ldots, z_N$ and be monic, i.e. the coefficient corresponding to the largest power of z is one. We denote by R the set of zeros of p and define a rational function $\varphi : \mathbb{C}\backslash R \mapsto \mathbb{C}$ as a function of the form

$$\varphi(z) = \frac{q(z)}{p(z)}, \tag{2.13}$$

The function φ is said to have degree n if $\max\{\deg(q), \deg(p)\} = n$. In the following we want to study rational functions contained in L^2.
Rational functions of the form (2.13) with $\deg(q) \leqslant \deg(p) = N$ admit the unique partial fraction expansion

$$\varphi(z) = \sum_{j=1}^{N} \frac{a_j}{z - z_j},$$

where z_j, $j = 1, \ldots, N$, are zeros of p and $a_j \in \mathbb{C}$ some constants. If $M = \deg(q) \geqslant \deg(p) = N$, then every rational function of the above form can be written as

$$\varphi(z) = r(z) + \frac{\tilde{q}(z)}{p(z)}$$

where $r(z)$ is a polynomial and $\deg(\tilde{q}) = N - 1$. Thus we obtain the generalized partial fraction expansion

$$\varphi(z) = r(z) + \sum_{j=1}^{N} \frac{a_j}{z - z_j}, \tag{2.14}$$

which turns out to be very useful for handling rational functions, particularly for integration.
Obviously polynomials of finite degree N

$$p(z) = \sum_{k=0}^{N} a_k z^k,$$

are contained in L^2. Since $p(z)$ has only non-negative Fourier coefficients $\hat{p}(k) = a_k$, $k = 0, \ldots, N$, it also belongs to H^2. We pass over to the rational functions on L^2 with poles and use the following useful property.

Proposition 2.7. *A rational function which belongs to L^2 has no pole on $\partial\mathbb{D}$.*

Proof. Assume that φ is a rational function in L^2 of the form (2.13) which has one pole $z_K := \mathrm{e}^{\mathrm{i}\gamma}$, $K \in \{1, \ldots, N\}$ on $\partial\mathbb{D}$. Then there exists an interval $I := [\gamma - \varepsilon, \gamma + \varepsilon]$ where the numerator $|q(\mathrm{e}^{\mathrm{i}t})|$ is bounded for $t \in I$ and we can find a constant c_1 such that $0 \neq |q(\mathrm{e}^{\mathrm{i}t})|^2 \geqslant c_1$, $t \in I$. Furthermore for the denominator we have $p(\mathrm{e}^{\mathrm{i}t}) = (\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}\gamma})\tilde{p}(\mathrm{e}^{\mathrm{i}t})$, where $\tilde{p}$ is a polynomial with $0 \neq |\tilde{p}(\mathrm{e}^{\mathrm{i}t})|^2 \leqslant c_2$ on I for some $c_2 \in \mathbb{R}$. For the L^2-norm of φ we obtain

$$\begin{aligned}
\int_0^{2\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}t})\right|^2 \mathrm{d}t &\geqslant \int_I \left|\frac{q(\mathrm{e}^{\mathrm{i}t})}{p(\mathrm{e}^{\mathrm{i}t})}\right|^2 \mathrm{d}t \geqslant c_1 \int_I \frac{1}{|\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}\gamma}|^2 |\tilde{p}(\mathrm{e}^{\mathrm{i}t})|^2} \mathrm{d}t \\
&\geqslant \frac{c_1}{c_2} \int_I \frac{1}{|\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}\gamma}|^2} \mathrm{d}t = \frac{c_1}{c_2} \int_I \frac{1}{|\mathrm{e}^{\mathrm{i}(t-\gamma)} - 1|^2} \mathrm{d}t \\
&= \frac{c_1}{c_2} \int_{-\varepsilon}^{\varepsilon} \frac{1}{|\mathrm{e}^{\mathrm{i}t} - 1|^2} \mathrm{d}t = \frac{c_1}{c_2} \int_{-\varepsilon}^{\varepsilon} \frac{1}{2 - \mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}} \mathrm{d}t \\
&= \frac{c_1}{2c_2} \int_{-\varepsilon}^{\varepsilon} \frac{1}{1 - \cos t} \mathrm{d}t = \frac{c_1}{2c_2} \int_{-\varepsilon}^{\varepsilon} \frac{(1 + \cos t)}{(1 - \cos t)(1 + \cos t)} \mathrm{d}t \\
&= \frac{c_1}{2c_2} \int_{-\varepsilon}^{\varepsilon} \frac{1 + \cos t}{\sin^2 t} \mathrm{d}t \geqslant \frac{c_1}{2c_2} \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sin^2 t} \mathrm{d}t,
\end{aligned}$$

where the last inequality holds due to the fact that cosine is positive around zero. Since the last integral is known to be divergent, it holds that $\varphi \notin L^2$. □

In this section we would like to characterize those rational functions which belong to the Hardy space H^2 and those which belong to L^2 but not to H^2. As we will see below, the crucial part of this characterization is about the question whether the poles of the function lie in $\mathbb{D}$ or not.
In the following two examples we consider "simple" rational functions with only one pole inside or outside of $\mathbb{D}$ first, trying to understand some simple rules about their Fourier coefficients. Afterwards we extend our results to general rational function of the form (2.13).

Example 2.8. Consider the function with one pole in $\mathbb{D}$

$$\varphi_1(z) := \frac{1}{z-\alpha}, \qquad \alpha \in \mathbb{D}.$$

Since $|\alpha e^{-it}| < 1$ for all $t \in [0, 2\pi)$, the Fourier coefficients of φ_1 are of the form

$$\begin{aligned}(\mathcal{F}\varphi_1)(k) = \hat{\varphi}_1(k) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikt}}{e^{it}-\alpha} dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikt}e^{-it}}{1-e^{-it}\alpha} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+1)t} \sum_{j=0}^{\infty} (\alpha e^{-it})^j dt \\ &= \sum_{j=0}^{\infty} \alpha^j \left[\frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+j+1)t} dt\right] \\ &= \begin{cases} \alpha^{-k-1}, & k \leqslant -1, \\ 0, & k \geqslant 0, \end{cases}\end{aligned}$$

since

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+j+1)t} dt = \delta_{k,-j-1} = \begin{cases} 1, & k+j+1 = 0, \\ 0 & \text{else.} \end{cases}$$

Thus φ_1 has only negative Fourier coefficients and does not belong to H^2. Furthermore, it holds for $z \in \partial\mathbb{D}$

$$|\varphi_1(z)| = \left|\sum_{k=-\infty}^{-1} \alpha^{-k-1} z^k\right| \leqslant \sum_{k=-\infty}^{-1} |\alpha|^{-k-1} = \sum_{k=0}^{\infty} |\alpha|^k = \frac{1}{1-|\alpha|}.$$

It follows that $\|\varphi_1\|_{L^\infty} < \infty$ and φ_1 is a function in L^∞. The sequence of Fourier coefficients $(\hat{\varphi}_1(k))_{k=-\infty}^{\infty}$ of φ_1 is in $\ell^1(\mathbb{Z})$, since

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}_1(k)| = \sum_{k=-\infty}^{-1} |\alpha|^{-k-1} = \sum_{k=0}^{\infty} |\alpha|^k = \frac{1}{1-|\alpha|} < \infty.$$

Note that if a rational function is given by the sum of partial fractions

$$\varphi(z) := \sum_{j=1}^{N} \frac{c_j}{z-\alpha_j} =: \sum_{j=1}^{N} \varphi_j(z), \qquad c_j \in \mathbb{C},\ \alpha_j \in \mathbb{D},$$

then the Fourier coefficients of φ are given by

$$(\mathcal{F}\varphi)(k) = \hat{\varphi}(k) = \sum_{j=1}^{N} \hat{\varphi}_j(k) = \begin{cases} \sum_{j=1}^{N} c_j \alpha_j^{-k-1}, & k \leqslant -1, \\ 0, & k \geqslant 0. \end{cases}$$

and also belong to $\ell^1(\mathbb{Z})$. Since φ is a finite sum of L^∞ functions, it clearly belongs to L^∞ as well.

△

Example 2.9. Consider the function with one pole in $\mathbb{C}\backslash\overline{\mathbb{D}}$

$$\varphi_2(z) := \frac{1}{z-\beta}, \qquad \beta \in \mathbb{C}\backslash\overline{\mathbb{D}}.$$

Since $|\beta^{-1}\mathrm{e}^{\mathrm{i}t}| < 1$ for all $t \in [0, 2\pi)$, the Fourier coefficients of φ_2 are given by

$$\begin{aligned}
(\mathcal{F}\varphi_2)(k) = \hat{\varphi}_2(k) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}kt}}{\mathrm{e}^{\mathrm{i}t}-\beta}\mathrm{d}t = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\beta^{-1}\mathrm{e}^{-\mathrm{i}kt}}{1-\beta^{-1}\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \beta^{-1}\mathrm{e}^{-\mathrm{i}kt}\sum_{j=0}^{\infty}(\beta^{-1}\mathrm{e}^{\mathrm{i}t})^j \mathrm{d}t \\
&= -\sum_{j=0}^{\infty}\beta^{-j-1}\left[\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}(k-j)t}\mathrm{d}t\right] \\
&= -\sum_{j=0}^{\infty}\beta^{-j-1}\delta_{j,k} \\
&= \begin{cases} 0, & k \leqslant -1, \\ -\beta^{-k-1}, & k \geqslant 0. \end{cases}
\end{aligned}$$

Thus φ_2 has only non-negative Fourier coefficients and is an H^2 function. Moreover, for $z \in \partial\mathbb{D}$ we have

$$|\varphi_2(z)| = \left|\sum_{k=0}^{\infty} -\beta^{-k-1}z^k\right| \leqslant \sum_{k=0}^{\infty}|\beta|^{-k-1} = \sum_{k=1}^{\infty}\left(\frac{1}{|\beta|}\right)^k = 1 + \frac{1}{1-1/|\beta|},$$

since $1/|\beta| \in \mathbb{D}$ and therefore the function φ_2 is in L^∞. In analogy to the previous example it follows that the sequence of Fourier coefficients $(\hat{\varphi}_2(k))_{k=-\infty}^{\infty}$ of φ_2 is in $\ell^1(\mathbb{Z})$.

△

A very important special case of a rational function with only one pole in $\mathbb{C}\backslash\overline{\mathbb{D}}$ is

$$\mathcal{K}_\alpha(z) := \frac{1}{1-\overline{\alpha}z}, \qquad \alpha \in \mathbb{D}.$$

It is easy to see that $\mathcal{K}_\alpha$ can be written as

$$\mathcal{K}_\alpha(z) = -\overline{\alpha}^{-1}\frac{1}{z-\overline{\alpha}^{-1}}.$$

Setting $\beta = \overline{\alpha}^{-1} \in \mathbb{C}\backslash\overline{\mathbb{D}}$ in the above computation we obtain for the Fourier coefficients

$$(\mathcal{F}\mathcal{K}_\alpha)(k) = \hat{\mathcal{K}}_\alpha(k) = \begin{cases} 0, & k \leqslant -1, \\ \overline{\alpha}^k & k \geqslant 0. \end{cases}$$

$\mathcal{K}_\alpha$ is frequently called the *reproducing kernel* of H^2, since every function $\phi \in H^2$ can be represented at a fixed value z_0 as a scalar product with $\mathcal{K}_{z_0}$

$$\phi(z_0) = \sum_{j=0}^{\infty} \hat{\phi}(j) z_0^j = \langle \hat{\phi}, \hat{\mathcal{K}}_{z_0} \rangle_{l^2(\mathbb{N})} = \langle \phi, \mathcal{K}_{z_0} \rangle_{H^2}. \tag{2.15}$$

Note that the last equality holds due to the Parseval's equation with scalar product on $\ell^2(\mathbb{N})$ defined as in (2.1). Using the examples above and the partial fraction decomposition (2.14) we can easily see that negative Fourier coefficients only appear if the function φ has poles z_j in $\mathbb{D}$, since

$$\hat{\varphi}(k) = \hat{r}(k) + \sum_{j=1}^{n} a_j \left(\frac{1}{\cdot - z_j} \right)^{\wedge} (k) = 0 \quad \text{for } k \leqslant -1$$

holds if and only if $z_j \in \mathbb{C}\backslash\overline{\mathbb{D}}$ for all $j = 1, \ldots, n$. We sum up this result in the following proposition.

Proposition 2.10. *A rational function of the form (2.13), which belongs to L^2 is in H^2 if and only if it has no poles in $\mathbb{D}$.*

Summarizing the above considerations we conclude: a rational function of degree n possesses only Fourier coefficients with non-negative index if it has no poles in $\mathbb{D}$. On the other hand, it possesses only Fourier coefficients with negative index if it has no poles in $\mathbb{C}\backslash\mathbb{D}$.

2.2.3. Hankel operator on L^2

We define the projection operator P_- as a mapping from L^2 to the orthogonal complement of H^2 in L^2. It maps every two-sided infinite Laurent series to its part with negative indexed coefficients, which is called *principal part*, that is

$$\mathrm{P}_- : L^2 \mapsto L^2 \ominus H^2, \qquad \mathrm{P}_- \left(\sum_{j=-\infty}^{\infty} a_j z^j \right) = \sum_{j=-\infty}^{-1} a_j z^j. \tag{2.16}$$

In the following we generally assume φ to be a function in $L^\infty \subseteq L^2$ and with Fourier coefficients $(\hat{\varphi}_k)_{k=-\infty}^{\infty}$ in $\ell^1(\mathbb{Z})$, if not defined otherwise. We define the multiplication operator induced by φ as

$$M_\varphi : L^2 \mapsto L^2, \qquad M_\varphi \vartheta := \varphi\vartheta.$$

Note that $\varphi\vartheta \in L^2$ for all $\vartheta \in L^2$ and M_φ indeed maps to L^2 as has been shown in Theorem 13.14 in [55]. Now we define the Hankel operator with respect to

$\varphi \in L^\infty$ as the projection P_- of the product with φ to the negative part of the Laurent polynomial

$$H_\varphi : H^2 \mapsto L^2 \ominus H^2, \qquad H_\varphi \vartheta := \mathrm{P}_-(M_\varphi \vartheta) = \mathrm{P}_-(\varphi\vartheta). \tag{2.17}$$

Let P_+ be the projection from L^2 to H^2, which preserves only non-negative coefficients, that is

$$\mathrm{P}_+ : L^2 \to H^2, \qquad \mathrm{P}_+\left(\sum_{j=-\infty}^{\infty} a_j z^j\right) = \sum_{j=0}^{\infty} a_j z^j. \tag{2.18}$$

Note that $\mathrm{P}_+ + \mathrm{P}_- = \mathrm{Id}$. Now we can show the following lemma about the adjoint Hankel operator.

Lemma 2.11. *Let H_φ be the Hankel operator as defined above. Then the adjoint operator $H_\varphi^* : L^2 \ominus H^2 \to H^2$ is given by*

$$H_\varphi^* \vartheta = \mathrm{P}_+(M_{\overline{\varphi}} \vartheta) = \mathrm{P}_+(\overline{\varphi}\vartheta). \tag{2.19}$$

Proof. We recall that the adjoint operator to H_φ is defined as an operator $H_\varphi^* : L^2 \ominus H^2 \to H^2$, for which it holds

$$\langle H_\varphi \vartheta, \xi \rangle_{L^2} = \langle \vartheta, H_\varphi^* \xi \rangle_{L^2}$$

for two arbitrary functions $\vartheta \in H^2$ and $\xi \in L^2 \ominus H^2$. First we compute the adjoint operator to the multiplication operator M_φ for some $\varphi \in L^\infty$. We find

$$\begin{aligned}
\langle M_\varphi \vartheta, \xi \rangle_{L^2} &= \langle \varphi\vartheta, \xi \rangle_{L^2} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}t}) \vartheta(\mathrm{e}^{\mathrm{i}t}) \overline{\xi(\mathrm{e}^{\mathrm{i}t})} \mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^{2\pi} \vartheta(\mathrm{e}^{\mathrm{i}t}) \overline{\overline{\varphi(\mathrm{e}^{\mathrm{i}t})} \xi(\mathrm{e}^{\mathrm{i}t})} \mathrm{d}t \\
&= \langle \vartheta, \overline{\varphi}\xi \rangle_{L^2} = \langle \vartheta, M_{\overline{\varphi}} \xi \rangle_{L^2}.
\end{aligned}$$

Thus the adjoint to M_φ is given by $M_\varphi^* = M_{\overline{\varphi}}$. Now, since ϑ has only Fourier coefficients with non-negative indices and ξ those with only negative indices, for the Hankel operator it holds

$$\begin{aligned}
\langle H_\varphi \vartheta, \xi \rangle_{L^2} &= \langle \mathrm{P}_-(M_\varphi \vartheta), \xi \rangle_{L^2} \\
&= \langle (\mathrm{Id} - \mathrm{P}_+)(M_\varphi \vartheta), \xi \rangle_{L^2} \\
&= \langle M_\varphi \vartheta, \xi \rangle_{L^2} - \langle \mathrm{P}_+(M_\varphi \vartheta), \xi \rangle_{L^2} \\
&= \langle \vartheta, M_{\overline{\varphi}} \xi \rangle_{L^2} - 0 \\
&= \langle \vartheta, \mathrm{P}_-(M_{\overline{\varphi}} \xi) \rangle_{L^2} + \langle \vartheta, \mathrm{P}_+(M_{\overline{\varphi}} \xi) \rangle_{L^2} \\
&= 0 + \langle \vartheta, \mathrm{P}_+(M_{\overline{\varphi}} \xi) \rangle_{L^2}.
\end{aligned}$$

□

How is the definition of a Hankel operator and the found representation of its adjoint operator connected to our Hankel matrix $\mathbf{\Gamma_f}$ and its adjoint matrix $\mathbf{\Gamma_f^*}$ from the Section 2.1? In order to shed light on this question, we will "translate" the above concepts to the Fourier domain.
We define the discrete projections $\mathbf{P}_-$ and $\mathbf{P}_+$ on the spaces of sequences in analogy to P_- and P_+

$$\mathbf{P}_- : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z}\backslash\mathbb{N}), \qquad \mathbf{P}_-(a_j)_{j=-\infty}^{\infty} = (a_j)_{j=-\infty}^{-1},$$
$$\mathbf{P}_+ : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{N}), \qquad \mathbf{P}_+(a_j)_{j=-\infty}^{\infty} = (a_j)_{j=0}^{\infty}.$$

Furthermore, we recall that the multiplication in L^2 corresponds to the convolution in $\ell^2(\mathbb{Z})$, i.e., for $\vartheta \in H^2$ and $\xi \in L^2 \ominus H^2$ we have

$$\mathcal{F}(M_\varphi \vartheta) = \mathcal{F}(\varphi\vartheta) = \hat{\varphi} * \hat{\vartheta} \quad \text{and} \quad \mathcal{F}(M_{\overline{\varphi}}\xi) = \mathcal{F}(\overline{\varphi}\xi) = \overline{\hat{\varphi}(-\cdot)} * \hat{\xi}.$$

Thus, the images of the Hankel operator H_φ and its adjoint operator H_φ^* in the Fourier domain are given as follows

$$\mathcal{F}(H_\varphi \vartheta) = \mathcal{F}(\mathrm{P}_-(M_\varphi \vartheta)) = \mathbf{P}_-(\mathcal{F}(M_\varphi \vartheta)) = \mathbf{P}_-(\hat{\varphi} * \hat{\vartheta}),$$

$$\mathcal{F}(H_\varphi^* \xi) = \mathcal{F}(\mathrm{P}_+(M_{\overline{\varphi}}\xi)) = \mathbf{P}_+(\mathcal{F}(M_{\overline{\varphi}}\xi)) = \mathbf{P}_+(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi}).$$

Since the discrete convolution can be written as a multiplication by a Toeplitz matrix, for

$$\mathbf{f} = (f_j)_{j=0}^{\infty} := (\hat{\varphi}_{-j-1})_{j=0}^{\infty} \quad \text{and} \quad \mathbf{v} = (v_j)_{j=0}^{\infty} := (\hat{\vartheta}_j)_{j=0}^{\infty} \tag{2.20}$$

we have

$$\left(\begin{array}{cccc|cccc}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\ddots & \hat{\varphi}_0 & \hat{\varphi}_{-1} & \hat{\varphi}_{-2} & \hat{\varphi}_{-3} & \ddots & \ddots & \\
\ddots & \hat{\varphi}_1 & \hat{\varphi}_0 & \hat{\varphi}_{-1} & \hat{\varphi}_{-2} & \hat{\varphi}_{-3} & \ddots & \ddots \\
\ddots & \hat{\varphi}_2 & \hat{\varphi}_1 & \hat{\varphi}_0 & \hat{\varphi}_{-1} & \hat{\varphi}_{-2} & \hat{\varphi}_{-3} & \ddots \\
\hline
\ddots & \hat{\varphi}_3 & \hat{\varphi}_2 & \hat{\varphi}_1 & \hat{\varphi}_0 & \hat{\varphi}_{-1} & \hat{\varphi}_{-2} & \ddots \\
\ddots & \ddots & \hat{\varphi}_3 & \hat{\varphi}_2 & \hat{\varphi}_1 & \hat{\varphi}_0 & \hat{\varphi}_{-1} & \ddots \\
 & \ddots & \ddots & \hat{\varphi}_3 & \hat{\varphi}_2 & \hat{\varphi}_1 & \hat{\varphi}_0 & \ddots \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right)
\left(\begin{array}{c} \vdots \\ 0 \\ \hline \hat{\vartheta}_0 \\ \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \\ \vdots \end{array}\right)
=
\left(\begin{array}{c} \vdots \\ (\hat{\varphi} * \hat{\vartheta})_{-3} \\ (\hat{\varphi} * \hat{\vartheta})_{-2} \\ (\hat{\varphi} * \hat{\vartheta})_{-1} \\ \hline (\hat{\varphi} * \hat{\vartheta})_0 \\ (\hat{\varphi} * \hat{\vartheta})_1 \\ (\hat{\varphi} * \hat{\vartheta})_2 \\ \vdots \end{array}\right)
=
\left(\begin{array}{c} \vdots \\ (\mathbf{\Gamma_f}\,\mathbf{v})_2 \\ (\mathbf{\Gamma_f}\,\mathbf{v})_1 \\ (\mathbf{\Gamma_f}\,\mathbf{v})_0 \\ \hline (\hat{\varphi} * \hat{\vartheta})_0 \\ (\hat{\varphi} * \hat{\vartheta})_1 \\ (\hat{\varphi} * \hat{\vartheta})_2 \\ \vdots \end{array}\right)$$

and hence we obtain

$$\mathcal{F}(H_\varphi \vartheta) = \mathbf{\Gamma_f}\mathbf{v}.$$

Analogously, for

$$\mathbf{f} = (f_j)_{j=0}^{\infty} := (\overline{\hat{\varphi}_{j+1}})_{j=0}^{\infty} \quad \text{and} \quad \mathbf{u} = (u_j)_{j=0}^{\infty} := (\hat{\xi}_{-j-1})_{j=0}^{\infty} \tag{2.21}$$

we have

$$
\left(\begin{array}{cccc|cccc}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\ddots & \overline{\hat{\varphi}_0} & \overline{\hat{\varphi}_1} & \overline{\hat{\varphi}_2} & \overline{\hat{\varphi}_3} & \ddots & \ddots & \\
\ddots & \overline{\hat{\varphi}_{-1}} & \overline{\hat{\varphi}_0} & \overline{\hat{\varphi}_1} & \overline{\hat{\varphi}_2} & \overline{\hat{\varphi}_3} & \ddots & \ddots \\
\ddots & \overline{\hat{\varphi}_{-2}} & \overline{\hat{\varphi}_{-1}} & \overline{\hat{\varphi}_0} & \overline{\hat{\varphi}_1} & \overline{\hat{\varphi}_2} & \overline{\hat{\varphi}_3} & \ddots \\
\hline
\ddots & \overline{\hat{\varphi}_{-3}} & \overline{\hat{\varphi}_{-2}} & \overline{\hat{\varphi}_{-1}} & \overline{\hat{\varphi}_0} & \overline{\hat{\varphi}_1} & \overline{\hat{\varphi}_2} & \ddots \\
\ddots & \ddots & \overline{\hat{\varphi}_{-3}} & \overline{\hat{\varphi}_{-2}} & \overline{\hat{\varphi}_{-1}} & \overline{\hat{\varphi}_0} & \overline{\hat{\varphi}_1} & \ddots \\
 & \ddots & \ddots & \overline{\hat{\varphi}_{-3}} & \overline{\hat{\varphi}_{-2}} & \overline{\hat{\varphi}_{-1}} & \overline{\hat{\varphi}_0} & \ddots \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right)
\left(\begin{array}{c}
\vdots \\ \hat{\xi}_{-3} \\ \hat{\xi}_{-2} \\ \hat{\xi}_{-1} \\ \hline 0 \\ \vdots
\end{array}\right)
$$

$$
= \left(\begin{array}{c}
\vdots \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-3} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-2} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-1} \\ \hline
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{0} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{1} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{2} \\
\vdots
\end{array}\right)
= \left(\begin{array}{c}
\vdots \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-3} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-2} \\
(\overline{\hat{\varphi}(-\cdot)} * \hat{\xi})_{-1} \\ \hline
(\mathbf{\Gamma}_{\mathbf{f}}^* \mathbf{u})_0 \\
(\mathbf{\Gamma}_{\mathbf{f}}^* \mathbf{u})_1 \\
(\mathbf{\Gamma}_{\mathbf{f}}^* \mathbf{u})_2 \\
\vdots
\end{array}\right)
$$

and thus we obtain

$$\mathcal{F}(H_\varphi^* \xi) = \mathbf{\Gamma}_{\mathbf{f}}^* \mathbf{u}.$$

Indeed, as it has been proven in Theorem 15.8 in [55], $\mathbf{\Gamma}_{\mathbf{f}}$ is the representation matrix of H_φ with respect to the standard orthonormal basis $1, z, z^2, \ldots$ in H^2 and $z^{-1}, z^{-2}, z^{-3}, \ldots$ in $L^2 \ominus H^2$.

Remark 2.12. Note that in both cases (2.20) and (2.21) the sequence $\mathbf{f}$ is in $\ell^1(\mathbb{N})$ due to the assumption we made in the beginning of this subsection. Thus, by Young's inequality, the convolution in terms of the multiplication by the Toeplitz matrices above is well-defined.

$\triangle$

As we will see below, due to the close connection between the infinite Hankel matrix and the Hankel operator via the Fourier transform, we preserve all the basic tools we used in the discrete setting. We recall from Example 2.8, that a rational function $\tilde{\varphi}$, which is given by the partial fractions

$$\tilde{\varphi}(z) := \sum_{j=1}^{N} \frac{c_j}{z - \alpha_j}, \qquad c_j \in \mathbb{C},\ \alpha_j \in \mathbb{D},$$

belongs to L^∞ and has Fourier coefficients of the form

$$(\mathcal{F}\tilde{\varphi})(k) = \begin{cases} \sum_{j=1}^{N} c_j \alpha_j^{-k-1}, & k \leqslant -1, \\ 0 & k \geqslant 0, \end{cases}$$

which build a sequence in $\ell^1(\mathbb{Z})$. Thus, the samples of the exponential sum,

$$f_k = \sum_{j=1}^{N} a_j z_j^k, \qquad k = 0, 1, 2, \ldots,$$

where $a_j \in \mathbb{C}$ and $z_j \in \mathbb{D}\backslash\{0\}$ can be seen as the Fourier coefficients with negative index of a rational function $\varphi \in L^\infty$ of the form

$$\varphi(z) := \sum_{j=1}^{N} \frac{a_j}{z - z_j} \tag{2.22}$$

by setting $(\mathcal{F}\varphi)(k) := f_{-k-1}$ for $k \leqslant -1$. Let us consider Hankel operators H_φ determined by such a rational function φ given in (2.22). First, the boundedness of $\mathbf{\Gamma_f}$ for all $\mathbf{f} \in \ell^1(\mathbb{N})$, which we saw in the beginning of Section 2.1, can be obviously carried over to the boundedness of the operator H_φ for all functions $\varphi \in L^\infty$.
Also Kronecker's Theorem 2.3 about Hankel matrices with finite rank can be formulated in terms of Hankel operators on the Hardy space. It can be found as Theorem 2.8 in [15] and states that for a function φ with N poles in $\mathbb{D}\backslash\{0\}$ given by (2.22) the Hankel operator H_φ has finite rank N.
Moving on to the concept of singular values of the Hankel operator, we recall their formal definition first. Note that we use the standard notation from the spectral theory for Hankel operators, which is outlined in Section 16.1 in [55]. Let $\mathcal{L}(H^2, L^2 \ominus H^2)$ denote the space of all linear bounded operators from H^2 to $L^2 \ominus H^2$. For all $k \in \mathbb{N}$ we call the non-negative numbers

$$\sigma_k := \inf_{\text{Rank}R \leqslant k} \{\|H_\varphi - R\| \ : \ R \in \mathcal{L}(H^2, L^2 \ominus H^2)\}$$

singular values of H_φ. For the bounded operator H_φ we always assume the singular values to be ordered non-increasingly, i.e. $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant 0$. Since H_φ is a compact operator if φ is given by (2.22), which has been proved by Hartman in [27], the above definition is equivalent to the solution of the eigenvalue equation

$$H_\varphi^* H_\varphi \vartheta_k = \sigma_k^2 \vartheta_k. \tag{2.23}$$

The functions ϑ_k satisfying the above equation are called *Schmidt vectors of* H_φ and form an orthonormal system in H^2. This result can be found as Theorem 16.4 in [55]. In analogy to the left and right singular vectors of a matrix, the functions ξ_k satisfying

$$H_\varphi H_\varphi^* \xi_k = \sigma_k^2 \xi_k. \tag{2.24}$$

are called *Schmidt vectors of* H_φ^*. Since the equations (2.23) and (2.24) are rather unhandy to work with, we introduce the following. A pair (ϑ_k, ξ_k) is called *Schmidt pair* corresponding to the singular value σ_k if

$$H_\varphi \vartheta_k = \sigma_k \xi_k \qquad \text{and} \qquad H_\varphi^* \xi_k = \sigma_k \vartheta_k. \tag{2.25}$$

Let $\sigma_k \neq 0$. Then multiplying both sides of the equation (2.23) by H_φ yields

$$H_\varphi H_\varphi^* H_\varphi \vartheta_k = \sigma_k^2 H_\varphi \vartheta_k.$$

and thus for $\sigma_k \xi_k = H_\varphi \vartheta_k$ we obtain (2.24). Conversely, multiplying (2.24) by H_φ^* we obtain (2.23) for $\sigma_k \vartheta_k = H_\varphi^* \xi_k$. We can formulate the following lemma.

Lemma 2.13. *Let φ be a function in L^∞, H_φ the corresponding Hankel operator and σ_k a non-zero singular value of H_φ. Then the following is equivalent:*

(1) *(ϑ_k, ξ_k) is a Schmidt pair of H_φ corresponding to σ_k.*

(2) *ϑ_k is a Schmidt vector of H_φ and it holds $H_\varphi \vartheta_k = \sigma_k \xi_k$.*

(3) *ξ_k is a Schmidt vector of H_φ^* and it holds $H_\varphi^* \xi_k = \sigma_k \vartheta_k$.*

Proof. The equivalence of (2) and (3) already has been shown above. Let now (ϑ_k, ξ_k) be a Schmidt pair of H_φ corresponding to σ_k. Then from (2.25) it follows

$$H_\varphi^* H_\varphi \vartheta_k = \sigma_k H_\varphi^* \xi_k = \sigma_k^2 \vartheta_k$$

as well as

$$H_\varphi H_\varphi^* \xi_k = \sigma_k H_\varphi \vartheta_k = \sigma_k^2 \xi_k.$$

□

In the following by writing "singular vector of H_φ" we will refer to a Schmidt vector of H_φ. Accordingly, by "singular pair of H_φ" we denote a tuple (σ_k, ϑ_k), where ϑ_k is a Schmidt vector of H_φ corresponding to the singular value σ_k.
Recalling the concept of con-eigenvalues and con-eigenvectors of a symmetric matrix in Section 2.1.3, we can easily see, that it corresponds to the definition of Schmidt vectors. Namely, if $(\sigma, \mathbf{v})$ is a con-eigenpair of $\mathbf{\Gamma_f}$ then it holds $\mathbf{\Gamma_f} \overline{\mathbf{v}} = \sigma \mathbf{v}$ as well as $\mathbf{\Gamma_f}^* \mathbf{v} = \sigma \overline{\mathbf{v}}$ and thus $(\mathbf{v}, \overline{\mathbf{v}})$ is a Schmidt pair of $\mathbf{\Gamma_f}$ corresponding to σ. This means that Schmidt vectors in a Schmidt pair of the Hankel matrix $\mathbf{\Gamma_f}$ have the special property to be the complex conjugate of each other. Our aim is to obtain a similar property for the Schmidt pairs of the Hankel operator H_φ. Let us make some useful definitions first. In the following we denote by

$$M_z : L^2 \to L^2, \qquad (M_z \vartheta)(z) := z\vartheta(z)$$

the *multiplication operator* by an independent variable z. Further we define the *(continuous) flip operator* $J : L^2 \to L^2$ as

$$J\vartheta = \overline{M_z \vartheta}, \qquad \vartheta \in L^2.$$

Evaluated at each $z \in \mathbb{C}$ we have

$$(J\vartheta)(z) = \overline{z\vartheta(z)}. \tag{2.26}$$

We collect the most useful properties of J in the following lemma.

Lemma 2.14. *Let J be the flip operator. Then the following holds.*

(1) *The multiplication by z and conjugation of a function ϑ in H^2 causes the conjugation and reflection of the Fourier coefficients of ϑ across the zero point, that is*

$$J(H^2) = L^2 \ominus H^2 \qquad \text{and} \qquad J(L^2 \ominus H^2) = H^2. \tag{2.27}$$

(2) *For the projection operators P_+ and P_- defined in (2.16) and (2.18) it holds*

$$J\mathrm{P}_+ = \mathrm{P}_- J \qquad \text{and} \qquad J\mathrm{P}_- = \mathrm{P}_+ J. \tag{2.28}$$

(3) *Let further φ be a function in L^∞ and H_φ be the corresponding Hankel operator. Then we have for $\vartheta_1 \in H^2$ and $\vartheta_2 \in L^2 \ominus H^2$*

$$JH_\varphi \vartheta_1 = H_\varphi^* J \vartheta_1 \qquad \text{and} \qquad H_\varphi J \vartheta_2 = J H_\varphi^* \vartheta_2$$

Proof.

1. Let the Fourier series representation of $\vartheta \in H^2$ be given by $\vartheta(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{j=0}^{\infty} \hat{\vartheta}(j) \mathrm{e}^{\mathrm{i}\omega j}$. Then we obtain

$$\begin{aligned}
\mathcal{F}(J\vartheta)(k) &= \frac{1}{2\pi} \int_0^{2\pi} \overline{\mathrm{e}^{\mathrm{i}t} \vartheta(\mathrm{e}^{\mathrm{i}t})} \mathrm{e}^{-\mathrm{i}tk} \mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^{\infty} \overline{\hat{\vartheta}(j)} \mathrm{e}^{-\mathrm{i}tj} \mathrm{e}^{-\mathrm{i}t(k+1)} \mathrm{d}t \\
&= \sum_{j=0}^{\infty} \overline{\hat{\vartheta}(j)} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t(k+1+j)} \mathrm{d}t \\
&= \sum_{j=0}^{\infty} \overline{\hat{\vartheta}(j)} \delta_{j,-k-1} \\
&= \begin{cases} \overline{\hat{\vartheta}(-k-1)}, & k \leqslant -1 \\ 0 & k \geqslant 0. \end{cases}
\end{aligned}$$

The second assertion $J(L^2 \ominus H^2) = H^2$ follows analogiously.

