The shifted convolution of generalized divisor functions

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CHAPTER 1

Introduction

Additive divisor problems have a rich history in analytic number theory. A classical example is the binary additive divisor problem, which asks for the asymptotic evaluation of the sum

$$D_{2,2}(x,h) := \sum_{n\le x} d(n)d(n+h), \quad h \ge 1,$$

where $d(n)$ is the usual divisor function. It is the presence of the shift parameter h, which makes the problem rather difficult, since many standard methods from analytic number theory cannot be applied then. Nevertheless, a lot of effort has been made to study the problem and it is well understood by now – for example, we know that, for any $\varepsilon > 0$,

$$D_{2,2}(x,h) = xP_{2,h}(\log x) + \mathcal{O}\Big(x^{\frac{2}{3}+\varepsilon}\Big) \quad \text{for} \quad h \ll x^{\frac{2}{3}},$$

with $P_{2,h}$ a quadratic polynomial depending on h, a result we have cited from Motohashi [**36**], where a detailed account of the history of this problem can be found as well. A similar asymptotic formula holds in fact also for much larger h (the best result in this respect is due to Meurman [**33**]).

One reason for the interest in this sum is its relation to the Riemann zeta function $\zeta(s)$. As a way of studying the behaviour of $\zeta(s)$ in the critical strip, the moments

$$I_k(T) := \int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt$$

have been subject to intense research. So far, asymptotic formulas have been established only for the cases $k = 1$ and $k = 2$ (see e.g. [**42**, Chapter VII]). While the asymptotic behaviour of the second moment $I_1(T)$ can be determined fairly easily, the fourth moment $I_2(T)$ is much more complicated, and it is here that the shifted convolution sums $D_{2,2}(x,h)$ come up and play an important role. For $k \ge 3$, the problem of finding an asymptotic formula for $I_k(T)$ – or even just getting non-trivial upper bounds – essentially remains unsolved.

A natural generalization of the binary additive divisor problem is given by the problem of determining the asymptotic behaviour of the sums

$$D_{k,\ell}(x,h) := \sum_{n\le x} d_k(n)d_\ell(n+h), \quad h \ge 1,$$

where $d_k(n)$ stands for the number of ways to write n as a product of k positive integers. In analogy to the case $k = 2$, the study of the shifted convolutions $D_{k,k}(x,h)$ might lead to a better understanding of the higher moments of the Riemann zeta function (see [**10, 22**]). However, the evaluation of the sums $D_{k,\ell}(x,h)$ is by no means an easy problem. In fact, as soon as $k, \ell \ge 3$, the problems in estimating the sums $D_{k,\ell}(x,h)$ get overwhelmingly hard, and even for the easiest case

$$D_{3,3}(x,1) = \sum_{n\le x} d_3(n)d_3(n+1)$$

no asymptotic formula is known.

The situation changes, however, when $k = 2$ or $\ell = 2$: The sums

$$D_k^+(x,h) := \sum_{n\le x} d_k(n)d(n+h) \quad \text{and} \quad D_k^-(x,h) := \sum_{n\le x} d_k(n+h)d(n), \quad h \ge 1,$$

can indeed be treated by current methods, and they form the main topic of this thesis. The best results for $D_2^\pm(x,h)$ have been obtained by employing spectral methods coming from the theory of automorphic forms. Here we want to show how these methods can be applied to the sums $D_k^\pm(x,h)$ with $k \ge 3$ in a way that enables us to obtain results considerably better than what has been achieved previously.

This will already become clear when we look at $D_3^\pm(x,h)$. The first asymptotic formula for this sum goes back to Hooley [**21**], who showed that, for h fixed,

$$D_3^\pm(x,h) = C_3(h)x\log^3 x + \mathcal{O}\big(x(\log x \log\log x)^2\big),$$

where $C_3(h)$ is some positive constant. We also want to mention Linnik [**31**] at this point, who used the dispersion method to treat the sums $D_k^\pm(x,h)$ for general $k \ge 3$, and whose results were subsequently improved by other authors. We will have to say more about this later – for the moment, however, we want to focus on $D_3^\pm(x,h)$, for which approaches specific to this case soon allowed to get considerably better results.

The first result with a power saving in the error term seems to be given by Deshouillers [**11**], who used spectral methods to attack a smoothed version of this problem, much in the spirit of his earlier joint work with Iwaniec [**12**] on the binary additive divisor problem. Naturally, Deshouillers' result can also be used to treat sums like $D_3^\pm(x,h)$ with sharp cut-off, although he did not work out the details. As Friedlander and Iwaniec [**18**] pointed out, a different approach was possible as a consequence on their work on the ternary divisor function in arithmetic progressions. Heath-Brown [**20**] improved their result, and showed that, for any $\varepsilon > 0$,

$$D^-(x,1) = xP_{3,1}(\log x) + \mathcal{O}\Big(x^{\frac{101}{102}+\varepsilon}\Big), \tag{1.1}$$

where $P_{3,1}$ is a polynomial of degree 3. The methods used in [**18**] and [**20**] depend ultimately on very deep results coming from algebraic geometry, and make no use of spectral theory.

Later, Bykovskiĭ and Vinogradov [**8**] returned to the spectral approach of Deshouillers [**11**] based on the Kuznetsov formula and stated (1.1) with an exponent $\frac{8}{9}$ in the error term. Unfortunately, not more than a few brief hints were given to support this claim, and our first result is a detailed proof of the following asymptotic formula, which yields in addition a substantial range of uniformity in the shift parameter h.

Theorem 1.1. *We have, for $h \ll x^{\frac{2}{3}}$,*

$$D_3^\pm(x;h) = xP_{3,h}(\log x) + \mathcal{O}\Big(x^{\frac{8}{9}+\varepsilon}\Big),$$

where $P_{3,h}$ is a cubic polynomial depending on h, and where the implied constants depend only on ε.

We also want to state the analogous result for the sum weighted by a smooth function.

Theorem 1.2. *Let $w : [1/2,1] \to \mathbb{R}$ be smooth and compactly supported. Then we have, for $h \ll x^{\frac{2}{3}}$,*

$$\sum_n w\Big(\frac{n}{x}\Big)d_3(n)d(n\pm h) = xP_{3,h,w}(\log x) + \mathcal{O}\Big(x^{\frac{5}{6}+\frac{\theta}{3}+\varepsilon}\Big),$$

where $P_{3,h,w}$ is a cubic polynomial depending on h and w, and where the implied constants depend only on w and ε.

By θ we denote here and in the following the bound in the Ramanujan-Petersson conjecture (see Section 3 in Chapter 2 for a precise definition). In any case, $\theta = \frac{7}{64}$ is admissible and with this value we get in Theorem 1.2 an error term which is $\ll x^{\frac{7}{8}}$, thus improving the result of Deshouillers [**11**].

Before going on to discuss the sums $D_k^{\pm}(x,h)$ with $k \geq 4$, we want to state a few related results which can be proven using the same methods as for the results above. Let φ be a holomorphic cusp form of weight κ for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Let $a(n)$ be its normalized Fourier coefficients, so that $\varphi(z)$ has the Fourier expansion

$$\varphi(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{\kappa-1}{2}} e(nz). \tag{1.2}$$

The divisor function and the Fourier coefficients $a(n)$ share a lot of similarities in their behaviour, so one might expect to get analogous results as in Theorems 1.1 and 1.2 for the sums

$$A_3^+(x,h) := \sum_{n \leq x} d_3(n) a(n+h) \quad \text{and} \quad A_3^-(x,h) := \sum_{n \leq x} d_3(n+h) a(n), \quad h \geq 1,$$

and their smooth counterparts, with the difference that we cannot expect a main term to appear anymore. Indeed Pitt [**39**] and Munshi [**38**] already obtained results of this sort. Using our method, we will be able to partially improve their results by showing the following theorem.

Theorem 1.3. *We have, for $h \ll x^{\frac{2}{3}}$,*

$$A^{\pm}(x;h) \ll x^{\frac{8}{9}+\varepsilon},$$

where the implied constants depend only on φ and ε.

Not surprisingly, the analogous result for the smoothed sum holds as well.

Theorem 1.4. *Let $w : [1/2, 1] \to \mathbb{R}$ be smooth and compactly supported. Then we have, for $h \ll x^{\frac{2}{3}}$,*

$$\sum_n w\Big(\frac{n}{x}\Big) d_3(n) a(n \pm h) \ll x^{\frac{5}{6}+\frac{\theta}{3}+\varepsilon},$$

where the implied constants depend only on w, φ and ε.

Another interesting problem is the following sum, which can be seen as a dual version to $D_3^{\pm}(x,h)$,

$$D_3(N) := \sum_{n=1}^{N-1} d_3(n) d(N-n).$$

In contrast to the analogous sum with two binary divisor functions (see [**36**, Theorem 2]), the main term in our case is a little bit more complicated. Our result is the following theorem.

Theorem 1.5. *We have*

$$D_3(N) = M_3(N) + \mathcal{O}\Big(N^{\frac{11}{12}+\varepsilon}\Big),$$

where the main term $M_3(N)$ has the form

$$M_3(N) = N \sum_{\substack{0 \leq i,j,k,\ell \leq 3 \\ i+j+k+\ell \leq 3}} c_{i,j,k,\ell} F^{(i,j,k,\ell)}(0,0,0,0),$$

with certain constants $c_{i,j,k,\ell}$ *and*

$$F(\alpha,\beta,\gamma,\delta) := N^{\alpha} \sum_{d|N} \frac{\chi_1(d)}{d^{1-\beta}} \sum_{c|d} \chi_2(c)\chi_3\Big(\frac{d}{c}\Big),$$

where the arithmetic functions χ_1*,* χ_2 *and* χ_3 *are defined by*

$$\begin{aligned} \chi_1(n) &:= \prod_{p|n}\Big(1 - \frac{1}{p^{3-\gamma-\beta} - p^{1-\gamma+\delta} - p^{1-\gamma} + 1}\Big), \\ \chi_2(n) &:= \prod_{p|n}\Big(1 + \frac{1}{p^{2-\beta-\delta} - p^{-\delta} - 1}\Big), \quad \chi_3(n) := \prod_{p|n}\Big(1 - \frac{1}{p^{1-\gamma-\delta}}\Big). \end{aligned} \tag{1.3}$$

The implied constant depends only on ε*.*

In particular, we have as leading term

$$D_3(N) = (1+o(1))C_0C(N)N\log^3 N,$$

where the constant C_0 is given by

$$C_0 := \frac{3}{\pi^2}\prod_p\Big(1 - \frac{1}{p(p+1)}\Big),$$

and where $C(N)$ is a multiplicative function defined on prime powers by

$$C\big(p^{\ell}\big) := 1 + \Big(1 - \frac{1}{p^{\ell}}\Big)\frac{2p^2+2p-1}{p^3-2p+1} - \frac{\ell}{p^{\ell}}\frac{p+1}{(p^2+p-1)}.$$

Of course, we can also look at the same problem with the divisor function $d(n)$ replaced by the Fourier coefficients $a(n)$,

$$A_3(N) := \sum_{n=1}^{N-1} d_3(n)a(N-n),$$

and it should not come as a surprise that an analogue of Theorem 1.5 holds in this situation as well.

THEOREM 1.6. *We have*

$$A_3(N) \ll N^{\frac{11}{12}+\varepsilon},$$

where the implied constant depends only on ε*.*

As indicated above, many of the methods used to treat $D_3^{\pm}(x,h)$ – in particular those leading to power savings in the error term – do not extend to the sums $D_k^{\pm}(x,h)$ with $k \geq 4$. We already mentioned the work of Linnik [**31**], who established an asymptotic formula for the first time by showing that, for $k \geq 3$,

$$D_k^{\pm}(x,1) = C_k(1)x\log^k x + \mathcal{O}\Big(x(\log x)^{k-1}(\log\log x)^{k^4}\Big),$$

where $C_k(1)$ is some positive constant. This result was improved subsequently by other authors, in particular by Motohashi [**35**], who gave an asymptotic formula including all lower-order terms. Specifically, he proved that, for each $k \geq 3$, there exists a constant c_k such that

$$D_k^{\pm}(x,1) = xP_{k,1}(\log x) + \mathcal{O}\big(x(\log x)^{-1}(\log\log x)^{c_k}\big),$$

where $P_{k,1}$ is a polynomial of degree k. Fouvry and Tenenbaum [**17**] were able to improve on this result and show that, for each $k \geq 4$, there exists a $\delta_k > 0$ such that

$$D_k^{+}(x,1) = xP_{k,1}(\log x) + \mathcal{O}\Big(xe^{-\delta_k\sqrt{\log x}}\Big). \tag{1.4}$$

In a recent preprint, Drappeau [**14**] refined their approach and used spectral methods to get a power saving in the error term. His result states that there exists a $\delta > 0$, such that

$$D_k^+(x,h) = xP_{k,h}(\log x) + \mathcal{O}\Big(x^{1-\frac{\delta}{k}}\Big) \quad \text{for} \quad h \ll x^{\delta}, \tag{1.5}$$

where $P_{k,h}$ is a polynomial of order k depending on h.

We also need to mention again the work of Bykovskiĭ and Vinogradov [**8**], where they state a result which is considerably better than (1.5). Unfortunately, their proposed proof is incomplete and does not seem to yield the error terms claimed in their paper[1]. Nevertheless, their initial approach turned out to be useful and led us, together with new crucial ingredients, to a proof of the following theorem, which improves on (1.4) and (1.5).

THEOREM 1.7. *We have, for $k \geq 4$ and $h \ll x^{\frac{15}{19}}$,*

$$D_k^{\pm}(x,h) = xP_{k,h}(\log x) + \mathcal{O}\Big(x^{1-\frac{4}{15k-9}+\varepsilon} + x^{\frac{56}{57}+\varepsilon}\Big),$$

where $P_{k,h}$ is a polynomial of degree k depending on h, and where the implied constants depend only on k and ε.

The analogous result for the sum weighted by a smooth function is as follows.

THEOREM 1.8. *Let $w : [1/2, 1] \to \mathbb{R}$ be smooth and compactly supported. Then we have, for $k \geq 4$ and $h \ll x^{\frac{15}{19}}$,*

$$\sum_n w\Big(\frac{n}{x}\Big) d_k(n) d(n \pm h) = xP_{k,h,w}(\log x) + \mathcal{O}\Big(x^{1-\frac{1}{3k-2}+\varepsilon} + x^{\frac{37}{38}+\frac{\theta}{19}+\varepsilon}\Big),$$

where $P_{k,h,w}$ is a polynomial of degree k depending on w and h, and where the implied constants depend only on w, k and ε.

At this point, we want to describe in broad terms the main ideas used to prove these results. The most direct way to handle shifted convolutions like $D_k^{\pm}(x,h)$ is to open one of the divisor functions, and then try to evaluate the arising divisor sums over arithmetic progressions in some way. This was the strategy followed in many of the works mentioned above, for example in [**18, 20**] on $D_3^{\pm}(x,h)$, and in [**14, 17, 31, 35**] on $D_k^{\pm}(x,h)$, and in all these works the choice was to open $d(n)$. In contrast to this, we have chosen to open $d_k(n)$ – although this approach is more difficult from a combinatorial point of view as we have to deal with more variables, the main advantage is that it is much easier to handle the divisor functions $d(n)$ over arithmetic progressions than the generalized divisor functions $d_k(n)$ with $k \geq 3$.

This way we arrive at sums of the form

$$\sum_{\substack{a_1,\ldots,a_k \\ a_i \asymp A_i}} d(a_1 \cdots a_k + h), \tag{1.6}$$

where we can assume that the variables $a_1, \ldots, a_k$ are supported in dyadic intervals $a_i \asymp A_i$. As long as some of the variables are supported in large intervals, we can average over one of them by use of the Voronoi summation formula, and then use the Kuznetsov formula to handle the sums of Kloosterman sums that appear at this point. If $k = 3$, this strategy goes through and eventually leads to the asymptotic formula for $D_3^{\pm}(x,h)$ stated in Theorem 1.1. The results concerning $A_3^{\pm}(x,h)$, $D_3(N)$ and $A_3(N)$ are proven the same way and differ only in technical details.