2. The second assertion follows immediately from the first due to the fact that for every function $\vartheta \in L^2$ it holds $\mathrm{P}_+\vartheta \in H^2$ and $\mathrm{P}_-\vartheta \in L^2 \ominus H^2$.

3. Let $\vartheta_1 \in H^2$ and $\vartheta_2 \in L^2 \ominus H^2$. Using (2) we have

$$JH_\varphi \vartheta_1 = J\mathrm{P}_-(\varphi\vartheta_1) = \mathrm{P}_+ J(\varphi\vartheta_1) = \mathrm{P}_+(\overline{\varphi} J \vartheta_1) = H_\varphi^* J \vartheta_1$$

and

$$H_\varphi J \vartheta_2 = \mathrm{P}_-(\varphi J \vartheta_2) = \mathrm{P}_- J(\overline{\varphi}\vartheta_2) = J\mathrm{P}_+(\overline{\varphi}\vartheta_2) = J H_\varphi^* \vartheta_2.$$

□

Remark 2.15. Note that the flip operator can also be defined in the discrete case, namely as

$$\mathbf{J} : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z}), \qquad \mathbf{Jv} := \mathbf{ASv},$$

where $\mathbf{v} \in \ell^2(\mathbb{N})$, S the shift operator and $\mathbf{A} := (a_{jk})_{j,k=-\infty}^{\infty}$ is the *counter-identity matrix* with

$$a_{jk} := \begin{cases} 1, & j = -k \\ 0 & \text{else.} \end{cases}$$

Furthermore, similar properties as in the previous lemma can be proven also for the discrete flip operator involving the spaces $\ell^2(\mathbb{Z})$, $\ell^2(\mathbb{N})$, $\ell^2(\mathbb{Z}\backslash\mathbb{N})$ and the discrete projection operators $\mathbf{P}_+$ and $\mathbf{P}_-$.

△

Now let us assume that (σ_k, ϑ_k) is a singular pair of H_φ, i.e. it holds $H_\varphi^* H_\varphi \vartheta_k = \sigma_k^2 \vartheta_k$. Then, using the above properties of the flip operator, we obtain

$$H_\varphi H_\varphi^* J\vartheta_k = H_\varphi J H_\varphi \vartheta_k = J H_\varphi^* H_\varphi \vartheta_k = \sigma_k^2 J \vartheta_k.$$

Hence $J\vartheta_k$ is the Schmidt vector of H_φ^* corresponding to σ_k and the Schmidt pairs of the Hankel operator H_φ are of the form $(\vartheta, J\vartheta)$.
Another concept we want to transfer from the discrete to the continuous setting is the one of the shift operator and shift-invariant subspaces. For this purpose we note that the shift S of a sequence $\mathbf{v} \in \ell^2(\mathbb{N})$ corresponds to the multiplication of the corresponding Laurent polynomial $P_{\mathbf{v}}(z)$ by z. Thus, in analogy to the shift invariant subspace $\mathcal{S}_{\mathbf{v}}$ generated by a sequence $\mathbf{v} \in \ell^2(\mathbb{N})$, we define the *multiplication invariant subspace*

$$\mathcal{M}_\vartheta := \text{clos}_{H^2}\text{span}\{M_z^k \vartheta \;:\; k \in \mathbb{N}\}$$

generated by a function ϑ in H^2. Now we can formulate a similar assertion to Lemma 2.1 about the kernel of the Hankel operator H_φ.

Lemma 2.16. *Let φ be a function in L^∞ and H_φ the corresponding Hankel operator. Then the following holds.*

(1) $\text{Ker}(H_\varphi) := \{\vartheta \in H^2 \;:\; H_\varphi \vartheta = 0\}$ *is multiplication invariant, i.e. for a function $\vartheta \in \text{Ker}(H_\varphi)$ it holds $M_z^k \vartheta \in \text{Ker}(H_\varphi)$ for all $k \in \mathbb{N}$.*

(2) *A function ϑ is in* $\text{Ker}(H_\varphi)$ *if and only if* $\overline{\varphi} \perp \mathcal{M}_\vartheta$.

Proof.

1. Let ϑ be in $\text{Ker}(H_\varphi)$. Then, since z^k is a H^2-function, we have

$$\begin{aligned} H_\varphi(M_z^k \vartheta)(z) &= \mathrm{P}_-(M_z^k \varphi\vartheta)(z) = \mathrm{P}_-(z^k \varphi(z)\vartheta(z)) \\ &= \mathrm{P}_-(z^k \mathrm{P}_+(\varphi(z)\vartheta(z))) + \mathrm{P}_-(z^k \mathrm{P}_-(\varphi(z)\vartheta(z))) \\ &= \mathrm{P}_-(M_z^k H_\varphi \vartheta)(z) = 0 \end{aligned}$$

for all $z \in \mathbb{C}$.

2. Using properties of the Fourier transform we find

$$\begin{aligned} H_\varphi \vartheta = 0 &\Leftrightarrow \mathrm{P}_-(\varphi\vartheta) = 0 \\ &\Leftrightarrow \mathcal{F}(\varphi\vartheta)(k) = 0 && \forall k \leqslant -1 \\ &\Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}t})\vartheta(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}tk}\mathrm{d}t = 0 && \forall k \geqslant 1 \\ &\Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}t})(M_z)^k\vartheta(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = 0 && \forall k \geqslant 1 \\ &\Leftrightarrow \langle \overline{\varphi}, (M_z)^k\vartheta\rangle_{L^2} = 0 && \forall k \geqslant 1 \end{aligned}$$

and therefore $\overline{\varphi} \perp \mathcal{M}_\vartheta$.

□

2.2.4. Finite Blaschke Products and the corresponding Toeplitz matrices

In this section we introduce special rational functions, the so-called finite Blaschke products, which are well-studied in the context of complex analysis. Detailed information can be found, e.g., in [36]. The infinite version of Blaschke products was originally introduced by Wilhelm Blaschke in 1915. However, in this section we will restrict ourselves to the finite version of Blaschke products and its properties, which are used in the proof of the AAK theorem in Chapter 3. For instance, as we will see below, finite Blaschke products appear to be unimodular. Thus, in the proof of the AAK theorem we will rewrite a rational function as a product, where one factor is a Blaschke product that does not change its norm. We will study the properties of Blaschke products also by means of their Fourier coefficients and the corresponding infinite triangular Toeplitz matrices. Finally we will show how these Toeplitz matrices are related to our main protagonist, the infinite Hankel matrix $\mathbf{\Gamma_f}$.
Let $\alpha_1, \ldots, \alpha_K$ be distinct complex numbers in $\mathbb{D}\backslash\{0\}$. A function of the form

$$B(z) := \begin{cases} \prod_{j=1}^{K} \frac{z-\alpha_j}{1-\overline{\alpha}_j z}, & K > 0, \\ 1, & K = 0 \end{cases} \tag{2.29}$$

is called a *finite Blaschke Product* of degree K. We denote by

$$b_{\alpha_j}(z) := \frac{z-\alpha_j}{1-\overline{\alpha_j}z}$$

the *Blaschke factor* corresponding to α_j. It holds

$$B(z) = \prod_{j=1}^{K} b_{\alpha_j}(z).$$

Defining the polynomial $p(z) := \prod_{j=1}^{K}(z-\alpha_j)$ another alternative representation of B on $\partial\mathbb{D}$ is given by

$$B(z) = \frac{p(z)}{\prod_{j=1}^{K}(1-\overline{\alpha_j}z)} = \frac{p(z)}{z^K\prod_{j=1}^{K}(\overline{z}-\overline{\alpha_j})} = \frac{p(z)}{z^K\overline{p(z)}}. \tag{2.30}$$

Some basic properties of finite Blaschke products are presented in a nutshell in the following lemma.

Lemma 2.17. *Let B be a finite Blaschke Product as given in (2.29). Then the following properties holds.*

(1) $B \in H^2$

(2) $|B| = B\overline{B} = 1$ *a.e. on* $\partial\mathbb{D}$ *and thus* $B \in L^\infty$.

(3) $\|B\|_{L^2} = 1$.

(4) *B has exactly K zeros in $\mathbb{D}$, namely $\alpha_1, \ldots, \alpha_K$.*

(5) *B has exactly K poles in $\mathbb{C}\backslash\mathbb{D}$, namely $1/\overline{\alpha_1}, \ldots, 1/\overline{\alpha_K}$.*

(6) *Every function $\phi \in H^2$ which has zeros at $\alpha_1, \ldots, \alpha_K$ can be represented on $\mathbb{D}$ as*

$$\phi = B \cdot \tilde{\phi},$$

for a suitable $\tilde{\phi} \in H^2$.

(7) *Let φ be an arbitrary function in L^2. Then it holds*

$$\mathrm{P}_-(B\varphi) = \mathrm{P}_-(B\mathrm{P}_-\varphi) \qquad \text{and} \qquad \mathrm{P}_+(\overline{B}\varphi) = \mathrm{P}_+(\overline{B}\mathrm{P}_+\varphi).$$

Proof. As (4) and (5) are obvious, we only prove the remaining assertions. Since B does not have any poles in $\mathbb{D}$, it is by Proposition 2.10 an H^2 function. Using the representation (2.30), for the modulus of B we obtain

$$|B(z)|^2 = B(z)\overline{B(z)} = \frac{p(z)}{z^K\overline{p(z)}}\frac{z^K\overline{p(z)}}{p(z)} = 1 \qquad \text{a.e. on } \partial\mathbb{D}.$$

Hence, for the L^2-norm of B it follows that

$$\|B\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi}|B(\mathrm{e}^{\mathrm{i}t})|^2\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}1\mathrm{d}t = 1.$$

Now, let ϕ be a function in H^2 with zeros at $\alpha_1, \ldots, \alpha_K$. Then ϕ can be represented as a product

$$\phi(z) = (z-\alpha_1)\cdot \ldots \cdot (z-\alpha_K)\cdot q(z) = p(z)\cdot\tilde{\phi}(z),$$

where $\tilde{\phi}$ is also in H^2. Hence, we can write ϕ in $\mathbb{D}$ as

$$\phi(z) = \frac{p(z)}{z^K\overline{p(z)}}\cdot z^K\overline{p(z)}q(z) = B(z)\cdot\tilde{\phi}(z)$$

with $\tilde{\phi}(z) := z^K\overline{p(z)}q(z) \in H^2$, since $z^K\overline{p(z)}$ only has the zeros $1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_K}$ outside of $\mathbb{D}$. Let the function φ be in L^2. The first assertion in (7) follows from the fact that $B \in H^2$ and from the properties of the projections, that is

$$\mathrm{P}_-(B\varphi) = \mathrm{P}_-(B\mathrm{P}_+\varphi) + \mathrm{P}_-(B\mathrm{P}_-\varphi) = \mathrm{P}_-(B\mathrm{P}_-\varphi).$$

Let $B(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=0}^{\infty} b(j)\mathrm{e}^{\mathrm{i}tj}$ be the Fourier series of B. For the second assertion we recall that on the unit circle $\overline{B}$ can be written as

$$\begin{aligned}
\overline{B(\mathrm{e}^{\mathrm{i}t})} &= \sum_{j=0}^{\infty} \overline{b(j)}\mathrm{e}^{-\mathrm{i}tj} \\
&= \sum_{j=-\infty}^{0} \overline{b(-j)}\mathrm{e}^{\mathrm{i}tj} \\
&= b(0) + \sum_{j=-\infty}^{-1} \overline{b(-j)}\mathrm{e}^{\mathrm{i}tj} \\
&= b(0) + \mathrm{P}_-(\overline{B})(\mathrm{e}^{\mathrm{i}t})
\end{aligned}$$

and thus we obtain

$$\begin{aligned}
\mathrm{P}_+(\overline{B}\varphi) &= \mathrm{P}_+(\overline{B}\mathrm{P}_-\varphi) + \mathrm{P}_+(\overline{B}\mathrm{P}_+\varphi) \\
&= \mathrm{P}_+(\mathrm{P}_-\overline{B}\mathrm{P}_-\varphi) + b(0)\cdot \mathrm{P}_+(\mathrm{P}_-\varphi) + \mathrm{P}_+(\overline{B}\mathrm{P}_+\varphi) \\
&= \mathrm{P}_+(\overline{B}\mathrm{P}_+\varphi).
\end{aligned}$$

□

From the assertion (2) in the previous lemma we can easily see that the product $B\varphi$ is in L^∞ for $\varphi \in L^\infty$. We need this fact for the next lemma, where we collect further properties of Blaschke products in connection with Hankel operators and their singular values. In particular, in Lemma 2.18(1) we formulate an analogous statement to Lemma 2.2(3). Note that in case of matrices this assertion holds more generally, whereas (1) only applies to a singular vector of the Hankel matrix.

Lemma 2.18. *Let φ be a function in L^∞ and B the Blaschke product of the form (2.29). Further let H_φ and $H_{B\varphi}$ be the Hankel operators corresponding to φ and $B\varphi$.*

(1) *Let (σ, ϑ) be a singular pair of H_φ with $\sigma \neq 0$ and multiplicity 1. Further let $\alpha_1, \dots, \alpha_K$ be zeros of ϑ in $\mathbb{D}$ and $B(z)$ the corresponding Blaschke product as given in (2.29), i.e. $\vartheta = B\tilde{\vartheta}$ for some $\tilde{\vartheta} \in H^2$. Then we have*

$$H_{B\varphi}\vartheta = BH_\varphi\vartheta.$$

In particular it holds $BH_\varphi\vartheta \in L^2 \ominus H^2$.

(2) *Let $\sigma_n(H_\varphi)$ and $\sigma_n(H_{B\varphi})$ be the n-th singular values of H_φ and $H_{B\varphi}$ ordered non-increasingly, i.e.,*

$$\begin{aligned}
&\sigma_0(H_\varphi) \geqslant \sigma_1(H_\varphi) \geqslant \dots \\
\text{and}\quad &\sigma_0(H_{B\varphi}) \geqslant \sigma_1(H_{B\varphi}) \geqslant \dots .
\end{aligned}$$

Then, for all $n \in \mathbb{N}$, we have

$$\sigma_n(H_{B\varphi}) \leqslant \sigma_n(H_\varphi).$$

Proof.

1. First we note that with (7) of Lemma 2.17 we have

$$H_{B\varphi}\vartheta = \mathrm{P}_-(B\varphi\vartheta) = \mathrm{P}_-(B \cdot \mathrm{P}_-(\varphi\vartheta)) = \mathrm{P}_-(BH_\varphi\vartheta)$$

and thus, in order to prove our assertion it suffices to show that $BH_\varphi\vartheta \in L^2 \ominus H^2$. Let J be the flip operator as defined in the previous section. Due to the properties of the Schmidt pairs of H_φ, it follows that $H_\varphi\vartheta = J\vartheta$ and thus $JH_\varphi\vartheta = \vartheta$. Hence $JH_\varphi\vartheta \in H^2$ is also a singular function of H_φ corresponding to the singular value σ and has the same zeros as ϑ. By Lemma 2.17 it can be written as

$$JH_\varphi\vartheta = B\tilde{\vartheta},$$

where $\tilde{\vartheta}$ is a function in H^2. Since $J^2 = \mathrm{Id}$ and $B\overline{B} = \mathrm{Id}$, we have

$$H_\varphi\vartheta = J(B\tilde{\vartheta}) = \overline{B}J\tilde{\vartheta} \in L^2 \ominus H^2$$

and thus

$$BH_\varphi\vartheta = J\tilde{\vartheta} \in L^2 \ominus H^2.$$

2. For the proof of the second assertion let $\mathcal{L}(H^2, L^2 \ominus H^2)$ be the space of linear bounded operators from H^2 to $L^2 \ominus H^2$. Using the definition of a singular value and the fact that the norm of the multiplication operator $\|M_B\| = 1$ we obtain

$$\begin{aligned}
\sigma_n(\mathbf{\Gamma_f}) &= \min\{\|H_\varphi - R\| \ : \ R \in \mathcal{L}(H^2, L^2 \ominus H^2),\ \mathrm{Rank}(R) \leqslant n\} \\
&= \min\{\|H_\varphi - R\|\|M_B\| \ : \ R \in \mathcal{L}(H^2, L^2 \ominus H^2),\ \mathrm{Rank}(R) \leqslant n\} \\
&\geqslant \min\{\|(H_\varphi - R)M_B\| \ : \ R \in \mathcal{L}(H^2, L^2 \ominus H^2),\ \mathrm{Rank}(R) \leqslant n\} \\
&= \min\{\|H_\varphi M_B - RM_B\| \ : \ R \in \mathcal{L}(H^2, L^2 \ominus H^2),\ \mathrm{Rank}(R) \leqslant n\} \\
&= \min\{\|H_{B\varphi} - \tilde{R}\| \ : \ \tilde{R} \in \mathcal{L}(H^2, L^2 \ominus H^2),\ \mathrm{Rank}(\tilde{R}) \leqslant n\} \\
&= \sigma_n(H_{B\varphi}),
\end{aligned}$$

since $\mathrm{Rank}(RM_B)$ remains at most n due to the fact that $\mathrm{Rank}(R) \leqslant n$ and multiplication with M_B it can not increase it.

□

We know already that a finite Blaschke product is an H^2-function and therefore analytic in $\mathbb{D}$. Now we can state the following lemma, in which the properties of B in the Fourier domain are examined.

Lemma 2.19. *For some $K \in \mathbb{N}$ let $\mathbf{b} = (b_k)_{k=0}^{\infty}$ be given as the Fourier coefficients of a Blaschke product*

$$B(e^{i\omega}) = \prod_{j=1}^{K} \frac{e^{i\omega} - \alpha_j}{1 - \overline{\alpha}_j e^{i\omega}} = \sum_{k=0}^{\infty} b_k e^{i\omega k} \tag{2.31}$$

where $\alpha_1, \ldots, \alpha_K \in \mathbb{D}$. Then $\mathbf{b} \in \ell^1(\mathbb{N})$ and the infinite triangular Toeplitz matrix $\mathbf{T_b}$ generated by $\mathbf{b}$ has the following properties.

(1) *Let $\mathbf{v}$ and $\mathbf{u}$ be two sequences in $\ell^2(\mathbb{N})$ with corresponding Laurent polynomials $P_{\mathbf{v}}(z)$ and $P_{\mathbf{u}}(z)$ as defined in (2.2). Furthermore let the equality*

$$P_{\mathbf{v}}(z) = B(z) \cdot P_{\mathbf{u}}(z),$$

be satisfied. Then it holds

$$\mathbf{v} = \mathbf{T_b} \mathbf{u}.$$

(2) $\mathbf{T_b^* T_b} = \mathbf{I}$*, i.e. $\mathbf{T_b^*}$ is the left inverse of $\mathbf{T_b}$.*

(3) *The operator $\mathbf{T_b} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ has the norm $\|\mathbf{T_b}\|_{\ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})} = 1$ for $p \in \{1, 2\}$.*

(4) *Let $\mathbf{\Gamma_f}$ be an infinite Hankel matrix generated by $\mathbf{f} \in \ell^1(\mathbb{N})$. In analogy to the previous lemma let $\sigma_n(\mathbf{\Gamma_f})$ and $\sigma_n(\mathbf{\Gamma_f T_b})$ be the n-th singular values of $\mathbf{\Gamma_f}$ and $\mathbf{\Gamma_f T_b}$ being ordered non-increasingly, i.e.,*

$$\begin{aligned} &\sigma_0(\mathbf{\Gamma_f}) &&\geqslant \sigma_1(\mathbf{\Gamma_f}) &&\geqslant \ldots \\ \text{and} \quad &\sigma_0(\mathbf{\Gamma_f T_b}) &&\geqslant \sigma_1(\mathbf{\Gamma_f T_b}) &&\geqslant \ldots \end{aligned}$$

Then, for all $n \in \mathbb{N}$, we have

$$\sigma_n(\mathbf{\Gamma_f T_b}) \leqslant \sigma_n(\mathbf{\Gamma_f}).$$

Proof. First we prove that $\mathbf{b} \in \ell^1(\mathbb{N})$. Let $b_{\alpha_j}(z) := \frac{e^{i\omega} - \alpha_j}{1 - \overline{\alpha}_j e^{i\omega}}$ be the Blaschke factor corresponding to α_j. Then, using the geometric series, b_{α_j} can be written as

$$\begin{aligned} b_{\alpha_j}(z) &= (e^{i\omega} - \alpha_j) \sum_{k=0}^{\infty} (\overline{\alpha}_j e^{i\omega})^k = \sum_{k=0}^{\infty} \overline{\alpha}_j^k e^{i\omega(k+1)} - \sum_{k=0}^{\infty} \alpha_j \overline{\alpha}_j^k e^{i\omega k} \\ &= \sum_{k=1}^{\infty} \overline{\alpha}_j^{(k-1)} e^{i\omega k} - \sum_{k=0}^{\infty} \alpha_j \overline{\alpha}_j^k e^{i\omega k} = -\alpha_j + \sum_{k=1}^{\infty} \overline{\alpha}_j^{(k-1)} (1 - \alpha_j \overline{\alpha}_j) e^{i\omega k}. \end{aligned}$$

Thus the Fourier coefficients of b_{α_j} are given by

$$\hat{b}_{\alpha_j}(k) = \begin{cases} -\alpha_j, & k = 0, \\ \overline{\alpha}_j^{(k-1)} (1 - |\alpha_j|^2), & k \geqslant 1 \end{cases}$$

and it holds

$$\begin{aligned}\|(\hat{b}_{\alpha_j}(k))_{k=0}^{\infty}\|_{\ell^1(\mathbb{N})} &= |\alpha_j| + \sum_{k=1}^{\infty} |\overline{\alpha}_j^{(k-1)}|(1-|\alpha_j|^2) \\ &\leqslant (1-|\alpha_j|^2) \sum_{k=0}^{\infty} |\overline{\alpha}_j|^k \\ &= (1-|\alpha_j|^2) \cdot \frac{1}{1-|\overline{\alpha}_j|} < \infty.\end{aligned}$$

1. This assertion follows immediately from the fact that the multiplication in terms of functions corresponds to the convolution of sequences in the Fourier domain. Since all the sequences of Fourier coefficients $\mathbf{v}, \mathbf{b}, \mathbf{u}$ have only non-negative indices, the convolution matrix is given by the truncated triangular Toeplitz matrix $\mathbf{T_b}$.

2. Obviously, $\mathbf{T_b^* T_b}$ is hermitian. For the (l,k)-th entry of $\mathbf{T_b}$ we obtain for $l \geqslant k$

   $$(\mathbf{T_b^* T_b})_{l,k} = \sum_{j=l}^{\infty} \overline{b}_{j-l} b_{j-k} = \sum_{j=0}^{\infty} \overline{b}_j b_{j+(l-k)}.$$

   The coefficients b_k are the Fourier coefficients of B,

   $$b_k = \frac{1}{2\pi} \int_0^{2\pi} B(\mathrm{e}^{\mathrm{i}\omega})\, \mathrm{e}^{-\mathrm{i}\omega k} \mathrm{d}\omega, \qquad k = 0, 1, 2, \ldots.$$

   Thus,

   $$\begin{aligned}\sum_{j=0}^{\infty} \overline{b}_j b_{j+(l-k)} &= \frac{1}{2\pi} \sum_{j=0}^{\infty} \overline{b}_j \int_0^{2\pi} B(\mathrm{e}^{\mathrm{i}\omega}) \mathrm{e}^{-\mathrm{i}\omega(j+l-k)} \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} B(\mathrm{e}^{\mathrm{i}\omega}) \mathrm{e}^{-\mathrm{i}\omega(l-k)} \sum_{j=0}^{\infty} \overline{b}_j \mathrm{e}^{-\mathrm{i}\omega j} \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} |B(\mathrm{e}^{\mathrm{i}\omega})|^2 \mathrm{e}^{-\mathrm{i}\omega(l-k)} \mathrm{d}\omega = \delta_{l,k}\end{aligned}$$

   due to property (2) from Lemma 2.17.

3. Now the third assertion follows directly from the equality above.

4. Let $\mathcal{L}(\ell^p(\mathbb{N}))$ be the set of all linear operators from $\ell^p(\mathbb{N})$ to $\ell^p(\mathbb{N})$. Analogously to the previous proof we obtain for $p \in \{1, 2\}$

   $$\begin{aligned}\sigma_n(\mathbf{\Gamma_f}) &= \min\{\|\mathbf{\Gamma_f} - \mathbf{R}\| \;:\; \mathbf{R} \in \mathcal{L}(\ell^p(\mathbb{N})), \operatorname{Rank}(\mathbf{R}) \leqslant n\} \\ &= \min\{\|\mathbf{\Gamma_f} - \mathbf{R}\| \|\mathbf{T_b}\| \;:\; \mathbf{R} \in \mathcal{L}(\ell^p(\mathbb{N})), \operatorname{Rank}(\mathbf{R}) \leqslant n\} \\ &\geqslant \min\{\|(\mathbf{\Gamma_f} - \mathbf{R})\mathbf{T_b}\| \;:\; \mathbf{R} \in \mathcal{L}(\ell^p(\mathbb{N})), \operatorname{Rank}(\mathbf{R}) \leqslant n\} \\ &= \min\{\|\mathbf{\Gamma_f T_b} - \mathbf{R T_b}\| \;:\; \mathbf{R} \in \mathcal{L}(\ell^p(\mathbb{N})), \operatorname{Rank}(\mathbf{R}) \leqslant n\} \\ &= \min\{\|\mathbf{\Gamma_f T_b} - \tilde{\mathbf{R}}\| \;:\; \tilde{\mathbf{R}} \in \mathcal{L}(\ell^p(\mathbb{N})), \operatorname{Rank}(\tilde{\mathbf{R}}) \leqslant n\} \\ &= \sigma_n(\mathbf{\Gamma_f T_b}),\end{aligned}$$

   since $\operatorname{Rank}(\mathbf{R T_b})$ is still at most n.

□

Remark 2.20. In the following table we sum up the connection between the mathematical objects presented in this section in the discrete and continuous setting. The connection usually employs the Fourier transform $\mathcal{F}$. Note that in this table we identify the vector in $\ell^2(\mathbb{Z})$ of the form

$$(\ldots, 0, 0, v_0, v_1, v_2, \ldots)^T$$

and the vector

$$(v_0, v_1, v_2, \ldots)^T$$

in $\ell^2(\mathbb{N})$. We also identify the vector

$$(\ldots, v_{-3}, v_{-2}, v_{-1}, 0, 0, 0, \ldots)^T$$

in $\ell^2(\mathbb{Z})$ and the vector

$$(\ldots, v_{-3}, v_{-2}, v_{-1})^T$$

in $\ell^2(\mathbb{Z}\backslash\mathbb{N})$. △

Discrete Setting	**Continuous Setting**	**Connection**
$\ell^2(\mathbb{Z}) := \left\{\mathbf{v} = (v_k)_{k=-\infty}^{\infty} \,:\, \sum_{k=-\infty}^{\infty} \lvert v_k\rvert^2 < \infty\right\}$	$L^2 := \left\{\vartheta : \partial\mathbb{D} \to \mathbb{C} \,:\, \int_0^{2\pi} \lvert\vartheta(\mathrm{e}^{\mathrm{i}t})\rvert^2 \mathrm{d}t < \infty\right\}$	$v_k = \hat{\vartheta}_k,\ k \in \mathbb{Z}$
$\ell^2(\mathbb{N}) := \{\mathbf{v} = (v_k)_{k=0}^{\infty} \,:\, \sum_{k=0}^{\infty} \lvert v_k\rvert^2 < \infty\}$	$H^2 := \left\{\vartheta \in L^2 \,:\, \vartheta(\mathrm{e}^{\mathrm{i}t}) = \sum_{k=0}^{\infty} \hat{\vartheta}_k \mathrm{e}^{\mathrm{i}tk}\right\}$	$v_k = \hat{\vartheta}_k,\ k \in \mathbb{N}$
$\ell^2(\mathbb{Z}\backslash\mathbb{N}) := \left\{\mathbf{v} = (v_k)_{k=-\infty}^{-1} \,:\, \sum_{k=-\infty}^{-1} \lvert v_k\rvert^2 < \infty\right\}$	$L^2 \ominus H^2 := \left\{\vartheta \in L^2 \,:\, \vartheta(\mathrm{e}^{\mathrm{i}t}) = \sum_{k=-\infty}^{-1} \hat{\vartheta}_k \mathrm{e}^{\mathrm{i}tk}\right\}$	$v_k = \hat{\vartheta}_k,\ k \in \mathbb{Z}\backslash\mathbb{N}$
Discrete projection operator: $\mathbf{P}_+ : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{N})$, $\mathbf{P}_+\mathbf{v} = \mathbf{P}_+(v_k)_{k=-\infty}^{\infty} := (v_k)_{k=0}^{\infty}$	Continuous projection operator: $\mathrm{P}_+ : L^2 \to H^2$, $\mathrm{P}_+\vartheta = \mathrm{P}_+\left(\sum_{k=-\infty}^{\infty} v_k z^k\right) := \sum_{k=0}^{\infty} v_k z^k$	$\mathbf{P}_+\mathbf{v} = \mathcal{F}(\mathrm{P}_+\vartheta)$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Discrete projection operator: $\mathbf{P}_- : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z}\backslash\mathbb{N})$, $\mathbf{P}_-\mathbf{v} = \mathbf{P}_-(v_k)_{k=-\infty}^{\infty} := (v_k)_{k=-\infty}^{-1}$	Continuous projection operator: $\mathrm{P}_- : L^2 \mapsto L^2 \ominus H^2$, $\mathrm{P}_-\vartheta = \mathrm{P}_-\left(\sum_{k=-\infty}^{\infty} v_k z^k\right) := \sum_{k=-\infty}^{-1} v_k z^k$	$\mathbf{P}_-\mathbf{v} = \mathcal{F}(\mathrm{P}_-\vartheta)$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Shift operator: $\mathrm{S} : \ell^p(\mathbb{N}) \mapsto \ell^p(\mathbb{N}),\ p = 1, 2,$ $\mathrm{S}\mathbf{v} := (0, v_0, v_1, v_2, \ldots)$ for $\mathbf{v} = (v_k)_{k=0}^{\infty}$.	Multiplication operator: $M_z : L^2 \mapsto L^2,\ (M_z\vartheta)(z) := z\vartheta(z)$	Let $\vartheta \in H^2$. Then we have $\mathrm{S}\mathbf{v} = \mathcal{F}(M_z\vartheta)$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Backward shift operator: $\mathrm{S}^* : \ell^p(\mathbb{N}) \mapsto \ell^p(\mathbb{N}),\ p = 1, 2,$ $\mathrm{S}^*\mathbf{v} := (v_1, v_2, v_3 \ldots)$ for $\mathbf{v} = (v_k)_{k=0}^{\infty}$.	Multiplication operator: $M_{\bar{z}} : L^2 \mapsto L^2,\ (M_{\bar{z}}\vartheta)(z) := \bar{z}\vartheta(z)$	Let $\vartheta \in H^2$. Then we have $\mathrm{S}^*\mathbf{v} = \mathbf{P}_+(\mathcal{F}(M_{\bar{z}}\vartheta))$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Shift invariant subspace of $\ell^p(\mathbb{N})$: $\mathcal{S}_{\mathbf{v}} := \mathrm{clos}_{\ell^2(\mathbb{N})}\mathrm{span}\,\{\mathrm{S}^k\mathbf{v} : k \in \mathbb{N}\}$ generated by $\mathbf{v} = (v_k)_{k=0}^{\infty} \in \ell^2(\mathbb{N})$	Multiplication invariant subspace of H^2: $\mathcal{M}_\vartheta := \mathrm{clos}_{H^2}\mathrm{span}\{M_z^k\vartheta \,:\, k \in \mathbb{N}\}$ generated by $\vartheta \in H^2$	$\mathcal{F}(\mathcal{M}_\vartheta) = \mathcal{S}_{\mathbf{v}}$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Discrete flip operator: $\mathbf{J} : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z}),\ \ \mathbf{J}\mathbf{v} := \mathbf{A}\mathrm{S}\mathbf{v}$ for $\mathbf{v} \in \ell^2(\mathbb{N})$ and $\mathbf{A}$ is the counter identity matrix	Continuous flip operator: $J : L^2 \mapsto L^2,\ \ J\vartheta = \overline{M_z\vartheta}$ for $\vartheta \in L^2$	$\mathbf{J}\mathbf{v} = \mathcal{F}(J\vartheta)$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=0}^{\infty}$

Discrete Setting	Continuous Setting	Connection
Infinite Toeplitz matrix: $\mathbf{T_p} : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z})$, $\mathbf{T_p} := (p_{j-k})_{j,k=-\infty}^{\infty}$ for $(p_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$	Multiplication operator: $M_\varphi : L^2 \mapsto L^2,\ M_\varphi\vartheta := \varphi\vartheta$ for $\varphi \in L^\infty$ with Fourier coefficients $(\hat{\varphi}_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$	$\mathcal{F}(M_\varphi\vartheta) = \mathbf{p} * \mathbf{v} = \mathbf{T_p}\mathbf{v}$ for $\mathbf{p} := (\hat{\varphi}_k)_{k=-\infty}^{\infty}$ and $\mathbf{v} := (\hat{\vartheta}_k)_{k=-\infty}^{\infty}$
Infinite triangular Toeplitz matrix: $\mathbf{T_g} : \ell^2(\mathbb{N}) \mapsto \ell^2(\mathbb{N})$, $\mathbf{T_g} := (g_{j-k})_{j,k=0}^{\infty}$ for $\mathbf{g} = (g_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$ with $g_k = 0$ for $k \leqslant -1$	Multiplication operator: $M_\varphi : H^2 \mapsto H^2,\ \ M_\varphi\vartheta := \varphi\vartheta$ for $\varphi \in L^\infty \cap H^2$ with Fourier coefficients $(\hat{\varphi}_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$	$\mathcal{F}(M_\varphi\vartheta) = \mathbf{g} * \mathbf{v} = \mathbf{T_g}\mathbf{v}$ for $\mathbf{g} := (\hat{\varphi}_k)_{k=0}^{\infty}$ and $\mathbf{v} := (\hat{\vartheta}_k)_{k=0}^{\infty}$
Samples of exponential sum: $\mathbf{f} := (f_k)_{k=0}^{\infty} = \left(\sum_{j=1}^{N} a_j z_j^k\right)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$ where $z_j \in \mathbb{D}$ and $a_j \in \mathbb{C}$.	Rational function: $\varphi(z) := \sum_{j=1}^{N} \frac{a_j}{z-z_j} \in L^\infty$, where $z_j \in \mathbb{D}$ and $a_j \in \mathbb{C}$.	$f_k = \hat{\varphi}(-k-1),\ k \in \mathbb{N}$
Infinite Hankel matrix: $\mathbf{\Gamma_f} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}),\ p = 1, 2$, $\mathbf{\Gamma_f} := (f_{j+k})_{j,k=0}^{\infty}$ for $\mathbf{f} := (f_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$	Hankel operator: $H_\varphi : H^2 \mapsto L^2 \ominus H^2$ for $\varphi \in L^\infty$ with Fourier coefficients $(\hat{\varphi}_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, $H_\varphi\vartheta := \mathrm{P}_-(M_\varphi\vartheta) = \mathrm{P}_-(\varphi\vartheta)$	$\mathcal{F}(H_\varphi\vartheta) = \mathbf{P}_-(\mathbf{f} * \mathbf{v}) = \mathbf{\Gamma_f}\mathbf{v}$ for $\mathbf{f} := (\hat{\varphi}_{-k-1})_{k=0}^{\infty}$ and $\mathbf{v} := (\hat{\vartheta}_k)_{k=0}^{\infty}$
Fourier coefficients of a Blaschke product $\mathbf{b} = (b_k)_{k=0}^{\infty} := (\widehat{B}_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$	Finite Blaschke product: $B(z) := \prod_{j=1}^{K} \frac{z-\alpha_j}{1-\overline{\alpha_j}z} \in H^2$ for $K > 0$	$\mathbf{b} = \mathcal{F}(B)$
Infinite triangular Toeplitz matrix: $\mathbf{T_b} : \ell^p(\mathbb{N}) \mapsto \ell^p(\mathbb{N}),\ p = 1, 2$, $\mathbf{T_b} := (b_{j-k})_{j,k=0}^{\infty}$ with $\mathbf{b} := (\widehat{B}_k)_{k=0}^{\infty}$	Multiplication operator: $M_B : H^2 \mapsto H^2,\ \ M_B\vartheta := B\vartheta$	$\mathcal{F}(M_B\vartheta) = \mathbf{b} * \mathbf{v} = \mathbf{T_b}\mathbf{v}$ for $\mathbf{v} := (\hat{\vartheta}_k)_{k=0}^{\infty}$
Product of Hankel and Toeplitz matrices: $\mathbf{\Gamma_f}\mathbf{T_b} := \left(\sum_{i=0}^{\infty} f_{k+i} b_{i-j}\right)_{k,j=0}^{\infty}$ with $\mathbf{b} := (\widehat{B}_k)_{k=0}^{\infty}$ and $\mathbf{f} := (f_k)_{k=0}^{\infty} \in \ell^1(\mathbb{N})$	Hankel operator: $H_{B\varphi} : H^2 \mapsto L^2 \ominus H^2$ for $\varphi \in L^\infty$ with Fourier coefficients $(\hat{\varphi}_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, $H_{B\varphi}\vartheta := \mathrm{P}_-(M_{B\varphi}\vartheta) = \mathrm{P}_-(B\varphi\vartheta)$	$\mathcal{F}(H_{B\varphi}\vartheta) = \mathbf{P}_-(\mathbf{b} * \mathbf{f} * \mathbf{v}) = \mathbf{\Gamma_f}\mathbf{T_b}\mathbf{v}$ for $\mathbf{f} := (\hat{\varphi}_{-k-1})_{k=0}^{\infty}$ and $\mathbf{v} := (\hat{\vartheta}_k)_{k=0}^{\infty}$

3. AAK Theory

In this chapter we will present a special case of the so-called AAK theorem, namely for infinite Hankel matrices with finite rank. We will introduce two versions of the theorem, in terms of infinite Hankel matrices and sequences as well as in terms of Hankel operators and functions on H^2. In Section 3.1 we provide a new proof for the discrete version of the theorem using only tools from linear algebra and Fourier analysis. In Section 3.2 the proof for the complex analysis version of the theorem is presented. Both proofs have similar structure and involve the same mathematical objects just on different "sides" of the Fourier transform. In contrast, the original result from [1] and all further representations of the AAK theory we are aware of, involve fundamental theorems in complex analysis for approximation of meromorphic functions, such as the Nehari theorem and the Beurling theorem, see e.g. [15, 40, 55, 34]. The new proof of the AAK Theorem in the linear algebra setting is one of the main new results of this thesis. The new insights which we achieve by these considerations may have important impact on the understanding and the computation of low rank approximations of Hankel matrices. Also the connection between the AAK theory and the Prony's method can be understood better due to these considerations.
Let us state the two different versions of the AAK theorem first, beginning with the discrete one.