[1]In particular, the step from (5.6) to (5.7) is not correct unless n_1 and n_2 are coprime, and it is unclear how their proposed treatment of $S(n_1, n_2)$ should work for general n_1 and n_2. See also the comments after Lemma 5.2 for another problematic issue.

However, if $k \geq 4$, this is not enough. The problem is that it can happen that all the intervals A_i are so small, that we cannot average over any of the variables a_i (for example, when all A_i are of the size $A_i \asymp x^{\frac{1}{k}}$). In this case, we follow an idea of Bykovskiĭ and Vinogradov [**8**], and insert the expected main term $\Phi_0(b)$ for the sum

$$\Phi(b) := \sum_{a_1 \asymp A_1} d(a_1 b + h)$$

manually into (1.6), so that the latter can be written as

$$\sum_{\substack{a_2,\dots,a_k \\ a_i \asymp A_i}} \Phi_0(a_2 \cdots a_k) - \sum_{\substack{a_2,\dots,a_k \\ a_i \asymp A_i}} (\Phi_0(a_2 \cdots a_k) - \Phi(a_2 \cdots a_k)).$$

While the first sum will be part of the eventual main term, we need to find an upper bound for the second sum. To do so, we use the Cauchy-Schwarz inequality to bound it by

$$\Bigg(\sum_{b \asymp A_2 \cdots A_k} d_{k-1}(b)^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{b \asymp A_2 \cdots A_k} (\Phi_0(b) - \Phi(b))^2 \Bigg)^{\frac{1}{2}},$$

which has the important effect that the variables $a_2, \dots, a_k$ are now merged into one large variable b. After opening the square in the right factor, we are faced with three different sums, the most difficult of them being

$$\sum_{b \asymp A_2 \cdots A_k} \Phi(b)^2 = \sum_{a_1, \tilde{a}_1 \asymp A_1} \sum_{b \asymp A_2 \cdots A_k} d(a_1 b + h) d(\tilde{a}_1 b + h).$$

The evaluation of the inner sum on the right hand side, a variation of the binary additive divisor problem with linear factors in the arguments, lies at the heart of our method. In a slightly more general form, we can state it as

$$D(x_1, x_2, r_1, r_2) := \sum_n w_1\left(\frac{r_1 n + f_1}{x_1}\right) w_2\left(\frac{r_2 n + f_2}{x_2}\right) d(r_1 n + f_1) d(r_2 n + f_2),$$

where $w_1, w_2 : [1/2, 1] \to \mathbb{R}$ are smooth and compactly supported weight functions, where r_1 and r_2 are positive integers, and where f_1 and f_2 are integers such that $r_1 f_2 - r_2 f_1 \neq 0$.

The case $r_1 = r_2 = 1$ is of course nothing else than a smooth version of $D_2^{\pm}(x, h)$, which has been studied extensively. A few results are also available when r_1 and r_2 are assumed to be coprime: Besides the implicit treatment in [**5**], there is the work of Duke, Friedlander and Iwaniec [**15**], who showed that

$$D(x_1, x_2, r_1, r_2) = (\text{main term}) + \mathcal{O}\Big((r_2 x_1 + r_1 x_2)^{\frac{1}{4}} (r_1 r_2 x_1 x_2)^{\frac{1}{4} + \varepsilon} \Big). \tag{1.7}$$

As they did not make use of spectral theory, the size of the error term is inferior compared to what can be achieved for $D_2^{\pm}(x, h)$. More importantly, the range in r_1 and r_2 where this formula is non-trivial is comparatively small and would not be sufficient for our purposes. For the sake of completeness, we want to mention that this result has been improved in the case $r_2 = 1$ in a preprint by Aryan [**1**].

Correlations of a more general type have been investigated by Matthiesen [**32**], but the methods used there do not apply to our case and do not give power savings in the error term. Similar problems, where the divisor functions are replaced by Fourier coefficients of automorphic forms, have been studied as well (see e.g. [**2**]). In particular, Pitt [**40**, Theorem 1.4] was able to prove an asymptotic estimate for an analogue of $D(x_1, x_2, r_1, r_2)$ for r_1, r_2 squarefree and $f_1 = f_2 = -1$, where the divisor functions are replaced by Fourier coefficients of holomorphic cusp forms. Unfortunately, his method relies on Jutila's variant of the circle method, which becomes ineffective when a main term is present, as is the case in our problem.

We could not find any results in the literature covering the sum $D(x_1, x_2, r_1, r_2)$ for general r_1 and r_2, and the following result seems to be new.

THEOREM 1.9. *Set*

$$r_0 := \min\{(r_1, {r_2}^\infty), (r_2, {r_1}^\infty)\} \quad \text{and} \quad h := r_1 f_2 - r_2 f_1.$$

Then we have, for $f_1 \ll {x_1}^{1-\varepsilon}$, $f_2 \ll {x_2}^{1-\varepsilon}$ and $h \neq 0$,

$$D(x_1, x_2, r_1, r_2) = M(x_1, x_2, r_1, r_2) + \mathcal{O}\Big(r_0 (r_2 x_1)^{\frac{1}{2}+\theta+\varepsilon}\Big),$$

where the main term is given by

$$M(x_1, x_2, r_1, r_2) := \int w_1\left(\frac{r_1\xi + f_1}{x_1}\right) w_2\left(\frac{r_2\xi + f_1}{x_2}\right) \cdot P_2(\log(r_1\xi + f_1), \log(r_2\xi + f_2))\, d\xi,$$

where $P_2(\xi_1, \xi_2)$ is a quadratic polynomial depending on r_1, r_2, f_1 and f_2. The implied constants depend only on w_1, w_2 and on ε.

We also want to state the following result for an analogue of $D(r_1, r_2, x_2, x_2)$ with sharp cut-off.

THEOREM 1.10. *Let r_0 and $h \neq 0$ be defined as above. Assume that*

$$f_1 \ll (r_1 x)^{1-\varepsilon}, \quad f_2 \ll (r_2 x)^{1-\varepsilon} \quad \text{and} \quad (r_0 r_1 r_2, h) h \ll {r_0}^{\frac{1}{3}} (r_1 r_2)^{\frac{5}{3}} x^{\frac{1}{3}-\varepsilon}.$$

Then we have

$$\sum_{\frac{x}{2} < n \leq x} d(r_1 n + f_1) d(r_2 n + f_2) = x P_2(\log x) + \mathcal{O}\Big((r_0 r_1 r_2, h)^\theta {r_0}^{\frac{2}{3}+\theta} (r_1 r_2)^{\frac{1}{3}} x^{\frac{2}{3}+\varepsilon}\Big),$$

where $P_2(\xi)$ is a quadratic polynomial depending on r_1, r_2, f_1 and f_2, and where the implied constants depend only on ε.

It seems likely that the dependance on r_0 in these results is not optimal, although it is not immediately clear how an improvement might be achieved. Compared to (1.7) our result has a better error term, and more importantly, it is non-trivial for much larger r_1 and r_2, which will be crucial when applying it to the sums $D_k^{\pm}(x, h)$. In the case $r_2 = 1$, our result is the same as [**1**, Theorem 0.3].

The proof of Theorems 1.9 and 1.10 follows standard lines: We split one of the divisor functions and use the Voronoi summation formula to deal with the divisor sums in arithmetic progressions. The main difficulty lies in the handling of the sums of Kloosterman sums entering the stage at this point. In a simplified form, we are faced with sums roughly of the shape

$$\sum_{\substack{c \\ (c, r_2)=1}} \frac{S(1 - r_1\overline{r_2}, 1; r_1 c)}{r_1 c} F(r_1 c),$$

where F is some weight function, and where $\overline{r_2}$ is understood to be mod c. We could bound the Kloosterman sums individually using Weil's bound, and the resulting error terms in our theorems would be of a size comparable to (1.7). However, as we already mentioned, this would not be sufficient for our purposes, and – once again – our aim is to use spectral methods to get results beyond that.

If r_1 and r_2 are coprime, we can use the Kuznetsov formula with an appropriate choice of cusps. Otherwise, it is not directly clear how the Kuznetsov formula might be put into use here. We solve the problem by splitting the variable $r_1 = tv$ into a factor t, which is coprime to r_2, and a factor v, which contains only the same prime factors as r_2. By twisted multiplicativity of Kloosterman sums, we have

$$\frac{S(1 - r_1\overline{r_2}, 1; r_1 c)}{r_1 c} = \frac{S(\overline{tc}, \overline{tc}; v)}{v} \frac{S\big(r_2 - r_1, \overline{v^2 r_2}; tc\big)}{tc},$$

where now all the inverses are understood to be modulo the respective modulus of the Kloosterman sum. Following an idea of Blomer and Milićević [**7**], we separate the variable c occuring in the first factor by exploiting the orthogonality of Dirchlet characters as follows,

$$\frac{S(\overline{tc},\overline{tc};v)}{v}=\frac{1}{\varphi(v)}\sum_{\chi \bmod v}\chi(tc)\hat{S}_v(\chi)\quad\text{with}\quad \hat{S}_v(\chi):=\sum_{\substack{y\,(v)\\(y,v)=1}}\overline{\chi}(y)\frac{S(\overline{y},\overline{y};v)}{v},$$

where the left sum runs over all Dirichlet characters mod v. This way we are led to sums of Kloosterman sums twisted by Dirichlet characters, which we can treat by spectral methods.

This thesis is organized as follows. In Chapter 2 we collect the tools needed in the subsequent chapters and fix the necessary notation. The treatment of $D_3^{\pm}(x,h)$ and $A_3^{\pm}(x,h)$ is carried out in Chapter 3, and afterwards, in Chapter 4, we deal with $D_3(N)$ and $A_3(N)$. In Chapter 5, we look at $D_k^{\pm}(x,h)$ for $k\geq 4$. We have put the treatment of $D(x_1,x_2,r_1,r_2)$ in a separate chapter, Chapter 6.

Last but not least, we want to mention that the contents of Chapters 3 and 4 have been published in [**44**], that the content of Chapter 5 has been made available online in [**45**], and that the content of Chapter 6 has been made available online in [**43**].

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CHAPTER 2

Preliminaries

In this chapter, we want to go through the main tools needed to prove our results and fix the necessary notation.

In the following, ε always stands for some positive real number, which can be chosen arbitrarily small. However, it need not be the same on every occurrence, even if it appears in the same equation. The letter p is reserved for prime numbers. When we write $A \asymp B$, this means $A \ll B \ll A$. Given a function $f : \mathbb{R} \to \mathbb{C}$, we will occasionally write

$$\operatorname{supp} f \asymp X$$

to mean that there exist constants $c_1, c_2 > 0$, such that $\operatorname{supp} f \subseteq [c_1 X, c_2 X]$. The expression (a, b) denotes the greatest common divisor of a and b. The summation

$$\sum_{a\,(c)} (\dots) := \sum_{a \bmod c} (\dots)$$

means that the variable a runs over some residue system mod c. Analogously, we will frequently write $n \equiv h\,(c)$ instead of $n \equiv h \bmod c$. As usual, $e(q) := e^{2\pi i q}$ and

$$S(m, n; c) := \sum_{\substack{a\,(c) \\ (a,c)=1}} e\left(\frac{ma + n\overline{a}}{c}\right) \quad \text{and} \quad c_q(n) := \sum_{\substack{a\,(q) \\ (a,q)=1}} e\left(\frac{na}{q}\right),$$

which are the usual notations for Kloosterman sums and Ramanujan sums (here $\overline{a}$ indicates a solution to $a\overline{a} \equiv 1 \bmod c$).

1. The Voronoi summation formula and Bessel functions

Using the well-known Voronoi formula for the divisor function (see [**24**, Chapter 4.5] or [**25**, Theorem 1.6]) and the identity

$$\sum_{n \equiv h\,(c)} d(n) f(n) = \frac{1}{c} \sum_{d|c} \sum_{\substack{b\,(d) \\ (b,d)=1}} e\left(\frac{-bh}{d}\right) \sum_{n=1}^{\infty} d(n) f(n) e\left(\frac{bn}{d}\right),$$

it is not hard to show the following summation formula for the divisor function in arithmetic progressions:

THEOREM 2.1. *Let h and $c \geq 1$ be integers. Let $f : (0, \infty) \to \mathbb{R}$ be smooth and compactly supported. Then*

$$\begin{aligned} \sum_{n \equiv h\,(c)} d(n) f(n) = \frac{1}{c} \int \lambda_{h,c}(\xi) f(\xi)\, d\xi \\ - \frac{2\pi}{c} \sum_{d|c} \sum_{n=1}^{\infty} d(n) \frac{S(h, n; d)}{d} \int Y_0\left(4\pi \frac{\sqrt{n\xi}}{d}\right) f(\xi)\, d\xi \\ + \frac{4}{c} \sum_{d|c} \sum_{n=1}^{\infty} d(n) \frac{S(h, -n; d)}{d} \int K_0\left(4\pi \frac{\sqrt{n\xi}}{d}\right) f(\xi)\, d\xi, \end{aligned}$$

with

$$\lambda_{h,c}(\xi) := \sum_{d|c} \frac{c_d(h)}{d}(\log \xi + 2\gamma - 2\log d). \tag{2.1}$$

If we define the differential operator

$$\Delta_\delta(\xi) := \left(\log \xi + 2\gamma + 2\frac{\partial}{\partial \delta}\right)\Bigg|_{\delta=0}, \tag{2.2}$$

we can rewrite $\lambda_{h,c}(\xi)$ as

$$\lambda_{h,c}(\xi) = \Delta_\delta(\xi) \sum_{d|c} \frac{c_d(h)}{d^{1+\delta}}.$$

Writing $\lambda_{h,c}(\xi)$ this way can be particularly useful when doing explicit calculations, as the expression on the right hand side is now multiplicative in c.

An analogue of Theorem 2.1 for the Fourier coefficients $a(n)$ defined in (1.2) can be obtained in the same way as above by using the corresponding Voronoi formula (see [**25**, Theorem 1.6]):

Theorem 2.2. *Let h and $c \geq 1$ be integers. Let $f : (0,\infty) \to \mathbb{R}$ be smooth and compactly supported. Then*

$$\sum_{n \equiv h\ (c)} a(n) f(n) = (-1)^{\frac{\kappa}{2}} \frac{2\pi}{c} \sum_{d|c} \sum_{n=1}^{\infty} a(n) \frac{S(h,n;d)}{d} \int_0^\infty J_{\kappa-1}\left(4\pi \frac{\sqrt{n\xi}}{d}\right) f(\xi)\, d\xi.$$

At this point, we also want to recall the bounds

$$d(n) \ll n^\varepsilon \quad \text{and} \quad a(n) \ll n^\varepsilon,$$

the latter following from the Ramanujan-Petersson conjecture proven by Deligne.

We want to sum up some well-known facts concerning the Bessel functions $J_\nu(\xi)$, $Y_\nu(\xi)$, $\nu \in \mathbb{Z}$, and $K_0(\xi)$ (see e.g. [**23**, Appendix B.4]). Regarding $K_0(\xi)$, it is known that, for $\xi \gg 1$,

$$K_0^{(\mu)}(\xi) \ll \frac{1}{e^\xi \sqrt{\xi}} \quad \text{for} \quad \mu \geq 0,$$

and that, for $\xi \ll 1$,

$$K_0(\xi) \ll |\log \xi| \quad \text{and} \quad K_0^{(\mu)}(\xi) \ll \frac{1}{\xi^\mu} \quad \text{for} \quad \mu \geq 1.$$

Regarding the other two Bessel functions, we know that, for $\xi \gg 1$,

$$J_\nu^{(\mu)}(\xi),\ Y_\nu^{(\mu)}(\xi) \ll \frac{1}{\sqrt{\xi}} \quad \text{for} \quad \nu \geq 0,\ \mu \geq 0.$$

For $\xi \ll 1$, we can bound $J_\nu(\xi)$ and its derivatives by

$$J_\nu^{(\mu)}(\xi) \ll \xi^{\nu-\mu} \quad \text{for} \quad \nu \geq 0,\ \mu \geq 0,$$

while we have the following bounds for $Y_0(\xi)$,

$$Y_0(\xi) \ll |\log \xi| \quad \text{and} \quad Y_0^{(\mu)}(\xi) \ll \frac{1}{\xi^\mu} \quad \text{for} \quad \mu \geq 1,$$

and the following for $Y_\nu(\xi)$,

$$Y_\nu^{(\mu)}(\xi) \ll \frac{1}{\xi^{\nu+\mu}} \quad \text{for} \quad \nu \geq 1,\ \mu \geq 0.$$

From the recurrence relations

$$(\xi^\nu B_\nu(\xi))' = \xi^\nu B_{\nu-1}(\xi) \quad \text{and} \quad B_{\nu-1}(\xi) - B_{\nu+1}(\xi) = 2B_\nu'(\xi), \tag{2.3}$$

which are true for $B_\nu(\xi) = J_\nu(\xi)$ and $B_\nu(\xi) = Y_\nu(\xi)$, we get

$$\int B_0\left(4\pi\frac{\sqrt{h\xi}}{c}\right)f(\xi)\,d\xi = \left(\frac{-2c}{4\pi\sqrt{h}}\right)^\nu \int \xi^{\frac{\nu}{2}} B_\nu\left(4\pi\frac{\sqrt{h\xi}}{c}\right)f^{(\nu)}(\xi)\,d\xi. \tag{2.4}$$

This identity is particularly useful when estimating the Bessel transforms occurring in Theorems 2.1 and 2.2. Furthermore, the Bessel functions $J_\nu(\xi)$ and $Y_\nu(\xi)$ oscillate for large values, and to make use of this behaviour we have the following lemma.

LEMMA 2.3. *For any $\nu \geq 0$, there are smooth functions $v_J, v_Y : (0,\infty) \to \mathbb{C}$ such that*

$$J_\nu(\xi) = 2\,\mathrm{Re}\left(e\left(\frac{\xi}{2\pi}\right)v_J\left(\frac{\xi}{\pi}\right)\right), \tag{2.5}$$

$$Y_\nu(\xi) = 2\,\mathrm{Re}\left(e\left(\frac{\xi}{2\pi}\right)v_Y\left(\frac{\xi}{\pi}\right)\right), \tag{2.6}$$

and such that, for any $\mu \geq 0$,

$$v_J^{(\mu)}, v_Y^{(\mu)} \ll \frac{1}{\xi^{\mu+\frac{1}{2}}} \quad \text{for} \quad \xi \gg 1, \tag{2.7}$$

where the implied constants depend on ν and μ.

PROOF. We start with the integral representations

$$J_0(\xi) = \frac{1}{\pi}\int_0^\infty \sin\left(\frac{x}{2\pi} + \frac{\pi\xi^2}{2x}\right)\frac{dx}{x} \quad \text{and} \quad Y_0(\xi) = -\frac{1}{\pi}\int_0^\infty \cos\left(\frac{x}{2\pi} + \frac{\pi\xi^2}{2x}\right)\frac{dx}{x},$$

which can be found in [**19**, 3.871]. Here we will only look at $Y_\nu(\xi)$, as the proof for $J_\nu(\xi)$ is almost identical. As in [**12**, Lemma 4], we use a substitution

$$y = \frac{\sqrt{x}}{2\pi} - \frac{\xi}{2\sqrt{x}}, \quad x = \pi^2\left(y + \sqrt{y^2 + \frac{\xi}{\pi}}\right)^2,$$

so that we can write the integral above as

$$Y_0(\xi) = -\frac{2}{\pi}\int_{-\infty}^\infty \cos\left(2\pi\left(y^2 + \frac{\xi}{2\pi}\right)\right)\left(y^2 + \frac{\xi}{\pi}\right)^{-\frac{1}{2}} dy.$$

Now writing the cosine function out as a sum of exponential functions, we get (2.6) for Y_0 with

$$v_Y(\xi) = -\frac{2}{\pi}\int_0^\infty \frac{e(y^2)}{\sqrt{y^2+\xi}}\,dy.$$

The estimate (2.7) can be shown by splitting the integral at 1 and repeatedly using partial integration on the part which goes to ∞. The statements for $Y_\nu(\xi)$ follow from (2.3). $\square$

2. The Hecke congruence subgroup and Kloosterman sums

Here and in the following sections we will go through some results from the theory of automorphic forms. For a general description of the spectral theory of automorphic forms, we refer to [**23**] and [**24**, Chapters 14–16]. A very nice introduction to Maaß forms of higher weight with arbitrary nebentypus can be found in [**16**]. We also want to cite [**5**] as a reference, where we borrow parts of the notation.

Let q be some positive integer, let $\kappa \in \{0,1\}$, and let χ be a Dirichlet character mod q_0, with $q_0 \mid q$, such that

$$\chi(-1) = (-1)^\kappa.$$

Let $\Gamma := \Gamma_0(q)$ be the Hecke congruence subgroup of level q. The character χ naturally extends to Γ by setting

$$\chi(\gamma) := \chi(d) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Every cusp $\mathfrak{a}$ of Γ is equivalent to some $\frac{u}{w}$ with $(u, w) = 1$ and $w \mid q$. It is called singular if

$$\chi(\gamma) = 1 \quad \text{for all } \gamma \in \Gamma_{\mathfrak{a}},$$

where $\Gamma_{\mathfrak{a}}$ is the stabilizer of $\mathfrak{a}$.

For any cusp $\mathfrak{a}$ of Γ, we can choose $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a} \quad \text{and} \quad {\sigma_{\mathfrak{a}}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty.$$

Given two singular cusps $\mathfrak{a}$, $\mathfrak{b}$, we define, for $n, m \in \mathbb{Z}$, the Kloosterman sum

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma) := \sum_{\delta \bmod \gamma\mathbb{Z}} \overline{\chi}\left(\sigma_{\mathfrak{a}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} {\sigma_{\mathfrak{b}}}^{-1}\right) e\left(m\frac{\alpha}{\gamma} + n\frac{\delta}{\gamma}\right),$$

where the sum runs over all δ mod $\gamma\mathbb{Z}$, for which there exist α, β such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in {\sigma_{\mathfrak{a}}}^{-1}\Gamma\sigma_{\mathfrak{b}}.$$

Note that this definition depends on the chosen scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$.

As an example, for $\mathfrak{a} = \mathfrak{b} = \infty$ and the choice $\sigma_\infty = 1$, the sum is non-empty exactly when $q \mid c$ and in this case it reduces to the usual twisted Kloosterman sum

$$S_{\infty\infty}(m, n; c) = S_\chi(m, n; c) := \sum_{\substack{a\,(c) \\ (a,c)=1}} \chi(a) e\left(\frac{ma + n\overline{a}}{c}\right).$$

A well-known result by Weil says that, for any prime p, this sum can be bound by

$$S_\chi(m, n; p) \leq 2(m, n, p)^{\frac{1}{2}} p^{\frac{1}{2}},$$

which, in case χ is the principal character, leads to the bound

$$S(m, n; c) \leq d(c)(m, n, c)^{\frac{1}{2}} c^{\frac{1}{2}}.$$

However, for general χ we have to account for its conductor as well, and in this case the following bound holds (see [**28**, Theorem 9.2]),

$$S_\chi(m, n; c) \ll (m, n, c)^{\frac{1}{2}} {q_0}^{\frac{1}{2}} c^{\frac{1}{2}+\varepsilon}.$$

Another important example is given for q having the form $q = rs$ with $(r, s) = 1$ and $q_0 \mid r$. Consider the two singular cusps ∞ and $\frac{1}{s}$, together with the choices

$$\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_{\frac{1}{s}} = \begin{pmatrix} \sqrt{r} & 1 \\ s\sqrt{r} & \sqrt{r}^{-1} \end{pmatrix}.$$

Now the sum $S_{\infty\frac{1}{s}}(m, n; \gamma)$ is non-empty exactly when γ may be written as

$$\gamma = \sqrt{r}sc, \quad \text{with} \quad c \in \mathbb{Z} \setminus \{0\}, \quad (c, r) = 1,$$

and in this case we have

$$S_{\infty\frac{1}{s}}(m, n; \gamma) = e\left(n\frac{\overline{s}}{r}\right)\overline{\chi}(c) S(m, n\overline{r}; sc).$$

3. Automorphic forms and their Fourier expansions

By $\mathcal{S}_k(q,\chi)$ we denote the finite-dimensional Hilbert space of holomorphic cusp forms of weight $k \equiv \kappa \bmod 2$ with respect to $\Gamma_0(q)$ and with nebentypus χ. Let $\theta_k(q,\chi)$ be its dimension. For each k, we choose an orthonormal Hecke eigenbasis $f_{j,k}$, $1 \le j \le \theta_k(q,\chi)$. Then the Fourier expansion of $f_{j,k}$ around a singular cusp $\mathfrak{a}$ (with associated scaling matrix $\sigma_\mathfrak{a}$) is given by

$$i(\sigma_\mathfrak{a}, z)^{-k} f_{j,k}(\sigma_\mathfrak{a} z) = \sum_{n=1}^{\infty} \psi_{j,k}(n,\mathfrak{a})(4\pi n)^{\frac{k}{2}} e(nz),$$

where we have set

$$i(\gamma, z) := cz + d \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Next, let $\mathcal{L}^2(q,\chi)$ be the Hilbert space of Maaß forms of weight κ with respect to $\Gamma_0(q)$ and with nebentypus χ, and let $\mathcal{L}^2_0(q,\chi) \subset \mathcal{L}^2(q,\chi)$ be its subspace of Maaß cusp forms. Let u_j, $j \ge 1$, run over an orthonormal Hecke eigenbasis of $\mathcal{L}^2_0(q,\chi)$ with corresponding real eigenvalues $\lambda_1 \le \lambda_2 \le \ldots$; we can assume each u_j to be either even or odd. We set ${t_j}^2 = \lambda_j - \frac{1}{4}$, where we choose the sign of t_j so that $it_j \ge 0$ if $\lambda_j < \frac{1}{4}$, and $t_j \ge 0$ if $\lambda_j \ge \frac{1}{4}$. Then the Fourier expansions of these functions around a singular cusp $\mathfrak{a}$ is given by

$$j(\sigma_\mathfrak{a}, z)^{-\kappa} u_j(\sigma_\mathfrak{a} z) = \sum_{n \ne 0} \rho_j(n,\mathfrak{a}) W_{\frac{n}{|n|}\frac{\kappa}{2}, it_j}(4\pi |n| y) e(nx),$$

where

$$j(\gamma, z) := \frac{cz+d}{|cz+d|} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Selberg eigenvalue conjecture says that $\lambda_1 \ge \frac{1}{4}$, which would imply that all t_j are real and non-negative. While for $\kappa = 1$ this is known to be true, it is still an open question for $\kappa = 0$. The eigenvalues with $0 < \lambda_j < \frac{1}{4}$, together with the corresponding values t_j, are called exceptional, and lower bounds for the exceptional λ_j imply upper bounds for the corresponding it_j. Let $\theta \in [0,\infty)$ be such that $it_j \le \theta$ for all exceptional t_j uniformly for all levels q and any nebentypus; by the work of Kim and Sarnak [**27**], we know that we can choose

$$\theta = \frac{7}{64}. \tag{2.8}$$

The orthogonal complement to $\mathcal{L}^2_0(q,\chi)$ in $\mathcal{L}^2(q,\chi)$ is the Eisenstein spectrum $\mathcal{E}(q,\chi)$, plus possibly the space of constant functions if χ is trivial. It can be described explicitly by means of the Eisenstein series

$$E_\mathfrak{c}(z;s) := \sum_{\gamma \in \Gamma_\mathfrak{c} \backslash \Gamma} \overline{\chi}(\gamma) j\big({\sigma_\mathfrak{c}}^{-1}\gamma, z\big)^{-\kappa} \operatorname{Im}\big({\sigma_\mathfrak{c}}^{-1}\gamma z\big)^{s},$$

where $\mathfrak{c}$ is a singular cusp. Although these series converge only for $\operatorname{Re}(s) > 1$, the functions $E_\mathfrak{c}(z;s)$ can be continued meromorphically to the whole complex plane. For s on the line $\operatorname{Re}(s) = \frac{1}{2}$, their Fourier expansions around a singular cusp $\mathfrak{a}$ are given by

$$\begin{aligned} j(\sigma_\mathfrak{a}, z)^{-\kappa} E_\mathfrak{c}\left(\sigma_\mathfrak{a} z; \frac{1}{2} + it\right) &= c_{\mathfrak{c},1}(t) y^{\frac{1}{2}+it} + c_{\mathfrak{c},2}(t) y^{\frac{1}{2}-it} \\ &\quad + \sum_{n \ne 0} \varphi_{\mathfrak{c},t}(n,\mathfrak{a}) W_{\frac{n}{|n|}\frac{\kappa}{2}, it}(4\pi|n|y) e(nx), \end{aligned}$$

where $t \in \mathbb{R}$.

Note that by the choice of our basis, we have that

$$|\rho_j(-n,\infty)| = |t_j|^\kappa |\rho_j(n,\infty)| \quad \text{for} \quad n \geq 1.$$

Furthermore, since all Eisenstein series are even, the same is true for their Fourier coefficients, namely

$$|\varphi_{\mathfrak{c},t}(-n,\infty)| = |t|^\kappa |\varphi_{\mathfrak{c},t}(n,\infty)| \quad \text{for} \quad n \geq 1.$$

4. The Kuznetsov formula

With the whole notation set up, we can now formulate the famous Kuznetsov formula, which in our case reads as follows.

THEOREM 2.4. *Let $f:(0,\infty)\to\mathbb{C}$ be smooth and compactly supported, let $\mathfrak{a}$, $\mathfrak{b}$ be singular cusps, and let m, n be positive integers. Then*

$$\begin{aligned}\sum_\gamma \frac{S_{\mathfrak{ab}}(m,n;\gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right) &= \sum_{j=1}^\infty \overline{\rho_j}(m,\mathfrak{a})\rho_j(n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t_j)}\tilde{f}(t_j)\\ &+ \sum_{\mathfrak{c}\ sing.} \frac{1}{4\pi}\int_{-\infty}^\infty \overline{\varphi_{\mathfrak{c},t}}(m,\mathfrak{a})\varphi_{\mathfrak{c},t}(n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t)}\tilde{f}(t)\,dt\\ &+ \sum_{\substack{k\equiv\kappa\,(2),\ k>\kappa\\ 1\leq j\leq\theta_k(q,\chi)}} (k-1)!\,\overline{\psi_{j,k}}(m,\mathfrak{a})\psi_{j,k}(n,\mathfrak{b})\sqrt{mn}\dot{f}(k),\end{aligned}$$

and

$$\begin{aligned}\sum_\gamma \frac{S_{\mathfrak{ab}}(m,-n;\gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right) &= \sum_{j=1}^\infty \overline{\rho_j}(m,\mathfrak{a})\rho_j(-n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t_j)}\check{f}(t_j)\\ &+ \sum_{\mathfrak{c}\ sing.} \frac{1}{4\pi}\int_{-\infty}^\infty \overline{\varphi_{\mathfrak{c},t}}(m,\mathfrak{a})\varphi_{\mathfrak{c},t}(-n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t)}\check{f}(t)\,dt,\end{aligned}$$

where γ runs over all positive real numbers for which $S_{\mathfrak{ab}}(m,n;\gamma)$ is non-empty, and where the Bessel transforms are defined by

$$\begin{aligned}\tilde{f}(t) &= \frac{2\pi i t^\kappa}{\sinh(\pi t)}\int_0^\infty (J_{2it}(\xi)-(-1)^\kappa J_{-2it}(\xi))f(\xi)\,\frac{d\xi}{\xi},\\ \check{f}(t) &= 8i^{-\kappa}\cosh(\pi t)\int_0^\infty K_{2it}(\xi)f(\xi)\,\frac{d\xi}{\xi},\\ \dot{f}(k) &= 4i^k\int_0^\infty J_{k-1}(\xi)f(\xi)\,\frac{d\xi}{\xi}.\end{aligned}$$

PROOF. For $\mathfrak{a}=\mathfrak{b}=\infty$, the first formula was proven in [**41**], the second formula in [**3**, Proposition 2]. The extension to our situation with general cusps is straightforward. ☐

For $\mathfrak{a}=\mathfrak{b}=\infty$, the sum of Kloosterman sums in the theorem above is just

$$\sum_\gamma \frac{S_{\infty\infty}(m,\pm n;\gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right) = \sum_{c\equiv 0\,(q)} \frac{S_\chi(m,\pm n;c)}{c} f\left(4\pi\frac{\sqrt{mn}}{c}\right), \tag{2.9}$$

while in the case $q = rs$ with $(r, s) = 1$ and $q_0 \mid r$ mentioned above, we have

$$\sum_{\gamma} \frac{S_{\infty\frac{1}{s}}(m, \pm n; \gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right)$$
$$= e\left(\frac{\pm n\overline{s}}{r}\right) \sum_{\substack{c \\ (c,r)=1}} \overline{\chi}(c) \frac{S(m, \pm n\overline{r}; sc)}{\sqrt{r}sc} f\left(4\pi\frac{\sqrt{mn}}{\sqrt{r}sc}\right). \quad (2.10)$$

To get some first estimates for the Bessel transforms appearing above, we refer to [**6**, Lemma 2.1], where the case $\kappa = 0$ is covered. The proofs carry over to the case $\kappa = 1$ with minimal changes.