Theorem 3.1. *Let the Hankel matrix* $\mathbf{\Gamma_f}$ *of rank* N *be generated by the sequence* $\mathbf{f}$ *of the form* (1.1) *with* $1 > |z_1| \geqslant \cdots \geqslant |z_N| > 0$. *Let the* N *non-zero singular values of* $\mathbf{\Gamma_f}$ *be ordered by size,* $\sigma_0 \geqslant \sigma_1 \ldots \geqslant \sigma_{N-1} > 0$. *Then, for each* $K \in \{0, \ldots, N-1\}$ *satisfying* $\sigma_K \neq \sigma_k$ *for* $K \neq k$, *the Laurent polynomial of the corresponding con-eigenvector* $\mathbf{v}^{(K)} = (v_l^{(K)})_{l=0}^{\infty}$,

$$P_{\mathbf{v}^{(K)}}(z) := \sum_{l=0}^{\infty} v_l^{(K)} z^l,$$

has exactly K *zeros* $z_1^{(K)}, \ldots, z_K^{(K)}$ *in* $\mathbb{D}\backslash\{0\}$, *repeated according to their multiplicity. Furthermore, if* $z_1^{(K)}, \ldots, z_K^{(K)}$ *are pairwise different, then there exist coefficients* $\tilde{a}_1, \ldots, \tilde{a}_K \in \mathbb{C}$ *such that for*

$$\tilde{\mathbf{f}}^{(K)} = \left(\tilde{f}_l^{(K)}\right)_{l=0}^{\infty} = \left(\sum_{j=1}^{K} \tilde{a}_j (z_j^{(K)})^l\right)_{l=0}^{\infty} \tag{3.1}$$

we have

$$\|\mathbf{\Gamma_f} - \mathbf{\Gamma}_{\tilde{\mathbf{f}}^{(K)}}\| = \sigma_K.$$

In order to formulate the theorem in terms of Hankel operators and functions on H^2, we recall from Example 2.8 that the samples of the exponential sum (1.1) can be seen as the Fourier coefficients with negative index of a rational function of the form

$$\varphi(z) := \sum_{j=1}^{N} \frac{a_j}{z - z_j}. \tag{3.2}$$

Therefore, the problem of reduction of the number of terms in the exponential sum (1.1) can be "translated" into the function domain. Namely, our goal is now to reduce the number of poles in $\mathbb{D}$ of the rational function φ.
This reformulation of the problem leads to the following definition. Let K be a natural number. We denote by $\mathcal{R}^{(K)}$ the space of rational functions in $(L^2 \ominus H^2) \cap L^\infty$ having at most K poles in $\mathbb{D}$. Note that a function ψ in $\mathcal{R}^{(K)}$ has only Fourier coefficients with negative index and hence no poles in $\mathbb{C}\backslash\mathbb{D}$. Now we can reformulate Theorem 3.1 in terms of rational functions and corresponding Hankel operators.

Theorem 3.2. *Let φ be a rational function with exactly N distinct poles in $\mathbb{D}$ given by (3.2) and H_φ the corresponding Hankel operator of rank N. Further, let the N non-zero singular values of H_φ be ordered non-increasingly, i.e. $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Then, for every $K \in \{0, \ldots, N-1\}$ satisfying $\sigma_K \neq \sigma_k$ for $K \neq k$, the corresponding singular function $\vartheta^{(K)}$ has exactly K zeros $z_1^{(K)}, \ldots, z_K^{(K)}$ in $\mathbb{D}$, repeated according to their multiplicity. Furthermore, if $z_j^{(K)}$ are pairwise different for $j = 1, \ldots, K$, then there exist coefficients $a_j^{(K)} \in \mathbb{C}$, $j = 1, \ldots, K$, such that for*

$$\tilde{\varphi}(z) = \sum_{j=1}^{K} \frac{a_j^{(K)}}{z - z_j^{(K)}} \in L^2 \ominus H^2 \tag{3.3}$$

we have

$$\inf_{\mathrm{Rank}(H_\psi) \leqslant K} \|H_\varphi - H_\psi\| = \|H_\varphi - H_{\tilde{\varphi}}\| = \sigma_K(H_\varphi).$$

3.1. Proof of the AAK Theorem for Hankel matrices with finite rank

In the following we will use the results about infinite Hankel and Toeplitz matrices from Section 2.1. Note that the proof of the AAK theorem similar to the one presented below can be found in our paper [43]. We start the preparations for the actual proof with some useful properties of special Hankel matrices. Let the sequence $\mathbf{f}$ be of the special form (1.1) with $z_j \in \mathbb{D}$. Then the structure of the (con)-eigenvectors corresponding to the zero-con-eigenvalues of $\mathbf{\Gamma_f}$ can be described as follows.

Theorem 3.3. *Let* $\mathbf{f}$ *be a vector of the form* (1.1). *Then* $\mathbf{v} \in \ell^2(\mathbb{N})$ *satisfies* $\mathbf{\Gamma_f}\overline{\mathbf{v}} = \mathbf{0}$ *if and only if the corresponding Laurent polynomial satisfies* $P_{\overline{\mathbf{v}}}(z_j) = 0$ *for* $j = 1, \ldots, N$, *where the* z_j *are given in (1.1).*

Proof. Observe first that $P_{\overline{\mathbf{v}}}(z)$ is well-defined for each $z \in \mathbb{D}$. The assertion $\mathbf{\Gamma_f}\overline{\mathbf{v}} = \mathbf{0}$ implies

$$0 = (\mathbf{\Gamma_f}\overline{\mathbf{v}})_k = \sum_{r=0}^{\infty} f_{k+r}\overline{v}_r = \sum_{r=0}^{\infty}\sum_{j=1}^{N} a_j z_j^{k+r}\overline{v}_r = \sum_{j=1}^{N} a_j z_j^k \sum_{r=0}^{\infty} \overline{v}_r z_j^r = \sum_{j=1}^{N} a_j P_{\overline{\mathbf{v}}}(z_j) z_j^k,$$

for all $k \in \mathbb{N}$ and hence, $P_{\overline{\mathbf{v}}}(z_j) = 0$ for $j = 1, \ldots, N$. In particular we have

$$0 = \sum_{k=0}^{\infty}\sum_{j=1}^{N} a_j P_{\overline{\mathbf{v}}}(z_j) z_j^k z^k = \sum_{j=1}^{N} \frac{a_j P_{\overline{\mathbf{v}}}(z_j)}{1 - z_j z}$$

for all $z \in \mathbb{D}$. Conversely, $P_{\overline{\mathbf{v}}}(z_j) = 0$ obviously implies that $\mathbf{\Gamma_f}\overline{\mathbf{v}} = \mathbf{0}$ is satisfied. □

Next, in Lemmas 3.4 and 3.5 we will construct an infinite Hankel matrix with operator norm 1 that possesses a predetermined con-eigenvector $\mathbf{v} \in \ell^1(\mathbb{N})$ to the con-eigenvalue 1. For that purpose, we first need to understand the image of an infinite Hankel matrix.

Lemma 3.4. *For given sequences* $\mathbf{f} \in \ell^1(\mathbb{N})$ *and* $\mathbf{v} \in \ell^1(\mathbb{N})$ *with corresponding Laurent polynomials* $P_{\mathbf{f}}(z)$ *and* $P_{\mathbf{v}}(z)$ *the vector* $\mathbf{w} = (w_k)_{k=0}^{\infty}$ *obtained by*

$$\mathbf{w} = \mathbf{\Gamma_f}\mathbf{v}$$

satisfies

$$w_k = \frac{1}{2\pi}\int_0^{2\pi} P_{\mathbf{f}}(\mathrm{e}^{\mathrm{i}t}) P_{\mathbf{v}}(\mathrm{e}^{-\mathrm{i}t}) \mathrm{e}^{-\mathrm{i}tk} \mathrm{d}t, \qquad \forall k \in \mathbb{N}.$$

Proof. Let $P_{\mathbf{w}}(z) := \sum_{k=0}^{\infty} w_k z^k$. Then, on the one hand, we find

$$P_{\mathbf{w}}(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_{k+j} v_j z^k = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} f_k v_j z^{(k-j)}.$$

On the other hand,

$$P_{\mathbf{f}}(z) P_{\mathbf{v}}(z^{-1}) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_k v_j z^{(k-j)} = P_{\mathbf{w}}(z) + \sum_{j=0}^{\infty}\sum_{k=0}^{j-1} f_k v_j z^{(k-j)},$$

where in the second sum only negative powers of z occur. Hence, $P_{\mathbf{w}}(z)$ possesses the Fourier coefficients

$$w_k = \frac{1}{2\pi}\int_0^{2\pi} P_{\mathbf{f}}(\mathrm{e}^{\mathrm{i}t}) P_{\mathbf{v}}(\mathrm{e}^{-\mathrm{i}t}) \mathrm{e}^{-\mathrm{i}tk} \mathrm{d}t$$

for $k \in \mathbb{N}$. □

Now we consider the construction of a special infinite Hankel matrix with operator norm 1.

Lemma 3.5. *Let* $\mathbf{v} \in \ell^1(\mathbb{N})$ *with the corresponding Laurent polynomial* $P_{\mathbf{v}}(z)$ *be given. Assume that* $P_{\mathbf{v}}(z) \neq 0$ *for all* $z \in \partial\mathbb{D}$*. Further, let* $\mathbf{w} = (w_k)_{k=0}^{\infty}$ *be given by*

$$w_k := \frac{1}{2\pi}\int_0^{2\pi} \frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}tk}\mathrm{d}t, \qquad k \in \mathbb{N}.$$

Then $\mathbf{w} \in \ell^2(\mathbb{N})$*, and* $\|\mathbf{w}\|_2 = 1$*. Furthermore, the Hankel operator* $\mathbf{\Gamma_w}$ *satisfies* $\mathbf{\Gamma_w}\overline{\mathbf{v}} = \mathbf{v}$ *and it holds*

$$\|\mathbf{\Gamma_w}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} := \sup_{\mathbf{u}\in\ell^2(\mathbb{N})\setminus\{\mathbf{0}\}} \frac{\|\mathbf{\Gamma_w}\mathbf{u}\|_2}{\|\mathbf{u}\|_2} = \frac{\|\mathbf{\Gamma_w}\overline{\mathbf{v}}\|_2}{\|\mathbf{v}\|_2} = 1.$$

Proof. First we note that $\overline{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})} = P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})$ and thus it holds $\left|\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\right| = 1$ for all $t \in [0, 2\pi)$. Now we observe that by Parseval's identity

$$\|\mathbf{w}\|_2^2 = \sum_{k=0}^{\infty}|w_k|^2 = \left\|\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}\cdot})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}\cdot})}\right\|^2_{L^2([0,2\pi))} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\right|^2\mathrm{d}t = 1$$

and thus $\mathbf{w} \in \ell^2(\mathbb{N})$. Further, we obtain

$$\begin{aligned}(\mathbf{\Gamma_w}\overline{\mathbf{v}})_k &= \sum_{j=0}^{\infty} w_{k+j}\bar{v}_j = \sum_{j=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}t(k+j)}\bar{v}_j\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}tk}\sum_{j=0}^{\infty}\bar{v}_j\mathrm{e}^{-\mathrm{i}tj}\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}tk}\mathrm{d}t = v_k\end{aligned}$$

for all $k \in \mathbb{N}$ and thus $\mathbf{\Gamma_w}\overline{\mathbf{v}} = \mathbf{v}$. The norm of $\mathbf{\Gamma_w}$ is indeed equal to 1 since for arbitrary $\mathbf{u} \in \ell^2(\mathbb{N})$ it follows by Lemma 3.4 and Parseval's identity

$$\begin{aligned}\|\mathbf{\Gamma_w}\overline{\mathbf{u}}\|_2^2 &= \sum_{k=0}^{\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}P_{\mathbf{w}}(\mathrm{e}^{\mathrm{i}t})P_{\overline{\mathbf{u}}}(\mathrm{e}^{-\mathrm{i}t})\,\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t\right|^2 \\ &\leqslant \sum_{k=-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}P_{\overline{\mathbf{u}}}(\mathrm{e}^{-\mathrm{i}t})\,\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t\right|^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\right|^2|P_{\overline{\mathbf{u}}}(\mathrm{e}^{-\mathrm{i}t})|^2\,\mathrm{d}t \\ &\leqslant \sup_{t\in[0,2\pi)}\left|\frac{P_{\mathbf{v}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}}(\mathrm{e}^{-\mathrm{i}t})}\right|^2\frac{1}{2\pi}\int_0^{2\pi}|P_{\overline{\mathbf{u}}}(\mathrm{e}^{-\mathrm{i}t})|^2\,\mathrm{d}t = \sum_{k=-\infty}^{\infty}|\bar{u}_k|^2 = \|\mathbf{u}\|_2^2,\end{aligned}$$

and thus the assertion holds. □

This result also immediately implies $\|\mathbf{\Gamma_w}\mathbf{u}\|_{\ell^2(\mathbb{N})} \leqslant \|\mathbf{w}\|_{\ell^2(\mathbb{N})}\|\mathbf{u}\|_{\ell^1(\mathbb{N})} = \|\mathbf{u}\|_{\ell^1(\mathbb{N})}$ for all $\mathbf{u} \in \ell^1(\mathbb{N})$ by Young's inequality.

Let us come back to the Hankel matrix $\mathbf{\Gamma_f}$ of rank N and its con-eigenvalues and con-eigenvectors. We can state the following lemma.

Lemma 3.6. *Let $\mathbf{\Gamma_f}$ be the Hankel matrix of rank N generated by the sequence $\mathbf{f}$ of the form (1.1) with $1 > |z_1| \geqslant \cdots \geqslant |z_N| > 0$, and with con-eigenvectors $\mathbf{v}^{(l)}$, $l = 0, \ldots, N-1$ corresponding to the non-zero con-eigenvalues (resp. singular values) $\sigma_0 \geqslant \sigma_1 \ldots \geqslant \sigma_{N-1} > 0$. Then the following holds.*

(1) *The con-eigenvectors $\mathbf{v}^{(l)}$ have the same structure as $\mathbf{f}$ for all $l = 0, \ldots, N-1$, i.e. it holds*

$$v_k^{(l)} = \sum_{i=1}^{N} b_i^{(l)} z_i^k, \tag{3.4}$$

where the coefficients $b_1^{(l)}, \ldots, b_N^{(l)}$ are given by

$$b_i^{(l)} = \frac{1}{\sigma_l} a_i P_{\bar{\mathbf{v}}^{(l)}}(z_i).$$

(2) *Further, the corresponding Laurent polynomials $P_{\mathbf{v}^{(l)}}(z)$ are rational functions of the form*

$$P_{\mathbf{v}^{(l)}}(z) = \frac{1}{\sigma_l} \sum_{j=1}^{N} \frac{a_j P_{\bar{\mathbf{v}}^{(l)}}(z_j)}{1 - z_j z} \tag{3.5}$$

for all $z \in \overline{\mathbb{D}}$, where the numerator of $P_{\mathbf{v}^{(l)}}(z)$ is a polynomial of degree $N-1$.

Proof.

1. For all $l = 0, \ldots, N-1$ it holds

$$\begin{aligned} \sigma_l v_k^{(l)} = (\mathbf{\Gamma_f} \bar{\mathbf{v}}^{(l)})_k = \sum_{j=0}^{\infty} f_{j+k} \bar{v}_j^{(l)} &= \sum_{j=0}^{\infty} \sum_{i=1}^{N} a_i z_i^{j+k} \bar{v}_j^{(l)} \\ = \sum_{i=1}^{N} a_i \left(\sum_{j=0}^{\infty} \bar{v}_j^{(l)} z_i^j \right) z_i^k &= \sum_{i=1}^{N} a_i P_{\bar{\mathbf{v}}^{(l)}}(z_i) z_i^k, \end{aligned}$$

and thus $\mathbf{v}^{(l)}$ can be written as a sequence of samples of an exponential sum with nodes z_j, $j = 1, \ldots, N$, and coefficients $b_i^{(l)} = \frac{1}{\sigma_l} a_i P_{\bar{\mathbf{v}}^{(l)}}(z_i)$.

2. Using (1) for the corresponding Laurent polynomials we obtain

$$\begin{aligned} P_{\mathbf{v}^{(l)}}(z) &= \sum_{k=0}^{\infty} v_k^{(l)} z^k = \sum_{k=0}^{\infty} \sum_{j=1}^{N} \frac{1}{\sigma_l} a_j P_{\bar{\mathbf{v}}^{(l)}}(z_j) z_j^k z^k \\ &= \frac{1}{\sigma_l} \sum_{j=1}^{N} a_j P_{\bar{\mathbf{v}}^{(l)}}(z_j) \sum_{k=0}^{\infty} (z_j z)^k = \frac{1}{\sigma_l} \sum_{j=1}^{N} \frac{a_j P_{\bar{\mathbf{v}}^{(l)}}(z_j)}{1 - z_j z}. \end{aligned} \tag{3.6}$$

□

We will come back to the above lemma in Chapter 5 in order to derive the main algorithm for the explicite computation of the approximation sequences $\tilde{\mathbf{f}}^{(K)}$ in (3.1). We want to show now that for each single non-zero con-eigenvalue σ_K of $\mathbf{\Gamma_f}$ the Laurent series of the corresponding con-eigenvector $\mathbf{v}^{(K)}$ possesses exactly K zeros in $\mathbb{D}$, and moreover, that these zeros $z_1^{(K)}, \ldots, z_K^{(K)}$ can be used to construct a new Hankel matrix $\mathbf{\Gamma}_{\tilde{\mathbf{f}}}$ of rank K with $\tilde{\mathbf{f}}$ of the form (3.1) and $\|\mathbf{\Gamma_f} - \mathbf{\Gamma}_{\tilde{\mathbf{f}}}\| = \sigma_K$.
Let n_K be the number of zeros of $P_{\mathbf{v}^{(K)}}(z)$ in $\mathbb{D}$, where $0 \leqslant n_K \leqslant N-1$. Further let us denote those zeros by $\alpha_1, \ldots, \alpha_{n_K}$. We first show that $n_K \leqslant K$. Since the Laurent polynomial $P_{\mathbf{v}^{(K)}}(z)$ is a function in H^2, we know from Lemma 2.17(6), that $P_{\mathbf{v}^{(K)}}(z)$ can be written as

$$P_{\mathbf{v}^{(K)}}(z) = B^{(K)}(z) \cdot P_{\mathbf{u}^{(K)}}(z),$$

where all the zeros $\alpha_1, \ldots, \alpha_{n_K}$ of $P_{\mathbf{v}^{(K)}}$ inside $\mathbb{D}$ are collected in the Blaschke product

$$B^{(K)}(z) := \prod_{j=1}^{n_K} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z} = \sum_{j=0}^{\infty} b_j^{(K)} z^j. \tag{3.7}$$

Further, combining Lemma 2.17(6) and (3.6) we also know that $P_{\mathbf{u}^{(K)}}(z)$ is a function in H^2 given by

$$P_{\mathbf{u}^{(K)}}(z) := \frac{1}{\sigma_K} \frac{\prod_{j=1}^{n_K}(1 - \overline{\alpha}_j z) \prod_{j=n_K+1}^{N-1}(z - \beta_j)}{\prod_{j=1}^{N}(1 - z_j z)} \tag{3.8}$$

with β_j denoting the zeros of $P_{\mathbf{v}^{(K)}}(z)$ outside $\mathbb{D}$. Note that the function $P_{\mathbf{u}^{(K)}}(z)$ defines a sequence $\mathbf{u} \in \ell^1(\mathbb{N})$. By Lemma 2.19(1) it follows that

$$\mathbf{v}^{(K)} = \mathbf{T}_{\mathbf{b}^{(K)}} \mathbf{u}^{(K)}, \tag{3.9}$$

where $\mathbf{T}_{\mathbf{b}^{(K)}}$ denotes the triangular Toeplitz matrix corresponding to the Fourier coefficients $\mathbf{b}^{(K)} := (b_j^{(K)})_{j=0}^{\infty}$ of $B^{(K)}(z)$. Now we can prove the following.

Theorem 3.7. *Let $\mathbf{\Gamma_f}$ be the infinite Hankel matrix of finite rank N generated by $\mathbf{f} = (f_k)_{k=0}^{\infty}$ of the form (1.1) with non-zero singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \mathbf{v}^{(K)})$ be the K-th con-eigenpair of $\mathbf{\Gamma_f}$ with $\sigma_K \neq \sigma_k$ for $K \neq k$. Let $\mathbf{T}_{\mathbf{b}^{(K)}}$ be the Toeplitz matrix corresponding to the Blaschke product $B^{(K)}(z)$ as above. Then $\mathbf{\Gamma_f}\mathbf{T}_{\mathbf{b}^{(K)}}$ possesses the singular value σ_K with multiplicity at least $n_K + 1$, where n_K denotes the number of zeros of $P_{\mathbf{v}^{(K)}}$ in $\mathbb{D}$. In particular, we have $n_K \leqslant K$.*

Proof. Considering the Blaschke product as in (3.7), we define the n_k partial products by

$$B_j^{(K)}(z) := \sum_{r=0}^{\infty} (b_{j,r}^{(K)}) z^r = \prod_{k=1}^{j} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \qquad j = 1, \ldots, n_K,$$

where $\alpha_1, \ldots, \alpha_{n_K}$ are the zeros of $P_{\mathbf{v}^{(K)}}(z)$ inside $\mathbb{D}$. We employ the notation $\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}$ for the triangular Toeplitz matrix generated by the sequence of Fourier coefficients of $\prod_{k=j+1}^{n_K} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}$ such that

$$\mathbf{T}_{\mathbf{b}^{(K)}} = \mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}} \cdot \mathbf{T}_{\mathbf{b}_j^{(K)}} = \mathbf{T}_{\mathbf{b}_j^{(K)}} \cdot \mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}},$$

where all $\mathbf{b}_j^{(K)}$ are $\ell^1(\mathbb{N})$ sequences according to Lemma 2.19. We show now that the $n_K + 1$ vectors

$$\mathbf{v}^{(K)},\ \mathbf{T}^*_{\mathbf{b}_1^{(K)}}\mathbf{v}^{(K)}, \ldots, \mathbf{T}^*_{\mathbf{b}_{n_K}^{(K)}}\mathbf{v}^{(K)} \tag{3.10}$$

are linearly independent singular vectors of $\mathbf{\Gamma_f T_{b^{(K)}}}$ corresponding to the singular value σ_K. For $j = 0, \ldots, n_K$ (with $\mathbf{T}_{\mathbf{b}_0^{(K)}} := \mathbf{I}$) we obtain by Lemma 2.2 and Lemma 2.19

$$\begin{aligned}
& (\mathbf{\Gamma_f T_{b^{(K)}}})^*(\mathbf{\Gamma_f T_{b^{(K)}}})\mathbf{T}^*_{\mathbf{b}_j^{(K)}}\mathbf{v}^{(K)} \\
= \;& \mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\mathbf{\Gamma_f}\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}_j^{(K)}}\mathbf{T}^*_{\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{u}^{(K)} \\
= \;& \mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\mathbf{T}^T_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{\Gamma_f}\mathbf{T}_{\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{u}^{(K)} \\
= \;& \mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\mathbf{T}^T_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{\Gamma_f}\mathbf{v}^{(K)} \\
= \;& \sigma_K\mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\mathbf{T}^T_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\overline{\mathbf{v}}^{(K)} \\
= \;& \sigma_K\mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\overline{\mathbf{T}^*_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}_j^{(K)}}}\overline{\mathbf{u}}^{(K)} \\
= \;& \sigma_K\mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{\Gamma}^*_{\mathbf{f}}\overline{\mathbf{T}_{\mathbf{b}_j^{(K)}}}\overline{\mathbf{u}}^{(K)} \\
= \;& \sigma_K\mathbf{\Gamma}^*_{\mathbf{f}}\overline{\mathbf{T}_{\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}^{(K)}/\mathbf{b}_j^{(K)}}\mathbf{T}_{\mathbf{b}_j^{(K)}}}\overline{\mathbf{u}}^{(K)} \\
= \;& \sigma_K\mathbf{T}\mathbf{b}_j^{(K)^*}\mathbf{\Gamma}^*_{\mathbf{f}}\overline{\mathbf{v}}^{(K)} = \sigma_K^2\mathbf{T}^*_{\mathbf{b}_j^{(K)}}\mathbf{v}^{(K)}.
\end{aligned}$$

To show that the vectors (3.10) are linearly independent, we recall that by Lemma 2.2(4) the equality

$$\sum_{j=0}^{n_K}\gamma_j\mathbf{T}^*_{\mathbf{b}_j^{(K)}}\mathbf{v}^{(K)} = \mathbf{0}$$

for some coefficients $\gamma_j \in \mathbb{C},\ j = 0, \ldots, n_K$, is equivalent to

$$\mathbf{\Gamma}_{\mathbf{v}^{(K)}}\left(\sum_{j=0}^{n_K}\gamma_j\mathbf{b}_j^{(K)}\right) = \mathbf{0}.$$

Therefore the vectors (3.10) are linearly dependent if and only if $\left(\sum_{j=0}^{n_K}\gamma_j\mathbf{b}_j^{(K)}\right)$ is a zero-(con)-eigenvector of $\mathbf{\Gamma}_{\mathbf{v}^{(K)}}$. In this case, by Theorem 3.3 and (3.5), the corresponding Laurent polynomial

$$\sum_{r=0}^{\infty}\sum_{j=0}^{n_K}\gamma_j(\mathbf{b}_j^{(K)})_r z^r = \gamma_0 + \sum_{j=1}^{n_K}\gamma_j\prod_{k=1}^{j}\frac{z-\alpha_k}{1-\bar{\alpha}_k z}$$

possesses all zeros $z_1, \ldots, z_N$, defined in (1.1). Since the degree of the polynomial in the numerator above is n_K+1 and it holds $n_K \leqslant N-1$, we conclude that $\gamma_0 = \ldots = \gamma_{n_K} = 0$. Therefore, the vectors (3.10) are linearly independent and the multiplicity of σ_K as a singular value of $\mathbf{\Gamma_f T_{b^{(K)}}}$ is at least n_K+1. Since $\sigma_K(\mathbf{\Gamma_f T_{b^{(K)}}}) \leqslant \sigma_K(\mathbf{\Gamma_f})$ by Lemma 2.19, it follows that $n_K \leqslant K$. □

In the next step, we construct a sequence $\tilde{\mathbf{f}}^{(K)} = \mathbf{f} - \mathbf{g}^{(K)}$ such that

$$\mathrm{Rank}(\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}) = K$$

and

$$\|\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\| = \|\mathbf{\Gamma}_{\mathbf{f}-\tilde{\mathbf{f}}^{(K)}}\| = \sigma_K.$$

Let $\mathbf{\Gamma_g} = \mathbf{\Gamma}_{\mathbf{g}^{(K)}}$ be the Hankel matrix generated by $\mathbf{g}^{(K)} = (g_l^{(K)})_{l=0}^{\infty}$ with

$$g_l^{(K)} := \frac{\sigma_K}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}^{(K)}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}^{(K)}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}tl}\mathrm{d}t \tag{3.11}$$

for $l \in \mathbb{N}$. Then by Lemma 3.5 it follows that $\|\mathbf{\Gamma_g}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \sigma_K$ and $\mathbf{\Gamma_g}\overline{\mathbf{v}}^{(K)} = \sigma_K\mathbf{v}^{(K)}$. Now we can show

Theorem 3.8. *Let $\mathbf{\Gamma_f}$ be a Hankel operator of finite rank N generated by $\mathbf{f} = (f_k)_{k=0}^{\infty}$ with f_k of the form* (1.1) *with the non-zero con-eigenvalues $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \mathbf{v}^{(K)})$ be the K-th con-eigenpair of $\mathbf{\Gamma_f}$ with $\sigma_K \neq \sigma_k$ for $K \neq k$. Then the shift-invariant space*

$$\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} := \mathrm{clos}_{\ell^2(\mathbb{N})}\mathrm{span}\,\{S^l\overline{\mathbf{v}}^{(K)} : l \in \mathbb{N}\}$$

has at least co-dimension K in $\ell^2(\mathbb{N})$, and the matrix $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$ with $\mathbf{g}^{(K)}$ determined by (3.11) *has at least rank K. Moreover, for the operator norm of $\mathbf{\Gamma}_{\mathbf{g}^{(K)}}$ we have*

$$\|\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \|\mathbf{\Gamma_f} - \mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \sigma_K.$$

Proof. 1. Similarly as in the proof of Lemma 3.5 we observe that

$$\begin{aligned}(\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(K)})_l &= \sum_{j=0}^{\infty} g_{l+j}\bar{v}_j^{(K)} = \sum_{j=0}^{\infty}\frac{\sigma_K}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}^{(K)}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}^{(K)}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}t(l+j)}\bar{v}_j^{(K)}\mathrm{d}t \\ &= \frac{\sigma_K}{2\pi}\int_0^{2\pi}\frac{P_{\mathbf{v}^{(K)}}(\mathrm{e}^{\mathrm{i}t})}{P_{\overline{\mathbf{v}}^{(K)}}(\mathrm{e}^{-\mathrm{i}t})}\mathrm{e}^{-\mathrm{i}tl}\sum_{j=0}^{\infty}\bar{v}_j^{(K)}\mathrm{e}^{-\mathrm{i}tj}\mathrm{d}t \\ &= \frac{\sigma_K}{2\pi}\int_0^{2\pi}P_{\mathbf{v}}^{(K)}(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}tl}\mathrm{d}t = \sigma_K v_l^{(K)},\end{aligned}$$

for all $k \in \mathbb{N}$ and thus $\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(K)} = \sigma \mathbf{v}^{(K)}$, resp. $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(K)} = \mathbf{0}$. Moreover, by Lemma 3.5 it follows that $\|\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \sigma_K$.
We consider now the operator $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$. By Lemma 2.1, the shift-invariant space $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}}$ is a subset of $\operatorname{Ker}\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$. On the other hand, we observe that for the singular vectors $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(K-1)}$ corresponding to the larger singular values $\sigma_0, \ldots, \sigma_{K-1}$ we have

$$\begin{aligned}\|\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(r)}\|_2 &= \|\mathbf{\Gamma}_{\mathbf{f}}\overline{\mathbf{v}}^{(r)} - \mathbf{\Gamma}_{\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(r)}\|_2 \\ &\geqslant |\|\mathbf{\Gamma}_{\mathbf{f}}\overline{\mathbf{v}}^{(r)}\|_2 - \|\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\overline{\mathbf{v}}^{(r)}\|_2| \\ &\geqslant (\sigma_r - \sigma_K)\|\mathbf{v}^{(r)}\|_2 > 0.\end{aligned}$$

Thus, the K linearly independent con-eigenvectors $\overline{\mathbf{v}}^{(0)}, \ldots, \overline{\mathbf{v}}^{(K-1)}$ to the larger con-eigenvalues $\sigma_0 \geqslant \ldots \geqslant \sigma_{K-1}$ are not contained in the kernel of $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$ and thus not in $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}}$. Hence, codim $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} \geqslant K$, and $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$ possesses at least rank K. □

Finally, we conclude the following theorem.

Theorem 3.9. *Let $\mathbf{\Gamma}_{\mathbf{f}}$ be the Hankel operator of finite rank N generated by $\mathbf{f} = (f_k)_{k=0}^{\infty}$ of the form* (1.1) *with non-zero singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \mathbf{v}^{(K)})$ be the K-th con-eigenpair of $\mathbf{\Gamma}_{\mathbf{f}}$. Then for each $K \in \{0, \ldots, N-1\}$ where σ_K is a simple singular value we have:*

(1) *The Laurent polynomial $P_{\mathbf{v}^{(K)}}(z)$ corresponding to the con-eigenvector $\mathbf{v}^{(K)}$ has exactly K zeros $z_1^{(K)}, \ldots, z_K^{(K)}$ in $\mathbb{D}$ repeated according to multiplicity.*

(2) *Considering the Hankel matrix $\mathbf{\Gamma}_{\mathbf{g}^{(K)}}$ given by the sequence $\mathbf{g}^{(K)} = (g_k)_{k=0}^{\infty}$ in* (3.11), *it follows that $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$ possesses the rank K and*

$$\|\mathbf{\Gamma}_{\mathbf{g}^{(K)}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \|\mathbf{\Gamma}_{\mathbf{f}} - \mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} = \sigma_K.$$

(3) *The kernel of $\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}$ has co-dimension K. If the zeros $z_1^{(K)}, \ldots, z_K^{(K)}$ are pairwise different, then it satisfies*

$$\operatorname{Ker}(\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}}) = \mathcal{S}_{\overline{\mathbf{v}}^{(K)}} = (\operatorname{clos}_{\ell^2(\mathbb{N})}\operatorname{span}\{((z_1^{(K)})^l)_{l=0}^{\infty}, \ldots, ((z_{n_K}^{(K)})^l)_{l=0}^{\infty}\})^{\perp},$$

where $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} := \operatorname{clos}_{\ell^2(\mathbb{N})}\operatorname{span}\{S^l\overline{\mathbf{v}}^{(K)} : l \in \mathbb{N}\}$.