LEMMA 2.5. *Let $f : (0, \infty) \to \mathbb{C}$ be a smooth and compactly supported function such that*

$$\operatorname{supp} f \asymp X \quad \textit{and} \quad f^{(\nu)}(\xi) \ll \frac{1}{Y^\nu} \quad \textit{for} \quad \nu \geq 0,$$

for positive X and Y with $X \gg Y$. Then

$$\tilde{f}(it), \check{f}(it) \ll \frac{1 + Y^{-2t}}{1 + Y} \qquad \textit{for} \quad 0 \leq t < \frac{1}{4}, \quad (2.11)$$

$$\frac{\tilde{f}(t)}{(1+t)^\kappa}, \check{f}(t), \dot{f}(t) \ll \frac{1 + |\log Y|}{1 + Y} \qquad \textit{for} \quad t \geq 0, \quad (2.12)$$

$$\frac{\tilde{f}(t)}{(1+t)^\kappa}, \check{f}(t), \dot{f}(t) \ll \left(\frac{X}{Y}\right)^2 \left(\frac{1}{t^{\frac{5}{2}}} + \frac{X}{t^3}\right) \qquad \textit{for} \quad t \gg \max(X, 1). \quad (2.13)$$

For certain oscillating functions, we can do better. Assume $w : (0, \infty) \to \mathbb{C}$ to be a smooth and compactly supported function such that

$$\operatorname{supp} w \asymp X \quad \text{and} \quad w^{(\nu)}(\xi) \ll \frac{1}{X^\nu} \quad \text{for} \quad \nu \geq 0,$$

and define, for $\alpha > 0$,

$$f(\xi) := e\left(\xi \frac{\alpha}{2\pi}\right) w(\xi).$$

Then the following two lemmas give bounds for the Bessel transforms of f depending on the sizes of X and α.

LEMMA 2.6. *Assume that*

$$X \ll 1 \quad \textit{and} \quad \alpha X \gg 1.$$

Then, for any $\nu, \mu \geq 0$,

$$\tilde{f}(it), \check{f}(it) \ll X^{-2t}\left(X^\mu + \frac{1}{(\alpha X)^\nu}\right) \qquad \textit{for} \quad 0 < t \leq \frac{1}{4}, \quad (2.14)$$

$$\frac{\tilde{f}(t)}{(1+t)^\kappa}, \check{f}(t), \dot{f}(t) \ll \frac{\alpha^\varepsilon}{\alpha X} \left(\frac{\alpha X}{t}\right)^\nu \qquad \textit{for} \quad t > 0. \quad (2.15)$$

PROOF. We will only look at the case $\kappa = 0$, since the proofs in the case $\kappa = 1$ can be done very similarly.

We begin with (2.14). Using the Taylor series of the J_ν-Bessel function we can write the Bessel transform $\tilde{f}(it)$ as

$$\tilde{f}(it) = 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!} \int_0^\infty e\left(\xi \frac{\alpha}{2\pi}\right) g(\xi, t, m) w(\xi) \xi^{2m-1} \, d\xi, \quad (2.16)$$

with

$$g(\xi, t, m) := -\frac{1}{\sin(\pi t)} \left(\frac{1}{\Gamma(m + 2t + 1)} \left(\frac{\xi}{2}\right)^{2t} - \frac{1}{\Gamma(m - 2t + 1)} \left(\frac{\xi}{2}\right)^{-2t} \right).$$

For $0 < t < \frac{1}{2}$, one can check that we have the bound

$$\frac{\partial^\nu}{\partial \xi^\nu} g(\xi, t, m) \ll \frac{X^{-2t-\nu}}{(m-1)!}.$$

By splitting the sum in (2.16) at $m = \frac{\mu}{2}$, and using partial integration for the finite part while estimating trivially the rest, we get that

$$\tilde{f}(it) \ll X^{-2t}\left(X^\mu + \frac{1}{(\alpha X)^\nu}\right).$$

The estimate for $\check{f}(it)$ follows in exactly the same way by using the corresponding Taylor series for $K_{2it}(\xi)$.

For the proof of (2.15), we follow [**26**, Lemma 3]. We begin with the following identity (see [**19**, 8.411.11]),

$$\frac{J_{2it}(\eta) - J_{-2it}(\eta)}{\sinh(\pi t)} = \frac{2}{\pi i}\int_{-\infty}^{\infty} \cos(\eta \cosh \zeta) \cos(2t\zeta)\, d\zeta, \tag{2.17}$$

which gives

$$\tilde{f}(t) = 4\int_{-\infty}^{\infty}\int \cos(\eta\cosh\zeta)\cos(2t\zeta) f(\eta)\, \frac{d\eta}{\eta} d\zeta =: -(I^+ + I^-),$$

with

$$I^\pm := \int_{-\infty}^{\infty}\int e\left(\eta\left(\frac{\alpha \pm \cosh\zeta}{2\pi}\right)\right)\frac{w(\eta)}{\eta}\cos(2t\zeta)\, d\eta d\zeta.$$

To bound I^+ we use partial integration μ times on the integral over η and get

$$I^+ \ll \frac{\alpha^\varepsilon}{(\alpha X)^\mu}.$$

The treatment of I^- is a little trickier, since the factor

$$\gamma(\zeta) := \alpha - \cosh\zeta$$

occuring in the exponent may vanish, so that we have to treat the integral differently depending on whether $\gamma(\zeta)$ is near 0 or not. Out of technical reasons, it is easier to use smooth weight functions to split the integral. Set

$$Z_1 := \operatorname{arcosh}(\alpha - A) \quad \text{and} \quad Z_2 := \operatorname{arcosh}(\alpha + A), \quad \text{with} \quad A := \frac{1}{X}.$$

Let $u_i : \mathbb{R} \to [0, \infty)$, $i = 1, 2$, be suitable weight functions such that

$$\begin{aligned} u_1(\xi) &= 1 \quad \text{for} \quad |\xi| \le \frac{1}{2}Z_1 && \text{and} && \operatorname{supp} u_1 \subseteq [-Z_1, Z_1], \\ u_2(\xi) &= 1 \quad \text{for} \quad |\xi| \ge 2Z_2 && \text{and} && \operatorname{supp} u_2 \subseteq [-\infty, -Z_2] \cup [Z_2, \infty], \end{aligned}$$

and define

$$u_3(\xi) := 1 - u_1(\xi) - u_2(\xi).$$

Note that for all $i = 1, 2, 3$,

$$u_i^{(\nu)}(\xi) \ll 1 \quad \text{for} \quad \nu \ge 0.$$

Then we have to consider the integrals

$$I_i^- := \int\int u_i(\zeta) e\left(\eta\frac{\gamma(\zeta)}{2\pi}\right)\frac{w(\eta)}{\eta}\cos(2t\zeta)\, d\eta d\zeta, \tag{2.18}$$

and using partial integration μ times over η we get

$$I_1^-, I_2^- \ll \frac{A}{\alpha(XA)^\mu} + \frac{\alpha^\varepsilon}{(\alpha X)^\mu},$$

whereas bounding I_3^- directly gives

$$I_3^- \ll \frac{A}{\alpha}.$$

This already proves (2.15) for $\nu = 0$. The result for $\nu \geq 1$ can be shown the same way by partially integrating ν times over ζ before estimating the integrals absolutely.

The estimate for $\check{f}(t)$ can be shown analogously by using the integral representation

$$K_{2it}(\eta) = \frac{1}{\cosh(\pi t)} \int_0^\infty \cos(\eta \sinh \zeta) \cos(2t\zeta)\, d\zeta$$

(see [**19**, 8.432.4]). Finally, the proof for $\dot{f}(k)$ also goes along the same lines – in this case we use the identity

$$J_k(\eta) = \frac{1}{\pi} \int_0^\pi \cos(k\zeta - \eta \sin \zeta)\, d\zeta,$$

which can be found, for instance, in [**19**, 8.411.1]. □

Lemma 2.7. *Assume that*

$$X \gg 1 \quad \textit{and} \quad |\alpha - 1| \ll \frac{X^\varepsilon}{X}.$$

Then, for any $\nu \geq 0$,

$$\tilde{f}(it), \check{f}(it) \ll 1 \qquad \textit{for} \quad 0 < t \leq \frac{1}{4}, \tag{2.19}$$

$$\frac{\tilde{f}(t)}{(1+t)^\kappa}, \dot{f}(t) \ll \frac{X^\varepsilon}{X^{\frac{1}{2}}} \left(\frac{X^{\frac{1}{2}}}{t} \right)^\nu \qquad \textit{for} \quad t > 0, \tag{2.20}$$

$$\check{f}(t) \ll \frac{X^\varepsilon}{X} \left(\frac{X}{t} \right)^\nu \qquad \textit{for} \quad t > 0. \tag{2.21}$$

Proof. We will again only look at the case $\kappa = 0$, since the proofs in the case $\kappa = 1$ can be done along the same lines.

The first bound (2.19) follows directly from (2.11). The proof of the other bounds follows the same path as in Lemma 2.6, so we only want to point out some differences. In the case of $\tilde{f}(t)$, we again use the identity (2.17). For I^+ we get here the bound

$$I^+ \ll \frac{1}{X^\mu}.$$

It is again necessary to split I^-, and in order to do so, we choose a suitable weight function $u_1(\xi)$ which satisfies

$$u_1(\xi) = 1 \quad \text{for} \quad |\xi| \geq 2Z, \quad u_1(\xi) = 0 \quad \text{for} \quad |\xi| \leq Z,$$

and

$$u_1^{(\nu)}(\xi) \ll \frac{1}{Z^\nu} \asymp \frac{1}{A^{\frac{\nu}{2}}} \quad \text{for} \quad \nu \geq 0,$$

where

$$A := \frac{X^\varepsilon}{X} \quad \text{and} \quad Z := \operatorname{arcosh}(2A + \alpha).$$

Set $u_2(\xi) := 1 - u_1(\xi)$. Then

$$I^- =: I_1^- + I_2^-$$

in the same way as in (2.18), and we get

$$I_1^- \ll \frac{A^{\frac{1}{2}}}{(XA)^\mu} + \frac{1}{X^\mu} \ll \frac{X^\varepsilon}{X^{\frac{1}{2}}} \quad \text{and} \quad I_2^- \ll A^{\frac{1}{2}} \ll \frac{X^\varepsilon}{X^{\frac{1}{2}}}.$$

This gives (2.20) for $\nu = 0$. By partially integrating over ζ, we get the result for higher ν. Finally, the results for $\check{f}(t)$ and $\dot{f}(k)$ can be deduced similarly by using the appropriate integral representations for the occuring Bessel functions. □

5. The large sieve inequalities and estimates for Fourier coefficients

Another important tool are the large sieve inequalities for Fourier coefficients of cusp forms and Eisenstein series, which were proven by Deshouillers and Iwaniec [**13**] with respect to Hecke congruence subgroups. Their results can be extended to the more general setting needed here, the details of which have luckily been worked out by Drappeau [**14**].

Let $\mathfrak{a}$ be singular cusp of Γ written in the form $\mathfrak{a} = \frac{u}{w}$ with $(u, w) = 1$. For a sequence a_n of complex numbers we set

$$\Sigma^{(1)}_{j,\pm}(N) := \frac{(1+|t_j|)^{\pm\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t_j)}} \sum_{N<n\leq 2N} a_n \rho_j(\pm n, \mathfrak{a})\sqrt{n},$$

$$\Sigma^{(2)}_{\mathfrak{c},t,\pm}(N) := \frac{(1+|t|)^{\pm\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t)}} \sum_{N<n\leq 2N} a_n \varphi_{\mathfrak{c},t}(\pm n, \mathfrak{a})\sqrt{n},$$

$$\Sigma^{(3)}_{j,k}(N) := \sqrt{(k-1)!} \sum_{N<n\leq 2N} a_n \psi_{j,k}(n, \mathfrak{a})\sqrt{n}.$$

Then the following bounds are known as the large sieve inequalities.

Theorem 2.8. *Let $T \geq 1$, $N \geq \frac{1}{2}$, and $\mathfrak{a}$ as above. Let a_n be a sequence of complex numbers. Then*

$$\sum_{|t_j|\leq T} \left|\Sigma^{(1)}_{j,\pm}(N)\right|^2 \ll \left(T^2 + q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\leq 2N} |a_n|^2,$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} \left|\Sigma^{(2)}_{\mathfrak{c},t,\pm}(N)\right|^2 dt \ll \left(T^2 + q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\leq 2N} |a_n|^2,$$

$$\sum_{\substack{k\leq T,\ k\equiv\kappa\,(2)\\ 1\leq j\leq\theta_k(q,\chi)}} \left|\Sigma^{(3)}_{k,j}(N)\right|^2 \ll \left(T^2 + q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\leq 2N} |a_n|^2,$$

where the implied constants depend only on ε.

Proof. With the appropriate changes, these bounds can be deduced essentially in the same way as it is done in [**13**, Section 5]. We refer to [**14**] for details. □

When there is no averaging over n, the following lemma gives useful bounds, especially when q or T is large.

Lemma 2.9. *Let $T \geq 1$, $n \geq 1$, and $\mathfrak{a}$ as above. Then*

$$\sum_{|t_j|\leq T} \frac{(1+|t_j|)^{\pm\kappa}}{\cosh(\pi t_j)} |\rho_j(\pm n, \mathfrak{a})|^2 n \ll T^2 + (qnT)^{\varepsilon}(q,n)^{\frac{1}{2}} q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{n^{\frac{1}{2}}}{q},$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} \frac{(1+|t|)^{\pm\kappa}}{\cosh(\pi t)} |\varphi_{\mathfrak{c},t}(\pm n, \mathfrak{a})|^2 n\, dt \ll T^2 + (qnT)^{\varepsilon}(q,n)^{\frac{1}{2}} q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{n^{\frac{1}{2}}}{q},$$

$$\sum_{\substack{k\leq T,\ k\equiv\kappa\,(2)\\ 1\leq j\leq\theta_k(q,\chi)}} (k-1)! |\psi_{j,k}(n, \mathfrak{a})|^2 n \ll T^2 + (qnT)^{\varepsilon}(q,n)^{\frac{1}{2}} q_0^{\frac{1}{2}}\left(w, \frac{q}{w}\right)\frac{n^{\frac{1}{2}}}{q},$$

where the implied constants depend only on ε.

PROOF. For the full modular group and trivial nebentypus, a proof for the first two bounds can be found for example in [**37**, Lemma 2.4]. Using an appropriate formula as starting point (e.g. [**16**, Proposition 5.2]), the proof carries over easily to our case. Except for the same kind of modifications, the proof of the last bound is a simpler variant of [**13**, Proposition 4]. □

For n large, the following bounds are often better.

LEMMA 2.10. *Let $T \geq 1$ and $n \geq 1$. Then*

$$\sum_{|t_j|\leq T} \frac{(1+|t_j|)^{\pm\kappa}}{\cosh(\pi t_j)} |\rho_j(\pm n,\infty)|^2 n \ll (qnT)^{\varepsilon} T^2 n^{2\theta}, \tag{2.22}$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} \frac{(1+|t|)^{\pm\kappa}}{\cosh(\pi t)} |\varphi_{\mathfrak{c},t}(\pm n,\infty)|^2 n\, dt \ll (qnT)^{\varepsilon} T, \tag{2.23}$$

$$\sum_{\substack{k\leq T,\ k\equiv\kappa\,(2)\\ 1\leq j\leq\theta_k(q,\chi)}} (k-1)! |\psi_{j,k}(n,\infty)|^2 n \ll (qnT)^{\varepsilon} T^2, \tag{2.24}$$

where the implied constants depend only on ε.

PROOF. The bounds (2.22) and (2.24) can be proven along the lines of [**34**, Proposition 2.3]. For (2.23) we refer to [**7**, Lemma 1]. □

Finally, in order to handle the exceptional eigenvalues, which occur in the case $\kappa = 0$, the following result will turn out to be useful.

LEMMA 2.11. *Let $X \geq 1$, $n \geq 1$, and $\mathfrak{a}$ as above. Assume that*

$$X \gg X_0 \quad \text{with} \quad X_0 := \frac{q}{(q,n)^{\frac12} q_0^{\frac12} (w, \frac{q}{w}) n^{\frac12}}.$$

Then

$$\sum_{t_j\ exc.} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} X^{4it_j} \ll (qnX)^{\varepsilon} \left(\frac{X}{X_0}\right)^{4\theta} \left(1 + (q,n)^{\frac12} q_0^{\frac12} \left(w, \frac{q}{w}\right) \frac{n^{\frac12}}{q}\right),$$

where the implied constants only depend on ε.

PROOF. We have that

$$\sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} X^{4it_j} \ll \left(\frac{X}{X_0}\right)^{4\theta} \sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} (1+X_0)^{4it_j}.$$

Now we use the fact that, for $Y \geq 1$,

$$\sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} Y^{4it_j} \ll 1 + (qnY)^{\varepsilon} (q,n)^{\frac12} q_0^{\frac12} \left(w, \frac{q}{w}\right) \frac{n^{\frac12} Y}{q},$$

which can be shown the same way as in [**24**, chapter 16.5], and the result follows. □

CHAPTER 3

Proof of Theorems 1.1, 1.2, 1.3 and 1.4

In this chapter we will look at the sums $D_3^{\pm}(x,h)$ and $A_3^{\pm}(x,h)$, and prove Theorems 1.1, 1.2, 1.3 and 1.4. Since our method applies to $D_3^{\pm}(x,h)$ and $A_3^{\pm}(x,h)$ in the same way, it will pose no further difficulty to treat both cases simultaneously. With this in mind, we let $\alpha(n)$ be a placeholder for $d(n)$ or $a(n)$.

Let $w : [1/2,1] \to \mathbb{R}$ be a smooth and compactly supported function which satisfies

$$w^{(\nu)}(\xi) \ll \frac{1}{\Omega^{\nu}} \quad \text{for} \quad \nu \geq 0, \quad \text{and} \quad \int \left| w^{(\nu)}(\xi) \right| d\xi \ll \frac{1}{\Omega^{\nu-1}}\, d\xi \quad \text{for} \quad \nu \geq 1,$$

for some $\Omega \leq 1$. We will then look at the sum

$$\Psi := \sum_n w\Big(\frac{n}{x}\Big) d_3(n)\alpha(n+h), \quad h \in \mathbb{Z} \setminus \{0\},$$

and, assuming that h is of the size

$$h \ll \Omega^2 x^{1-\varepsilon}, \tag{3.1}$$

we will prove the following asymptotic formula for Ψ.

Lemma 3.1. *The sum Ψ can be written asymptotically as*

$$\Psi = M + \mathcal{O}\left(x^{\frac{5}{6}+\varepsilon} \left(x^{\frac{\theta}{3}} + \frac{1}{\Omega^{\frac{1}{2}}} \right) \left(1 + \left(\frac{|h|}{x^{\frac{2}{3}}} \right)^{\frac{1}{4}} \right) \right),$$

where M is the possible main term, which vanishes if $\alpha(n) = a(n)$ and otherwise has the form

$$M = xP_{3,h,w}(\log x),$$

with a cubic polynomial $P_{3,h,w}$.

Recall that θ was defined in (2.8). The choice $\Omega = 1$ gives Theorems 1.2 and 1.4, while the choice $\Omega = x^{-\frac{1}{9}}$, together with suitable weight functions, gives Theorems 1.1 and 1.3.

1. A decomposition of the ternary divisor function

We need a smooth decomposition of the ternary divisor function, for which we want to use a similar construction as the one used by Meurman [**33**] (which originally goes back to Heath-Brown). Let $u_0 : \mathbb{R} \to [0,\infty)$ be a smooth and compactly supported function such that

$$u_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1, \quad \text{and} \quad u_0(\xi) = 0 \quad \text{for} \quad |\xi| \geq 2,$$

and define

$$u_1(\xi) := u_0\left(\frac{\xi}{x^{\frac{1}{3}}}\right), \quad u_2(\xi) := u_0\left(\frac{\xi}{\sqrt{\frac{x}{a}}}\right).$$

If $abc \leq x$, then obviously

$$(u_1(a)-1)(u_1(b)-1)(u_1(c)-1) = 0 \quad \text{and} \quad (u_2(b)-1)(u_2(c)-1) = 0,$$

and hence

$$d\Big(\frac{n}{a}\Big) = \sum_{bc=\frac{n}{a}} u_2(b)(2-u_2(c)),$$

as well as

$$\begin{aligned} d_3(n) &= \sum_{abc=n} (u_1(a)u_1(b)u_1(c) - 3u_1(a)u_1(b) + 3u_1(a)) \\ &= \sum_{abc=n} (u_1(a)u_1(b)u_1(c) - 3u_1(a)u_1(b)) + 3\sum_{a|n} u_1(a) d\Big(\frac{n}{a}\Big) \\ &= \sum_{abc=n} h(a,b,c), \end{aligned} \tag{3.2}$$

where we have set

$$h(a,b,c) := u_1(a)u_1(b)u_1(c) - 3u_1(a)u_1(b) + 3u_1(a)u_2(b)(2-u_2(c)).$$

Note that this function is non-zero only when $a, b \ll c$.

Moreover, we will use a partition of unity on $(0,\infty)$ constructed as follows. Let $u_X : (0,\infty) \to \mathbb{R}$ be smooth and compactly supported functions such that

$$\operatorname{supp} u_X \subset \Big[\frac{X}{2}, 2X\Big] \quad \text{and} \quad u_X^{(\nu)}(\xi) \ll \frac{1}{X^\nu} \quad \text{for} \quad \nu \geq 0,$$

and such that

$$\sum_X u_X(\xi) = 1 \quad \text{for} \quad \xi \in (0,\infty),$$

where the last sum runs over powers of 2. Then we set

$$h_{ABC}(a,b,c) := h(a,b,c) u_A(a) u_B(b) u_C(c)$$

and

$$\Psi_{ABC} := \sum_{a,b,c} w\Big(\frac{abc}{x}\Big) h_{ABC}(a,b,c) \alpha(abc+h), \tag{3.3}$$

so that

$$\Psi = \sum_{A,B,C} \Psi_{ABC}, \tag{3.4}$$

where again A, B and C run over powers of 2.

In the following, we will evaluate Ψ_{ABC} asymptotically and show that

$$\Psi_{ABC} = M_{ABC} + \mathcal{O}\Bigg(x^{\frac{5}{6}+\varepsilon}\Big(\frac{1}{\Omega^{\frac{1}{2}}} + x^{\frac{\theta}{3}}\Big)\Big(1 + \frac{|h|^{\frac{1}{4}}}{x^{\frac{1}{6}}}\Big)\Bigg), \tag{3.5}$$

where M_{ABC} vanishes if $\alpha(n) = a(n)$ and otherwise is given by

$$M_{ABC} := \sum_{a,b} \frac{1}{ab} \int \lambda_{h,ab}(\xi + h) w\Big(\frac{\xi}{x}\Big) h_{ABC}\Big(a, b, \frac{\xi}{ab}\Big)\, d\xi,$$

with $\lambda_{h,ab}(\xi+h)$ defined as in (2.1). In view of (3.4), this proves Lemma 3.1 after evaluating the possible main term, which we will do in Section 6.

2. Use of the Voronoi summation formula

In (3.3) it will be sufficient to look at the sums running over the variables b and c alone, since this is where the saving in the error term actually will come from. We will do the evaluation of this sum in a slightly more general form than actually needed here, since we will need these results in Chapter 5 again. With this in mind, we define

$$\Phi_v(a) := \sum_{b,c} w\left(\frac{abc}{x}\right) v(a,b,c)\alpha(abc+h), \tag{3.6}$$

where $v : \mathbb{R}^3 \to \mathbb{R}$ is a smooth and compactly supported function, such that

$$\operatorname{supp} v \asymp A \times B \times C,$$

and

$$\frac{\partial^{\nu_1+\nu_2+\nu_3}}{\partial a^{\nu_1}\partial b^{\nu_2}\partial c^{\nu_3}} v(a,b,c) \ll \frac{1}{A^{\nu_1}B^{\nu_2}C^{\nu_3}} \quad \text{for} \quad \nu_1,\nu_2,\nu_3 \geq 0.$$

In the coming sections, we will prove the following lemma, which gives an asymptotic formula for $\Phi_v(a)$.

Lemma 3.2. *Let $A \ll x^{\frac{1}{2}}$ and $h \ll \Omega^2 x^{1-\varepsilon}$. Then $\Phi_v(a)$ can be written asymptotically as*

$$\Phi_v(a) = M_v(a) + R_v(a),$$

where the main term $M_v(a)$ vanishes if $\alpha(n) = a(n)$ and otherwise has the form

$$M_v(a) := \frac{1}{a}\sum_b \frac{1}{b}\int \lambda_{h,ab}(\xi+h) w\left(\frac{\xi}{x}\right) v\left(a,b,\frac{\xi}{ab}\right) d\xi, \tag{3.7}$$

and where the error term $R_v(a)$ is bounded by

$$R_v(a) \ll (a,h)^{\frac{1}{2}} x^{\frac{1}{2}+\varepsilon}\left(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{x^\theta}{A^{2\theta}}\right)\left(1+\frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}}\right)\left(1+\frac{|h|^{\frac{1}{4}}}{A^{\frac{1}{2}}}\right). \tag{3.8}$$

Choosing $v = h_{ABC}$, and recalling that

$$A, B \ll C \quad \text{and} \quad A \ll x^{\frac{1}{3}},$$

this result then immediately leads to (3.5).

In order to prove Lemma 3.2, we write the sum $\Phi_v(a)$ as

$$\begin{aligned}\Phi_v(a) &= \sum_b \sum_{m\equiv h\,(ab)} \alpha(m) w\left(\frac{m-h}{x}\right) v\left(a,b,\frac{m-h}{ab}\right)\\ &= \sum_b \sum_{m\equiv h\,(ab)} \alpha(m) f(m;a,b),\end{aligned}$$

where we have set

$$f(\xi;a,b) := w\left(\frac{\xi-h}{x}\right) v\left(a,b,\frac{\xi-h}{ab}\right).$$

Note that

$$\operatorname{supp} f(\,\bullet\,;a,b) \asymp x \quad \text{and} \quad \frac{\partial^{\nu_1+\nu_2}}{\partial \xi^{\nu_1}\partial b^{\nu_2}} f(\xi;a,b) \ll \frac{1}{(x\Omega)^{\nu_1}B^{\nu_2}} \quad \text{for} \quad \nu_1,\nu_2 \geq 0.$$

Here we use Theorem 2.1 in case $\alpha(n) = d(n)$,

$$\begin{aligned}\Phi_v(a) = \frac{1}{a}\sum_b \frac{1}{b}\int \lambda_{h,ab}(\xi) f(\xi; a, b)\, d\xi \\ - \frac{2\pi}{a}\sum_{\substack{b,c\\ c\mid ab}} \frac{1}{b}\sum_{m=1}^{\infty} d(m)\frac{S(h,m;c)}{c}\int_0^\infty Y_0\left(4\pi\frac{\sqrt{m\xi}}{c}\right) f(\xi; a, b)\, d\xi \\ + \frac{4}{a}\sum_{\substack{b,c\\ c\mid ab}} \frac{1}{b}\sum_{m=1}^{\infty} d(m)\frac{S(h,-m;c)}{c}\int_0^\infty K_0\left(4\pi\frac{\sqrt{m\xi}}{c}\right) f(\xi; a, b)\, d\xi,\end{aligned}$$

and Theorem 2.2 in case $\alpha(n) = a(n)$,

$$\Phi_v(a) = (-1)^{\frac{\kappa}{2}}\frac{2\pi}{a}\sum_{\substack{b,c\\ c\mid ab}} \frac{1}{b}\sum_{m=1}^{\infty} a(m)\frac{S(h,m;c)}{c}\int_0^\infty J_{\kappa-1}\left(4\pi\frac{\sqrt{m\xi}}{c}\right) f(\xi; a, b)\, d\xi.$$

The possible main term of $\Phi_v(a)$ is given by

$$M_v(a) := \frac{1}{a}\sum_b \frac{1}{b}\int \lambda_{h,ab}(\xi) f(\xi; a, b)\, d\xi,$$

which is identical to (3.7). It remains to evaluate the remaining terms and show that they satisfy the bound (3.8).

We restate the outer sum as follows,

$$\sum_{\substack{b,c\\ c\mid ab}} (\dots) = \sum_{\substack{b,c,d\\ ab=cd}} (\dots) = \sum_d \sum_{\frac{a}{(a,d)}\mid c} (\dots), \tag{3.9}$$

and set

$$F^\pm(c,m) := \frac{1}{c}\int_0^\infty B^\pm\left(4\pi\frac{\sqrt{m\xi}}{c}\right) f\left(\xi; a, \frac{cd}{a}\right) d\xi,$$

with

$$\begin{aligned} &B^+(\xi) = Y_0(\xi), && B^-(\xi) = K_0(\xi), && \text{if } \alpha(n) = d(n),\\ &B^+(\xi) = J_{\kappa-1}(\xi), && B^-(\xi) = 0, && \text{if } \alpha(n) = a(n),\end{aligned} \tag{3.10}$$

so that the sums we have to deal with can be written in the form

$$R^\pm_{ABC} := \sum_d \frac{1}{d}\sum_m \alpha(m) \sum_{\frac{a}{(a,d)}\mid c} \frac{S(h,\pm m;c)}{c} F^\pm(c,m).$$

Note that we have now $c \asymp \frac{AB}{d}$.

The function $F^\pm(c,m)$ can be bounded by

$$F^\pm(c,m) \ll x^{1+\varepsilon}\frac{d}{AB},$$

however, when $m \gg \frac{c^2}{x}$, we can use (2.4) to get the better bounds

$$F^+(c,m) \ll \frac{1}{x^{\frac{\nu}{2}-\frac{3}{4}}\Omega^{\nu-1}}\frac{c^{\nu-\frac{1}{2}}}{m^{\frac{\nu}{2}+\frac{1}{4}}} \quad \text{and} \quad F^-(c,m) \ll \frac{1}{x^{\frac{\nu}{2}-\frac{3}{4}}}\frac{c^{\nu-\frac{1}{2}}}{m^{\frac{\nu}{2}+\frac{1}{4}}}.$$

We set

$$M_0^- := \frac{x^\varepsilon}{x}\left(\frac{AB}{d}\right)^2 \quad \text{and} \quad M_0^+ := \frac{x^\varepsilon}{x\Omega^2}\left(\frac{AB}{d}\right)^2,$$

and a standard exercise then shows that we can cut the sum over m in $R_{ABC}^{\pm}$ at $M_0^{\pm}$, so that it is sufficient to look at the sums

$$R_{ABC}^{\pm}(M) := \sum_{M<m\leq 2M} \alpha(m) \sum_{\frac{a}{(a,d)}|c} \frac{S(h,\pm m;c)}{c} F^{\pm}(c,m), \tag{3.11}$$

where we have divided the range of summation over m into dyadic intervals $[M,2M]$ with $M = \frac{M_0^{\pm}}{2^k}$, where k runs over positive integers.