Proof. First we show that $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} = (\operatorname{clos}_{\ell^2(\mathbb{N})}\operatorname{span}\{((z_1^{(K)})^l)_{l=0}^{\infty}, \ldots, ((z_{n_K}^{(K)})^l)_{l=0}^{\infty}\})^{\perp}$, where $z_1^{(K)}, \ldots, z_{n_K}^{(K)}$ are all pairwise different zeros of $P_{\mathbf{v}^{(K)}}(z)$ inside $\mathbb{D}$. Indeed for all $l \in \mathbb{N}$ and for $j = 1, \ldots, n_K$,

$$\begin{aligned}\langle((z_j^{(K)})^r)_{r=0}^{\infty}, S^l\overline{\mathbf{v}}^{(K)}\rangle_{\ell^2(\mathbb{N})} &= \langle(S^*)^l((z_j^{(K)})^r)_{r=0}^{\infty}, \overline{\mathbf{v}}^{(K)}\rangle_{\ell^2(\mathbb{N})} \\ &= \sum_{r=0}^{\infty}(z_j^{(K)})^{r+l}\, v_r^{(K)} \\ &= (z_j^{(K)})^l \sum_{r=0}^{\infty}(z_j^{(K)})^r\, v_r^{(K)} = (z_j^{(K)})^l\, P_{\mathbf{v}^{(K)}}(z_j^{(K)}) = 0.\end{aligned}$$

Thus,

$$\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} \perp \operatorname{span}\{((z_1^{(K)})^l)_{l=0}^{\infty}, \ldots, ((z_{n_K}^{(K)})^l)_{l=0}^{\infty}\}.$$

Assume now, that $\mathbf{u} \in \ell^2(\mathbb{N})$ satisfies $\mathbf{u} \perp \operatorname{span}\{((z_1^{(K)})^l)_{l=0}^{\infty}, \ldots, ((z_{n_K}^{(K)})^l)_{l=0}^{\infty}\}$, i.e., that $P_{\overline{\mathbf{u}}}(z_j^{(K)}) = 0$ for $j = 1, \ldots, n_K$. We show that $\mathbf{u} \in \mathcal{S}_{\overline{\mathbf{v}}^{(K)}}$. We can rewrite

$$P_{\overline{\mathbf{u}}}(\mathrm{e}^{\mathrm{i}\omega}) = \prod_{j=1}^{n_K} \frac{(\mathrm{e}^{\mathrm{i}\omega} - z_j^{(K)})}{(1 - \overline{z}_j^{(K)}\mathrm{e}^{\mathrm{i}\omega})} P_{\mathbf{w}}(\mathrm{e}^{\mathrm{i}\omega}) = B^{(K)}(\mathrm{e}^{\mathrm{i}\omega}) P_{\mathbf{w}}(\mathrm{e}^{\mathrm{i}\omega})$$

with the same Blaschke product as in (3.7), where $P_{\mathbf{w}}(\mathrm{e}^{\mathrm{i}\omega})$ still corresponds to a sequence $\mathbf{w} = (w_l)_{l=0}^{\infty} \in \ell^1(\mathbb{N})$. Equivalently, we have $\overline{\mathbf{u}} = \mathbf{T}_{\mathbf{b}^{(K)}}\mathbf{w}$. Since $\mathbf{T}_{\overline{\mathbf{v}}^{(K)}}$ contains the columns $\overline{\mathbf{v}}^{(K)}$, $S\overline{\mathbf{v}}^{(K)}, \ldots$, the assertion $\mathbf{u} \in \mathcal{S}_{\overline{\mathbf{v}}^{(K)}}$ is equivalent to the assertion that there exists a sequence $\mathbf{y} \in \ell^2(\mathbb{N})$ such that

$$\overline{\mathbf{u}} = \mathbf{T}_{\mathbf{v}^{(K)}}\mathbf{y}.$$

By Lemma 2.2 and (3.9) this is equivalent to

$$\mathbf{T}_{\mathbf{b}^{(K)}}\mathbf{w} = \mathbf{T}_{\mathbf{b}^{(K)}}\mathbf{T}_{\mathbf{u}^{(K)}}\,\mathbf{y},$$

and thus to

$$\mathbf{w} = \mathbf{T}^*_{\mathbf{b}^{(K)}}\mathbf{T}_{\mathbf{b}^{(K)}}\mathbf{w} = \mathbf{T}_{\mathbf{u}^{(K)}}\,\mathbf{y}.$$

Now we observe that (3.8) implies

$$P_{\mathbf{u}^{(K)}}(z) = \frac{1}{\sigma_K} \frac{\prod\limits_{j=1}^{n_K}(1 - \overline{z}_j^{(K)}z) \prod\limits_{j=n_K+1}^{N-1}(-\beta_j^{(K)})(1 - (\beta_j^{(K)})^{(-1)}z)}{\prod\limits_{j=1}^{N}(1 - z_j z)},$$

and thus

$$\mathbf{T}^{-1}_{\mathbf{u}^{(K)}} = \frac{\sigma_K}{\prod_{j=n_K+1}^{N-1}(-\beta_j^{(K)})} \left(\prod_{j=1}^{n_K} \mathbf{T}_{\overline{z}_j^{(K)}}\right) \left(\prod_{j=n_K+1}^{N-1} \mathbf{T}_{(\beta_j^{(K)})^{-1}}\right) \mathbf{T}_{\tilde{\mathbf{p}}},$$

where $\mathbf{T}_{\overline{z}_j^{(K)}}$, $\mathbf{T}_{(\beta_j^{(K)})^{-1}}$ and $\mathbf{T}_{\tilde{\mathbf{p}}}$ are the infinite Toeplitz matrices generated by the sequences $((\overline{z}_j^{(K)})^r)_{r=0}^{\infty}$, $((\beta_j^{(K)})^{-r})_{r=0}^{\infty}$ and by the finite sequence $\tilde{\mathbf{p}} = (1, p_{N-1}, \ldots, p_0)$ containing the coefficients of the Prony polynomial in (2.4). The assertion (1) now follows since $\mathbf{T}_{\mathbf{u}^{(K)}}$ is invertible.
By Theorem 3.7 we have $n_K \leqslant K$, i.e., $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}}$ possesses at most co-dimension K. On the other hand, $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} \subseteq \operatorname{Ker}(\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}})$ and $\operatorname{Ker}(\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}})$ has at least co-dimension K by Theorem 3.8. Thus, $n_K = K$, i.e., $P_{\mathbf{v}^{(K)}}(z)$ possesses exactly K zeros in $\mathbb{D}$, and $\mathcal{S}_{\overline{\mathbf{v}}^{(K)}} = \operatorname{Ker}(\mathbf{\Gamma}_{\mathbf{f}-\mathbf{g}^{(K)}})$ has co-dimension K, which proves (3). Assertion (2) follows directly from Theorem 3.8. □

Theorem 3.1 is now a corollary of Theorem 3.9. In this last theorem the new sequence $\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{g}^{(K)}$ is explicitely given. Furthermore from Theorem 3.9(3) and Lemma 2.1 it follows that

$$\tilde{\mathbf{f}} \in \operatorname{clos}_{\ell^2(\mathbb{N})} \operatorname{span}\{((z_1^{(K)})^l)_{l=0}^{\infty}, \dots, ((z_K^{(K)})^l)_{l=0}^{\infty}\},$$

i.e., it can be written as a linear combination of the form (3.1).

Remark 3.10. The proof given in this subsection does not explicitly use the Theorems of Beurling and Nehari for Hankel operators. Nehari's result states that the norm of the operator $\mathbf{\Gamma_f}$ is equal to the infimum of the L^∞-norm over all bounded 2π-periodic functions whose Fourier coefficients coincide with f_k for $k \in \mathbb{N}$, see e.g. [55]. This result is "hidden" in Lemma 3.5, where a sequence $\mathbf{w}$ is constructed from the Fourier coefficients of a special function with norm 1 in L^∞.
Beurling's theorem essentially states that the linear span of all shifts of a given sequence $\mathbf{v}$ in $\ell^2(\mathbb{N})$ is characterized by the inner factor of its corresponding Laurent polynomial $P_{\mathbf{v}}(z)$. Thus assertion (3) of Theorem 3.9 is a direct consequence of Beurling's theorem. We have proven it directly by showing the invertibility of the Toeplitz matrix $\mathbf{T}_{\mathbf{u}^{(K)}}$.

△

3.2. Proof of the AAK Theorem for finite rank Hankel operators on the Hardy space

We start the preparations for the proof of Theorem 3.2 with the following characterization of the kernel of a Hankel operator, which is the continuous version of Theorem 3.3. Note that a result similar to the assertion of the next theorem can be found, e.g., in [55], see Lemma 16.11.

Theorem 3.11. *For $N \in \mathbb{N}$ let φ be a rational function of the form*

$$\varphi(z) := \sum_{j=1}^{N} \frac{a_j}{z - z_j},$$

where $z_j \in \mathbb{D} \backslash \{0\}$ and $a_j \in \mathbb{C}$. Then a function $\vartheta \in H^2$ satisfies $H_\varphi \vartheta = 0$ if and only if $\vartheta(z_j) = 0$ for all $j = 1, \dots, N$.

Proof. By definition of the Hankel operator $H_\varphi \vartheta = 0$ holds if and only if $P_-(\varphi\vartheta) = 0$. This means that the function $\varphi\vartheta$ has no Fourier coefficients with negative indices and therefore $\varphi\vartheta \in H^2$. By Proposition 2.10 it follows that $\varphi\vartheta$ has no poles in $\mathbb{D}$. However we recall that φ can be written in the form $\varphi(z) = q(z)/p(z)$, where $p(z) = (z - z_1) \cdot \ldots \cdot (z - z_N)$ and $q(z)$ is some polynomial of degree $N - 1$ with $q(z_j) \neq 0$ for all z_j. Since $\varphi\vartheta = \frac{q}{p}\vartheta \in H^2$ we can conclude that the poles $z_j \in \mathbb{D}$ of φ have to be canceled out by the zeros of ϑ. Hence $\vartheta(z_j) = 0$ for all $j = 1, \dots, N$. □

Now, similarly to the previous section, we construct a special Hankel operator with operator norm 1 and a predetermined Schmidt pair corresponding to the singular value $\sigma = 1$. The operator-term analog to Lemmas 3.4 and 3.5 is presented below.

Lemma 3.12. *Let φ be given as in (3.2) and ϑ be a non-zero function in H^2. Further we define*

$$\psi := \frac{J\vartheta}{\vartheta} = \frac{\overline{M_z\vartheta}}{\vartheta} \in L^2.$$

Then the Hankel operator H_ψ has ϑ as a singular function corresponding to the singular value $\sigma = 1$. Furthermore we have

$$\|H_\psi\| := \sup_{\xi \in H^2\setminus\{0\}} \frac{\|H_\psi \xi\|_{L^2}}{\|\xi\|_{L^2}} = \frac{\|H_\psi \vartheta\|_{L^2}}{\|\vartheta\|_{L^2}} = 1.$$

Proof. We recall from Section 2.2.3 that the multiplication by z and conjugation of a function ϑ in H^2 causes the conjugation and reflection of the Fourier coefficients of ϑ across the zero point, i.e., $\overline{z\vartheta} \in L^2 \ominus H^2$, and by definition of ψ we have

$$H_\psi \vartheta = \mathrm{P}_-(\psi\vartheta) = \mathrm{P}_-\left(\frac{(J\vartheta)\vartheta}{\vartheta}\right) = \mathrm{P}_-(J\vartheta) = J\vartheta. \tag{3.12}$$

Further, on the unit circle $\partial\mathbb{D}$ it holds $z\bar{z} = 1$, and using the formula (2.19) for the adjoint Hankel operator and the definition of J we obtain

$$H_\psi^*(J\vartheta) = \mathrm{P}_+(\overline{\psi}J\vartheta) = \mathrm{P}_+\left(\frac{M_z\vartheta\overline{M_z\vartheta}}{\overline{\vartheta}}\right) = \mathrm{P}_+(M_z\overline{M}_z\vartheta) = \vartheta.$$

Thus, ϑ is a singular function of H_ψ corresponding to the singular value $\sigma = 1$ and $(\vartheta, J\vartheta)$ is a Schmidt pair to σ. Furthermore, analogously to the proof of Lemma 3.5, for all functions $\xi \in L^2$ we have

$$\|H_\psi\xi\|_{L^2} = \|\mathrm{P}_-(\psi\xi)\|_{L^2} \leqslant \|\psi\xi\|_{L^2} \leqslant \|\xi\|_{L^2},$$

since ψ is unimodular on the unit circle, that is

$$|\psi(\mathrm{e}^{\mathrm{i}t})| = \left|\frac{\mathrm{e}^{-\mathrm{i}t}\overline{\vartheta(\mathrm{e}^{\mathrm{i}t})}}{\vartheta(\mathrm{e}^{\mathrm{i}t})}\right| = 1 \qquad \forall t \in [0, 2\pi).$$

Therefore the assertion holds. □

For the proof of the AAK Theorem we consider again the Hankel operator H_φ of rank N with symbol φ given as in (3.2). Let $\vartheta^{(l)} \in H^2$, $l = 0, \dots, N-1$ be the singular functions of H_φ corresponding to the singular values $\sigma_0 \geqslant \sigma_1 \geqslant \dots \geqslant \sigma_{N-1} > 0$. We want to show that for each non-zero singular value σ_K of H_φ the corresponding singular function $\vartheta^{(K)}$ possesses exactly K zeros in $\mathbb{D}\setminus\{0\}$. Moreover we show that these zeros $\tilde{z}_1^{(K)}, \dots, \tilde{z}_K^{(K)}$ can be used to construct a new rational function $\tilde{\varphi}$ of the form (3.3) and the corresponding Hankel operator

$H_{\tilde{\varphi}}$ of rank K with $\|H_\varphi - H_{\tilde{\varphi}}\| = \sigma_K$.
Let n_K be the number of zeros of $\vartheta^{(K)}$ in $\mathbb{D}$, where $0 \leqslant n_K \leqslant N-1$. We denote these zeros by $\alpha_1, \dots, \alpha_{n_K}$. Imitating the structure of the discrete case from the previous section and in analogy to Theorem 3.7, we first show that $n_K \leqslant K$.

Theorem 3.13. *Let H_φ be the Hankel operator of finite rank N with symbol φ of the form (2.22) and non-zero singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \vartheta^{(K)})$ be the K-th singular pair of H_φ with $\sigma_K \neq \sigma_k$ for $K \neq k$. Let n_K be the number of zeros of $\vartheta^{(K)}$ in $\mathbb{D}$ and $B^{(K)}(z)$ the corresponding Blaschke product as in (3.7). Then $H_{\varphi B^{(K)}}$ possesses the singular value σ_K with multiplicity at least $n_K + 1$. In particular it holds $n_K \leqslant K$.*

Proof. Let $\alpha_1 \dots, \alpha_{n_K}$ denote the zeros of $\vartheta^{(K)}$ inside $\mathbb{D}$ such that

$$\vartheta^{(K)} = B^{(K)} \cdot \tilde{\vartheta}^{(K)}, \tag{3.13}$$

where $\tilde{\vartheta}^{(K)}$ is an H^2-function. Further let

$$B_j^{(K)}(z) = \prod_{k=1}^{j} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \qquad j = 1, \dots, n_K$$

be the partial Blaschke product of length j, and $B_{n_K - j}^{(K)} := B^{(K)} / B_j^{(K)}$ such that it holds

$$B^{(K)} = B_j^{(K)} \cdot B_{n_K - j}^{(K)}.$$

Similarly as in the proof of Theorem 3.7 we show that the $n_K + 1$ functions

$$\vartheta^{(K)}, \overline{B}_1^{(K)} \vartheta^{(K)}, \dots, \overline{B}_{n_K}^{(K)} \vartheta^{(K)} \tag{3.14}$$

are linearly independent singular functions of $H_{\varphi B^{(K)}}$ to the singular value σ_K. Note that the above functions are all in H^2, since with (3.13) we have

$$\overline{B}_j^{(K)} \vartheta^{(K)} = \overline{B}_j^{(K)} B_j^{(K)} B_{n_K - j}^{(K)} \tilde{\vartheta}^{(K)} = B_{n_K - j}^{(K)} \tilde{\vartheta}^{(K)} \in H^2 \quad \forall j = 1, \dots, n_K.$$

With $\overline{B}_0^{(K)} := 1$ we obtain by Lemma 2.17 and 2.18

$$\begin{aligned}
H^*_{\varphi B^{(K)}} H_{\varphi B^{(K)}} (\overline{B}_j^{(K)} \vartheta^{(K)}) &= H^*_{\varphi B^{(K)}} \mathrm{P}_- (\varphi B_{n_K - j}^{(K)} B_j^{(K)} \overline{B}_j^{(K)} \vartheta^{(K)}) \\
&= H^*_{\varphi B^{(K)}} \mathrm{P}_- (B_{n_K - j}^{(K)} \varphi \vartheta^{(K)}) \\
&= H^*_{\varphi B^{(K)}} \mathrm{P}_- (B_{n_K - j}^{(K)} \mathrm{P}_- (\varphi \vartheta^{(K)})) \\
&= H^*_{\varphi B^{(K)}} \mathrm{P}_- (B_{n_K - j}^{(K)} H_\varphi \vartheta^{(K)}) \\
&= H^*_{\varphi B^{(K)}} B_{n_K - j}^{(K)} H_\varphi \vartheta^{(K)} \\
&= \mathrm{P}_+ (\overline{\varphi} \overline{B}_j^{(K)} \overline{B}_{n_K - j}^{(K)} B_{n_K - j}^{(K)} H_\varphi \vartheta^{(K)})
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{P}_+(\overline{B}_j^{(K)}\overline{\varphi}H_\varphi\vartheta^{(K)}) \\
&= \mathrm{P}_+(\overline{B}_j^{(K)}\mathrm{P}_+(\overline{\varphi}H_\varphi\vartheta^{(K)})) \\
&= \mathrm{P}_+(\overline{B}_j^{(K)}H_\varphi^*H_\varphi\vartheta^{(K)}) \\
&= \sigma_K^2\mathrm{P}_+(\overline{B}_j^{(K)}\vartheta^{(K)}) \\
&= \sigma_K^2\overline{B}_j^{(K)}\vartheta^{(K)}.
\end{aligned}
$$

Now, for some complex coefficients γ_j, $j = 0, \dots, n_K$, it follows from (3.13) that

$$\sum_{j=0}^{n_K} \gamma_j \overline{B}_j^{(K)}\vartheta^{(K)} = \tilde{\vartheta}^{(K)} \sum_{j=0}^{n_K} \gamma_j B_{n_K-j}^{(K)} = 0$$

if and only if

$$\sum_{j=0}^{n_K} \gamma_j B_j^{(K)} = 0.$$

The numerator of the last sum is a non-zero polynomial of degree $n_K + 1$. Thus, the above equality leads to $\gamma_j = 0$ $\forall j = 0, \dots, n_K$ and the functions (3.14) are linearly independent. We have shown that σ_K is a singular value of $H_{\varphi B^{(K)}}$ with multiplicity at least $n_K + 1$. On the other hand, since $\sigma_K(H_{\varphi B^{(K)}}) \leqslant \sigma_K(H_\varphi)$ by Lemma 2.18, it follows that $n_K \leqslant K$. □

Now we construct a function $\tilde{\varphi} = \varphi - \psi^{(K)}$ in L^∞ such that

$$\mathrm{Rank}(H_{\tilde{\varphi}}) = \mathrm{Rank}(H_{\varphi-\psi^{(K)}}) = K$$

and

$$\|H_{\psi^{(K)}}\| = \|H_{\varphi-\tilde{\varphi}}\| = \sigma_K.$$

Let $H_\psi = H_{\psi^{(K)}}$ be the Hankel operator with symbol $\psi^{(K)} \in L^\infty$ given by the K-th singular value and singular function of H_φ, namely

$$\psi^{(K)} = \sigma_K \frac{\overline{M_z\vartheta^{(K)}}}{\vartheta^{(K)}}. \tag{3.15}$$

Similarly to Theorem 3.8 we can show the following.

Theorem 3.14. *Let H_φ be the Hankel operator of finite rank N with symbol φ of the form (2.22) and non-zero singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \vartheta^{(K)})$ be the K-th singular pair of H_φ with $\sigma_K \neq \sigma_k$ for $K \neq k$. Then the multiplication invariant space*

$$\mathcal{M}_{\vartheta^{(K)}} := \mathrm{clos}_{H^2}\mathrm{span}\{M_z^l\vartheta^{(K)} \ : \ l \in \mathbb{N}\}$$

has at least co-dimension K in H^2. Moreover the operator $H_{\psi^{(K)}}$ with $\psi^{(K)}$ determined by (3.15) has at least rank K. For the operator norm of $H_{\psi^{(K)}}$ we have

$$\|H_{\psi^{(K)}}\| = \|H_\varphi - H_{\varphi-\psi^{(K)}}\| = \sigma_K.$$

Proof. Let $\psi^{(K)}$ be given by (3.15). By Lemma 3.12 it follows that $\|H_{\psi^{(K)}}\| = \sigma_K$ and $\vartheta^{(K)}$ is a singular vector of $H_{\psi^{(K)}}$ corresponding to the singular value σ_K. From Section 2.2.3 we know that $(\vartheta^{(K)}, J\vartheta^{(K)})$ is the Schmidt pair of $H_{\psi^{(K)}}$ corresponding to σ_K. Since $(\vartheta^{(K)}, J\vartheta^{(K)})$ is also the Schmidt pair of H_φ corresponding to σ_K, it holds

$$H_{\psi^{(K)}}\vartheta^{(K)} = \sigma_K J\vartheta^{(K)} \qquad \text{and} \qquad H_\varphi\vartheta^{(K)} = \sigma_K J\vartheta^{(K)}$$

and thus we have

$$H_\varphi\vartheta^{(K)} - H_{\psi^{(K)}}\vartheta^{(K)} = H_{\varphi-\psi^{(K)}}\vartheta^{(K)} = 0.$$

We consider now the operator $H_{\varphi-\psi^{(K)}}$. By Lemma 2.16 the multiplication invariant space $\mathcal{M}_{\vartheta^{(K)}}$ is a subset of $\mathrm{Ker}(H_{\varphi-\psi^{(K)}})$. On the other hand, we observe that for $r = 0, \ldots, K-1$

$$\begin{aligned}\|H_{\varphi-\psi^{(K)}}\vartheta^{(r)}\|_{L^2\ominus H^2} &= \|H_\varphi\vartheta^{(r)} - H_{\psi^{(K)}}\vartheta^{(r)}\|_{L^2\ominus H^2} \\ &\geqslant |\|H_\varphi\vartheta^{(r)}\|_{L^2\ominus H^2} - \|H_{\psi^{(K)}}\vartheta^{(r)}\|_{L^2\ominus H^2} \\ &\geqslant (\sigma_r - \sigma_K)|\|J\vartheta^{(r)}\|_{L^2\ominus H^2} > 0.\end{aligned}$$

Therefore, the K linearly independent Schmidt vectors $\vartheta^{(0)}, \ldots, \vartheta^{(K-1)}$ to the larger singular values $\sigma_0 \geqslant \ldots \geqslant \sigma_{K-1}$ are not contained in the kernel of $H_{\varphi-\psi^{(K)}}$ and thus not in $\mathcal{M}_{\vartheta^{(K)}}$. Hence, $\mathrm{codim}\mathcal{M}_{\vartheta^{(K)}} \geqslant K$ and $H_{\varphi-\psi^{(K)}}$ possesses at least rank K. □

Finally, the analog to Theorem 3.9 in the discrete case in terms of Hankel operators and their singular functions can be formulated as follows.

Theorem 3.15. *Let H_φ be the Hankel operator of finite rank N with symbol φ of the form (2.22) and non-zero singular values $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1} > 0$. Further, let $(\sigma_K, \vartheta^{(K)})$ be the K-th singular pair of H_φ with $\sigma_K \neq \sigma_k$ for $K \neq k$. Then we have the following.*

(1) *The singular function $\vartheta^{(K)}$ has exactly K zeros $z_1^{(K)}, z_2^{(K)}, \ldots, z_K^{(K)}$ in $\mathbb{D}\backslash\{0\}$ repeated according to multiplicity.*

(2) *Considering the Hankel operator $H_{\psi^{(K)}}$ with $\psi^{(K)}$ given as in (3.15), it follows that $H_{\varphi-\psi^{(K)}}$ possesses rank K and*

$$\|H_{\psi^{(K)}}\| = \|H_\varphi - H_{\varphi-\psi^{(K)}}\| = \sigma_K.$$

(3) *The kernel of $H_{\varphi-\psi^{(K)}}$ has co-dimension K. If the zeros $z_1^{(K)}, z_2^{(K)}, \ldots, z_K^{(K)}$ are pairwise different, then it satisfies*

$$\mathrm{Ker}(H_{\varphi-\psi^{(K)}}) = \mathcal{M}_{\vartheta^{(K)}} = \mathrm{clos}_{H^2}\mathrm{span}\{\mathcal{K}_{z_1^{(K)}}, \mathcal{K}_{z_2^{(K)}}, \ldots, \mathcal{K}_{z_K^{(K)}}\}^\perp,$$

where $\mathcal{M}_{\vartheta^{(K)}} := \mathrm{clos}_{H^2}\mathrm{span}\{M_z^k \vartheta^{(K)} \ : \ k \in \mathbb{N}\}$ *is the multiplication invariant space generated by* $\vartheta^{(K)}$ *and*

$$\mathcal{K}_{z_j^{(K)}}(z) := \frac{1}{1 - \overline{z}_j^{(K)} z}, \qquad j = 1, \ldots, K$$

are the reproducing kernels of H^2 *introduced in Section 2.2.2.*

Proof. First we show that

$$\mathcal{M}_{\vartheta^{(K)}} = \mathrm{clos}_{H^2}\mathrm{span}\{\mathcal{K}_{z_1^{(K)}}, \mathcal{K}_{z_2^{(K)}}, \ldots, \mathcal{K}_{z_{n_K}^{(K)}}\}^{\perp},$$

where $z_1^{(K)}, z_2^{(K)}, \ldots, z_{n_K}^{(K)}$ are the pairwise different zeros of $\vartheta^{(K)}$ in $\mathbb{D}\backslash\{0\}$. Indeed, using the reproducing property (2.15) of $\mathcal{K}_{z_j^{(K)}}$ we obtain for all $l \in \mathbb{N}$

$$\langle M_z^l \vartheta^{(K)}, \mathcal{K}_{z_j^{(K)}} \rangle_{H^2} = (M_z^l \vartheta^{(K)})(z_j^{(K)}) = (z_j^{(K)})^l \vartheta^{(K)}(z_j^{(K)}) = 0,$$

since $\vartheta^{(K)}(z_j^{(K)}) = 0$ for all $j = 1, \ldots, n_K$. Thus,

$$\mathcal{M}_{\vartheta^{(K)}} \perp \mathrm{span}\{\mathcal{K}_{z_1^{(K)}}, \mathcal{K}_{z_2^{(K)}}, \ldots, \mathcal{K}_{z_{n_K}^{(K)}}\}.$$

Assume now that $\phi \in H^2$ satisfies $\phi \perp \mathrm{span}\{\mathcal{K}_{z_1^{(K)}}, \mathcal{K}_{z_2^{(K)}}, \ldots, \mathcal{K}_{z_{n_K}^{(K)}}\}$, i.e., $\phi(z_j^{(K)}) = 0$ for all $j = 1, \ldots, n_K$. We show that $\phi \in \mathcal{M}_{\vartheta^{(K)}}$. First we recall that by (3.13) $\vartheta^{(K)}$ can be written as

$$\vartheta^{(K)} = B^{(K)} \cdot \tilde{\vartheta}^{(K)},$$

where $\tilde{\vartheta}^{(K)}$ is a non-zero function in H^2. Since ϕ has the same zeros in $\mathbb{D}$ as $\vartheta^{(K)}$, we can rewrite

$$\phi(z) = \prod_{j=1}^{n_K} \frac{z - z_j^{(K)}}{1 - \overline{z}_j^{(K)} z} \cdot \tilde{\phi}(z) = B^{(K)}(z) \cdot \tilde{\phi}(z)$$

with $\tilde{\phi} \in H^2$ and the same Blaschke product $B^{(K)}$ as in (3.13). Thus,

$$\phi(z) = B^{(K)}(z) \cdot \tilde{\vartheta}^{(K)}(z) \cdot \frac{\tilde{\phi}(z)}{\tilde{\vartheta}^{(K)}(z)} = \vartheta^{(K)}(z) \cdot \frac{\tilde{\phi}(z)}{\tilde{\vartheta}^{(K)}(z)},$$

where $\tilde{\phi}/\tilde{\vartheta}^{(K)}$ is a function in H^2, since $\tilde{\vartheta}^{(K)}$ has no zeros in $\mathbb{D}$. Let $(c_j)_{j=0}^{\infty}$ be the coefficients of the Laurent polynomial of $\tilde{\phi}/\tilde{\vartheta}^{(K)}$. Then it holds

$$\phi(z) = \vartheta^{(K)}(z) \cdot \frac{\tilde{\phi}(z)}{\tilde{\vartheta}^{(K)}(z)} = \sum_{j=0}^{\infty} c_j z^j \vartheta^{(K)}(z) = \sum_{j=0}^{\infty} c_j (M_z^j \vartheta^{(K)})(z)$$

and hence $\phi \in \mathcal{M}_{\vartheta^{(K)}}$.
Now, by Theorem 3.13, we have $n_K \leqslant K$, i.e. $\mathcal{M}_{\vartheta^{(K)}}$ possesses at most co-dimension K. On the other hand, $\mathcal{M}_{\vartheta^{(K)}} \subset \mathrm{Ker}(H_{\varphi-\psi^{(K)}})$ and $\mathrm{Ker}(H_{\varphi-\psi^{(K)}})$ has at least co-dimension K by the previous theorem. Thus $n_K = K$, i.e., $\vartheta^{(K)}$ possesses exactly K zeros in $\mathbb{D}\backslash\{0\}$ and $\mathrm{Ker}(H_{\varphi-\psi^{(K)}}) = \mathcal{M}_{\vartheta^{(K)}}$ has co-dimension K. Assertion (2) is now a direct consequence of the previous theorem. □

Now again the AAK Theorem 3.2 is a corollary of Theorem 3.15. The new rational function $\tilde{\varphi} = \varphi - \psi^{(K)}$ is given by Theorem 3.15(3) and Lemma 2.16, since we have

$$\overline{\tilde{\varphi}} = \overline{\varphi - \psi^{(K)}} \perp \mathcal{M}_{\vartheta^{(K)}}$$

and thus

$$\tilde{\varphi} \in \mathrm{clos}_{H^2}\mathrm{span}\{\overline{\mathcal{K}}_{z_1^{(K)}}, \overline{\mathcal{K}}_{z_2^{(K)}}, \ldots, \overline{\mathcal{K}}_{z_K^{(K)}}\}.$$

Hence, on $\partial\mathbb{D}$ $\tilde{\varphi}$ can be written as a linear combination

$$\tilde{\varphi}(z) = \sum_{j=1}^{K} \frac{\tilde{a}_j}{1 - z_j^{(K)}\overline{z}} = \sum_{j=1}^{K} \frac{z\tilde{a}_j}{z - z_j^{(K)}},$$

i.e., $\tilde{\varphi}$ is a rational function with K poles in $\mathbb{D}$ given by $z_1^{(K)}, \ldots, z_K^{(K)}$. According to Kronecker's theorem $H_{\tilde{\varphi}}$ has rank K and the assertion of Theorem 3.2 holds.

4. An overview over Prony-like methods

In this chapter a brief overview over the so-called Prony-like methods for reconstruction and approximation of exponential sums is given. These methods aim to solve the following problem. Let the function f be given by an exponential sum

$$f(x) = \sum_{j=1}^{N} a_j z_j^x, \tag{4.1}$$

where $a_j \neq 0$ are complex or real weights and $z_j := \mathrm{e}^{T_j}$ are pairwise different exponentials with $\mathrm{Im}(T_j) \in [0, 2\pi)$ $\forall j = 1, \ldots, N$. Given the length of the sum N and the samples

$$f_k := f(k) = \sum_{j=1}^{N} a_j z_j^k = \sum_{j=1}^{N} a_j \mathrm{e}^{T_j k}$$

for $k = 0, \ldots, 2N$ we want to reconstruct the parameters T_j or the nodes z_j and the coefficients a_j.

As we will observe later, the original method introduced by G. de Prony [48] in 1795 is known to be numerically unstable, see e.g. [45]. Therefore there have been several modifications of the original version developed in the last decades. In the following two sections we will give a summary of the original method, which can be found e.g. in [42], and one of the so far best performing modifications called *Approximate Prony's Method (APM)* by D. Potts and M. Tasche [46].

In this chapter we will deal with the finite setting. For $N \in \mathbb{N}$ and a vector $\mathbf{f} := (f_k)_{k=0}^{2N}$ of length $2N+1$ we denote by

$$\mathbf{H}_{\mathbf{f}}^{(N+1)} := \begin{pmatrix} f_0 & f_1 & \cdots & f_N \\ f_1 & f_2 & \cdots & f_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_{N+1} & \cdots & f_{2N} \end{pmatrix} = \left(f_{k+j}\right)_{k,j=0}^{N} \in \mathbb{C}^{(N+1)\times(N+1)}$$

the leading principal minor of $\mathbf{\Gamma}_{\mathbf{f}}$.

4.1. Classical Prony's method

Following the idea of G. de Prony we consider the function of the form (4.1). Let N and the samples f_k for $k = 0, \dots, 2N$ be given. We introduce the *Prony polynomial*

$$P(z) := \prod_{j=1}^{N}(z - z_j) = \sum_{k=0}^{N} p_k z^k, \tag{4.2}$$

defined by it's zeros $z_1, \dots, z_N$. Note that in the monomial representation the coefficients p_k are normalized, i.e. $p_N = 1$. We observe, that the vector $\mathbf{p} := (p_k)_{k=0}^{N}$ is an eigenvector of the Hankel matrix $\mathbf{H}_\mathbf{f}^{(N+1)}$ corresponding to the zero-eigenvalue, since

$$\sum_{k=0}^{N} p_k f_{j+k} = \sum_{k=0}^{N} p_k \sum_{l=1}^{N} a_l z_l^{(j+k)} = \sum_{l=1}^{N} a_l z_l^j \sum_{k=0}^{N} p_k z_l^k = \sum_{l=1}^{N} a_l z_l^j P(z_l) = 0$$

and therefore

$$\mathbf{H}_\mathbf{f}^{(N+1)} \mathbf{p} = \mathbf{0}. \tag{4.3}$$

It means, $\mathbf{p}$ can be obtained by solving the homogeneous linear equation system (4.3). Note that the above system has indeed a unique solution, since the nodes z_j were chosen pairwise different and thus $\mathrm{Rank}(\mathbf{H}_\mathbf{f}^{(N+1)}) = N$. Once we have the coefficients $p_0, \dots, p_N$ of the Prony polynomial, its zeros $z_1, \dots, z_N$ can be computed by solving the eigenvalue problem with the companion matrix

$$\mathbf{C}_\mathbf{p} := \begin{pmatrix} 0 & 0 & \cdots & 0 & p_0 \\ 1 & 0 & \cdots & 0 & p_1 \\ 0 & 1 & \cdots & 0 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & p_{N-1} \end{pmatrix}.$$

Note that the nodes T_j can be computed easily from the zeros z_j by taking the principle branch of the logarithm. Given the exponentials z_j the corresponding coefficients $a_1, \dots, a_N$ can be obtained by solving the overdetermined system

$$\sum_{j=1}^{N} a_j z_j^k = f_k, \qquad k = 0, \dots, 2N \tag{4.4}$$

for instance with the least squares method. This leads to the following algorithm.

Algorithm 4.1 (Prony's Method)

Input: N and the samples f_k, $k = 0, \dots, 2N$.

1. Compute an eigenvector $\mathbf{p}$ of $\mathbf{H}_\mathbf{f}^{(N+1)}$ corresponding to the zero-eigenvalue.

2. Compute all zeros $z_j,\ j = 1, \ldots, N$ of the Prony polynomial $P(x)$.
3. Solve the overdetermined linear system (4.4) and obtain the weights $a_j,\ j = 1, \ldots, N$.

Output: Nodes z_j and weights a_j of the exponential sum (4.1).

Note that since $p_N = 1$, the homogeneous linear system (4.3) can be rewritten as

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_{N-1} \\ f_1 & f_2 & \cdots & f_N \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1} & f_N & \cdots & f_{2N-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{pmatrix} = - \begin{pmatrix} f_N \\ f_{N+1} \\ \vdots \\ f_{2N} \end{pmatrix}.$$

Thus it represents a homogeneous difference equation

$$-f_{j+N} = \sum_{k=0}^{N-1} p_k f_{j+k},$$

as introduced in Section 2.1.2, with the same coefficients p_k as in the Prony polynomial (4.2).

Let us proceed with the stability issues of Prony's method. In many applications noisy data

$$\tilde{\mathbf{f}} = (\tilde{f}_k)_{k=0}^{2N-1} = (f_k + e_k)_{k=0}^{2N-1} = \mathbf{f} + \mathbf{e},$$

instead of exact samples f_k is given. Usually the perturbed Hankel matrix $\mathbf{H}_{\tilde{\mathbf{f}}}^{N+1}$ is not singular anymore and the linear system (4.3) has no exact solution. In many modified Prony's methods, such as [47] and [46], this problem is approached by finding an eigenvector $\mathbf{v}$ corresponding to a small eigenvalue of $\mathbf{H}_{\tilde{\mathbf{f}}}^{(N+1)}$ and setting $\mathbf{p} \approx \mathbf{v}$. This can cause another difficulty in the next step of the Prony algorithm, namely the instabilities arising while the roots of Prony polynomial are computed. The error $\mathbf{e}$ in the data is propagated throughout the whole algorithm and leads to large perturbations in the output. Another drawback of the classical Prony's method is the fact that the a priori knowledge of the sum length N is required. As presented in [25], in many applications the length of the exponential sum is not known a priori and has to be reconstructed as well as the coefficients a_j and the nodes z_j. Moreover, in case of sparse approximation of certain functions by exponential sums, one is interested in adapting the length of the sum and ideally obtaining small N. Therefore, in order to get rid of these disadvantages in our algorithm, we use an approximate Prony's method presented in the next section.

4.2. Approximate Prony's method

In the following we present the so-called Approximate Prony's Method (APM) for non-increasing exponentials z_j, first introduced by D. Potts and M. Tasche

in [46]. APM can substitute Prony's method for parameter reconstruction of exponential sums. In this case the main goal of APM is to improve the poor numerical performance of the classical Prony's method for noisy data. Furthermore, the authors get rid of the necessity of the a priori knowledge of N. For the Approximate Prony's Method only an upper bound for N is required. However APM is able to do more than just to reconstruct. It can also be applied to samples of functions, which are not necessarily exponential sums. In this case APM approximates the function by a sum of non-increasing exponentials. This method can be seen as a modification of the classical Prony's method as well as the generalization of [8] to a perturbed rectangular Hankel matrix.
First, we assume the underlying structure of the original function to be an exponential sum (4.1) of length N. Let the exponentials $z_j^k,\ j = 1, \dots, N$ be non-increasing and non-vanishing for $k \geqslant 0$, i.e. $z_j \in \mathbb{D} \backslash \{0\}\ \forall j$. The authors of [46] deal with noisy data and consider the vector of samples to be of the form

$$\tilde{\mathbf{f}} = (\tilde{f}_k)_{k=0}^{2M} = (f_k + e_k)_{k=0}^{2M} = \mathbf{f} + \mathbf{e},$$

where the noise is bounded by a hopefully small number ε_1, i.e. $|e_k| \leqslant \varepsilon_1$, and $M \geqslant N$ ensures a certain oversampling of the system. Furthermore they assume the upper bound L for N to be known a priori. In contrast to the classical Prony's method APM uses the perturbed rectangular Hankel matrix

$$\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}} := (\tilde{f}_{k+j})_{k,j=0}^{2M-L,L}$$

and it's singular value decomposition. However the general structure of both algorithms is similar, although APM adapts some steps from the classical Prony's method to the noisy data. Such a modification already occurs in the very beginning of the algorithm. Due to the noise it can not be assumed, that there exist a zero singular value of $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$. Thus in the first step of APM a singular value σ close to zero is evaluated, i.e. $\sigma \in (0, \varepsilon_2]$ for a certain accuracy $\varepsilon_2 > 0$. Let $\mathbf{u}$ be the corresponding right singular vector. Since $\mathbf{u} \in \mathbb{C}^{L+1}$, the polynomial $P_{\mathbf{u}}(x)$ has $L + 1$ zeros instead of N. The question is, how to sort out the superfluous zeros. The first approach would be to take only zeros in $\mathbb{D} \backslash \{0\}$, due to the a priori knowledge that the exponential sum has non-increasing exponentials. The problem is that the samples $\tilde{f}_k$ are noisy, so we cannot hope to find exactly N zeros in $\mathbb{D} \backslash \{0\}$. To solve this problem, the authors of [46] introduce the extended disk $\tilde{\mathbb{D}}(r) := \{z \in \mathbb{C} \ :\ |z| \leqslant r\}$ with radius $r \geqslant 1$ and take only zeros $\tilde{z}_j \in \tilde{\mathbb{D}}(r) \backslash \{0\},\ j = 1, \dots, \tilde{N}$ of $P_{\mathbf{u}}(x)$. Note that if r is sufficiently larger than 1, then $\tilde{N} \geqslant N$. In the next step of the algorithm, similarly to the classical Prony's method, the overdetermined linear system

$$\sum_{j=1}^{\tilde{N}} \tilde{a}_j \tilde{z}_j^k = f_k, \qquad k = 0, \dots, 2M$$

is solved and the weights $\tilde{a}_j,\ j = 1, \dots, \tilde{N}$ are obtained. Since it can happen that we still have too many nodes $\tilde{z}_j$, another attempt to reduce their number is done at this stage, namely by looking at the coefficients $\tilde{a}_j$. We remember that the bound of the noise $\mathbf{e}$ is given by ε_1. Thus it can be assumed, that the superfluous nodes $\tilde{z}_j$ correspond to the coefficients $\tilde{a}_j$, which have range less than the noise level, i.e. $|\tilde{a}_j| < \varepsilon_1$. Such nodes don't contain any substantial information about the original signal and can be discarded in this step. With the remaining set of nodes denoted by $\{z_j \; : \; j = 1, \ldots, N\}$ with $N \leqslant \tilde{N}$ the overdetermined linear system

$$\sum_{j=1}^{N} a_j z_j^k = f_k, \qquad k = 0, \ldots, 2M$$

is solved again, in order to obtain new weights a_j, $j = 1, \ldots, N$. The complete algorithm can be summarized as follows.

Algorithm 4.2 (APM)

Input: L, M as before, f_k, $k = 0, \ldots, 2M$, accuracy bounds $\varepsilon_1, \varepsilon_2$, radius r.

1. Compute a right singular vector $\mathbf{u} \in \mathbb{C}^{L+1}$ corresponding to a singular value $\sigma \in (0, \varepsilon_2]$ of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}} \in \mathbb{C}^{(2M-L+1)\times(L+1)}$.

2. Compute all zeros $\tilde{z}_j \in \tilde{\mathbb{D}}(r) \backslash \{0\}$, $j = 1, \ldots, \tilde{N}$ of the corresponding polynomial $P_{\mathbf{u}}(x)$.

3. Solve the overdetermined linear system

$$\sum_{j=1}^{\tilde{N}} \tilde{a}_j \tilde{z}_j^k = f_k, \qquad k = 0, \ldots, 2M$$

   and obtain the weights $\tilde{a}_j$, $j = 1, \ldots, \tilde{N}$.

4. Remove all the $\tilde{z}_j$ corresponding to $|\tilde{a}_j| < \varepsilon_1$ and denote the remaining set of nodes by $\{z_j \; : \; j = 1, \ldots, N\}$ with $N \leqslant \tilde{N}$.

5. Solve the overdetermined linear system

$$\sum_{j=1}^{N} a_j z_j^k = f_k, \qquad k = 0, \ldots, 2M$$

   again with respect to the new set of nodes and obtain the weights a_j, $j = 1, \ldots, N$.

Output: Nodes z_j, weights a_j and the length N of the exponential sum (4.1).

Even though APM is a rather stable method, it has several drawbacks, mostly caused by the choice of the parameters. For instance, in the first step, it can happen that for very small ε_2 there exist no singular value σ in the interval

$(0, \varepsilon_2]$, since the singular values of $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ don't decay fast enough. Thus, in practice one would first compute the singular values of $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ and then choose the parameter ε_2 accordingly. Furthermore there is no possibility to control the number of terms in the approximation sum.
Note that the APM algorithm can also be used for sparse approximation of decaying functions by exponential sums, since a suitable choice of parameters affects the resulting sum length of the output. We will see the performance of the approximation approach in Chapter 6.

5. Algorithms for sparse approximation of exponential sums

In this chapter we present two algorithms solving the following sparse approximation problem, where the first algorithm is new and is one of the main contributions of this work. Let the function f be of the form

$$f(x) = \sum_{j=1}^{N} a_j z_j^x$$

where $a_j \neq 0$ are complex or real coefficients and $z_j := \mathrm{e}^{T_j}$ pairwise different exponentials with $\mathrm{Im}(T_j) \in [0, 2\pi)$ $\forall j = 1, \ldots, N$. Further let the sequence of samples

$$f_k := f(k) = \sum_{j=1}^{N} a_j z_j^k = \sum_{j=1}^{N} a_j \mathrm{e}^{T_j k} \tag{5.1}$$

be denoted by $\mathbf{f} := (f_k)_{k=0}^{\infty}$. Given $L \geqslant N$ samples of f find new coefficients $\tilde{a}_j$ and new nodes $\tilde{z}_j$, such that for the sequence $\tilde{\mathbf{f}} = (\tilde{f}_k)_{k=0}^{\infty}$ with

$$\tilde{f}_k = \sum_{j=1}^{\tilde{N}} \tilde{a}_j \tilde{z}_j^k = \sum_{j=1}^{\tilde{N}} \tilde{a}_j \mathrm{e}^{\tilde{T}_j k} \tag{5.2}$$

it holds

$$\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \varepsilon,$$

where $\tilde{N} < N$ and ε some target accuracy.

The first algorithm in Section 5.1 solves the above approximation problem for decreasing exponentials $z_j \in \mathbb{D}\backslash\{0\}$, which implies $\mathrm{Re}(T_j) < 0$. It employs the AAK theory and Prony's method presented in the previous sections and can be found in our paper [43, 44]. The algorithm involves an idea of dimension reduction of the con-eigenvalue problem of an infinite Hankel matrix $\mathbf{\Gamma_f}$. We remark that in [8] a similar finite-dimensional approximation problem has been considered. The authors use $2N+1$ equidistant samples of a continuous function f to approximate it with a shorter exponential sum. However, in [8] neither Prony's method nor the AAK theorem are used explicitely and the dimension reduction is applied to a finite Hankel matrix of size $L \times L$ for $L \geqslant N$.

The second algorithm is presented in Section 5.3 and is known from the context of low rank approximation, for instance from [34], p. 68. It deals with finite Hankel matrices of special structure and involves the singular value decomposition to obtain a structured low rank approximation of the matrix. In the context of the AAK theory it can be seen as a special case of approximation of exponential sum with nodes on the unit circle. As an output it has a shorter sum, again with nodes on the unit circle, and the approximation error given by a singular value of the corresponding Hankel matrix.

5.1. Sparse approximation of exponential sums with AAK theorem and Prony-like method

In this section we want to apply the AAK Theorem 3.1 for infinite Hankel matrices with finite rank to compute a sparse approximation $\tilde{\mathbf{f}}$ of $\mathbf{f}$. Although the theorem involves computation of singular values and vectors of an infinite matrix as well as the computation of zeros of an infinite Laurent polynomial. In order to make these computations numerically feasible and with high precision, we introduce the dimension reduction approach below. Since the approximation error in Theorem 3.1 is given in terms of the operator norm of Hankel matrices, we shed light on the connection of this norm to the more common $\ell^2(\mathbb{N})$ norm of sequences.
In this section we consider the sequence $\mathbf{f}$ to be generated as in (5.1) with N complex coefficients $a_j \neq 0$ and pairwise different nodes $z_j \neq 0$ inside the unit disc $\mathbb{D}$.
Structure of the con-eigenvectors of $\mathbf{\Gamma_f}$. First, we need some useful properties concerning the structure of the con-eigenvectors of $\mathbf{\Gamma_f}$ which we already used in the proof of Theorem 3.7. We sum up these properties in the following lemma.

Lemma 5.1. *Let $\mathbf{f} = (f_j)_{j=0}^{\infty} \in \ell^1(\mathbb{N})$ be given as in (5.1) and let $\sigma_l \neq 0$ be a fixed con-eigenvalue of the infinite Hankel matrix $\mathbf{\Gamma_f} = (f_{j+k})_{j,k=0}^{\infty}$ with the corresponding con-eigenvector $\mathbf{v}^{(l)} := (v_k^{(l)})_{k=0}^{\infty}$. Then $\mathbf{v}^{(l)}$ can be represented by*

$$v_k^{(l)} = \sum_{j=1}^{N} b_j^{(l)} z_j^k, \qquad k = 0, 1, \dots,$$

where z_j, $j = 1, \dots, N$ are the same as in (5.1) and the coefficients $b_j^{(l)}$, $j = 1, \dots, N$ satisfy

$$b_j^{(l)} = \frac{1}{\sigma_l} a_j P_{\bar{\mathbf{v}}^{(l)}}(z_j) = \frac{1}{\sigma_l} a_j \overline{P_{\mathbf{v}^{(l)}}(\bar{z}_j)}. \tag{5.3}$$

Proof. Let $(\sigma_l, \mathbf{v}^{(l)})$ be a con-eigenpair of $\mathbf{\Gamma_f}$ for a fixed $l \in \{0, 1, \dots, N-1\}$, i.e. it holds $\mathbf{\Gamma_f} \bar{\mathbf{v}}^{(l)} = \sigma \mathbf{v}^{(l)}$. With the notation $P_{\bar{\mathbf{v}}^{(l)}}(z) = \sum_{j=0}^{\infty} \bar{v}_j^{(l)} z^j$ and (5.1) we obtain

$$\begin{aligned}\sigma_l v_k^{(l)} &= (\mathbf{\Gamma_f}\bar{\mathbf{v}}^{(l)})_k = \sum_{j=0}^{\infty} f_{j+k}\bar{v}_j^{(l)} = \sum_{j=0}^{\infty}\sum_{i=1}^{N} a_i z_i^{j+k}\bar{v}_j^{(l)} \\ &= \sum_{i=1}^{N} a_i \left(\sum_{j=0}^{\infty} \bar{v}_j^{(l)} z_i^j\right) z_i^k = \sum_{i=1}^{N} a_i P_{\bar{\mathbf{v}}^{(l)}}(z_i) z_i^k\end{aligned}$$

for all $k = 0, 1, 2, \ldots$. □

Dimension reduction for the con-eigenvalue problem. In the next theorem we present an alternative formulation of the con-eigenvalue problem for the infinite Hankel matrix $\mathbf{\Gamma_f}$. The new con-eigenvalue equation involves a finite Cauchy matrix of size $N \times N$ and can be solved numerically.

Theorem 5.2. *Let $\mathbf{f} = (f_j)_{j=0}^{\infty} \in \ell^1(\mathbb{N})$ be given as in (5.1) and let $(\sigma_l, \mathbf{v}^{(l)})$ be a fixed con-eigenpair of the infinite Hankel matrix $\mathbf{\Gamma_f} = (f_{j+k})_{j,k=0}^{\infty}$, where $l < N$, i.e., $\sigma_l \neq 0$. Then the con-eigenvalue equation*

$$\mathbf{\Gamma_f}\overline{\mathbf{v}}^{(l)} = \sigma_l \mathbf{v}^{(l)} \tag{5.4}$$

is equivalent to the following con-eigenvalue problem of the dimension N,

$$\mathbf{A}\mathbf{Z}\overline{\mathbf{b}}^{(l)} = \sigma_l \mathbf{b}^{(l)}, \tag{5.5}$$

where

$$\mathbf{A} := \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_N \end{pmatrix}, \quad \mathbf{Z} := \begin{pmatrix} \frac{1}{1-|z_1|^2} & \frac{1}{1-z_1\overline{z}_2} & \cdots & \frac{1}{1-z_1\overline{z}_N} \\ \frac{1}{1-\overline{z}_1 z_2} & \frac{1}{1-|z_2|^2} & \cdots & \frac{1}{1-z_2\overline{z}_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\overline{z}_1 z_N} & \frac{1}{1-\overline{z}_2 z_N} & \cdots & \frac{1}{1-|z_N|^2} \end{pmatrix}$$

and

$$\mathbf{b}^{(l)} := (b_1^{(l)}, \ldots, b_N^{(l)})^{\top} = \frac{1}{\sigma_l}(a_1 P_{\overline{\mathbf{v}}^{(l)}}(z_1), \ldots, a_N P_{\overline{\mathbf{v}}^{(l)}}(z_N))^{\top}.$$

Proof. Using (5.1) and Lemma 5.1 the con-eigenvalue problem (5.4) leads to

$$\begin{aligned} & \sum_{j=0}^{\infty} f_{j+k}\bar{v}_j^{(l)} = \sigma_l v_k^{(l)}, \quad \forall k = 0, 1, 2, \ldots \\ \Leftrightarrow \quad & \sum_{j=0}^{\infty}\left(\sum_{r=1}^{N} a_r z_r^{k+j}\right)\left(\overline{\sum_{s=1}^{N} b_s^{(l)} z_s^j}\right) = \sigma_l \sum_{r=1}^{N} b_r^{(l)} z_r^k \\ \Leftrightarrow \quad & \sum_{r=1}^{N} z_r^k \left(\sum_{s=1}^{N} a_r \overline{b}_s^{(l)} \sum_{j=0}^{\infty} (z_r\overline{z}_s)^j\right) = \sigma_l \sum_{r=1}^{N} b_r^{(l)} z_r^k. \\ \Leftrightarrow \quad & \sum_{r=1}^{N} z_r^k \left(\sum_{s=1}^{N} \frac{a_r \overline{b}_s^{(l)}}{1 - z_r\overline{z}_s}\right) = \sigma_l \sum_{r=1}^{N} b_r^{(l)} z_r^k. \end{aligned}$$

by use of the geometric series, since $|z_r\overline{z}_s| < 1$ for all $l, s = 1, \ldots, N$. Note that the vector $\mathbf{b} = (b_j)_{j=1}^N$ is given explicitely in Lemma 5.1. Moreover, the above equality is satisfied for all $k = 0, 1, 2, \ldots$, therefore we obtain

$$\sum_{s=1}^{N} \frac{a_r \overline{b}_s^{(l)}}{1 - z_r\overline{z}_s} = \sigma_l b_r^{(l)} \quad \forall r = 1, \ldots, N,$$

which is exactly the entry-wise representation of (5.5). □

Thus, instead of considering the infinite con-eigenvalue problem we can solve the equation (5.5). Due to Lemma 5.1, the con-eigenvector $\mathbf{v}^{(l)}$ of $\mathbf{\Gamma_f}$ is completely determined by it's coefficient vector $\mathbf{b}^{(l)}$.

Computation of the roots of a con-eigenpolynomial of $\mathbf{\Gamma_f}$. Now, as a consequence of Theorem 5.2, we can show that the infinite Laurent polynomial $P_{\mathbf{v}^{(l)}}(x)$ corresponding to a con-eigenvector $\mathbf{v}^{(l)}$ for $l < N$ has a finite rational function representation.

Corollary 5.3. *Let $\mathbf{f} = (f_j)_{j=0}^{\infty} \in \ell^1(\mathbb{N})$ be given as in (5.1) and let $(\sigma_l, \mathbf{v}^{(l)})$ be a fixed con-eigenpair of the infinite Hankel matrix $\mathbf{\Gamma_f} = (f_{j+k})_{j,k=0}^{\infty}$, where $l < N$, i.e., $\sigma_l \neq 0$. Then for the corresponding Laurent polynomial $P_{\mathbf{v}^{(l)}}(x)$ it holds*

$$P_{\mathbf{v}^{(l)}}(x) = \sum_{j=1}^{N} \frac{b_j^{(l)}}{1 - z_j x}, \tag{5.6}$$

where $b_j^{(l)}$ is the coefficient vector given in (5.3).

Proof. Using the structure of the con-eigenvectors $\mathbf{v}^{(l)}$ of $\mathbf{\Gamma_f}$ from Lemma 5.1, the Laurent polynomial $P_{\mathbf{v}^{(l)}}(x)$ can be written as

$$\begin{aligned} P_{\mathbf{v}^{(l)}}(x) &= \sum_{k=0}^{\infty} v_k^{(l)} x^k = \sum_{k=0}^{\infty} \sum_{j=1}^{N} b_j^{(l)} z_j^k x^k \\ &= \sum_{j=1}^{N} b_j^{(l)} \sum_{k=0}^{\infty} (z_j x)^k = \sum_{j=1}^{N} \frac{b_j^{(l)}}{1 - z_j x} \end{aligned}$$

for $|x| < \frac{1}{|z_j|}$ $\forall j = 1, \ldots, N$, and in particular for $x \in \mathbb{D}$, since $z_j \in \mathbb{D} \backslash \{0\}$ for all $j = 1, \ldots, N$. □

The above corollary indicates that $P_{\mathbf{v}^{(l)}}(x)$ is a rational function with numerator of degree at most $N - 1$, which is given by

$$\sum_{j=1}^{N} b_j^{(l)} z_j^{-1} \prod_{k=1, k\neq j}^{N} (z_k^{-1} - x) \tag{5.7}$$

Thus the zeros of $P_{\mathbf{v}^{(l)}}(x)$ in $\mathbb{D}$ can be obtained by computing the roots of the above polynomial.

On the approximation norm. Note that Theorem 3.1 provides the approximation error in terms of the operator norm of the corresponding Hankel matrix $\mathbf{\Gamma}_{\mathbf{f}-\tilde{\mathbf{f}}}$. The following considerations give us a connection to the $\ell^2(\mathbb{N})$ norm of sequences. For $\mathbf{e}_0 := (1, 0, 0, \ldots) \in \ell^2(\mathbb{N})$ it holds

$$\|\mathbf{f}\|_{\ell^2(\mathbb{N})} = \frac{\|\mathbf{\Gamma}_{\mathbf{f}}\mathbf{e}_0\|_{\ell^2(\mathbb{N})}}{\|\mathbf{e}_0\|_{\ell^2(\mathbb{N})}} \leqslant \|\mathbf{\Gamma}_{\mathbf{f}}\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})}.$$

Thus the norm assertion of the AAK theorem can be reformulated as

$$\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \|\mathbf{\Gamma}_{\mathbf{f}-\tilde{\mathbf{f}}}\| = \sigma_K.$$

The above considerations lead to the following algorithm.

Algorithm 5.1 (Algorithm for sparse approximation of exponential sums with decreasing exponentials)

Input: samples f_k, $k = 0, \ldots, M$ for sufficiently large $M \geqslant 2N - 1$, target approximation error ϵ or the desired length of the approximation sum $0 < K < N$.

1. Find the parameters $z_j \in \mathbb{D}\backslash\{0\}$ and $a_j \in \mathbb{C}\backslash\{0\}$, $j = 1, \ldots, N$ of the exponential representation of $\mathbf{f}$ in (5.1) using a Prony-like method.

2. Solve the con-eigenvalue problem (5.5) for the matrix $\mathbf{AZ}$ to compute the non-zero con-eigenvalues $\sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{N-1}$ of $\mathbf{\Gamma}_{\mathbf{f}}$. Determine the largest con-eigenvalue σ_K with $\sigma_K < \epsilon$ and the corresponding con-eigenvector $\mathbf{v}^{(K)}$. If a certain length of the approximation sum is desired, chose σ_K directly such that $0 < K < N$.

3. Compute the K zeros $z_j^{(K)}$, $j = 1, \ldots, K$ inside the unit disc $\mathbb{D}\backslash\{0\}$ of the con-eigenpolynomial $P_{\mathbf{v}^{(K)}}(z)$ of $\mathbf{\Gamma}_{\mathbf{f}}$ using it's rational function representation obtained in Corollary 5.3. More precisely, the zeros are obtained by computing the roots of the numerator polynomial (5.7). Note that the AAK Theorem 3.1 ensures that the number of zeros $z_j^{(K)}$ in $\mathbb{D}\backslash\{0\}$ is exactly K.

4. Compute the coefficients $\tilde{a}_j$ by solving the minimization problem

$$\min_{\tilde{a}_1,\ldots,\tilde{a}_K} \|\mathbf{f} - \tilde{\mathbf{f}}\|^2_{\ell^2(\mathbb{N})} = \min_{\tilde{a}_1,\ldots,\tilde{a}_K} \sum_{k=0}^{\infty} |f_k - \sum_{j=1}^{K} \tilde{a}_j (z_j^{(K)})^k|^2.$$

Output: sequence $\tilde{\mathbf{f}}$ of the form (5.1) such that $\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\ell^2(\mathbb{N})} \leqslant \sigma_K < \epsilon$.

Remark 5.4. Note that the $\ell^2(\mathbb{N})$-minimization problem in step 4 of the algorithm can be reformulated as

$$\min_{\tilde{a}_1,\ldots,\tilde{a}_K} \sum_{k=0}^{\infty} |f_k - \tilde{f}_k|^2 = \min_{\tilde{a}_1,\ldots,\tilde{a}_K} \left[\sum_{k=0}^{\infty} |f_k|^2 - 2\mathrm{Re}(\tilde{f}_k \overline{f}_k) + |\tilde{f}_k|^2 \right]$$
$$= \min_{\tilde{a}_1,\ldots,\tilde{a}_K} \left[-2\mathrm{Re} \sum_{k=0}^{\infty} \tilde{f}_k \overline{f}_k + \sum_{k=0}^{\infty} |\tilde{f}_k|^2 \right],$$

since $|f_k|^2$ does not depend on the coefficients $\tilde{a}_1, \ldots, \tilde{a}_K$. Furthermore, using the structure of f_k and $\tilde{f}_k$ the above problem breaks down to a least squares problem with complex coefficients as follows

$$\min_{\tilde{a}_1,\ldots,\tilde{a}_K} \left[-2\mathrm{Re} \sum_{k=0}^{\infty} \tilde{f}_k \overline{f}_k + \sum_{k=0}^{\infty} |\tilde{f}_k|^2 \right]$$
$$= \min_{\tilde{a}_1,\ldots,\tilde{a}_K} \left[-2\mathrm{Re} \sum_{j=1}^{K} \sum_{i=1}^{N} \tilde{a}_j \overline{a}_i \sum_{k=0}^{\infty} (z_j^{(K)} \overline{z}_i)^k + \sum_{j=1}^{K} \sum_{i=1}^{K} \tilde{a}_j \overline{\tilde{a}}_i \sum_{k=0}^{\infty} (z_j^{(K)} \overline{z}_i^{(K)})^k \right]$$
$$= \min_{\tilde{a}_1,\ldots,\tilde{a}_K} \left[-2\mathrm{Re} \sum_{j=1}^{K} \sum_{i=1}^{N} \frac{\tilde{a}_j \overline{a}_i}{1 - z_j^{(K)} \overline{z}_i} + \sum_{j=1}^{K} \sum_{i=1}^{K} \frac{\tilde{a}_j \overline{\tilde{a}}_i}{1 - z_j^{(K)} \overline{z}_i^{(K)}} \right].$$

△

The complexity of the above algorithm is determined by the computation of the con-eigenvalues in step 2 and the zeros of the con-eigenpolynomial in step 3. Further singular values and zero computation problems are usually part of the Prony-like method in step 1.

5.2. Stability of the algorithm for sparse approximation of exponential sums

In this section we discuss the stability issues of the Algorithm 5.1 and summarize two methods which try to overcome those. In order to improve the readability of this thesis, only the main procedure of each method is given below in form of a pseudocode. All the other algorithms, which occur in the main method as modules, are given in the appendix.
The problems with the stability of the Algorithm 5.1 mostly appear in the computation of the singular values of the matrix **AZ** in the second step, especially if the nodes z_j are close to each other. As we will see in Chapter 6, the singular values of **AZ** decay very fast, i. e. the matrix is badly conditioned
There exist several possibilities to improve the stability of the necessary SVD computation in step 2. The simplest way to do so is using the high precision floating point arithmetic, which is used in Chapter 6. Another way to overcome the stability issues is to use the structure of **AZ**, which is well known in

the literature as the so-called Cauchy matrix, defined as a matrix of the form

$$\mathbf{C} := \left(\frac{c_l d_s}{x_l + y_s}\right)_{l,s=1}^{N} = \begin{pmatrix} \frac{c_1 d_1}{x_1+y_1} & \frac{c_1 d_2}{x_1+y_2} & \cdots & \frac{c_1 d_N}{x_1+y_N} \\ \frac{c_2 d_1}{x_2+y_1} & \frac{c_2 d_2}{x_2+y_2} & \cdots & \frac{c_1 d_N}{x_1+y_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_N d_1}{x_N+y_1} & \frac{c_N d_2}{x_N+y_2} & \cdots & \frac{c_N d_N}{x_N+y_N} \end{pmatrix}, \tag{5.8}$$

where $c_j, d_j, x_j, y_j \in \mathbb{C}$ for $j = 1, \ldots, N$. It can be easily seen, that $\mathbf{C}$ of the form (5.8) can be written as the matrix product

$$\mathbf{C} = \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_N \end{pmatrix} \begin{pmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_N} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_1+y_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_N+y_1} & \frac{1}{x_N+y_2} & \cdots & \frac{1}{x_N+y_N} \end{pmatrix} \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_N \end{pmatrix}.$$

Thus, for $c_j := a_j/z_j$, $d_j = 1$ and $x_j := 1/z_j$, $y_j := -\bar{z}_j$ for $j = 1, \ldots, N$ our matrix $\mathbf{AZ}$ is a Cauchy matrix, since

$$(\mathbf{AZ})_{jk} = \frac{a_j}{1 - z_j \bar{z}_k} = \frac{a_j/z_j}{1/z_j - \bar{z}_k} \quad \text{for } k, j = 1, \ldots, N.$$

Below we summarize two attempts to solve the stability problem for Cauchy matrices, see [18] and [28]. The first paper only deals with the singular value decomposition of structured matrices, whereas the second explicitly computes the con-eigenvalues and con-eigenvectors. In both papers even the small singular values of a Cauchy matrix $\mathbf{C}$ are computed with high relative accuracy, i.e. the relative error $|\sigma_j - \tilde{\sigma}_j|$ between the exact con-eigenvalues σ_j and the computed con-eigenvalues $\tilde{\sigma}_j$ is small for all $j = 1, \ldots, N$.
In contrast, the classical numerical algorithms such as QR iteration, bisection method, traditional Jacobi, etc. compute the singular value decomposition only with high absolute accuracy. This means that the error estimation is bounded by $\mathcal{O}(\varepsilon)\sigma_1$, where ε is machine precision and σ_1 is the largest singular value. In our case, where the difference between σ_1 and σ_{N-1} is too large, this error estimation guarantees no accuracy at all.
The first paper [18] by J. Demmel proposes an algorithm for accurate singular value decomposition of Cauchy matrices with complexity $\mathcal{O}(N^3)$. The main idea in [18] is to compute the so-called rank revealing decomposition (RRD) $\mathbf{C} = \mathbf{XDY}^\top$, where $\mathbf{D}$ is diagonal and $\mathbf{X}$ and $\mathbf{Y}$ are well conditioned, prior to the singular value decomposition of $\mathbf{C}$. The author consideres such a decomposition to be provided by a modified Gaussian elimination with complete pivoting (GECP). It can be found as Algorithm 3 in [18] and uses the well known formula for the determinant of a Cauchy matrix. The complexity of the proposed GECP algorithm is $\frac{4}{3}N^3$. The computation of the SVD from an RRD was first published in [19] and involves a modified one-sided Jacobi method, which is first proposed in [20]. The full algorithm can be found as Algorithm

1 in [18] and reads as follows.

Algorithm 5.2 (High accuracy SVD for Cauchy matrices)

Input: Cauchy matrix $\mathbf{C}$ of the form (5.8).

1. Compute an RRD of $\mathbf{C} = \mathbf{XDY}^\top$ using GECP (Algorithm A.1).
2. Compute the QR factorization with pivoting on $\mathbf{XD}$ to obtain $\mathbf{XD} = \mathbf{QRP}$, where $\mathbf{P}$ is a permutation matrix. It holds $\mathbf{C} = \mathbf{QRPY}^\top$.
3. Multiply $\mathbf{W} = \mathbf{RPY}^\top$ (it must be conventional matrix multiplication) to get $\mathbf{C} = \mathbf{QW}$.
4. Compute the SVD of W using the one-sided Jacobi (Algorithm A.2) and obtain $\mathbf{W} = \overline{\mathbf{U}}\mathbf{\Sigma}\mathbf{V}^\top$. Thus $\mathbf{C} = \mathbf{Q}\overline{\mathbf{U}}\mathbf{\Sigma}\mathbf{V}^\top$.
5. Multiply $\mathbf{U} = \mathbf{Q}\overline{\mathbf{U}}$ to obtain the desired SVD $\mathbf{C} = \mathbf{U\Sigma V}^\top$.

Output: $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$, where $\mathbf{C} = \mathbf{U\Sigma V}^\top$ is the SVD of $\mathbf{C}$.

Remark 5.5. Note that in [18] the author transfers his algorithm for Cauchy matrices to the Vandermonde matrices. This is possible due to the observation that a Vandermonde matrix $\mathbf{V}$ can be transformed to a Cauchy matrix of the above structure via the discrete Fourier transform of length N, i.e. $\mathbf{VF}$ is of Cauchy structure. The detailed derivation of this connection can be found in [18] in Section 6.

$\triangle$

The second attempt to overcome the stability issues of Cauchy matrices can be found in [28]. Even though it is based on the main ideas from [18], in contrast to [18] it aims to explicitly solve the con-eigenvalue problem for a positive definite Cauchy matrix $\mathbf{C}$ from the singular value decomposition of $\overline{\mathbf{C}}\mathbf{C}$. Further the algorithm from [28] reduces the complexity of [18], since it computes the m-th con-eigenvalue in $\mathcal{O}(m^2N)$ operations due to the fact that some matrix factorizations are only computed partially.