3. Auxiliary estimates

We want to treat the inner sum in (3.11) with the Kuznetsov formula in the form (2.9) with $q = \frac{a}{(a,d)}$ and trivial nebentypus χ_0. To bring the functions $F^{\pm}(c,m)$ into the right shape, we define

$$\tilde{F}^{\pm}(c,m) := h(m) \frac{c}{4\pi\sqrt{|h|m}} \int_0^{\infty} B^{\pm}\left(c\sqrt{\frac{\xi}{|h|}}\right) f\left(\xi; a, 4\pi\frac{\sqrt{|h|m}}{c}\frac{d}{a}\right) d\xi,$$

where $h(m)$ is a smooth and compactly supported bump function, such that

$$\operatorname{supp} h \asymp M \quad \text{and} \quad h^{(\nu)}(m) \ll \frac{1}{M^{\nu}} \quad \text{for} \quad \nu \geq 0,$$

and

$$h(m) \equiv 1 \quad \text{for} \quad m \in [M, 2M].$$

Then we have

$$F^{\pm}(c,m) = \tilde{F}^{\pm}\left(4\pi\frac{\sqrt{|h|m}}{c}, m\right) \quad \text{for} \quad m \in [M, 2M].$$

In order to seperate the variable m we use Fourier inversion. First define

$$G_0(\lambda) := x^{1+\varepsilon} \frac{d}{AB} \min\left(M, \frac{1}{\lambda}, \frac{1}{M\lambda^2}\right),$$

which is just a normalization factor. We have

$$\tilde{F}^{\pm}(c,m) = \int G_0(\lambda) G_{\lambda}^{\pm}(c) e(\lambda m)\, d\lambda, \quad G_{\lambda}^{\pm}(c) := \frac{1}{G_0(\lambda)} \int \tilde{F}^{\pm}(c,m) e(-\lambda m)\, dm,$$

so that

$$R_{ABC}^{\pm}(M) = \int G_0(\lambda) \sum_{M<m\leq 2M} \alpha(m) e(\lambda m) \sum_{\frac{a}{(a,d)}|c} \frac{S(h,\pm m;c)}{c} G_{\lambda}^{\pm}\left(4\pi\frac{\sqrt{|h|m}}{c}\right) d\lambda.$$

Before going on, we need some good estimates for the Bessel transforms occuring in the Kuznetsov formula. For convenience set

$$W := \sqrt{|h|M}\frac{d}{AB} \quad \text{and} \quad Z := \sqrt{xM}\frac{d}{AB},$$

and note that $W \ll 1$, which follows from the assumption in (3.1).

Lemma 3.3. *If $M \ll M_0^-$, we have*

$$\tilde{G}_{\lambda}^{\pm}(it), \check{G}_{\lambda}^{\pm}(it) \ll W^{-2t} \qquad \textit{for} \quad 0 \leq t < \frac{1}{4}, \tag{3.12}$$

$$\tilde{G}_{\lambda}^{\pm}(t), \check{G}_{\lambda}^{\pm}(t), \dot{G}_{\lambda}^{\pm}(t) \ll \frac{x^{\varepsilon}}{1+t^{\frac{5}{2}}} \qquad \textit{for} \quad t \geq 0. \tag{3.13}$$

If $M_0^- \ll M \ll M_0^+$, we have, for any $\nu \geq 0$,

$$\tilde{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) \ll x^{-\nu} \qquad \textit{for} \quad 0 \leq t < \frac{1}{4}, \tag{3.14}$$

$$\tilde{G}_\lambda^\pm(t), \check{G}_\lambda^\pm(t), \dot{G}_\lambda^\pm(t) \ll \frac{x^\varepsilon}{Z^{\frac{5}{2}}}\left(\frac{Z}{t}\right)^\nu \qquad \textit{for} \quad t \geq 0. \tag{3.15}$$

PROOF. Since all occurring integrals can be interchanged, we can look directly at the Bessel transforms of $\tilde{F}^\pm(c,m)$ and its first two partial derivatives in m. We will confine ourselves with the treatment of $\tilde{F}^\pm(c,m)$, since the corresponding estimates for the derivatives can be shown the same way.

First we want to use Lemma 2.5 to prove the first two bounds. Again we can look directly at the function inside the integral over ξ, given by

$$H_1(c) := cB^\pm\left(c\sqrt{\frac{\xi}{|h|}}\right) f\left(\xi; a, 4\pi\frac{\sqrt{|h|m}}{c}\frac{d}{a}\right),$$

for which we have the bounds

$$\operatorname{supp} H_1 \asymp W \quad \text{and} \quad H_1^{(\nu)}(c) \ll x^\varepsilon W\left(\frac{x^\varepsilon}{W}\right)^\nu \quad \text{for} \quad \nu \geq 0.$$

Hence, by the mentioned lemma,

$$\tilde{H}_1(it), \check{H}_1(it) \ll W^{1-2t} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$

$$\tilde{H}_1(t), \check{H}_1(t), \dot{H}_1(t) \ll \frac{x^\varepsilon W}{1+t^{\frac{5}{2}}} \qquad \text{for} \quad t \geq 0,$$

from which we get (3.12) and (3.13).

When $M \gg M_0^-$, oscillation effects come into play. By using Lemma 2.3 and partially integrating once over ξ, we get

$$\tilde{F}^+(c,m) = -\frac{h(m)}{\pi i\sqrt{|h|m}}\operatorname{Re}\left(\int_0^\infty e\left(\frac{c}{2\pi}\sqrt{\frac{\xi}{|h|}}\right)\tilde{w}(c)\, d\xi\right),$$

with

$$\tilde{w}(c) := \frac{\partial}{\partial\xi}\left(\sqrt{\xi|h|}v_Y\left(\frac{c}{\pi}\sqrt{\frac{\xi}{|h|}}\right) f\left(\xi; a, 4\pi\frac{d\sqrt{|h|m}}{ac}\right)\right).$$

Note that $\operatorname{supp}\tilde{w} \asymp W$, and that we have the bounds

$$\tilde{w}^{(\nu)}(\xi) \ll \frac{W^{1-\nu}}{Z^{\frac{3}{2}}}C(\xi) \quad \text{for} \quad \nu \geq 0, \quad \text{with} \quad C(\xi) := 1 + \left|w'\left(\frac{\xi-h}{x}\right)\right|.$$

It is now enough to look at

$$H_2(c) := e\left(\frac{c}{2\pi}\sqrt{\frac{\xi}{|h|}}\right)\tilde{w}(c).$$

We use Lemma 2.6 with $\alpha = \sqrt{\frac{\xi}{|h|}}$ and $X = W$, which is possible since

$$W \ll \frac{x^\varepsilon}{\Omega}\sqrt{\frac{|h|}{x}} \ll \frac{1}{x^\varepsilon} \quad \text{and} \quad \alpha W \asymp Z \gg x^\varepsilon,$$

and so we get

$$\tilde{H}_2(it), \check{H}_2(it) \ll x^{-\nu} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$

$$\tilde{H}_2(t), \check{H}_2(t), \dot{H}_2(t) \ll x^\varepsilon C(\xi)\frac{W}{Z^{\frac{5}{2}}}\left(\frac{Z}{t}\right)^\nu \qquad \text{for} \quad t \geq 0,$$

which then give (3.14) and (3.15). $\square$

4. Use of the Kuznetsov formula

Now we are ready to apply the Kuznetsov formula. We will only look at the sum $R^+_{ABC}(M)$, and we will assume that $h \geq 1$, since all other cases can be treated in very similar ways. As indicated, we use the Kuznetsov formula in the form (2.9) on the inner sum,

$$\begin{aligned}\sum_{\frac{a}{(a,d)}|c} \frac{S(h,m;c)}{c} G^+_\lambda\left(4\pi\frac{\sqrt{hm}}{c}\right) &= \sum_{j=1}^{\infty} \overline{\rho_j}(h,\infty)\rho_j(m,\infty)\frac{\sqrt{hm}}{\cosh(\pi t_j)}\tilde{G}^+_\lambda(t_j)\\ &+ \sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\varphi_{\mathfrak{c},t}}(h,\infty)\varphi_{\mathfrak{c},t}(m,\infty)\frac{\sqrt{hm}}{\cosh(\pi t_j)}\tilde{G}^+_\lambda(t)\,dt\\ &+ \sum_{\substack{k\equiv 0\,(2)\\ 1\leq j\leq \theta_k\left(\frac{a}{(a,d)},\chi_0\right)}} (k-1)!\,\overline{\psi_{j,k}}(h,\infty)\psi_{j,k}(m,\infty)\sqrt{hm}\,\dot{G}^+_\lambda(k-1),\end{aligned}$$

so that we can then write $R^+_{ABC}(M)$ as

$$R^+_{ABC}(M) = \int G_0(\lambda)(\Xi_{\mathrm{exc.}}(M) + \Xi_1(M) + \Xi_2(M) + \Xi_3(M))\,d\lambda,$$

where

$$\begin{aligned}\Xi_{\mathrm{exc.}}(M) &:= \sum_{t_j\ \mathrm{exc.}} \tilde{G}^+_\lambda(t_j)\left(\frac{\overline{\rho_j}(h,\infty)\sqrt{h}}{\sqrt{\cosh(\pi t_j)}}\right)\Sigma^{(\mathrm{exc.})}_j(M),\\ \Xi_1(M) &:= \sum_{t_j\geq 0} \tilde{G}^+_\lambda(t_j)\left(\frac{\overline{\rho_j}(h,\infty)\sqrt{h}}{\sqrt{\cosh(\pi t_j)}}\right)\Sigma^{(1)}_j(M),\\ \Xi_2(M) &:= \sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{G}^+_\lambda(t)\left(\frac{\overline{\varphi_{\mathfrak{c},t}}(h,\infty)\sqrt{h}}{\sqrt{\cosh(\pi t)}}\right)\Sigma^{(2)}_{\mathfrak{c},t}(M)\,dt,\\ \Xi_3(M) &:= \sum_{\substack{k\equiv 0\,(2)\\ 1\leq j\leq \theta_k\left(\frac{a}{(a,d)},\chi_0\right)}} \tilde{G}^+_\lambda(k-1)\Big(\sqrt{(k-1)!}\,\overline{\psi_{j,k}}(h,\infty)\sqrt{h}\Big)\Sigma^{(3)}_{j,k}(M),\end{aligned}$$

and

$$\begin{aligned}\Sigma^{(\mathrm{exc.})}_j(M) &:= \frac{1}{\sqrt{\cosh(\pi t_j)}} \sum_{M<m\leq 2M} \alpha(m)e(\lambda m)\rho_j(m,\infty)\sqrt{m},\\ \Sigma^{(1)}_j(M) &:= \frac{1}{\sqrt{\cosh(\pi t_j)}} \sum_{M<m\leq 2M} \alpha(m)e(\lambda m)\rho_j(m,\infty)\sqrt{m},\\ \Sigma^{(2)}_{\mathfrak{c},t}(M) &:= \frac{1}{\sqrt{\cosh(\pi t_j)}} \sum_{M<m\leq 2M} \alpha(m)e(\lambda m)\varphi_{\mathfrak{c},t}(m,\infty)\sqrt{m},\\ \Sigma^{(3)}_{j,k}(M) &:= \sqrt{(k-1)!} \sum_{M<m\leq 2M} \alpha(m)e(\lambda m)\psi_{j,k}(m,\infty)\sqrt{m}.\end{aligned}$$

The sum $\Xi_{\mathrm{exc.}}(M)$ needs a special treatment, which we will do in the following section. First, we want to look at the other sums, and here we will restrict ourselves to $\Xi_1(M)$, since the treatment of $\Xi_2(M)$ and $\Xi_3(M)$ can be done along the same lines.

First assume $M \ll M_0^-$. We divide $\Xi_1(M)$ into two parts:

$$\Xi_1(M) = \sum_{t_j\leq 1}(\dots) + \sum_{1<t_j}(\dots) =: \Xi_{1\mathrm{a}}(M) + \Xi_{1\mathrm{b}}(M).$$

For $\Xi_{1\mathrm{a}}(M)$ we get using (3.13), Cauchy-Schwarz, Theorem 2.8 and Lemma 2.9,

$$\begin{aligned}\Xi_{1\mathrm{a}}(M) &\ll \max_{0\le t_j\le 1}\left|\tilde{G}_\lambda^+(t_j)\right|\sum_{t_j\le 1}\frac{|\rho_j(h,\infty)|\sqrt{h}}{\sqrt{\cosh(\pi t_j)}}\left|\Sigma_j^{(1)}(M)\right|\\ &\ll x^\varepsilon\left(1+\frac{(a,h)^{\frac12}(a,d)h^{\frac12}}{a}\right)^{\frac12}\left(1+(a,d)\frac{M}{a}\right)^{\frac12}M^{\frac12}\\ &\ll \frac{(a,h)^{\frac14}(a,d)^{\frac12}}{d}\frac{x^\varepsilon AB}{x^{\frac12}}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right),\end{aligned}$$

so that

$$\int G_0(\lambda)\Xi_{1\mathrm{a}}(M)\,d\lambda \ll (a,h)^{\frac14}(a,d)^{\frac12}x^{\frac12+\varepsilon}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right).$$

We split up the remainig sums into dyadic segments

$$\Xi_1(M,T) := \sum_{T<t_j\le 2T}\tilde{G}_\lambda^+(t_j)\frac{\overline{\rho_j}(h,\infty)\sqrt{h}}{\sqrt{\cosh(\pi t_j)}}\Sigma_j^{(1)}(M),$$

and in the same way as above, we get

$$\Xi_1(M,T) \ll \frac{(a,h)^{\frac14}(a,d)^{\frac12}}{d}\frac{x^\varepsilon AB}{x^{\frac12}T^{\frac12}}\left(1+\frac{B^{\frac12}}{C^{\frac12}T}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}T}\right),$$

which then gives

$$\int G_0(\lambda)\Xi_{1\mathrm{b}}(M)\,d\lambda \ll (a,h)^{\frac14}(a,d)^{\frac12}x^{\frac12+\varepsilon}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right).$$

The case $M \gg M_0^-$ is handled the same way: We again divide $\Xi_1(M)$ into two parts

$$\Xi_1(M) = \sum_{t_j\le Z}(\dots) + \sum_{Z<t_j}(\dots),$$

and this time we have to use the bound (3.15), which eventually leads to

$$\int G_0(\lambda)\Xi_1(M)\,d\lambda \ll (a,h)^{\frac14}(a,d)^{\frac12}\frac{x^{\frac12+\varepsilon}}{\Omega^{\frac12}}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right).$$

The same bounds can be proven very similarly for $\Xi_2(M)$ and $\Xi_3(M)$, so that we end up with

$$\begin{aligned}R_{ABC}^+(M) = &\int G_0(\lambda)\Xi_{\mathrm{exc.}}(M)\,d\lambda\\ &+\mathcal{O}\left((a,h)^{\frac14}(a,d)^{\frac12}\frac{x^{\frac12+\varepsilon}}{\Omega^{\frac12}}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right)\right).\end{aligned}\tag{3.16}$$

5. Treatment of the exceptional eigenvalues

For $M \gg M_0^-$, the exceptional eigenvalues pose no problem at all, since the Bessel transforms $\tilde{G}_\lambda^+(t_j)$ are very small, as can be seen from (3.14). Hence, in (3.16), the contribution of $\Xi_{\mathrm{exc.}}(M)$ certainly does not lead to a larger error term.