The procedure of [28] works as follows. First, similar to [18], a rank revealing decomposition of $\mathbf{C}$ is computed by a modified LDL factorization, i.e. $\mathbf{P}^*\mathbf{CP} = \mathbf{LD}^2\mathbf{L}$, where $\mathbf{L}$ is a lower triangular square matrix with unity diagonal elements, $\mathbf{P}$ is a permutation matrix and $\mathbf{D}$ is diagonal. Note that this decomposition is equivalent to

$$\mathbf{C} = (\mathbf{PL})\mathbf{D}^2(\mathbf{PL})^*$$

and is of the form $\mathbf{C} = \mathbf{XD}^2\mathbf{X}^*$ for $\mathbf{X} := \mathbf{PL}$. To obtain this representation the authors use a modified GECP, similar to the Algorithm A.1, which is given as Algorithm A.4 in the Appendix. Since in [28] only a con-eigenvalue of a

certain size δ is of interest, the authors equip Demmel's algorithm with a stopping criterion with respect to the target size δ of the desired con-eigenvalue. The iteration terminates when the diagonal elements D_{ii}^2 are smaller than the product of δ^2 and the machine precision. Therefore the modified GECP can be computed in fewer than $\mathcal{O}(N^3)$ operations. At the end only a partial Cholesky decomposition is obtained i.e. $\mathbf{C} \approx \tilde{\mathbf{C}} = (\mathbf{PL})\mathbf{D}^2(\mathbf{PL})^*$.
Next, the authors develope an accurate algorithm to obtain the con-eigenpair of interest of a Cauchy matrix $\mathbf{C}$ from a rank revealing factorization. Let $\mathbf{C}$ be of the form $\mathbf{C} = \mathbf{XD}^2\mathbf{X}^*$ and define the $(m \times m)$ matrix

$$\mathbf{G} := \mathbf{D}(\mathbf{X}^\top\mathbf{X})\mathbf{D}.$$

Considering the SVD of $\mathbf{G}$ given by $\mathbf{G} = \mathbf{W\Sigma V}^*$ it holds $\mathbf{G}^*\mathbf{G} = \mathbf{V\Sigma V}^*$. Now, let $\mathbf{v}_i$ be the i-th column of $\mathbf{V}$ for $1 \leqslant i \leqslant m$, i.e. the singular vector of $\mathbf{G}$ corresponding to the i-th singular value. Then we have

$$\mathbf{G}^*\mathbf{G}\mathbf{v}_i = (\mathbf{DX}^*\overline{\mathbf{X}}\mathbf{D})(\mathbf{DX}^\top\mathbf{XD})\mathbf{v}_i = \mathbf{\Sigma}_{ii}^2\mathbf{v}_i.$$

It follows that $\mathbf{z}_i := \mathbf{XD}\mathbf{v}_i$ is an eigenvector of $\mathbf{C}\overline{\mathbf{C}}$ corresponding to the eigenvalue $\mathbf{\Sigma}_{ii}$, since

$$\mathbf{C}\overline{\mathbf{C}}\mathbf{z}_i = (\mathbf{XD}^2\mathbf{X}^*)(\overline{\mathbf{X}}\overline{\mathbf{D}}^2\mathbf{X}^\top)\mathbf{z}_i = \mathbf{XD}(\mathbf{DX}^*\overline{\mathbf{X}}\mathbf{D})(\mathbf{DX}^\top\mathbf{XD})\mathbf{v}_i = \mathbf{\Sigma}_{ii}^2\mathbf{XD}\mathbf{v}_i. = \mathbf{\Sigma}_{ii}^2\mathbf{z}_i.$$

It means that a con-eigenvector $\mathbf{z}_i$ of $\overline{\mathbf{C}}\mathbf{C}$ is given by $\overline{\mathbf{z}}_i = \overline{\mathbf{X}}\mathbf{D}\overline{\mathbf{v}}_i\mathbf{\Sigma}_{ii}^{-1/2}$, where $\mathbf{v}_i$ is the i-th singular vector of $\mathbf{G} := \mathbf{D}(\mathbf{X}^\top\mathbf{X})\mathbf{D}$. The corresponding i-th con-eigenvalue is given by $\mathbf{\Sigma}_{ii}$, which is at the same time the i-th singular value of $\mathbf{G}$. The con-eigenvalue algorithm can be summarized as follows.

Algorithm 5.3 (Con-eigenvalue algorithm for Cauchy matrices)

Input: The parameters b, d, x, y of the positive definite Cauchy matrix $\mathbf{C}$ given by (5.8), target size δ of the con-eigenvalue.

1. Compute partial Cholesky factors $\mathbf{L}, \mathbf{D}$ and $\mathbf{P}$ such that $\mathbf{C} = (\mathbf{PL})\mathbf{D}^2(\mathbf{PL})^*$ (Algorithm A.4). Set $\mathbf{X} = \mathbf{PL}$.

2. Form $\mathbf{G} = \mathbf{D}(\mathbf{X}^\top\mathbf{X})\mathbf{D}$.

3. Compute the QR factorization of $\mathbf{G}$ with the Householder QR algorithm (with optional pivoting) and obtain $\mathbf{G} = \mathbf{QR}$, where $\mathbf{Q}$ is a unitary and $\mathbf{R}$ an upper triangular matrix. Note that $\mathbf{U}_R = \mathbf{R}^{-1}\mathbf{U}_L\mathbf{\Sigma}$.

4. Compute the SVD factors $\mathbf{U}_L, \mathbf{\Sigma}$ and $\mathbf{U}_R$ of $\mathbf{R}$ using the one-sided Jacobi method (Algorithm A.3) and obtain $\mathbf{G} = \mathbf{QR} = \mathbf{QU}_L\mathbf{\Sigma}\mathbf{U}_R$.

5. Compute $\mathbf{R}_1 = \mathbf{D}^{-1}\mathbf{RD}^{-1}$, $\mathbf{X}_1 = \mathbf{D}^{-1}\mathbf{U}_L\mathbf{\Sigma}^{1/2}$ and $\mathbf{Y}_1 = \mathbf{R}_1^{-1}\mathbf{X}_1$.

6. Form the matrix of con-eigenvectors $\mathbf{T} = \overline{\mathbf{XY}_1}$, since

   $\mathbf{XY}_1 = \mathbf{XR}_1^{-1}\mathbf{X}_1 = \mathbf{XDR}^{-1}\mathbf{DD}^{-1}\mathbf{U}_L\mathbf{\Sigma}^{1/2} = \mathbf{XDR}^{-1}\mathbf{U}_L\mathbf{\Sigma}^{1/2} = \mathbf{XDU}_R\mathbf{\Sigma}^{-1/2}$

   is indeed the a matrix containing the conjugated con-eigenvectors as it has been shown before. The con-eigenvalues are already given in $\mathbf{\Sigma}$.

7. Select the largest l such that $\mathbf{\Sigma}_{ll} \geqslant \delta$.

Output: Matrix $\mathbf{\Sigma}(1:l, 1:l)$ with con-eigenvalues on the diagonals and matrix $\mathbf{T}(1:N, 1:l)$ containing the corresponding con-eigenvectors in the columns.

5.3. Sparse approximation of exponential sums by means of SVD

In this section we want to solve the sparse approximation problem presented in the beginning of this chapter for a special case. We remember from Chapter 3 that the sparse approximation problem for exponential sums is equivalent to the structured low rank approximation of the Hankel matrix $\mathbf{\Gamma_f}$. Classically the best rank K approximation of a matrix is obtained by use of the singular value decomposition. It gives the best low rank approximation in the Euclidean and Frobenius norm. The approximation error in Euclidean norm is given by the K-th singular value. This result can be found in [23] as the *Eckart-Young-Mirsky Theorem*. It is a known fact that the SVD approach applied to a Hankel matrix in general destroys the Hankel structure, see e.g. Subsection 4.2.4 in [15]. In this subsection we present a special case of finite Hankel matrices, where the Hankel structure is preserved by the singular value decomposition and the low rank approximation can be computed efficiently using the Fast Fourier Transform.

We consider a periodic sequence of the form

$$f_k = \sum_{j=1}^{N} a_j z_j^k, \qquad k = 0, 1, 2, \ldots, \tag{5.9}$$

where $f_{k+L} = f_k$ for some $L \geqslant N$. This condition means that the sequence is determined by the first L values. In this section we denote by $\mathbf{f}$ the vector $(f_k)_{k=0}^{L-1}$. Due to the structure of the sequence $(f_k)_{k=0}^{\infty}$ we have

$$\sum_{j=1}^{N} a_j z_j^k z_j^L = \sum_{j=1}^{N} a_j z_j^k \qquad \text{for all } k = 0, \ldots, L-1$$

and thus $z_j^L = 1$ are L-th roots of unity for all $j = 1, \ldots, N$. Therefore, for $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_N \leqslant L-1$ the sum (5.9) can be written as

$$f_k = \sum_{j=1}^{N} a_j (\omega_L^{\nu_j})^k, \qquad k = 0, 1, 2, \ldots,$$

where $\omega_L := \mathrm{e}^{-2\mathrm{i}\pi/L}$ and $z_j = \omega_L^{\nu_j}$. We define the periodic Hankel matrix with respect to $\mathbf{f}$ by

$$\mathbf{H}_{\mathbf{f}}^{\mathrm{per}} := \begin{bmatrix} f_0 & f_1 & \cdots & f_{L-2} & f_{L-1} \\ f_1 & f_2 & \cdots & f_{L-1} & f_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{L-2} & f_{L-1} & \cdots & f_{L-4} & f_{L-3} \\ f_{L-1} & f_0 & \cdots & f_{L-3} & f_{L-2} \end{bmatrix}. \tag{5.10}$$

As we have seen in Section 2.1.4, a Hankel matrix becomes circulant by multiplication with the flip matrix $\mathbf{U}$ defined in (2.11) of the same size. It holds

$$\mathbf{U}\mathbf{H}_{\mathbf{f}}^{\mathrm{per}} = \begin{bmatrix} f_0 & f_1 & \cdots & f_{L-2} & f_{L-1} \\ f_{L-1} & f_0 & \cdots & f_{L-3} & f_{L-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_2 & f_3 & \cdots & f_0 & f_1 \\ f_1 & f_2 & \cdots & f_{L-1} & f_0 \end{bmatrix} = \mathrm{circ}\left(f_0, f_1, f_2, \dots, f_{L-1}\right).$$

Moreover, according to the Proposition 2.6, a circulant matrix in $\mathbb{C}^{L\times L}$ can be diagonalized by the Fourier matrix of size L, which is given by

$$\mathbf{F}_L := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_L & \cdots & \omega_L^{L-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_L^{L-2} & \cdots & \omega_L^{(L-1)(L-2)} \\ 1 & \omega_L^{L-1} & \cdots & \omega_L^{(L-1)(L-1)} \end{bmatrix}.$$

Let $\mathbf{D} = \mathrm{diag}(d_k)_{k=0}^{L-1}$ be a diagonal matrix containing the eigenvalues d_k of $\mathbf{U}\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}$. Using the fact that $\bar{\mathbf{F}}_L\mathbf{F}_L = L\mathbf{I}_L$ the equation

$$\mathbf{U}\mathbf{H}_{\mathbf{f}}^{\mathrm{per}} = \frac{1}{L}\bar{\mathbf{F}}_L\mathbf{D}\mathbf{F}_L,$$

is equivalent to

$$\mathbf{U}\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}\bar{\mathbf{F}}_L = \bar{\mathbf{F}}_L\mathbf{D}$$

and since $\mathbf{U}\mathbf{U} = \mathbf{I}_L$ and $\mathbf{U}\bar{\mathbf{F}}_L = \mathbf{F}_L$ it holds

$$\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}\bar{\mathbf{F}}_L = \mathbf{U}\bar{\mathbf{F}}_L\mathbf{D} = \mathbf{F}_L\mathbf{D}.$$

Thus, the con-eigenvalues of $\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}$ are given by the eigenvalues d_k of $\mathbf{U}\mathbf{H}_{\mathbf{f}}^{\mathrm{per}}$ and the normalized con-eigenvectors are the columns (or rows) $\frac{1}{\sqrt{L}}(\omega_L^{kl})_{l=0}^{L-1}$ of the Fourier matrix $\frac{1}{\sqrt{L}}\mathbf{F}_L$. The Proposition 2.6 also states that the con-eigenvalues d_j are of the form

$$d_j = \sum_{k=0}^{L-1} f_k \omega_L^{-kj}, \qquad j = 0, \dots, L-1,$$

i.e., they can be computed easily by applying the discrete Fourier transform to the vector $\mathbf{f}$,

$$\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{L-1} \end{bmatrix} = \bar{\mathbf{F}}_L \cdot \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{bmatrix}. \tag{5.11}$$

We order the modulus of the con-eigenvalues $|d_k|$, $k = 0, \dots, L-1$ non-increasingly

$$|d_{k_0}| \geqslant |d_{k_1}| \geqslant \dots \geqslant |d_{k_{L-1}}|$$

and define $\lambda_j := d_{k_j}$ s.t.

$$|\lambda_0| \geqslant |\lambda_1| \geqslant \dots \geqslant |\lambda_{L-1}|.$$

Let $\mathbf{v}_j := \frac{1}{\sqrt{L}}(\omega_L^{k_j l})_{l=0}^{L-1}$ be the corresponding con-eigenvectors, i.e.

$$\mathbf{H}_{\mathbf{f}}^{\text{per}} \bar{\mathbf{v}}_j = \lambda_j \mathbf{v}_j \qquad \text{for} \qquad j = 0, \dots, L-1.$$

We remember from the Section 2.1.3 that every con-eigenpair $(\lambda_j, \mathbf{v}_j)$ is also a singular pair of $\mathbf{H}_{\mathbf{f}}^{\text{per}}$. For a fixed $K \leqslant N$ the truncated singular value decomposition of $\mathbf{H}_{\mathbf{f}}^{\text{per}}$ is given by the matrix

$$\tilde{\mathbf{H}} := \sum_{j=0}^{K-1} \lambda_j \mathbf{v}_j \mathbf{v}_j^{\top}.$$

The classical low rank approximation by means of SVD, known as Eckart-Young-Mirsky, states that $\tilde{\mathbf{H}}$ provides the best rank K approximation of $\mathbf{H}_{\mathbf{f}}^{\text{per}}$ in the spectral and Frobenius norm. In the spectral norm the approximation error is given by the $(K+1)$-st sigular value λ_K, i.e.

$$\|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \tilde{\mathbf{H}}\|_2 = \inf_{\text{Rank}\mathbf{A} \leqslant K} \|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{A}\|_2 = \lambda_K.$$

For the Frobenius norm it holds

$$\|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \tilde{\mathbf{H}}\|_F = \inf_{\text{Rank}\mathbf{A} \leqslant K} \|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{A}\|_F = \left(\sum_{j=K}^{L-1} \lambda_j^2 \right)^{1/2}.$$

Note that such a truncated SVD doesn't preserve the Hankel structure in the non-periodic case. The following theorem gives us the structure preserving properties of $\tilde{\mathbf{H}}$.

Theorem 5.6. *Let $\tilde{\mathbf{H}}$ be the best rank K approximation of the periodic Hankel matrix $\mathbf{H}_{\mathbf{f}}^{\text{per}}$ given in (5.10). Then $\tilde{\mathbf{H}}$ is periodic and has Hankel structure, i.e. there exist a vector $\tilde{\mathbf{f}}$ such that $\tilde{\mathbf{H}} = \mathbf{H}_{\tilde{\mathbf{f}}}^{\text{per}}$. Moreover $\tilde{\mathbf{f}}$ is given by*

$$\tilde{\mathbf{f}} = \sum_{j=0}^{K-1} \lambda_j \mathbf{v}_j, \tag{5.12}$$

where $\mathbf{v}_j := \frac{1}{\sqrt{L}}(\omega_L^{k_j l})_{l=0}^{L-1}$ are the con-eigenvectors of $\mathbf{H}_{\mathbf{f}}^{\text{per}}$ corresponding to the largest con-eigenvalues $d_{k_j} = \lambda_j$ for $j = 0, \dots, L-1$.

Proof. We observe that for all $j = 0, \dots, L-1$ it holds

$$\begin{aligned}
\mathbf{v}_j \mathbf{v}_j^\top &= \frac{1}{L} (\omega_L^{k_j r})_{r=0}^{L-1} \cdot (\omega_L^{k_j s})_{s=0}^{L-1} \\
&= \frac{1}{L} \begin{bmatrix} 1 \\ \omega_L^{k_j} \\ \vdots \\ \omega_L^{k_j(L-1)} \end{bmatrix} \cdot \begin{bmatrix} 1 & \omega_L^{k_j} & \cdots & \omega_L^{k_j(L-1)} \end{bmatrix} \\
&= \frac{1}{L} \begin{bmatrix} 1 & \omega_L^{k_j} & \cdots & \omega_L^{k_j(L-1)} \\ \omega_L^{k_j} & \omega_L^{2k_j} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_L^{k_j(L-1)} & 1 & \cdots & \omega_L^{k_j(L-2)} \end{bmatrix}.
\end{aligned}$$

Thus the sum of the matrices of the above form is periodic and Hankel. The vector $\tilde{\mathbf{f}}$ is given by the first row of $\tilde{\mathbf{H}}$. Entry-wise it holds

$$\tilde{f}_l = \frac{1}{L} \sum_{j=0}^{K-1} \lambda_j \omega_L^{k_j l}, \qquad \text{for } l = 0, \dots, L-1,$$

which corresponds to (5.12) in the vector notation. □

Using the Eckart-Young-Mirsky Theorem combined with the above result we obtain

$$\|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{H}_{\tilde{\mathbf{f}}}^{\text{per}}\|_2 = \inf_{\text{Rank}\mathbf{A} \leqslant K} \|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{A}\|_2 = \lambda_K$$

in the spectral norm and

$$\|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{H}_{\tilde{\mathbf{f}}}^{\text{per}}\|_F = \inf_{\text{Rank}\mathbf{A} \leqslant K} \|\mathbf{H}_{\mathbf{f}}^{\text{per}} - \mathbf{A}\|_F = \left(\sum_{j=K}^{L-1} \lambda_j^2 \right)^{1/2}$$

in the Frobenius norm. In order to study the connection between the matrix norm of $\mathbf{H}_{\mathbf{f}}^{\text{per}}$ and the vector norm of $\mathbf{f} - \tilde{\mathbf{f}}$ we observe the following. For the spectral norm it holds

$$\|\mathbf{f} - \tilde{\mathbf{f}}\|_2 = \frac{\|(\mathbf{H}_{\mathbf{f}-\tilde{\mathbf{f}}}^{\text{per}})\mathbf{e}_1\|_2}{\|\mathbf{e}_1\|_2} \leqslant \|\mathbf{H}_{\mathbf{f}-\tilde{\mathbf{f}}}^{\text{per}}\|_2 = \lambda_K,$$

where $\mathbf{e}_1 := (1, 0, \dots, 0)^\top \in \mathbb{C}^L$ is the first unit vector of length L. In the Frobenius norm we even obtain the best K-term approximation $\tilde{\mathbf{f}}$ of the sequence $\mathbf{f}$, since

$$\|\mathbf{H}_{\mathbf{f}-\tilde{\mathbf{f}}}^{\text{per}}\|_F^2 = \sum_{j=0}^{L-1} \sum_{k=0}^{L-1} |(\mathbf{H}_{\mathbf{f}-\tilde{\mathbf{f}}}^{\text{per}})_{jk}|^2 = L \sum_{j=0}^{L-1} |f_j - \tilde{f}_j|^2 = L\|\mathbf{f} - \tilde{\mathbf{f}}\|_2^2$$

and thus

$$\|\mathbf{f}-\tilde{\mathbf{f}}\|_2 = \frac{1}{\sqrt{L}}\|\mathbf{H}^{\text{per}}_{\mathbf{f}-\tilde{\mathbf{f}}}\|_F = \left(\frac{1}{L}\sum_{j=K}^{L-1}\lambda_j^2\right)^{1/2}.$$

Hence the computation of the best K-term approximation of $\mathbf{f}$ can be divided into three essential steps given in the following algorithm.

Algorithm 5.4 (Structured SVD approximation)

Input: Sequence $\mathbf{f} = (f_k)_{k=0}^{L-1}$.

1. Compute the eigenvalues d_j of $\mathbf{U}\mathbf{H}_{\mathbf{f}}{}^{\text{per}}$ by $\mathbf{d} = \bar{\mathbf{F}}_L\mathbf{f}$ using the Fast Fourier Transform (FFT).
2. Sort the modulus of d_j decreasingly and obtain the con-eigenvalues λ_j and the corresponding con-eigenvectors $\mathbf{v}_j$ in the right order.
3. Take the first K pairs $(\lambda_j, \mathbf{v}_j)$ and build the sum (5.12).

Output: Nodes $\omega_L^{k_j}$ and weights λ_j, such that for $\tilde{\mathbf{f}} = (f_l)_{l=0}^{L-1}$ with

$$\tilde{f}_l = \frac{1}{L}\sum_{j=0}^{K-1}\lambda_j\omega_L^{k_j l}, \qquad \text{for } l = 0, \ldots, L-1$$

it holds

$$\|\mathbf{f}-\tilde{\mathbf{f}}\|_2 = \left(\frac{1}{L}\sum_{j=K}^{L-1}\lambda_j^2\right)^{1/2}.$$

The complexity of the above algorithm is determined by the Fast Fourier Transform of length L in the first step. FFT algorithm was first published by Cooley and Tukey in 1965 [16] and is still the most common algorithm for the computation of the discrete Fourier transform. The complexity of the FFT of length L is $\mathcal{O}(L\log L)$.

Remark 5.7. Note that the approximation method presented in this subsection is equivalent to the classical approximation approach. Namely, first the original signal is transformed into the Fourier domain and represented in the Fourier basis. Additionally, the basis elements corresponding to the coefficients under certain threshold are eliminated and the remaining signal is transformed back to the time domain. In our context Algorithm 5.4 can be seen as a special case of the AAK approximation, namely for finite Hankel matrices with periodic structure. △

6. Numerical experiments and applications

In this chapter we present various numerical experiments demonstrating the performance of the Algorithms 5.1 (AAK) and 5.4 (SVD). The AAK algorithm from the previous chapter is tested for approximation of exponential sums in Section 6.1. In Sections 6.2 and 6.3 the approximation of other decaying functions, such as $1/x$, is considered. The approximation is conducted for equidistant and non-equidistant sampling, respectively. Algorithm 5.1 requires a Prony-like method for Step 1. Here we use the APM algorithm, which was presented in Section 4.2. In order to overcome the instabilities of Algorithm 5.1 we chose to increase the precision of all computations in Section 6.1 to 128 significant decimal digits using the Symbolic Math toolbox in Matlab. Further, in Section 6.5 we present the performance of Algorithm 5.4 for one- and two-dimensional signals. All experiments in this chapter were implemented in Matlab 2016b and conducted with a 2,4 GHz Intel Core 2 Duo processor and 4GB 1067 MHz DDR3.

6.1. Approximation of exponential sums

Example 6.1. In the following example we test the performance of Algorithm 5.1 with an exponential sum of length $N = 10$ using $M = 50$ equidistant samples

$$f_k = \sum_{j=1}^{10} a_j z_j^k \tag{6.1}$$

for $k = 0, 1, \dots, 49$. In addition, we compare our results to the performance of the APM Algorithm 4.2. The complex nodes z_j and the coefficients a_j of the above sum were chosen randomly according to the uniform distribution in $\mathbb{D}$

n	σ_n	$\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$	$\frac{\max_j \|f_j - \tilde{f}_j^{(n)}\|}{\max_j \|f_j\|}$	$(\sigma_A)_n$	$\|\mathbf{f} - \mathbf{f_A}^{(n)}\|_2$	$\frac{\max_j \|f_j - (f_A)_j^{(n)}\|}{\max_j \|f_j\|}$
1	7.2335e-01	7.0434e-01	3.9596e-01	7.2335e-01	7.0434e-01	3.9596e-01
2	2.7143e-01	2.5776e-01	9.8862e-02	2.7143e-01	2.5776e-01	9.8862e-02
3	1.3522e-01	1.3464e-01	7.2974e-02	1.3522e-01	1.3464e-01	7.2974e-02
4	6.1020e-02	6.0843e-02	2.1650e-02	6.1020e-02	6.0844e-02	2.1650e-02
5	9.5748e-03	9.5732e-03	4.8520e-03	9.5748e-03	9.5732e-03	4.8520e-03
6	2.2792e-03	2.2790e-03	1.0436e-03	2.2792e-03	2.2790e-03	1.0436e-03
7	1.1236e-04	1.1236e-04	4.3713e-05	1.1236e-04	1.1235e-04	4.3707e-05
8	3.4296e-06	3.4291e-06	1.1382e-06	3.4295e-06	3.4291e-06	1.1381e-06
9	9.0150e-07	9.0150e-07	3.4800e-07	9.0149e-07	9.0152e-07	3.4776e-07

Table 6.1: Example 6.1: The error of the n-term approximation $\tilde{\mathbf{f}}^{(n)}$ by Algorithm 5.1 and $\mathbf{f_A}^{(n)}$ by the APM algorithm. Also the con-eigenvalues σ_n of the matrix $\mathbf{AZ}$ from Algorithm 5.1 and the singular values $(\sigma_A)_n$ of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ from Algorithm 4.2 are given.

and are given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} 0.5441 - 0.1213\mathrm{i} \\ 0.0491 - 0.1019\mathrm{i} \\ 0.2157 - 0.3831\mathrm{i} \\ -0.5561 - 0.2935\mathrm{i} \\ 0.3710 - 0.5754\mathrm{i} \\ 0.2983 + 0.5084\mathrm{i} \\ -0.4558 + 0.1844\mathrm{i} \\ 0.0301 + 0.5191\mathrm{i} \\ -0.2090 - 0.4038\mathrm{i} \\ 0.0557 + 0.5053\mathrm{i} \end{pmatrix}, \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix} = \begin{pmatrix} 0.3536 + 0.1705\mathrm{i} \\ 0.0929 - 0.0971\mathrm{i} \\ -0.0720 - 0.1311\mathrm{i} \\ -0.2909 + 0.3794\mathrm{i} \\ 0.3023 - 0.2191\mathrm{i} \\ -0.3256 + 0.3774\mathrm{i} \\ -0.5230 + 0.3469\mathrm{i} \\ 0.3208 + 0.4227\mathrm{i} \\ 0.2054 + 0.0068\mathrm{i} \\ 0.2583 + 0.1628\mathrm{i} \end{pmatrix}.$$

For the initial N-term approximation with APM of Algorithm 5.1 the parameters $L = 20$, $\varepsilon_2 =$ 1e-14, $\varepsilon_1 =$ 1e-15 and $r = 1$ were used. The obtained approximation error is $\|\mathbf{f} - \mathbf{f_A}\|_2 =$ 1.3619e-15, where $\mathbf{f} = (f_k)_{k=0}^{49}$ denotes the original sequence of samples and $\mathbf{f_A} = ((f_A)_k)_{k=0}^{49}$ the N-term APM approximation. The condition of the matrix $\mathbf{AZ}$ in this example is 9.1644e+06. The nodes $\tilde{z}_j^{(n)}$ and the coefficients $\tilde{a}_j^{(n)}$ of the AAK approximation are provided in Table 6.2. For better visualization the nodes $\tilde{z}_j^{(n)}$ are also plotted in Figure 6.1 for all sum length $n = 1, \ldots, N-1$ together with the original nodes z_j. We observe that the AAK nodes tend to the original ones as the length of the sum n grows larger. Remarkable is also the fact that the new nodes sometimes even exceed the range of the original nodes. This can be seen for instance in case $n = 2$, where the upper AAK node is closer to the unit circle than any of the original nodes z_j.

The comparison of our algorithm to the performance of the "pure" APM method is shown in Table 6.1. We consider the same sampling sequence $\mathbf{f}$ as in (6.1), and compute the n-term approximation by exponential sums with both

n	1	2	3	4	5	6	7	8	9
$\tilde{z}_1^{(n)}$	−0.1212 + 0.0295ßi	−0.2794 − 0.4335i	−0.5200 − 0.2539i	−0.3552 − 0.2721i	0.4393 − 0.0086i	−0.0437 + 0.3604i	−0.1383 − 0.3360i	0.1459 − 0.3986i	−0.0550 − 0.3371i
$\tilde{z}_2^{(n)}$		0.2980 + 0.7104i	0.1600 − 0.6425i	−0.0413 + 0.6390i	0.1670 + 0.4981i	−0.4924 + 0.0683i	−0.4481 + 0.1905i	−0.1943 − 0.4123i	−0.2270 − 0.4156ii
$\tilde{z}_3^{(n)}$			0.1946 + 0.6604i	0.4740 + 0.4809i	−0.4484 + 0.2946i	0.2415 + 0.4417i	0.0342 + 0.5236i	−0.4560 + 0.1853i	−0.4557 + 0.1848i
$\tilde{z}_4^{(n)}$				0.4774 − 0.6782i	−0.5076 − 0.3328i	−0.5158 − 0.2666i	0.5499 − 0.1276i	0.0350 + 0.5114i	0.2088 − 0.4454i
$\tilde{z}_5^{(n)}$					0.4252 − 0.5576i	0.6138 − 0.1642i	0.2903 + 0.5113i	0.5432 − 0.1219i	0.0353 + 0.5118i
$\tilde{z}_6^{(n)}$						0.3880 − 0.5909i	−0.5573 − 0.2885i	0.2982 + 0.5078i	0.5438 − 0.1217i
$\tilde{z}_7^{(n)}$							0.3749 − 0.5738i	−0.5562 − 0.2932i	0.2980 + 0.5081i
$\tilde{z}_8^{(n)}$								0.3713 − 0.5749i	−0.5559 − 0.2935i
$\tilde{z}_9^{(n)}$									0.3708 − 0.5752i
n	1	2	3	4	5	6	7	8	9
$\tilde{a}_1^{(n)}$	0.3021 + 1.4417i	0.4389 + 0.9889i	−0.1016 + 0.7654i	0.1115 + 1.3007i	0.7213 − 0.1881i	1.5286 − 0.8668i	0.4303 − 0.1585i	0.0169 − 0.1600i	0.0941 − 0.1463i
$\tilde{a}_2^{(n)}$		−0.0858 + 0.4274i	0.3388 + 0.0617i	0.4897 + 0.1734i	0.0450 + 1.3682i	0.1232 + 0.7535i	−0.5754 + 0.3031i	0.2022 − 0.0412i	0.1388 + 0.0225i
$\tilde{a}_3^{(n)}$			0.0796 + 0.5930i	−0.3413 + 0.0399i	−0.4610 − 0.0156i	−1.1556 + 0.9076i	0.4926 + 0.5273i	−0.5220 + 0.3399i	−0.5242 + 0.3432i
$\tilde{a}_4^{(n)}$				0.0623 − 0.0953i	−0.1012 + 0.5665i	−0.5141 + 0.6836i	0.3186 + 0.1844i	0.5904 + 0.5704i	−0.0111 − 0.0875i
$\tilde{a}_5^{(n)}$					0.1178 − 0.3118i	0.1278 + 0.1363i	−0.3039 + 0.4382i	0.3579 + 0.1761i	0.5851 + 0.5715i
$\tilde{a}_6^{(n)}$						0.2120 − 0.1950i	−0.3228 + 0.3617i	−0.3313 + 0.3791i	0.3549 + 0.1740i
$\tilde{a}_7^{(n)}$							0.2825 − 0.2370i	−0.2938 + 0.3783i	−0.3291 + 0.3811i
$\tilde{a}_8^{(n)}$								0.3015 − 0.2236i	−0.2919 + 0.3808i
$\tilde{a}_9^{(n)}$									0.3053 − 0.2201i

Table 6.2: Example 6.1: The nodes $\tilde{z}_j^{(n)}$ and the corresponding coefficients $\tilde{a}_j^{(n)}$ of the approximating exponential sum computed with Algorithm 5.1.

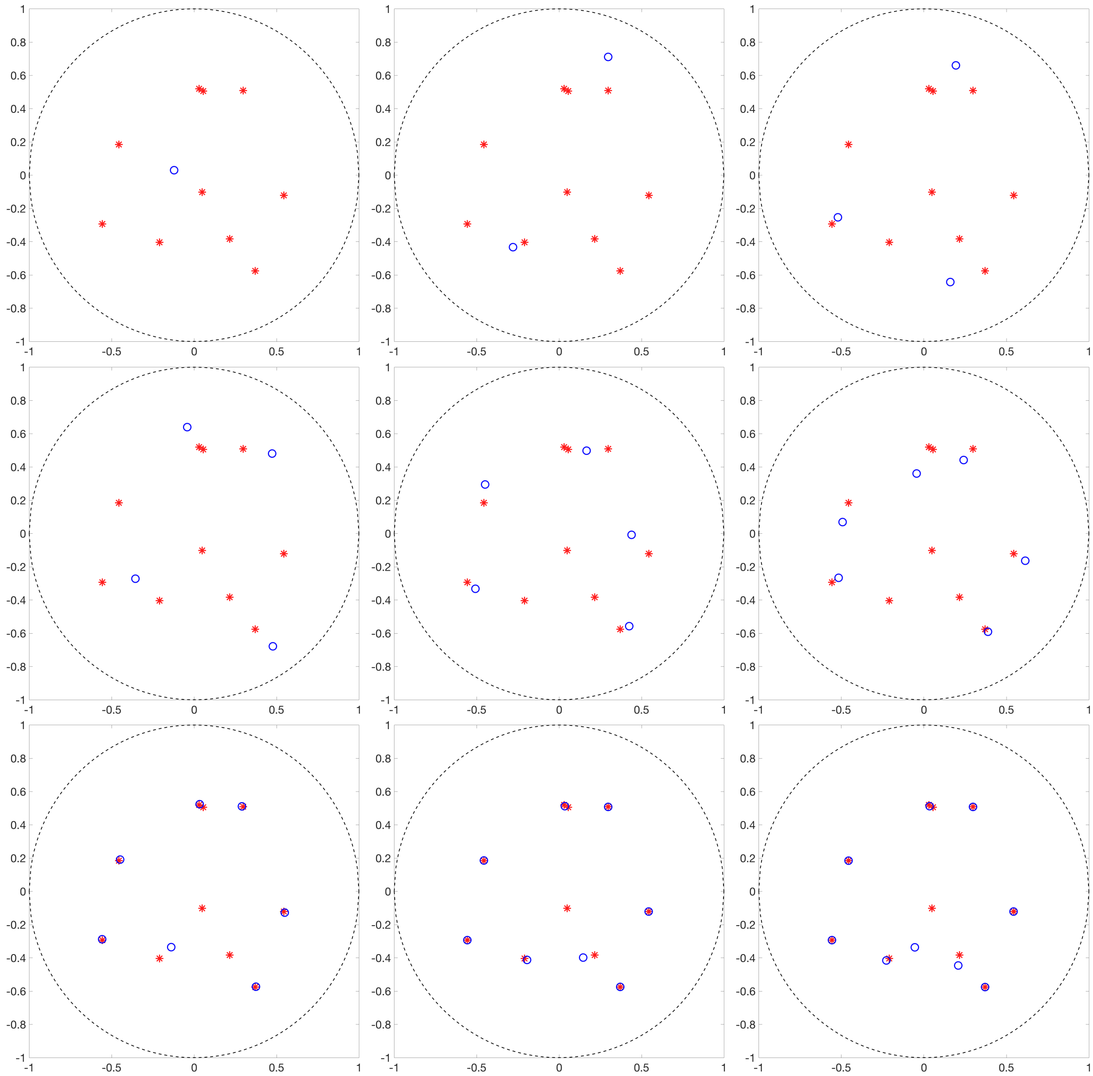

Figure 6.1: Example 6.1: The nodes $\tilde{z}_j^{(n)}$ (blue circles) of the approximation of an exponential sum (6.1) with Algorithm 5.1 for all sum length $n = 1, \ldots, 9$ (from top left to bottom right) together with the original nodes z_j (red asterisk).

algorithms for $n = 1, \ldots, 9$. For APM the parameters $L = 20$, $\varepsilon_1 = 1\text{e-}10$ and $r = 1$ were used. The tolerance ε_2 was adapted such that we obtain the desired sum length. In Table 6.1 we can see that the con-eigenvalues σ_n of the Cauchy matrix $\mathbf{AZ}$ from Algorithm 5.1 and the singular values $(\sigma_A)_n$ of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ from Algorithm 4.2 are almost identical for all n. Also the approximation error in terms of the Euclidean norm as well as the relative errors of both algorithms do not differ a lot. Although, due to the space constrains, the nodes and the coefficients of the n-term APM approximation are not listed here, it can be observed that also those are almost the same as the ones obtained by our method. We assume that this behavior can be explained by perturbation theory. The rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ is simply the truncated infinite Hankel matrix $\mathbf{\Gamma}_{\mathbf{f}}$. We know that for a fast decreasing sequence $\mathbf{f}$ the singular values of the Hankel matrix also decay rapidly. Thus, in this case,

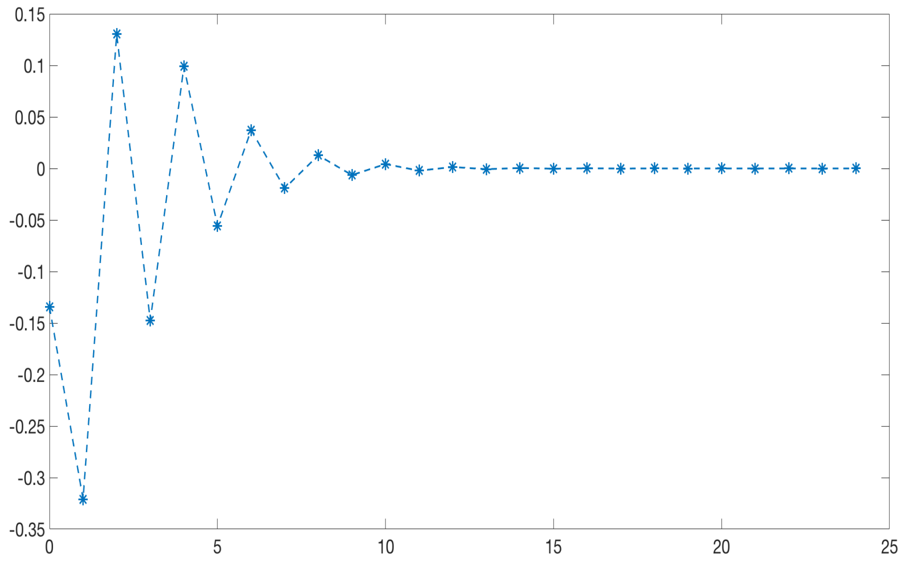

Figure 6.2: The first 25 samples of the sequence **f** from Example 6.2.

the SVD of $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ and the truncated SVD of $\mathbf{\Gamma}_{\mathbf{f}}$ do not differ a lot.
However, the advantage of our algorithm is noticeable when a fixed length K of the approximation sum is desired from the beginning. Algorithm 5.1 guarantees the approximation to have sum length K by choosing the K-th con-eigenvalue σ_K in Step 2. Furthermore, in this case an iterative algorithm for solving a con-eigenvalue problem with reduced computational costs can be used as presented in Section 5.2, since we can stop the computation at the K-th con-eigenvalue. Fixing the sum length with the APM algorithm requires rather special constellation of the parameters ε_1, ε_2 and r. Especially in case of noisy data this is not easy to achieve.

Example 6.2. In this example we consider $M = 50$ samples f_k, $k = 0, 1, \ldots, 49$ of an exponential sum of the same form (6.1), i.e., with $N = 10$. This time we choose real nodes z_j and coefficients a_j, $j = 1, \ldots, N$, randomly uniformly in the interval $[-1, 1]$,

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} -0.2090 \\ 0.0557 \\ -0.1213 \\ -0.1019 \\ -0.3831 \\ -0.2935 \\ -0.5754 \\ 0.5084 \\ 0.1844 \\ 0.5191 \end{pmatrix}, \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix} = \begin{pmatrix} -1.3460 \\ 1.6844 \\ 1.1786 \\ 0.3096 \\ -0.2399 \\ -0.9695 \\ 1.0078 \\ -1.0853 \\ -1.7433 \\ 1.0693 \end{pmatrix}.$$

For the purpose of illustration the first 25 values of the sequence of samples **f** are plotted in Figure 6.2. For the initial APM approximation in Step 1 of Algorithm 5.1 we use the parameters $L = 20$, $\varepsilon_1 =$ 1e-15, $\varepsilon_2 =$ 1e-16 and $r = 1$. The

n	σ_n	$\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$	$\frac{\max_j \|f_j - \tilde{f}_j^{(n)}\|}{\max_j \|f_j\|}$	$(\sigma_A)_n$	$\|\mathbf{f} - \mathbf{f_A}^{(n)}\|_2$	$\frac{\max_j \|f_j - (f_A)_j^{(n)}\|}{\max_j \|f_j\|}$
1	3.7324e-01	3.6585e-01	7.9042e-01	3.7324e-01	3.6585e-01	7.9042e-01
2	4.3827e-02	4.3484e-02	8.0874e-02	4.3827e-02	4.3484e-02	8.0874e-02
3	1.5447e-02	1.5443e-02	2.8169e-02	1.5447e-02	1.5443e-02	2.8169e-02
4	2.2440e-03	2.2440e-03	3.7798e-03	2.2440e-03	2.2440e-03	3.7798e-03
5	2.2308e-05	2.2308e-05	4.4868e-05	2.2308e-05	2.2308e-05	4.4868e-05
6	2.4205e-06	2.4205e-06	4.8623e-06	2.4205e-06	2.4205e-06	4.8623e-06
7	1.2358e-07	1.2358e-07	2.4582e-07	1.2358e-07	1.2358e-07	2.4580e-07
8	8.7787e-10	8.7787e-10	1.5171e-09	8.7787e-10	8.7787e-10	1.5169e-09
9	2.0828e-13	2.0829e-13	3.6771e-13	2.0828e-13	2.0828e-13	3.6809e-13

Table 6.3: Example 6.2: The error of the n-term approximation $\tilde{\mathbf{f}}^{(n)}$ by Algorithm 5.1 and $\mathbf{f_A}^{(n)}$ by the APM algorithm. Also the con-eigenvalues σ_n of the matrix $\mathbf{AZ}$ from Algorithm 5.1 and the singular values $(\sigma_A)_n$ of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ from Algorithm 4.2 are given.

error of the initial APM approximation is $\|\mathbf{f} - \mathbf{f_A}\|_2 = 1.9140\text{e-}15$, where, again, $\mathbf{f} = (f_k)_{k=0}^{49}$ denotes the original sequence of samples and $\mathbf{f_A} = ((f_A)_k)_{k=0}^{49}$ the APM approximation. The condition of the matrix $\mathbf{AZ}$ in this example is extremely bad, namely $\text{cond}(\mathbf{AZ}) = 6.9986\text{e+}13$.
Similar to Example 6.1 we summarize the approximation errors of an n-term approximation computed with Algorithm 5.1 compared to Algorithm 4.2 in Table 6.3. We observe that, again, the errors and the singular values are almost the same for both algorithms. Looking at the nodes $\tilde{z}_j^{(n)}$ and the coefficients $\tilde{a}_j^{(n)}$ obtained by Algorithm 5.1 given in Table 6.4 we make a remarkable observation. Since we do not enforce the nodes or the coefficients to be real, it appears that for some n (namely 3, 4 and 7) the nodes and the coefficients become complex in order to achieve the desired approximation in the $\ell^2(\mathbb{N})$ norm below the con-eigenvalue σ_n. Further, we can observe that the nodes and the coefficients appear to only become complex together, which can be confirmed by further numerical tests conducted in the course of this work. This behavior is a known property of polynomials with real coefficients.

Example 6.3. In this example we consider a very special structure of our exponential sum. Similar to the examples before we choose $N = 10$ and use $M = 50$ samples. However, this time we set the nodes to be real and equidistantly dis-

n	1	2	3	4	5	6	7	8	9
$\tilde{z}_1^{(n)}$	−0.9345	0.1743	0.3358	0.3233 + 0.2896i	−0.0403	0.0590	−0.0069	0.0702	0.0557
$\tilde{z}_2^{(n)}$		−0.7097	−0.6596 − 0.1709i	0.3233 − 0.2896i	0.2461	0.1144	0.2043	−0.0793	−0.1161
$\tilde{z}_3^{(n)}$			−0.6596 + 0.1709i	−0.5254 − 0.0861i	−0.2859	−0.3013	−0.2870	0.1820	0.1844
$\tilde{z}_4^{(n)}$				−0.5254 + 0.0861i	−0.5768	0.3769	−0.4463	−0.2589	−0.2107
$\tilde{z}_5^{(n)}$					0.5815	0.5593	0.5221 − 0.0285i	−0.3716	−0.2940
$\tilde{z}_6^{(n)}$						−0.5759	0.5221 + 0.0285i	0.5067	−0.3832
$\tilde{z}_7^{(n)}$							−0.5751	0.5205	0.5084
$\tilde{z}_8^{(n)}$								−0.5754	0.5191
$\tilde{z}_9^{(n)}$									−0.5754
n	1	2	3	4	5	6	7	8	9
$\tilde{a}_1^{(n)}$	0.0720	−0.4692	−0.3457	−0.0909 + 0.1302i	1.7859	5.1544	1.9955	1.5867	1.6857
$\tilde{a}_2^{(n)}$		0.3338	0.1056 − 0.1918i	−0.0909 − 0.1302i	−1.0634	−4.4424	−1.3209	1.2120	1.4507
$\tilde{a}_3^{(n)}$			0.1056 + 0.1918i	0.0238 − 0.9392i	−1.8985	−1.6721	−1.7362	−1.8532	−1.7431
$\tilde{a}_4^{(n)}$				0.0238 + 0.9392i	0.9806	−0.2645	−0.0866	−1.7026	−1.3242
$\tilde{a}_5^{(n)}$					0.0612	0.0943	−0.0003 + 0.1695i	−0.3683	−0.9563
$\tilde{a}_6^{(n)}$						0.9961	−0.0003 − 0.1695i	−0.8513	−0.2389
$\tilde{a}_7^{(n)}$							1.0146	0.8348	−1.0855
$\tilde{a}_8^{(n)}$								1.0076	1.0695
$\tilde{a}_9^{(n)}$									1.0078

Table 6.4: Example 6.2: The nodes $\tilde{z}_j^{(n)}$ and the corresponding coefficients $\tilde{a}_j^{(n)}$ of the approximation exponential sum computed with Algorithm 5.1.

n	1	2	3	4	5	6	7	8	9
$\tilde{z}_1^{(n)}$	−0.0000	−0.7307	−0.8544	−0.8867	−0.8965	−0.8993	−0.8999	−0.9000	−0.9000
$\tilde{z}_2^{(n)}$		0.7307	−0.000	−0.4184	−0.5895	−0.6605	−0.6888	−0.6979	−0.6998
$\tilde{z}_3^{(n)}$			0.8544	0.4184	−0.000	−0.2592	−0.3991	−0.4679	−0.4946
$\tilde{z}_4^{(n)}$				0.8867	0.5895	0.2592	−0.000	−0.1688	−0.2637
$\tilde{z}_5^{(n)}$					0.8965	0.6605	0.3991	0.1688	−0.000
$\tilde{z}_6^{(n)}$						0.8993	0.6888	0.4679	0.2637
$\tilde{z}_7^{(n)}$							0.8999	0.6979	0.4946
$\tilde{z}_8^{(n)}$								0.9000	0.6998
$\tilde{z}_9^{(n)}$									0.9000

Table 6.5: Example 6.3: The nodes $\tilde{z}_j^{(n)}$ of the approximation exponential sum computed with Algorithm 5.1. The entries of the table which appear as zeroes are all below 1e-14.

tributed inside the interval $[-1, 1]$ as follows:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} -0.9 \\ -0.7 \\ -0.5 \\ -0.3 \\ -0.1 \\ 0.1 \\ 0.3 \\ 0.5 \\ 0.7 \\ 0.9 \end{pmatrix}.$$

In order to simplify the example, the coefficients are set to be $a_j = 1$ for all $j = 1, \ldots, N$. In Table 6.5 the nodes $\tilde{z}_j^{(n)}$ computed with Algorithm 5.1 are given. We observe that in this example the nodes are interlacing, i.e., considering an order

$$\tilde{z}_{k_1}^{(n)} < \tilde{z}_{k_2}^{(n)} \ldots < \tilde{z}_{k_N}^{(n)}$$

for all $n = 1, \ldots, N$ it holds

$$\tilde{z}_{k_1}^{(n+1)} < \tilde{z}_{k_1}^{(n)} < \ldots < \tilde{z}_{k_n}^{(n+1)} < \tilde{z}_{k_n}^{(n)} < \tilde{z}_{k_{n+1}}^{(n+1)}.$$

Although this behavior has not been studied more closely in this work, it can be assumed that the interlacing has its origin in the fundamental properties of polynomials. Even a connection to orthogonal polynomials can be presumed in this case and is yet to be studied.

Example 6.4. Finally we present a representative example for noisy data. We use the same nodes z_j and coefficients a_j as in Example 6.1, i.e. the sum length

n	σ_n	$\Vert\mathbf{f}-\tilde{\mathbf{f}}^{(n)}\Vert_2$	$\frac{\max_j \lvert f_j-\tilde{f}_j^{(n)}\rvert}{\max_j \lvert f_j\rvert}$	$(\sigma_A)_n$	$\Vert\mathbf{f}-\mathbf{f_A}^{(n)}\Vert_2$	$\frac{\max_j \lvert f_j-(f_A)_j^{(n)}\rvert}{\max_j \lvert f_j\rvert}$
1	7.2335e-01	7.0434e-01	3.9596e-01	7.2335e-01	7.0434e-01	3.9596e-01
2	2.7143e-01	2.5776e-01	9.8862e-02	2.7143e-01	2.5776e-01	9.8862e-02
3	1.3522e-01	1.3464e-01	7.2974e-02	1.3522e-01	1.3464e-01	7.2974e-02
4	6.1020e-02	6.0843e-02	2.1650e-02	6.1020e-02	6.0843e-02	2.1650e-02
5	9.5748e-03	9.5732e-03	4.8520e-03	9.5748e-03	9.5732e-03	4.8520e-03
6	2.2792e-03	2.2790e-03	1.0436e-03	2.2792e-03	2.2790e-03	1.0436e-03
7	1.1236e-04	1.1236e-04	4.3713e-05	1.1236e-04	1.1236e-04	4.3712e-05
8	3.4301e-06	3.4303e-06	1.1398e-06	3.4295e-06	3.4303e-06	1.1408e-06
9	8.9825e-07	8.9845e-07	3.4865e-07	8.9734e-07	8.9941e-07	3.4856e-07
10	4.6693e-08	4.6079e-08	1.4741e-08	3.9961e-08	3.0247e-08	1.0967e-08
11	3.0380e-08	3.0678e-08	1.0022e-08	2.3912e-08	2.8829e-08	6.5616e-09
12	1.7420e-08	2.3472e-08	6.4942e-09	1.7218e-08	2.3756e-08	5.3718e-09
13	1.7420e-08	2.4063e-08	6.6276e-09	1.4609e-08	2.1247e-08	4.3500e-09
14	1.4917e-08	2.5550e-08	7.3254e-09	1.3192e-08	2.2261e-08	5.2587e-09
15	1.3977e-08	1.8756e-08	5.4749e-09	1.2803e-08	2.2379e-08	5.0981e-09
16	8.6763e-09	1.8419e-08	5.5655e-09	1.2071e-08	2.3144e-08	4.9552e-09
17	8.0903e-09	1.8539e-08	5.4616e-09	1.0748e-08	2.3844e-08	5.3377e-09
18	5.1834e-09	1.8952e-08	5.3407e-09	1.0294e-08	2.3699e-08	5.2083e-09
19	2.9653e-09	1.8898e-08	5.1181e-09	9.0372e-09	2.2557e-08	4.9929e-09
20	9.0714e-10	2.1824e-08	5.0089e-09	7.1365e-09	2.3084e-08	5.0272e-09
21	4.6724e-10	2.2586e-08	5.0477e-09	5.5830e-09	2.3559e-08	4.9684e-09
22	2.6402e-10	2.3049e-08	4.9597e-09	3.3391e-09	2.3307e-08	5.3393e-09

Table 6.6: Example 6.4: The error of the n-term approximation $\tilde{\mathbf{f}}^{(n)}$ by Algorithm 5.1 and $\mathbf{f_A}^{(n)}$ by the APM algorithm. Also the con-eigenvalues σ_n of the matrix $\mathbf{AZ}$ from Algorithm 5.1 and the singular values $(\sigma_A)_n$ of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ from Algorithm 4.2 are given.

remains $N = 10$. We use $M = 50$ equidistant samples again, which we equip with additive noise as

$$\mathbf{f}_{\text{noisy}} = \mathbf{f} + s\mathbf{e},$$

where $\mathbf{e} := (e_k)_{k=0}^{49}$ a randomly generated vector using a normal distribution with mean 1 and standard deviation 2 and s is some scaling factor which determines the magnitude of the noise. For this example we choose $s =$ 1e-09. For the initial approximation with APM in Algorithm 5.1 the parameters $L = 25$, $\varepsilon_1 =$ 1e-14, $\varepsilon_2 =$ 1e-08 and $r = 1.2$ were used. In order to demonstrate the development of the approximation error we chose the parameters such that the APM algorithm initially approximates our sum by a longer sum, namely with 23 terms. The error of the initial APM approximation is $\Vert\mathbf{f} - \mathbf{f_A}\Vert_2 =$ 2.3370e-08, where $\mathbf{f} = (f_k)_{k=0}^{49}$ is the original sequence of samples and $\mathbf{f_A} = ((f_A)_k)_{k=0}^{49}$ the APM approximation. The matrix $\mathbf{AZ}$ has condition 1.3209e+14 in this example.

In Table 6.6 we compare the errors of our algorithm and Algorithm 4.2. For the APM method the parameters $\varepsilon_1 = 1\text{e-}10$ and $r = 1.1$ were used for all $n = 1, \ldots, 22$. Only the tolerance ε_2 was adapted in every step in order to obtain an n-term approximaton. It was chosen such that for the n-term approximation we use the n-th singular value of the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$.
We see that the performance of both algorithms appears to be almost the same also in the noisy case. However, we observe that the errors presented in Table 6.6 begin to stagnate as soon as the magnitude of the noise is achieved. This is caused by the fact that the signal itself also stagnates once the function values get below the noise level. Note that in noisy case neither the infinite Hankel matrix $\mathbf{\Gamma_f}$ nor the rectangular Hankel matrix $\tilde{\mathbf{H}}_{\tilde{\mathbf{f}}}$ are singular anymore. Apart from this, we see that all values in Table 6.6 for $n \leqslant 9$ are almost identical to the values in Table 6.1. A further effect we can see in this example is that the gap between the con-eigenvalue of $\mathbf{AZ}$ and the actual Euclidean norm grows larger with n.
Concerning the nodes obtained by Algorithm 5.1 it could be observed that in this case the number of zeros of the con-eigenpolynomial in Step 3 does not correspond to the index of the con-eigenvalue. The noise destroys the exponential structure of the signal and the AAK Theorem does not hold in this case. Thus we used the K zeros, which have the smallest absolute value for this experiment in order to achieve the K-term approximation for $K = 1, \ldots, 23$. These zeros appear to be almost the same as the ones computed with APM. As expected, due to the noise perturbations, also the APM algorithm had difficulties to fix the sum length of the approximation.

6.2. Approximation of $1/x$ by exponential sums

In this section we approximate the function $f(x) = 1/x$ in the interval $[1, 50]$ by exponential sums and compare our results to the ones obtained by Hackbusch in [25]. Note that the interval is chosen to start at one in order to keep distance from the singularity of f at zero. This approximation result can be used, for instance, to evaluate some high-dimensional integrals by products of one-dimensional integrals as can be found in [12]. Further, a d-variate function of the form $f_d(x) := 1/(x_1 + x_2 + \ldots + x_d)$ can also be approximated by an exponential sum, since the d variables can be separated in the exponential representation. Functions of the form $f_d(x)$ appear in the so-called Coupled Cluster (CC) method for solving many-body systems in quantum chemistry and nuclear physics. In this application x_j represent energies, related to the occupied orbitals and virtual orbitals. Thus, the separation of variables x_j enables coupling of the orbitals, since the exponential sum representation $1/x \approx \sum_{j=1}^{N} a_j z_j^x$ leads to

$$\frac{1}{x_1 + x_2 + \ldots + x_d} \approx \sum_{j=1}^{N} a_j z_j^{x_1} \cdots z_j^{x_d}$$

n	σ_n	$\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$	$\|\mathbf{f} - \mathbf{f}_H^{(n)}\|_2$
1	1.5789e-00	1.0479e-00	-
2	4.3137 e-01	3.7340e-01	1.4145e-01
3	9.9203e-02	9.4372e-02	2.4771e-02
4	1.9627e-02	1.9207e-02	4.4988e-03
5	3.3233e-03	3.2870e-03	7.8479e-04
6	4.7360e-04	4.6840e-04	1.3138e-04
7	5.5123e-05	5.4309e-05	2.2138e-05
8	4.9665e-06	4.8884e-06	3.6552e-06
9	3.1299e-07	3.1581e-07	5.9684e-07
10	1.0840e-08	4.5328e-08	9.8033e-08

Table 6.7: Approximation of $1/x$ by exponential sums: Singular values σ_n of the matrix $\mathbf{AZ}$, the approximation error $\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$ of Algorithm 5.1 and the approximation error $\|\mathbf{f} - \mathbf{f}_H^{(n)}\|_2$ obtained with nodes and coefficients from [25] for length of the sum $n = 1, \ldots, 10$.

and the computational costs of CC can be reduced. For further details about coupled cluster analysis in quantum chemistry we refer to [33].
In this example we use $M = 100$ equidistant samples of $1/x$ in the interval $[1, 50]$. Let $\mathbf{f} := (f_k)_{k=0}^{99}$ be the vector of samples of f and $\tilde{\mathbf{f}}^{(n)} := (\tilde{f}_k^{(n)})_{k=0}^{99}$ the output vector of its n-term approximation by Algorithm 5.1. The initial $N = 11$ nodes z_j and weights a_j were obtained with parameters $\varepsilon_1 = \varepsilon_2 = 10^{-10}, r = 1.1$ and $L = 23$ by applying APM2 and are given as follows:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \\ z_{11} \end{pmatrix} = \begin{pmatrix} 0.9959 \\ 0.9781 \\ 0.9443 \\ 0.8919 \\ 0.8178 \\ 0.7198 \\ 0.5981 \\ 0.4568 \\ 0.3060 \\ 0.1634 \\ 0.0533 \end{pmatrix}, \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{pmatrix} = \begin{pmatrix} 0.0214 \\ 0.0507 \\ 0.0818 \\ 0.1137 \\ 0.1422 \\ 0.1597 \\ 0.1585 \\ 0.1339 \\ 0.0895 \\ 0.0405 \\ 0.0082 \end{pmatrix}.$$

In Table 6.7 we compare the approximation error of Algorithm 5.1 with exponential sum approximation obtained in [25], where the following approximation problem is considered. Let $E_n(x)$ be an exponential sum of the form

$$E_n(x) = \sum_{j=1}^{n} c_j \mathrm{e}^{-a_j x}, \tag{6.2}$$

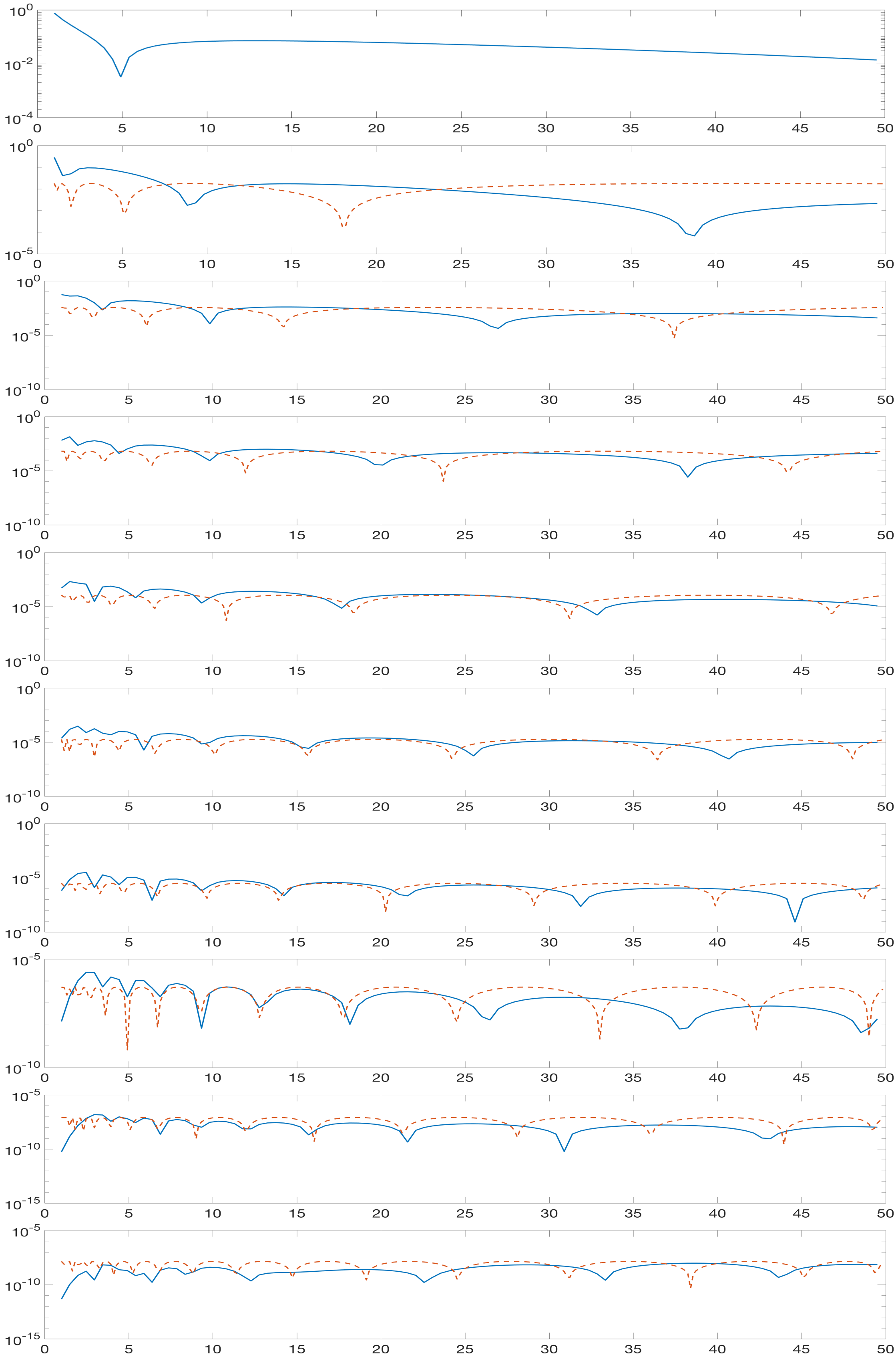

Figure 6.3: Plot of the approximation error $|f_k - \tilde{f}_k^{(n)}|$ (blue, solid line) and $|f_k - (f_k)_H^{(n)}|$ (red, dashed line) on the logarithmic y-axis and the index k of the sequence on the x-axis for $n = 1, \ldots, 10$ (from top to bottom) computed for approximation of $1/x$ with Algorithm 5.1.

n	1	2	3	4	5	6	7	8	9	10
$\tilde{z}_1^{(n)}$	0.9804	0.8725	0.6982	0.5254	0.3856	0.2816	0.2063	0.1516	0.1112	0.0802
$\tilde{z}_2^{(n)}$		0.9933	0.9545	0.8706	0.7544	0.6279	0.5079	0.4022	0.3123	0.2355
$\tilde{z}_3^{(n)}$			0.9953	0.9710	0.9187	0.8386	0.7391	0.6309	0.5220	0.4157
$\tilde{z}_4^{(n)}$				0.9958	0.9760	0.9358	0.8731	0.7901	0.6917	0.5814
$\tilde{z}_5^{(n)}$					0.9959	0.9776	0.9421	0.8869	0.8113	0.7154
$\tilde{z}_6^{(n)}$						0.9959	0.9780	0.9439	0.8911	0.8171
$\tilde{z}_7^{(n)}$							0.9959	0.9780	0.9443	0.8919
$\tilde{z}_8^{(n)}$								0.9959	0.9781	0.9443
$\tilde{z}_9^{(n)}$									0.9959	0.9781
$\tilde{z}_{10}^{(n)}$										0.9959
n	1	2	3	4	5	6	7	8	9	10
$\tilde{a}_1^{(n)}$	0.2419	0.6728	0.7437	0.5561	0.3499	0.2045	0.1158	0.0645	0.0354	0.0186
$\tilde{a}_2^{(n)}$		0.0434	0.1717	0.3228	0.3886	0.3592	0.2832	0.2011	0.1320	0.0797
$\tilde{a}_3^{(n)}$			0.0290	0.0921	0.1756	0.2434	0.2669	0.2474	0.2014	0.1455
$\tilde{a}_4^{(n)}$				0.0225	0.0637	0.1173	0.1685	0.1988	0.2001	0.1753
$\tilde{a}_5^{(n)}$					0.0217	0.0542	0.0929	0.1318	0.1597	0.1686
$\tilde{a}_6^{(n)}$						0.0215	0.0514	0.0841	0.1172	0.1444
$\tilde{a}_7^{(n)}$							0.0214	0.0508	0.0821	0.1140
$\tilde{a}_8^{(n)}$								0.0214	0.0507	0.0818
$\tilde{a}_9^{(n)}$									0.0214	0.0507
$\tilde{a}_{10}^{(n)}$										0.0214

Table 6.8: The nodes $\tilde{z}_j^{(n)}$ and the corresponding coefficients $\tilde{a}_j^{(n)}$ from the approximation of $1/x$ computed with Algorithm 5.1.

where c_j, $a_j \in \mathbb{R}$, $a_j > 0$ and x is restricted to non-negative numbers. Further let $\mathcal{E}_n := \{E_n \ : \ E_n(x) \text{ is of the form (6.2)}\}$ denote the set of all exponential sums $E_n(x)$ of the above form of length n. Find an exponential sum $E_n^* \in \mathcal{E}_n$, such that

$$\max_{x \in I} |E_n^* - 1/x| = \min_{E_n \in \mathcal{E}_n} \max_{x \in I} |E_k(x) - 1/x|,$$

where I is a closed interval in $\mathbb{R}^+ \backslash \{0\}$. Note that in [25] an algorithm for function approximation based on the Remez algorithm was used, which is considered a classical L^∞ approximation algorithm with equioscillating approximation error, whereas we approximate sequences of samples.
We denote the vector of samples of the n-term approximation of f from [25] by $\mathbf{f}_H^{(n)} = ((f_k)_H^{(n)})_{k=0}^{99}$. The nodes and weights obtained by our algorithm are given in Table 6.8 and the nodes and weights from [25] can be downloaded from `http://www.mis.mpg.de/scicomp/EXP_SUM/1_x/`. It can be observed that the nodes $\tilde{z}_j^{(n)}$ and weights $\tilde{a}_j^{(n)}$ for $j = 1, \ldots, 10$ obtained by Hackbusch are

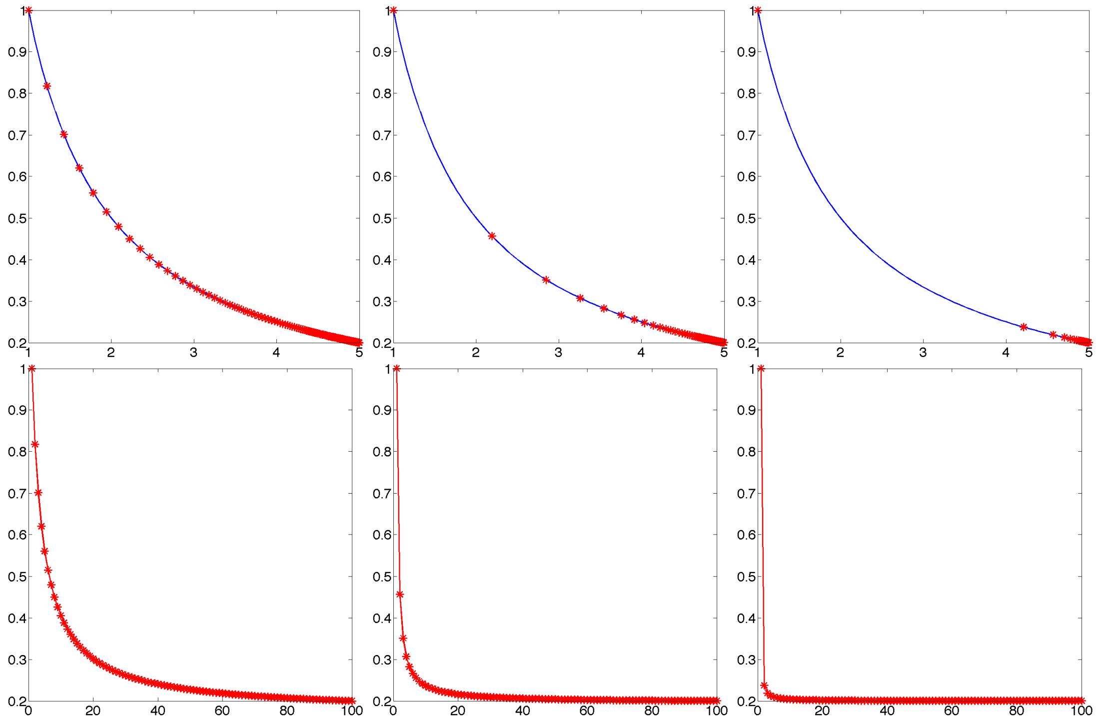

Figure 6.4: Distribution of 100 non-equidistant sampling points in the interval $[1,5]$ with sampling function $\phi(x)$ given by (6.3) for $p_2 = 0.01$ (left), $p_2 = 0.1$ (center) and $p_2 = 1$ (right). The y-axis is the function $f(\phi(k))$ at $k = 0, 1, \ldots, M$. First row: the x-axis is $\phi(k)$ for $k = 0, 1, \ldots, M-1$. Second row: the x-axis is the vector $(0, 1, \ldots, M-1)$ itself.

very different from the ones computed by our algorithm, especially for small n. Even though, for the most values of n we obtain the same order of the 2-error. For $n \geqslant 9$ the error of our algorithm is even slightly better.
Another interesting observation can be made by looking at Figures 6.3, where the approximation errors $|f_k - \tilde{f}_k^{(n)}|$ and $|f_k - (\tilde{f}_k)_H^{(n)}|$ at each sampling point $k = 1, 2, \ldots, 100$ are pictured for our algorithm and the algorithm from [25]. We see that the error from [25] rather increases towards the interval $[1, 50]$. In contrast, the error made by the AAK Algorithm is mostly concentrated in the beginning of the interval.

6.3. Equidistant vs. non-equidistant sampling

Observing a different error behavior in parts of the interval when $1/x$ is approximated by an exponential sum, we want to investigate whether the error can be decreased by using different sampling. Since Prony-like methods require equidistant samples, we will employ a special substitution.
In the following we consider approximation of the function $f(x) = 1/x$ in the interval $[a, b]$ with non-equidistant sampling. Our goal is first to construct a sampling function $\phi : [0, M-1] \mapsto [a, b]$ with $\phi(M) = b$ and $\phi(0) = a$ such that

we sample

$$f_k := f(\phi(k)) \qquad \text{for } \ k = 0, 1, \ldots, M$$

and the sampling points are distributed more densely in the end of the interval. Therefore we choose ϕ to be of the form

$$\phi(x) = p_1 - \frac{1}{p_2 x + p_3},$$

where p_1, p_2 and p_3 are some parameters in $\mathbb{C}$. We want the following conditions to be satisfied:

$$\phi(M) = p_1 - \frac{1}{p_2 M + p_3} = b \quad \text{and} \quad \phi(0) = p_1 - \frac{1}{p_3} = a.$$

Note that the second condition is equivalent to $p_1 = a + \frac{1}{p_3}$ and thus, substituting p_1 in the first condition, we obtain

$$\begin{aligned}
& \frac{1}{p_3} + a - \frac{1}{p_2 M + p_3} = b. \\
\Leftrightarrow \quad & \frac{p_2 M}{p_3(p_2 M + p_3)} = b - a \\
\Leftrightarrow \quad & p_2 M = (b-a) p_2 M p_3 + (b-a) p_3^2 \\
\Leftrightarrow \quad & p_3^2 + p_2 M p_3 - \frac{p_2 M}{b-a} = 0.
\end{aligned}$$

Now, if we choose p_2 to be a free parameter, the solution of the quadratic equation above is given by

$$p_3 = -\frac{p_2 M}{2} \pm \sqrt{\left(\frac{p_2 M}{2}\right)^2 + \frac{p_2 M}{b-a}}.$$

and the function ϕ is given as

$$\phi(x) = \frac{1 + a p_3}{p_3} - \frac{1}{p_2 x + p_3} = \frac{(1 + a p_3)(p_2 x + p_3)}{p_3(p_2 x + p_3)}. \tag{6.3}$$

Some examples of the sampling with function ϕ above are given in the first row of Figure 6.4 for $a = 1$ and $b = 5$. We observe that for small p_2 the sampling is more dense in the beginning, whereas for larger p_2 the sampling points are more and more shifted to the end of the interval $[a, b]$. Note that for $p_2 \to 0$ we will approach equidistant sampling. Of course it has to be said that if we chose p_2 too large, the full function $f(x) = 1/x$ is not represented properly by the obtained samples.