For $M \ll M_0^-$, this is a totally different story. If we would bound $\Xi_{\mathrm{exc.}}(M)$ the same way as in the section above using (3.12), we would end up with

$$\int G_0(\lambda)\Xi_{\mathrm{exc.}}(M)\,d\lambda \ll (a,h)^{\frac14}(a,d)^{\frac12}\frac{x^{\frac12+\theta+\varepsilon}}{h^\theta}\left(1+\frac{B^{\frac12}}{C^{\frac12}}\right)\left(1+\frac{h^{\frac14}}{A^{\frac12}}\right).\tag{3.17}$$

With the currently best value for θ, this would weaken our result considerably. However, we can reduce the effect of the exceptional eigenvalues by exploiting the fact that these eigenvalues appear infrequently. Cauchy-Schwarz and (3.12) give

$$\Xi_{\text{exc.}}(M) \ll \left(\sum_{t_j \text{ exc.}} W^{-4it_j} |\rho_j(h,\infty)|^2 h \right)^{\frac{1}{2}} \left(\sum_{t_j \text{ exc.}} \left|\Sigma_j^{\text{exc.}}(M)\right|^2 \right)^{\frac{1}{2}}.$$

The second factor can be treated with the large sieve inequalities. Because of

$$W^{-1} \gg \frac{x^{\frac{1}{2}}}{h^{\frac{1}{2}}} \gg \frac{a}{(a,d)h^{\frac{1}{2}}},$$

we can use Lemma 2.11 to bound the first factor. So,

$$\Xi_{\text{exc.}}(M) \ll (a,h)^{\frac{1}{2}} \frac{(a,d)}{d} \frac{x^{\varepsilon} A^{1-2\theta} B}{x^{\frac{1}{2}-\theta}} \left(1 + \frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}}\right)\left(1 + \frac{h^{\frac{1}{4}}}{A^{\frac{1}{2}}}\right),$$

and hence

$$\int G_0(\lambda) \Xi_{\text{exc.}}(M)\, d\lambda \ll (a,h)^{\frac{1}{2}} (a,d) \frac{x^{\frac{1}{2}+\theta+\varepsilon}}{A^{2\theta}} \left(1 + \frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}}\right)\left(1 + \frac{h^{\frac{1}{4}}}{A^{\frac{1}{2}}}\right),$$

which is a substantial improvement to (3.17).

Eventually we get

$$R^{\pm}_{ABC}(M) \ll (a,h)^{\frac{1}{2}} (a,d) x^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{x^{\theta}}{A^{2\theta}} \right) \left(1 + \frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}}\right)\left(1 + \frac{h^{\frac{1}{4}}}{A^{\frac{1}{2}}}\right),$$

which as a consequence gives (3.8). This completes the proof of Lemma 3.2.

6. The main term

Here we will evaluate the main term of Ψ, which appears only when $\alpha(n) = d(n)$, and arises in this case after summing over all the main terms M_{ABC}. It is given by

$$\begin{aligned} M_0 &:= \sum_{a,b} \frac{1}{ab} \int \lambda_{h,ab}(\xi + h) w\left(\frac{\xi}{x}\right) h\left(a, b, \frac{\xi}{ab}\right) d\xi \\ &= x \int w(\xi) \sum_{a,b} \frac{\lambda_{h,ab}(x\xi)}{ab} h\left(a, b, \frac{x\xi}{ab}\right) d\xi + \mathcal{O}(x^{\varepsilon} h), \end{aligned}$$

so that effectively we only need to look at

$$M_{\mathrm{a}}(\xi) := \sum_{a,b} \frac{\lambda_{h,ab}(x\xi)}{ab} H_{\mathrm{a}}(a,b;\xi),$$

where

$$H_{\mathrm{a}}(a,b;\xi) := h\left(a, b, \frac{x\xi}{ab}\right).$$

Using Mellin inversion this sum can be written as

$$M_{\mathrm{a}}(\xi) = \frac{1}{2\pi i} \sum_a \frac{1}{a} \int_{(\sigma)} \hat{H}_{\mathrm{a}}(a,s;\xi) \sum_{b=1}^{\infty} \frac{\lambda_{h,ab}(x\xi)}{b^{1+s}}\, ds, \quad \sigma > 0,$$

where the Mellin transform of $H_{\mathrm{a}}(a,b;\xi)$ is given by

$$\hat{H}_{\mathrm{a}}(a,s;\xi) := \int_0^{\infty} H_{\mathrm{a}}(a,b;\xi) b^{s-1}\, db, \quad \mathrm{Re}(s) > 0.$$

A routine calculation then shows that, for $\mathrm{Re}(s) > 0$,

$$\sum_{b=1}^{\infty} \frac{\lambda_{h,ab}(x\xi)}{b^{1+s}} = \zeta(1+s) \sum_{d=1}^{\infty} \frac{c_d(h)(\log(x\xi) + 2\gamma - 2\log d)(a,d)^{1+s}}{d^{2+s}},$$

so that it is sufficient to look at

$$M_{\mathrm{b}}(\xi, d) := \frac{1}{2\pi i} \sum_a \frac{(a,d)}{a} \int_{(\sigma)} \hat{H}_{\mathrm{a}}(a,s;\xi)\zeta(1+s)\frac{(a,d)^s}{d^s}\, ds. \tag{3.18}$$

Here we want to use the residue theorem. $\hat{H}_{\mathrm{a}}(a,s;\xi)$ can be continued meromorphically to the whole complex plane with a simple pole at $s=0$, and its Laurent series is given by

$$\hat{H}_{\mathrm{a}}(a,s;\xi) = 3v_1(a)\frac{1}{s} + 3v_1(a)\left(\log\frac{x\xi}{a} + C(a)\right) + \mathcal{O}(s),$$

where

$$C(a) := \int_0^{\infty} v_1'(b)\log b\, db + \frac{1}{3}\int_0^{\infty} v_1'(b)v_1\left(\frac{x\xi}{ab}\right)\log\frac{x\xi}{ab^2}\, db.$$

We also have that

$$\hat{H}_{\mathrm{a}}(a,s;\xi) \ll \frac{1}{|s||s+1|} x^{\frac{1}{3}\mathrm{Re}(s)}.$$

Now we shift the line of integration in (3.18) to $\mathrm{Re}(s) = -1+\varepsilon$, and the residue theorem gives

$$M_{\mathrm{b}}(\xi,d) = 3M_{\mathrm{c}}(d) + 3M_{\mathrm{d}}(\xi,d) + \mathcal{O}\left(\frac{d^{1-\varepsilon}}{x^{\frac{1}{3}-\varepsilon}}\right),$$

where

$$M_{\mathrm{c}}(d) := \sum_a \frac{(a,d)}{a}\log\frac{(a,d)}{a}H_{\mathrm{c}}(a), \quad M_{\mathrm{d}}(\xi,d) := \sum_a \frac{(a,d)}{a}H_{\mathrm{d}}(a,\xi),$$

and

$$H_{\mathrm{c}}(a) := v_1(a), \quad H_{\mathrm{d}}(a;\xi) := v_1(a)\left(\log\frac{x\xi}{d} + \gamma + C_1(a)\right).$$

The evaluation of these two sums can be done the same way as above using Mellin inversion and the residue theorem. The appearing Dirichlet series can be continued meromorphically via

$$\sum_a \frac{(a,d)}{a^{1+s}}\log\frac{(a,d)}{a} = \sum_{r|d}\frac{\mu(r)}{r}\sigma_s\left(\frac{d}{r}\right)\frac{(\zeta'(1+s) - \zeta(1+s)\log r)}{d^s},$$

$$\sum_a \frac{(a,d)}{a^{1+s}} = \frac{\zeta(1+s)}{d^s}\sum_{r|d}\frac{\mu(r)}{r}\sigma_s\left(\frac{d}{r}\right),$$

which are identites for $\mathrm{Re}(s) > 0$. The Mellin transforms $\hat{H}_{\mathrm{c}}(s)$ and $\hat{H}_{\mathrm{d}}(s;\xi)$ too have a meromorphic continuation to the whole complex plane, both with a simple pole at $s=0$, and with Laurent series of the form

$$\hat{H}_{\mathrm{c}}(s) = \frac{1}{s} + P_{1\mathrm{c}}(\log x) + sP_{2\mathrm{c}}(\log x) + \mathcal{O}(s^2),$$
$$\hat{H}_{\mathrm{d}}(s) = \frac{1}{s}P_{1\mathrm{d}}(\log x, \log\xi) + P_{2\mathrm{d}}(\log x, \log\xi) + \mathcal{O}(s),$$

where $P_{1\mathrm{c}}$ and $P_{1\mathrm{d}}$ are linear polynomials, and $P_{2\mathrm{c}}$ and $P_{2\mathrm{d}}$ quadratic ones (which may depend on d and v_1). We also have the bounds

$$\hat{H}_{\mathrm{c}}(s), \hat{H}_{\mathrm{d}}(s;\xi) \ll \frac{1}{|s||s+1|}x^{\frac{1}{3}\mathrm{Re}(s)+\varepsilon}.$$

Applying the residue theorem the same way as before, we get

$$M_{\mathrm{b}}(\xi, d) = P_{2,d}(\log x, \log \xi) + \mathcal{O}\left(\frac{d^{1-\varepsilon}}{x^{\frac{1}{3}-\varepsilon}}\right),$$

where $P_{2,d}$ is a quadratic polynomial depending only on d, which as a consequence shows that the main term is of the form as stated in Lemma 3.1.

CHAPTER 4

Proof of Theorems 1.5 and 1.6

In this chapter we will look at $D_3(N)$ and $A_3(N)$, and prove Theorems 1.5 and 1.6. As before, we can consider both sums simultaneously, so that we will stick to the convention that $\alpha(n)$ is a placeholder for $d(n)$ or $a(n)$.

1. Construction of a smooth partition of unity

We first construct a smooth decomposition of the unit interval in a form suiting our needs. There exist functions $w_j : \mathbb{R} \to [0, \infty)$, $j \geq 1$, which are smooth and compactly supported and which satisfy

$$\operatorname{supp} w_j \subset \left[\frac{1}{2^{j+2}}, \frac{1}{2^j}\right] \quad \text{and} \quad w_j^{(\nu)}(\xi) \ll 2^{(j+1)\nu} \quad \text{for} \quad \nu \geq 0,$$

and

$$\sum_{j=1}^{\infty} w_j(\xi) = 1 \quad \text{for} \quad \xi \in (0, 1/4].$$

For $j \geq 1$ we then define

$$w_{-j}(\xi) := w_j(1-\xi) \quad \text{and} \quad w_0(\xi) := 1 - w_1(\xi) - w_{-1}(\xi),$$

so that by construction

$$\sum_{j \in \mathbb{Z}} w_j(\xi) = 1 \quad \text{for} \quad \xi \in (0, 1).$$

We can write our sum now as

$$\sum_{n=1}^{N-1} d_3(n)\alpha(N-n) = \sum_{j \in \mathbb{Z}} \sum_{n} w_j\left(\frac{n}{N-1}\right) d_3(n)\alpha(N-n),$$

hence it is enough to look at

$$\Psi_j := \sum_{n} w_j\left(\frac{n}{N-1}\right) d_3(n)\alpha(N-n). \tag{4.1}$$

The evaluation of these sums follows the same path as in Chapter 3, and we will therefore use in large parts the same notation and omit many details.

For the sake of easier notation, we will leave out the j-subscript from now on, so that, for example, $w(\xi) := w_j(\xi)$. Note that the variable n in (4.1) is supported in

$$n \in \left[\frac{x}{2}, 2x\right] \quad \text{for} \quad j \geq 0, \quad \text{and} \quad n \in \left[N-1-2x, N-1-\frac{x}{2}\right] \quad \text{for} \quad j < 0,$$

where

$$x := \frac{N-1}{2^{|j|+1}}.$$

A first trivial bound for $\Psi := \Psi_j$ is now given by

$$\Psi \ll N^{\varepsilon} x.$$

The decomposition we use for $d_3(n)$ is the same as in (3.2), but with a different normalization, namely

$$u_1(\xi) := u_0\left(\frac{\xi}{(N-1)^{\frac{1}{3}}}\right) \quad \text{and} \quad u_2(\xi) := u_0\left(\frac{\xi}{\sqrt{\frac{N-1}{a}}}\right),$$

and as in Chapter 3, we then need to evaluate the sums

$$\Psi_{ABC} := \sum_{a,b,c} w\left(\frac{abc}{N-1}\right) h_{ABC}(a,b,c)\alpha(N-abc).$$

Our result will be the following asymptotic formula.

Lemma 4.1. *We have*

$$\Psi_{ABC} = M_{ABC} + \mathcal{O}\left(N^{\frac{11}{12}} + \frac{N^{\frac{4}{3}+\varepsilon}}{x^{\frac{1}{2}}}\right),$$

where M_{ABC} vanishes if $\alpha(n) = a(n)$, and otherwise is given by

$$M_{ABC} := \sum_{a,b} \frac{1}{ab} \int \lambda_{N,ab}(\xi) h_{ABC}\left(a, b, \frac{N-\xi}{ab}\right) w\left(\frac{N-\xi}{N-1}\right) d\xi,$$

with $\lambda_{N,ab}(\xi)$ defined as in (2.1).

We use Lemma 4.1 when $x \gg N^{\frac{8}{9}}$, and otherwise just bound trivially. After evaluating the possible main term, which we will do in Section 4, this proves Theorems 1.5 and 1.6.

2. Use of the Voronoi summation formula

The saving in the error term comes again primarily from averaging over the variables b and c, and it is hence sufficient to look at

$$\begin{aligned}\Phi_{ABC}(a) &:= \sum_{b,c} w\left(\frac{abc}{N-1}\right) h_{ABC}(a,b,c)\alpha(N-abc) \\ &= \sum_{b} \sum_{m \equiv N(ab)} \alpha(m) f(m; a, b),\end{aligned}$$

where

$$f(\xi; a, b) := h_{ABC}\left(a, b, \frac{N-\xi}{ab}\right) w\left(\frac{N-\xi}{N-1}\right).$$

Note that

$$\operatorname{supp} f(\bullet; a, b) \subset \left[N-1-2x, N-1-\frac{x}{2}\right] \quad \text{for} \quad j \geq 0,$$

and

$$\operatorname{supp} f(\bullet; a, b) \subset \left[\frac{x}{2}, 2x\right] \qquad\qquad \text{for} \quad j < 0,$$

and that the derivatives of $f(\xi; a, b)$ are bounded by

$$\frac{\partial^\nu}{\partial \xi^\nu} f(\xi; a, b) \ll \frac{1}{x^\nu} \quad \text{for} \quad \nu \geq 0.$$

Furthermore, we have that

$$ABC \asymp x \quad \text{for} \quad j \geq 0, \quad \text{and} \quad ABC \asymp N \quad \text{for} \quad j < 0.$$

After using the Voronoi formula, we get as a possible main term for $\Phi_{ABC}(a)$,

$$M_{ABC}(a) := \frac{1}{a} \sum_{b} \frac{1}{b} \int \lambda_{N,ab}(\xi) f(\xi; a, b)\, d\xi,$$

and as error terms we have to deal with

$$R_{ABC}(a) := \frac{1}{a}\sum_{\substack{b,c\\ c|ab}}\frac{1}{b}\sum_{m=1}^{\infty}\alpha(m)\frac{S(N,\pm m;c)}{c}\int B^{\pm}\left(4\pi\frac{\sqrt{m\xi}}{c}\right)f(\xi;a,b)\,d\xi,$$

where $B^{\pm}(\xi)$ are defined as in (3.10). We rearrange the variables in the same way as in (3.9), so that

$$R_{ABC}(a) = \sum_{d}\frac{1}{d}\sum_{m=1}^{\infty}\alpha(m)\sum_{\frac{a}{(a,d)}|c}\frac{S(N,\pm m;c)}{c}F^{\pm}(c,m),$$

where $F^{\pm}(c,m)$ is defined as

$$F^{\pm}(c,m) := \frac{1}{c}\int B^{\pm}\left(4\pi\frac{\sqrt{m\xi}}{c}\right)f\left(\xi;a,\frac{cd}{a}\right)d\xi.$$

It is not hard to see that we can cut the sum over m in $R_{ABC}(a)$ at $M_0^{\pm}$, where

$$M_0^{+} := \frac{N^{1+\varepsilon}}{x^2}\left(\frac{AB}{d}\right)^2, \quad M_0^{-} := \frac{N^{\varepsilon}}{N}\left(\frac{AB}{d}\right)^2 \qquad \text{for} \quad j \geq 0,$$

and

$$M_0^{+} := M_0^{-} := \frac{N^{\varepsilon}}{x}\left(\frac{AB}{d}\right)^2 \qquad \text{for} \quad j < 0,$$

so that we eventually have to look at

$$R^{\pm}_{ABC}(M) := \sum_{M<m\leq 2M}\alpha(m)\sum_{\frac{a}{(a,d)}|c}\frac{S(N,\pm m;c)}{c}F^{\pm}(c,m).$$

3. Use of the Kuznetsov formula

We bring again everything into the right shape for the use of the Kuznetsov formula by setting

$$\tilde{F}^{\pm}(c,m) := F^{\pm}\left(4\pi\frac{\sqrt{Nm}}{c}, m\right),$$

and using Poisson inversion to separate the variable m, so that

$$R^{\pm}_{ABC}(M) = \int G_0(\lambda)\sum_{M<m\leq 2M}\alpha(m)e(\lambda m)\sum_{ar|c}\frac{S(N,\pm m;c)}{c}G^{\pm}_{\lambda}\left(4\pi\frac{\sqrt{Nm}}{c}\right)d\lambda,$$

where

$$G^{\pm}_{\lambda}(c) := \frac{1}{G_0(\lambda)}\int \tilde{F}^{\pm}(c,m)e(-\lambda m)\,dm$$

with

$$G_0(\lambda) := N^{\varepsilon}x\frac{d}{AB}\min\left(M,\frac{1}{\lambda},\frac{1}{M\lambda^2}\right).$$

We also set

$$W := \sqrt{NM}\frac{d}{AB}.$$

When bounding the Bessel transforms of $G^{\pm}_{\lambda}(c)$, we have to distinguish between the cases $j \geq 0$ and $j < 0$.