The second row of Figure 6.4 shows the same sampling points plotted at the original equidistant points $0, 1, \ldots, M-1$. We can see that the larger we choose p_2, the faster the function $f(\phi(x))$ tends to zero. This causes fast decay of the con-eigenvalues of the matrix $\mathbf{AZ}$ and the resulting approximation error, as we can see in Figures 6.5 and 6.6. Comparing to the approximation error obtained

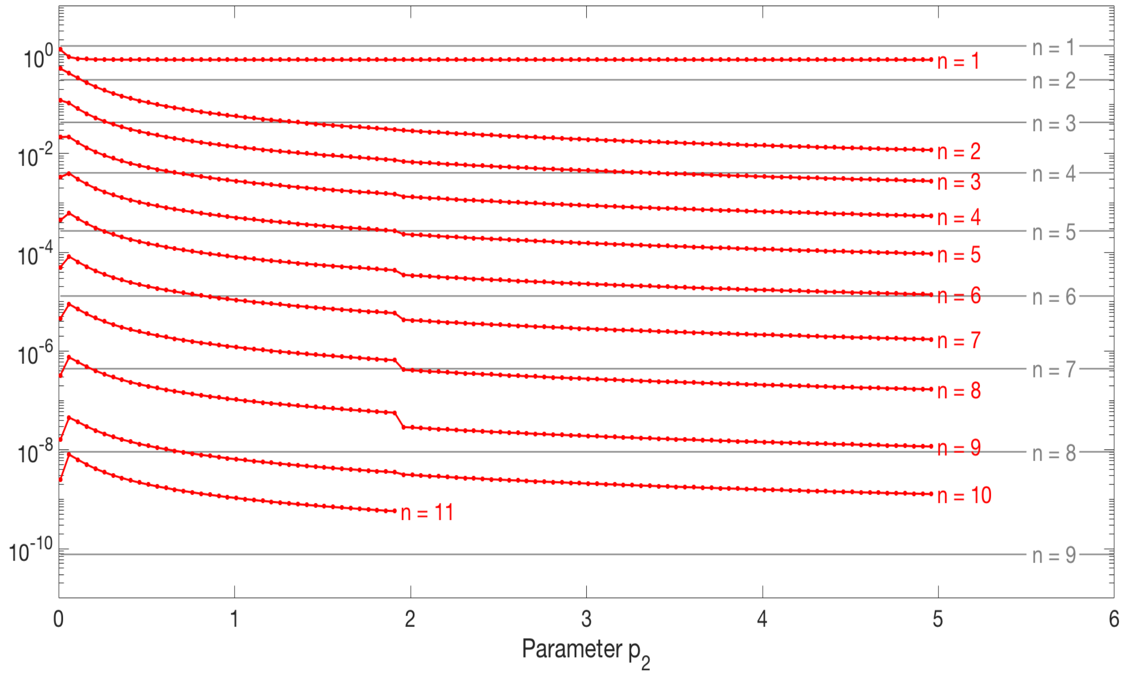

Figure 6.5: Approximation error $\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$ on a logarithmic scale obtained with equidistant sampling (gray) and non-equidistant sampling (red) for different parameter p_2 and for different sum length n.

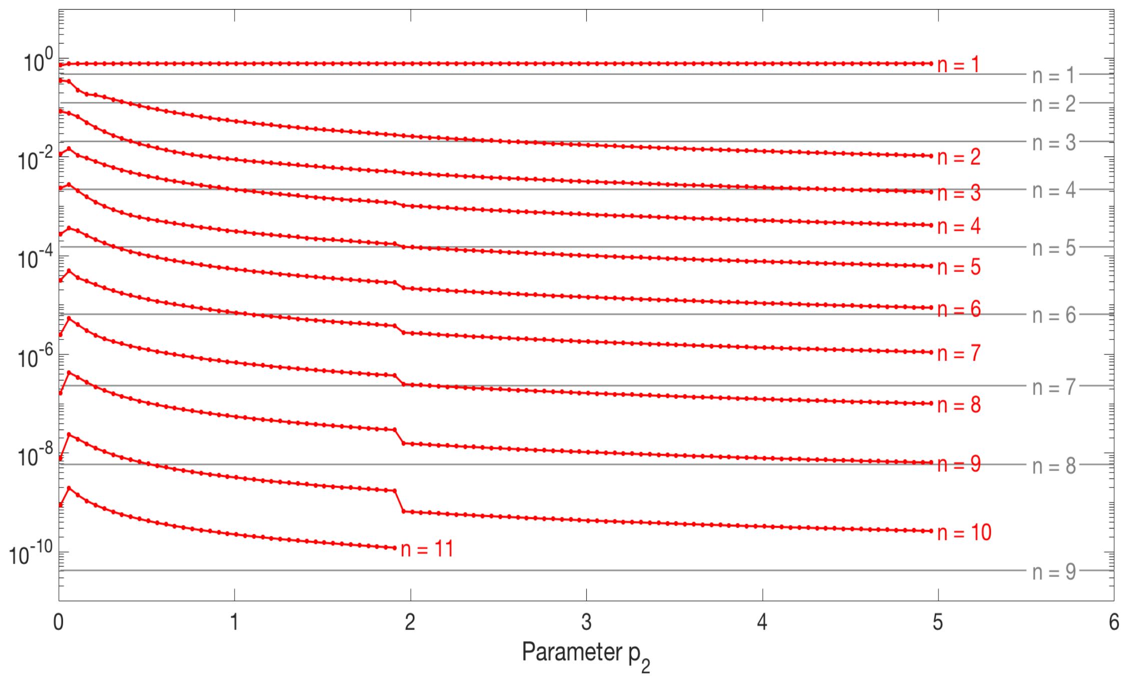

Figure 6.6: Relative approximation error $\frac{\max_j |f_j - \tilde{f}_j^{(n)}|}{\max_j |f_j|}$ on a logarithmic scale obtained with equidistant sampling (gray) and non-equidistant sampling (red) for different parameter p_2 and for different sum length n.

with equidistant sampling (gray lines in the figures) we observe that in particular for sum length $n \leqslant 5$ non-equidistant sampling gives us much better results already with rather small p_2. An exception is the case $n = 1$ for the relative error in Figure 6.6. Note that from Figure 6.4 we also see that if the sampling points would be distributed more dense in the beginning of the interval, the function $f(\phi(x))$ would decay slowly. Thus we cannot expect small approximation error in this case.

6.4. Comparison to the Greedy approach

In this chapter we compare our Algorithm 5.1 to the so-called greedy approach. A greedy algorithm is an iterative method of obtaining a globally good solution by making the optimal choice in each stage. For the approximation by exponential sums this means that in the first step we approximate our sequence of samples $\mathbf{f} = (f_k)_{k=0}^M$ by only one exponential z_0 and the corresponding coefficient c_0. We obtain

$$f_k \approx f_k^{(0)} := c_o z_o^k, \qquad k = 0, 1, \ldots, M.$$

Then the residual term

$$\mathbf{f}^{(1)} := \mathbf{f} - \mathbf{f}^{(0)} = \mathbf{f} - (c_0 z_0^k)_{k=0}^M$$

is built, which is again approximated by one exponential z_1 and its coefficient c_1 in the next step. The iteration stops when the global approximation error is small enough, i.e., for a certain tolerance *epsilon* > 0 the condition

$$\|\mathbf{f} - \mathbf{f}^{(n)}\|_2 < \epsilon$$

is satisfied for some $n \in \mathbb{N}$. The approximation at each iteration step is performed as follows. Choosing the ansatz

$$f_k^{(n)} = c_n z_n^k$$

for $k = 1, 2, \ldots, M$ the Prony polynomial is given by

$$P^{(n)}(x) = x - z_n =: p_0^{(n)} + p_1^{(n)} x$$

with coefficients $p_0^{(n)} = -z_n$ and $p_1^{(n)} = 1$. This leads to the linear overdetermined system of equations

$$\begin{pmatrix} f_0^{(n)} & f_1^{(n)} \\ f_1^{(n)} & f_2^{(n)} \\ \vdots & \vdots \\ f_{M-1}^{(n)} & f_M^{(n)} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = 0,$$

which is equivalent to

$$\begin{pmatrix} \overline{f}_0^{(n)} & \overline{f}_1^{(n)} & \cdots & \overline{f}_{M-1}^{(n)} \\ \overline{f}_1^{(n)} & \overline{f}_2^{(n)} & \cdots & \overline{f}_M^{(n)} \end{pmatrix} \begin{pmatrix} f_0^{(n)} & f_1^{(n)} \\ f_1^{(n)} & f_2^{(n)} \\ \vdots & \vdots \\ f_{M-1}^{(n)} & f_M^{(n)} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = 0$$

and thus

$$\begin{pmatrix} \sum_{k=0}^{M-1} |f_k^{(n)}|^2 & \sum_{k=0}^{M-1} \overline{f}_k^{(n)} f_{k+1}^{(n)} \\ \sum_{k=0}^{M-1} \overline{f}_{k+1}^{(n)} f_k^{(n)} & \sum_{k=0}^{M-1} |f_{k+1}^{(n)}|^2 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = 0.$$

Since we assumed $p_1^{(n)} = 1$, from the above equality we can compute $p_0^{(n)} = -z_n$ by solving the equation

$$z_n = -p_0^{(n)} = \frac{\sum_{k=0}^{M-1} \overline{f}_k^{(n)} f_{k+1}^{(n)}}{\sum_{k=0}^{M-1} |f_{k+1}^{(n)}|^2} \tag{6.4}$$

from the first row or

$$z_n = -p_0^{(n)} = \frac{\sum_{k=0}^{M-1} |f_{k+1}^{(n)}|^2}{\sum_{k=0}^{M-1} \overline{f}_{k+1}^{(n)} f_k^{(n)}} \tag{6.5}$$

from the second row. Once the node z_n is computed, we know that

$$\mathbf{f}^{(n)} = c_n \cdot \mathbf{z}^{(n)},$$

where $\mathbf{z}^{(n)} := (1, z_n, \ldots, z_n^M)^\top$ and thus

$$(\mathbf{f}^{(n)})^\top \mathbf{f}^{(n)} = c_0 (\mathbf{f}^{(n)})^\top \mathbf{z}^{(n)}.$$

Therefore the corresponding coefficient c_n can be computed by

$$c_n = \frac{(\mathbf{f}^{(n)})^\top \mathbf{f}^{(n)}}{(\mathbf{f}^{(n)})^\top \mathbf{z}^{(n)}}. \tag{6.6}$$

The full greedy algorithm can be summarized as follows.

Algorithm 6.1 (Greedy algorithm for approximation by exponential sums)

Input: Sequence of samples $\mathbf{f} = (f_k)_{k=0}^M$, error tolerance ϵ.

1. Initialize $\mathbf{f}^{(0)} := \mathbf{f}$ and $n := 0$.
2. **while** $\|\mathbf{f} - \mathbf{f}^{(n)}\|_1 > \epsilon$ or $n = 0$
   - 2.1 Compute the node z_n using (6.4) or (6.5).
   - 2.2 Compute the corresponding coefficient c_n using (6.6).
   - 2.3 Update the residual term
     $$\mathbf{f}^{(n+1)} := \mathbf{f} - \mathbf{f}^{(n)} = \mathbf{f} - (c_n z_n^k)_{k=0}^M$$

n	σ_n	$\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$
1	7.1762e+00	1.4824e+00
2	8.2868e-01	2.8978e-01
3	7.1421e-02	3.6591e-02
4	4.7100e-03	3.0648e-03
5	2.3362e-04	1.7532e-04
6	8.2387e-06	6.8243e-06
7	1.8143e-07	1.6518e-07
8	1.7681e-09	1.7105e-09

Table 6.9: Approximation of $1/x$ by exponential sums: singular values σ_n of the matrix **AZ** and the approximation error $\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2$ obtained with Algorithm 5.1 for different sum length $n = 1, \ldots, 8$.

n	$\|\mathbf{f} - \tilde{\mathbf{f}}_{\mathbf{G}}^{(n)}\|_2$	n	$\|\mathbf{f} - \tilde{\mathbf{f}}_{\mathbf{G}}^{(n)}\|_2$
1	6.6975e-01	11	6.3553e-06
2	2.0419e-01	12	1.5547e-06
3	8.5992e-02	13	4.0673e-07
4	1.4476e-02	14	1.2588e-07
5	8.9278e-03	15	3.8642e-08
6	1.7453e-03	16	1.3680e-08
7	4.3659e-04	17	4.8130e-09
8	2.4497e-04	18	5.9306e-09
9	2.3027e-05	19	5.9306e-09
10	1.1232e-05	20	5.9306e-09

Table 6.10: Approximation of $1/x$ by exponential sums: approximation error $\|\mathbf{f} - \tilde{\mathbf{f}}_{\mathbf{G}}^{(n)}\|_2$ obtained by Greedy Algorithm 6.1 for different sum lengths $n = 1, \ldots, 20$.

and

$$n := n + 1.$$

end

In the following example we test the algorithm above for approximation of the function $f(x) = 1/x$ and compare it to the performance of Algorithm 5.1. We sample the function $1/x$ on the interval $[1, 5]$ with 100 equidistant sampling points, i.e., the initial sequence of samples is of the form $\mathbf{f} = (f_k)_{k=0}^{99}$. The parameters for the APM algorithm are chosen as follows: $L = 23$, $\varepsilon_2 =$ 1e-11, $\varepsilon_1 =$ 1e-11 and $r = 1$. This leads to the exponential sum of length 9 with approximation error $\|\mathbf{f} - \mathbf{f_A}\|_2 =$ 2.3178e-10 for the initial APM approximation, where $\mathbf{f_A}$ is the approximation sequence of length 100. For $\tilde{\mathbf{f}}^{(n)} = (\tilde{f}_k^{(n)})_{k=0}^{99}$ the

AAK approximation sequence for the sum length n the approximation error is given in Table 6.9.
Note that the condition of the **AZ** matrix in this example is 1.4100e+11. For comparison, Algorithm 6.1 is used to approximate the same sampling sequence $\mathbf{f}$, where the function $1/x$ is sampled equidistantly on the same interval $[1,5]$ with the same number of samples 100. The approximation error of the greedy algorithm is given in Table 6.10.
For sum length $n = 1$ and $n = 2$ Algorithm 6.1 indeed provides a better performance. But for $n \geqslant 3$ we observe that the approximation error of the greedy approach is worse than the one obtained with Algorithm 5.1. In general, for larger n we see that much more terms in the approximation sum are necessary to obtain the same order of approximation error as with our algorithm. Furthermore, for $n \geqslant 18$ the error of Algorithm 6.1 does not change substantially, i.e., the best possible approximation seems to be already achieved.
Note that the relatively weak performance of Algorithms 5.1 for small n in comparison can be improved by the parameter choice. First, the number of samples M has to be lowered for approximation with small sum length n. Further, the upper estimate L for the sum length should be adapted for extreme short approximating sums. Thus, considering $1/x$ on the same interval as above with only 5 samples and APM parameters $L = 2$, $\varepsilon_2 =$ 1e-02, $\varepsilon_1 =$ 1e-14 and $r = 1$ leads to the approximation error $\|\mathbf{f} - \tilde{\mathbf{f}}^{(n)}\|_2 =$ 2.2140e-01 for $n = 1$. Note that for this new sampling also the error of Greedy algorithm for $n = 1$ improves, namely $\|\mathbf{f} - \tilde{\mathbf{f}}_{\mathbf{G}}^{(n)}\|_2 =$ 1.2668e-01. However both errors have become comparable in this case.
Comparing the nodes computed with our algorithm and those of Algorithm 6.1 we also observe big differences. In the Tables 6.11 and 6.12 the nodes $\tilde{z}_j$ of Algorithm 5.1 are given for every n as well as the nodes z_j^G of Greedy algorithm. The first main difference, which is rather of conceptional matter, is that the Greedy algorithm does not change the nodes computed in the previous steps, whereas in Algorithm 5.1 the nodes are obtained independently for every sum length n. Further we see that for $n \in \{2,4,6,8\}$ the nodes of Algorithm 6.1 are not inside the unit disk $\mathbb{D}$. However with both algorithms we obtain quite large nodes, i.e., close to the unit disk, which can be explained by a rather moderate decay of the function $1/x$.

6.5. Structured low rank approximation with SVD

In this section we show the performance of Algorithm 5.4 for one dimensional signal $\mathbf{f} = (f_k)_{k=0}^{L-1}$ given by the 100-th row of the image "trui". The signal has length $L = 256$, i.e. we consider the samples to be an exponential sum of the

n	1	2	3	4	5	6	7	8
$\tilde{z}_1^{(n)}$	0.9953	0.9762	0.9443	0.9031	0.8548	0.8005	0.7401	0.6702
$\tilde{z}_2^{(n)}$		0.9973	0.9851	0.9624	0.9298	0.8879	0.8363	0.7728
$\tilde{z}_3^{(n)}$			0.9975	0.9867	0.9667	0.9370	0.8968	0.8445
$\tilde{z}_4^{(n)}$				0.9975	0.9870	0.9675	0.9382	0.8981
$\tilde{z}_5^{(n)}$					0.9976	0.9870	0.9675	0.9383
$\tilde{z}_6^{(n)}$						0.9976	0.9870	0.9675
$\tilde{z}_7^{(n)}$							0.9976	0.9870
$\tilde{z}_8^{(n)}$								0.9976

Table 6.11: The nodes $\tilde{z}_j$ from the approximation of $1/x$ computed with Algorithm 5.1.

z_1^G	0.9762	z_6^G	1.0339	z_{11}^G	0.9817	z_{16}^G	0.7829
z_2^G	1.0125	z_7^G	0.8065	z_{12}^G	0.8627	z_{17}^G	0.7419
z_3^G	0.8756	z_8^G	1.0383	z_{13}^G	0.8193	z_{18}^G	0.7324
z_4^G	1.0297	z_9^G	0.8731	z_{14}^G	0.8302	z_{19}^G	0.7333
z_5^G	0.8944	z_{10}^G	0.7480	z_{15}^G	0.7777	z_{20}^G	0.7333

Table 6.12: The nodes z_j^G from the approximation of $1/x$ computed with Algorithm 6.1.

form

$$f_k = \sum_{j=0}^{L-1} a_j z_j^k = \sum_{j=0}^{L-1} a_j \omega_L^{jk}, \qquad k = 0, 1, 2, \ldots,$$

where the original nodes z_j are the 256-th roots of unity $z_j = \omega_L^j$ for $j = 1, 2, \ldots, L$. In this case our periodic Hankel matrix $\mathbf{H}_\mathbf{f}^{\text{per}} \in \mathbb{R}^{L \times L}$ has period L.
In Figure 6.7 we present the results of the rank K approximation for $K = 10$, $K = 30$ and $K = 50$. In the right column we see the nodes on the unit circle of the shorter sum $\tilde{\mathbf{f}}$. We observe that for $K = 50$ we already obtain a really good sparse approximation of the initial signal. This can be seen as a signal compression, which means that instead of 256 values it suffices to store 50 nodes z_j and 50 coefficients a_j.
As shown in Figure 6.8 the Algorithm 5.4 can also be applied to images, namely row- or column-wise, although the correlation between the rows or columns gets lost. Thus we observe horizontal and vertical artifacts after the row- and column-wise approximation, respectively. This can be improved a little by taking the average between the row- and the column-wise approximated images.

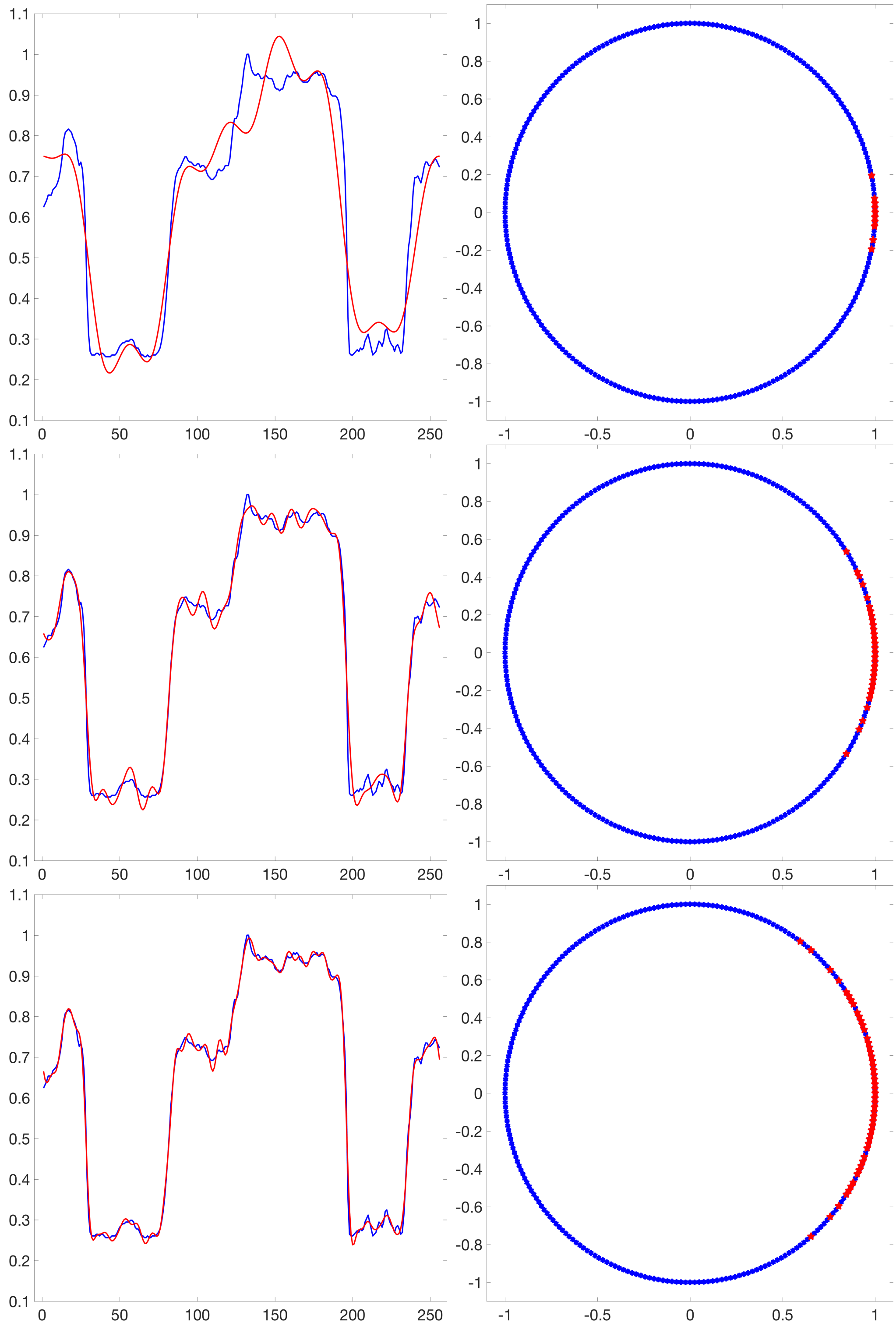

Figure 6.7: Sparse SVD approximation (red) of the original signal (blue) in the left column and the corresponding nodes $z_j = \omega_L^{k_j}$ (red) on the unit circle in the right column. The length of the approximation sum is $K = 10$, $K = 30$ and $K = 50$ (from top to bottom) and $L = 256$.

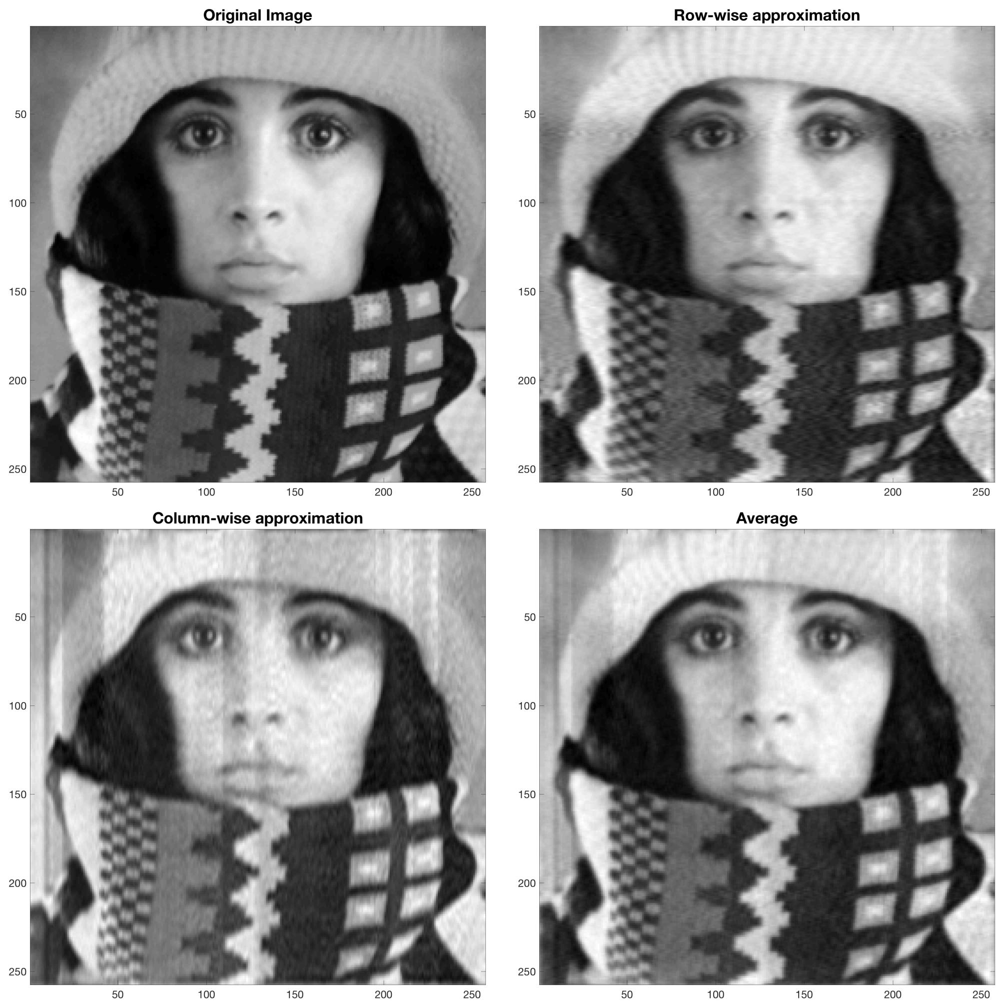

Figure 6.8: Sparse SVD approximation of an image with $K = 50$.

Conclusions and future work

In this dissertation a new algorithm solving the approximation problem (1.2) based on Prony-like methods and AAK theory was presented. The main idea of the algorithm is the dimension reduction procedure for the con-eigenvalue problem of infinite Hankel matrix $\mathbf{\Gamma_f}$ with finite rank. This procedure is based on the investigation of the structure of the con-eigenvectors of $\mathbf{\Gamma_f}$ corresponding to the non-zero con-eigenvalues. Further, a new proof of the AAK theorem for Hankel matrices with finite rank was provided in Chapter 3, where only tools from linear algebra and Fourier analysis were used. The mathematical objects involved in our proof were characterized in the discrete framework and the connection to the operator theoretical setting was established in Chapter 2. Also a structure preserving "translation" of our proof to the continuous setting was provided in Section 3.2. With this work a solid groundwork for understanding the AAK theory from the perspective of linear algebra and Prony methods was accomplished.

However there exist further open questions in the field connecting the AAK theory and Prony's method. As it has been pointed out in [8] and [3], an open problem concerning the AAK theory is whether a similar result can be obtained for finite Hankel matrices. In [8] the numerical experiments support the presumption that the ℓ^2-error of the K-term approximation using finite Hankel matrices and the K-th con-eigenvalue correlate. A theoretical foundation for this fact is yet to be developed.

A further interesting object of investigation would be the two-dimensional case. As we mentioned in the Introduction, a lot of efford has been made to obtain a Prony-like method in several variables, see [32, 52, 51]. Also attempts to apply the AAK theory for function approximation by exponential sums in two dimensions can be found for instance in [2]. In this context the question arises, whether the algorithm presented in this thesis can be extended to the two-dimensional case.

From Chapter 3.1 we know that the AAK Theorem provides the best approximation in the operator norm of the infinite Hankel matrix $\mathbf{\Gamma_f}$. In view of this fact the question arises, whether a best approximation is possible to achieve with an algorithm using a minimization approach, such as [31] or [13], which we mentioned in the Introduction. All earlier attempts to solve the structured low rank approximation problem for Hankel matrices using tools from convex analysis cannot guarantee to obtain a global optimal solution.

Also the location of the nodes z_j is a critical point of the algorithm presented in this work. Some applications consider approximation by exponential sums with exponents outside of the unit disc $\mathbb{D}$ or mixed nodes. Thus, based on the groundwork of this thesis, the possibility for a related algorithm of sparse approximation by exponential sums with non-decaying exponents is to be discovered.

Finally we want to empathize the great performance of the APM method (used with suitable parameters). While our new algorithm based on AAK theory provides provable good error bounds in terms of the con-eigenvalues of Hankel matrices, the APM algorithm often achieves numerical result which are as good as those obtained by our algorithm for approximation with short sums of exponentials. The reason for this excellent behavior of APM is not yet completely understood.

Appendices

A. Pseudocodes

Algorithm A.1 (High accuracy GECP of a Cauchy matrix of the form (5.8))

Input: Cauchy matrix C of size $N \times N$, x, y from (5.8)

for $k = 1, \ldots, N-1$

1. Find the largest absolute entry in $C(k:n, k:n)$
2. Swap rows and columns of C, and entries of x and y, such that C_{kk} is largest
3. **for** $r = k, \ldots, N-1$ and $s = k+1, \ldots, N$ overwrite:
   $C_{rs} = C_{rs} * (x_r - x_k) * (y_s - y_k) \, / \, [(x_k - y_s) * (x_r + y_k)]$
   endfor

endfor

4. Build the matrices for SVD:
   $D = \text{diag}(C)$
   $X = \text{tril}(C) * D^{-1} + Id$, where tril means strict lower triangle
   $Y = (D^{-1} * \text{trui}(C) + Id)^T$, where trui means strict upper triangle

Output: The rank revealing decomposition $C = XDY^T$.

Algorithm A.2 (One-sided Jacobi for the singular value problem)

Input: Matrix C of size $N \times N$, tol (for the stopping criterion)

repeat until convergence (all $|c|/\sqrt{ab} \leqslant tol$)

1. Compute the (i,j)-th submatrix $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ of $C^T C$:

   $a = \sum_{k=1}^{N} C_{ki}^2$
   $b = \sum_{k=1}^{N} C_{kj}^2$
   $c = \sum_{k=1}^{N} C_{ki} * C_{kj}$

2. Compute the Jacobi rotation which diagonalizes $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$:

   $\zeta = (b-a)/(2c)$
   $t = sign(\zeta)/(|\zeta| + \sqrt{1+\zeta^2})$
   $cs = 1/\sqrt{1+t^2}$
   $sn = cs * t$

3. Update columns i and j of C:

   for $k = 1, \dots, N$
   $tmp = C_{ki}$
   $C_{ki} = cs * tmp - sn * C_{kj}$
   $C_{kj} = sn * tmp + cs * C_{kj}$
   endfor

4. Update tha matrix V of right singular vectors:

   for $k = 1, \dots, N$
   $tmp = C_{ki}$
   $V_{ki} = cs * tmp - sn * V_{kj}$
   $V_{kj} = sn * tmp + cs * V_{kj}$
   endfor

end

Output: The computed singular values are the norms of the columns of the final C and the left singular vectors are the normalized columns of the final C.

Algorithm A.3 (Pivot Order)

Input: The parameters b, d, x, y of the positive definite Cauchy matrix C given by (5.8), target size δ of the con-eigenvalue.

1. Form the vector $c_i := a_i b_i / (x_i + y_i)$ for $i = 1, \ldots, N$.
2. Set the cutoff parameter for GECP termination: $\eta := \epsilon \delta^2$.
3. Initialize permutation matrix $\tilde{P} = I(N, N)$ ($N \times N$ identity).
4. Compute correctly pivoted vectors:

    $m := 1$

    while $|g(m)| \geqslant \eta$ or $m = N - 1$

    4.1. Find $m \leqslant l \leqslant N$ such that $|c(l)| = \max|c(m : N)|$.

    4.2. Swap elements:

    $c(l) \leftrightarrow c(m),\ x(l) \leftrightarrow x(m),\ y(l) \leftrightarrow y(m),$

    $b(l) \leftrightarrow b(m),\ d(l) \leftrightarrow d(m)$

    4.3. Swap rows of the permutation matrix:

    $\tilde{P}(l, :) \leftrightarrow \tilde{P}(m, :)$

    4.4. Update the diagonal of Schur complement:

    $c(m+1 : N) := \frac{x(m+1:N) - x(m)}{y(m+1:N) - y(m)} * c(m+1 : N)$

    $m := m + 1$

    end

5. Overwrite $\tilde{P} = \tilde{P}(1 : m, 1 : N)$.

Output: Correctly pivoted vectors b, d, x, y, truncation size m and the permutation matrix $\tilde{P}$ of size $m \times N$.

Algorithm A.4 (Cholesky Cauchy)

Input: The parameters b, d, x, y of the positive definite Cauchy matrix C given by (5.8), target size δ of the con-eigenvalue.

1. Compute the pivoted vectors b, d, x, y, permutation matrix $\tilde{P}$ and the matrix size m with Algorithm A.3.
2. Initialize generators: $\alpha := b,\ \beta := d$.
3. Compute the first column of the Shur complement: $G(:,1) := \alpha * \beta/(x+y)$
4. **for** $k = 2, \ldots, m$ update generators:
   $\alpha(k:N) := \alpha(k:N) * \frac{x(k:N)-x(k-1)}{x(k:N)+y(k-1)}$
   $\beta(k:N) := \beta(k:N) * \frac{y(k:N)-y(k-1)}{y(k:N)+y(k-1)}$
   end
5. Extract the k-th column for the Cholesky factors:
   $G(k:N,k) := \alpha(k:N) * \beta(k:N)/(x(k:N)+y(k:N))$
6. Compute the partial Cholesky factors:
   $\tilde{D} = \text{diag}(G(1:N,1:m))^{1/2}$
   $\tilde{L} = \text{tril}(G(1:N,1:m))\tilde{D}^{-2} + I(N,m)$, where tril means strict lower diagonal

Output: Matrices $\tilde{L}$ of size $N \times m$, $\tilde{D}$ of size $m \times m$ and $\tilde{P}$ of size $m \times N$, such that $C = \tilde{L}\tilde{D}\tilde{P}$ is a partial Cholesky factorization.

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