3.1. The case $j \geq 0$. In this case, we have the following bounds if $M \ll M_0^-$,

$$\hat{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) \ll N^\varepsilon W^{-2t} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$
$$\hat{G}_\lambda^\pm(t), \check{G}_\lambda^\pm(t), \tilde{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{1+t^{\frac{5}{2}}} \qquad \text{for} \quad t \geq 0,$$

and if $M_0^- \ll M \ll M_0^+$, we have, for any $\nu \geq 0$,

$$\hat{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) \ll \frac{N^\varepsilon}{W} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$
$$\hat{G}_\lambda^\pm(t), \tilde{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{W}\left(\frac{W^{\frac{1}{2}}}{t}\right)^\nu \qquad \text{for} \quad t \geq 0,$$
$$\check{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{W^{\frac{3}{2}}}\left(\frac{W}{t}\right)^\nu \qquad \text{for} \quad t \geq 0.$$

All these bounds can be derived the same way as in Lemma 3.3. There are two slight differences, though: Applying partial integration once over ξ is useless here, and instead of Lemma 2.6 we need to use Lemma 2.7.

Now applying the Kuznetsov formula and the large sieve inequalities, we get the bounds

$$R_{ABC}^+(M) \ll (a,N)^{\frac{1}{4}}(a,d)^{\frac{1}{2}}\frac{N^{\frac{3}{4}+\varepsilon}}{A^{\frac{1}{2}}}, \tag{4.2}$$
$$R_{ABC}^-(M) \ll (a,N)^{\frac{1}{4}}(a,d)^{\frac{1}{2}}\frac{N^{\frac{3}{4}+\varepsilon}}{A^{\frac{1}{2}}}\left(1+\frac{A^{\frac{1}{2}}N^{\frac{1}{4}}}{x^{\frac{1}{2}}}\right). \tag{4.3}$$

In contrast to Chapter 3, the exceptional eigenvalues cause no problems at all.

3.2. The case $j < 0$. Here the following bounds hold if $M \ll \frac{N^\varepsilon}{N}\left(\frac{AB}{d}\right)^2$,

$$\hat{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) \ll N^\varepsilon W^{-2t} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$
$$\hat{G}_\lambda^\pm(t), \check{G}_\lambda^\pm(t), \tilde{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{1+t^{\frac{5}{2}}} \qquad \text{for} \quad t \geq 0,$$

and if $\frac{N^\varepsilon}{N}\left(\frac{AB}{d}\right)^2 \ll M \ll M_0^\pm$, we have that

$$\hat{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) \ll \frac{N^\varepsilon}{W} \qquad \text{for} \quad 0 \leq t < \frac{1}{4},$$
$$\hat{G}_\lambda^\pm(t), \check{G}_\lambda^\pm(t), \tilde{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{W} \qquad \text{for} \quad t \geq 0,$$
$$\hat{G}_\lambda^\pm(t), \check{G}_\lambda^\pm(t), \tilde{G}_\lambda^\pm(t) \ll \frac{N^\varepsilon}{t^{\frac{5}{2}}}\left(1+\frac{W}{t^{\frac{1}{2}}}\right) \qquad \text{for} \quad t \gg W.$$

The use of the Kuznetsov formula this time gives

$$R_{ABC}^\pm(M) \ll (a,N)^{\frac{1}{4}}(a,d)^{\frac{1}{2}}\frac{N^{\frac{3}{4}+\varepsilon}}{A^{\frac{1}{2}}}.$$

As before, the exceptional eigenvalues do not lead to any difficulties.

This bound, together with the other bounds (4.2) and (4.3), then leads to

$$\Phi_{ABC}(a) = M_{ABC}(a) + \mathcal{O}\left((a,N)^{\frac{1}{4}}(a,d)^{\frac{1}{2}}\frac{N^{\frac{3}{4}+\varepsilon}}{A^{\frac{1}{2}}}\left(1+\frac{A^{\frac{1}{2}}N^{\frac{1}{4}}}{x^{\frac{1}{2}}}\right)\right),$$

and together with the fact that

$$A \ll B \ll C \quad \text{and} \quad A \ll N^{\frac{1}{3}},$$

we get Lemma 4.1.

4. The main term

To finish the proof of Theorem 1.5, we have to evaluate the main term, which occurs in case $\alpha(n) = d(n)$ and is then given by

$$\begin{aligned} M_0 &:= \sum_{a,b} \frac{1}{ab} \int_1^{N-1} \lambda_{N,ab}(\xi) h\left(a, b, \frac{N-\xi}{ab}\right) d\xi \\ &= N \int_0^1 \sum_{a,b} \frac{\lambda_{N,ab}(N(1-\xi))}{ab} h\left(a, b, \frac{(N-1)\xi}{ab}\right) d\xi + \mathcal{O}(N^{\varepsilon}). \end{aligned}$$

This, too, can be done in the same way as in Chapter 3, so we will just state a few intermediate results. It is enough to look at

$$M_{\mathrm{a}}(\xi) := \sum_{a,b} \frac{\lambda_{N,ab}(N(1-\xi))}{ab} h\left(a, b, \frac{(N-1)\xi}{ab}\right) d\xi,$$

and this sum can be evaluated by using Mellin inversion and the residue theorem, so that we get

$$\begin{aligned} M_{\mathrm{a}}(\xi) = 3 \sum_{d=1}^{\infty} \frac{c_d(N)(\log(N(1-\xi)) + 2\gamma - 2\log d)}{d^2} (M_{\mathrm{c}}(d) + M_{\mathrm{d}}(\xi, d)) \\ + \mathcal{O}\left(\frac{1}{N^{\frac{1}{3}-\varepsilon}\xi^{1-\varepsilon}}\right), \end{aligned}$$

with

$$\begin{aligned} M_{\mathrm{c}}(d) &:= \sum_a \frac{(a,d)}{a} \log\left(\frac{(a,d)}{a}\right) v_1(a), \\ M_{\mathrm{d}}(\xi, d) &:= \sum_a \frac{(a,d)}{a} v_1(a) \left(\log\left(\frac{N\xi}{d}\right) + \gamma + C(a)\right), \end{aligned}$$

and

$$C(a) := \int_0^{\infty} v_1'(b) \log b \, db + \frac{1}{3} \int_0^{\infty} v_1'(b) v_1\left(\frac{(N-1)\xi}{ab}\right) \log\left(\frac{(N-1)\xi}{ab^2}\right) db.$$

The evaluation of $M_{\mathrm{c}}(d)$ and $M_{\mathrm{d}}(\xi, d)$ follows the usual pattern, and as result we get

$$M_{\mathrm{c}}(d) + M_{\mathrm{d}}(\xi, d) = \sum_{r|d} \frac{\mu(r)}{r} \sum_{m|\frac{d}{r}} P_{2,\xi}(\log N, \log d, \log r, \log m) + \mathcal{O}\left(\frac{d^{1-\varepsilon}}{N^{\frac{1}{3}-\varepsilon}\xi^{1-\varepsilon}}\right),$$

where $P_{2,\xi}$ is a quadratic polynomial depending on ξ. From this, we see that the main term of $D_3(N)$ has the form

$$M := N \sum_{d=1}^{\infty} \frac{c_d(N)}{d^2} \sum_{r|d} \frac{\mu(r)}{r} \sum_{m|\frac{d}{r}} P_3(\log N, \log d, \log r, \log m) + \mathcal{O}\left(N^{\frac{2}{3}+\varepsilon}\right),$$

with a cubic polynomial P_3.

We want to reshape this result a little bit. Set

$$G(\alpha, \beta, \gamma, \delta) := N^{\alpha} \sum_{d=1}^{\infty} \frac{c_d(N)}{d^{2-\beta}} \sum_{r|d} \frac{\mu(r)}{r^{1-\gamma}} \sigma_{\delta}\left(\frac{d}{r}\right),$$

so that the main term can be stated in terms of the partial derivatives of G up to third order evaluated at $(0, 0, 0, 0)$. A lengthy but elementary calculation shows that

$$\begin{aligned} G(\alpha,\beta,\gamma,\delta) &= N^{\alpha} \sum_{d|N} \sum_{\substack{c|d \\ b|c}} \mu\left(\frac{d}{c}\right) \frac{c^{1-\gamma+\delta}}{d^{2-\gamma-\beta} b^{\delta}} \sum_{(r,d)=1} \sum_{\substack{(s,br)=1 \\ (d,rs)=1}} \frac{\mu^2(r)\mu(s)\mu(d)}{r^{3-\gamma-\delta} s^{2-\beta-\delta} d^{2-\beta}} \\ &= C(\beta,\gamma,\delta) N^{\alpha} \sum_{d|N} \frac{\chi_1(d)}{d^{1-\beta}} \sum_{c|d} \chi_2\left(\frac{d}{c}\right) \chi_3(c), \end{aligned}$$

with

$$C(\beta,\gamma,\delta) := \frac{1}{\zeta(2-\delta)} \prod_{p} \left(1 - \frac{p^{1-\gamma+\delta}-1}{p^{1-\gamma}(p^{2-\beta}-1)}\right),$$

and χ_1, χ_2, and χ_3 as defined in (1.3). This finally proves Theorem 1.5.

CHAPTER 5

Proof of Theorems 1.7 and 1.8

In this chapter we will look at the sums $D_k^{\pm}(x,h)$ with $k \geq 4$, and prove Theorems 1.7 and 1.8.

Let $w : [1/2, 1] \to [0, \infty)$ be a smooth and compactly supported function satisfying

$$w^{(\nu)}(\xi) \ll \frac{1}{\Omega^{\nu}} \quad \text{for} \quad \nu \geq 0, \quad \text{and} \quad \int \left| w^{(\nu)}(\xi) \right| d\xi \ll \frac{1}{\Omega^{\nu-1}} \quad \text{for} \quad \nu \geq 1,$$

for some $\Omega \leq 1$. We will look at the sum

$$\Psi := \sum_n w\left(\frac{n}{x}\right) d_k(n) d(n+h), \quad h \in \mathbb{Z} \setminus \{0\},$$

and, assuming that h is of the size

$$h \ll \Omega^2 x^{1-\varepsilon},$$

we will prove the following lemma, which gives an asymptotic formula for Ψ.

Lemma 5.1. *We have the asymptotic formula*

$$\Psi = M + R,$$

where the main term M is given by

$$M := \int w\left(\frac{\xi}{x}\right) P_{k,h}(\log x, \log \xi, \log(\xi + h))\, d\xi,$$

with a polynomial $P_{k,h}$ of degree k, and where we have the following estimate for the error term R,

$$R \ll \frac{x^{1-\frac{1}{3k-2}+\varepsilon}}{\Omega^{\frac{1}{4}+\frac{1}{4(3k-2)}}} + x^{\frac{37}{38}+\varepsilon}\left(\frac{1}{\Omega^{\frac{1}{2}}} + x^{\frac{\theta}{19}}\right)\left(1 + \left(\frac{|h|}{x^{\frac{15}{19}}}\right)^{\frac{1}{16}}\right). \tag{5.1}$$

Remember that the constant θ, which appears in the estimate for R, was defined in (2.8).

Theorem 1.8 follows directly from Lemma 5.1 with the choice $\Omega = 1$. Moreover, with the choice $\Omega = x^{-\frac{1}{57}}$ we get

$$R \ll x^{\frac{229}{228}-\frac{227}{228(3k-2)}+\varepsilon} + x^{\frac{56}{57}+\varepsilon}\left(1 + \left(\frac{|h|}{x^{\frac{15}{19}}}\right)^{\frac{1}{16}}\right),$$

while the choice $\Omega = x^{-\frac{4}{15k-9}}$ leads to

$$R \ll x^{1-\frac{4}{15k-9}+\varepsilon} + x^{\frac{37}{38}+\varepsilon}\left(x^{\frac{2}{15k-9}} + x^{\frac{\theta}{19}}\right)\left(1 + \left(\frac{|h|}{x^{\frac{15}{19}}}\right)^{\frac{1}{16}}\right).$$

We use the former bound for $k \leq 15$ and the latter bound for $k \geq 16$, so that

$$R \ll x^{1-\frac{4}{15k-9}+\varepsilon} + x^{\frac{56}{57}+\varepsilon}\left(1 + \left(\frac{|h|}{x^{\frac{15}{19}}}\right)^{\frac{1}{16}}\right).$$

After choosing appropriate weight functions, this proves Theorem 1.7.

1. Opening the divisor function $d_k(n)$

In order to prove Lemma 5.1, we will open $d_k(n)$ and then dyadically split the supports of the appearing variables. This will be carried out rigorously in the following section – for the moment, just assume that we have a sum of the form

$$\Psi_{v_1,\ldots,v_k} := \sum_{a_1,\ldots,a_k} w\Big(\frac{a_1\cdots a_k}{x}\Big) v_1(a_1)\cdots v_k(a_k) d(a_1\cdots a_k+h), \tag{5.2}$$

where $v_1,\ldots,v_k$ are smooth and compactly supported functions satisfying

$$\operatorname{supp} v_i \asymp A_i \quad \text{and} \quad v_i^{(\nu)}(\xi) \ll \frac{1}{A_i{}^{\nu}} \quad \text{for} \quad \nu \geq 0.$$

Our main results are three asymptotic estimates for $\Psi_{v_1,\ldots,v_k}$, which we state together in the following lemma.

Lemma 5.2. *We have the asymptotic formula*

$$\Psi_{v_1,\ldots,v_k} = M_{v_1} + R_{v_1},$$

where M_{v_1} is the main term given by

$$M_{v_1} := \int w\Big(\frac{\xi}{x}\Big) \sum_{\substack{a_2,\ldots,a_k\\ d\mid a_2\cdots a_k}} \frac{v_2(a_2)\cdots v_k(a_k)}{a_2\cdots a_k} v_1\Big(\frac{\xi}{a_2\cdots a_k}\Big)\lambda_{h,d}(\xi+h)\,d\xi, \tag{5.3}$$

with $\lambda_{h,d}(\xi+h)$ defined as in (2.1), and where we have the following bounds for the error term R_{v_1},

$$R_{v_1} \ll \frac{x^{\frac{3}{2}+\varepsilon}}{A_1{}^{\frac{3}{2}}\Omega^{\frac{1}{2}}}, \tag{5.4}$$

$$R_{v_1} \ll \frac{x^{\frac{3}{2}+\varepsilon}}{A_1A_2}\bigg(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{(A_1A_2)^{2\theta}}{x^{\theta}}\bigg)\bigg(1+\frac{A_2{}^{\frac{1}{2}}}{A_1{}^{\frac{1}{2}}}\bigg)\bigg(1+|h|^{\frac{1}{4}}\frac{A_1{}^{\frac{1}{2}}A_2{}^{\frac{1}{2}}}{x^{\frac{1}{2}}}\bigg), \tag{5.5}$$

$$R_{v_1} \ll \frac{x^{1+\varepsilon}}{A_1{}^{\frac{1}{2}}\Omega^{\frac{1}{4}}} + A_1{}^{\frac{3}{8}}x^{\frac{7}{8}+\varepsilon}\bigg(\frac{1}{\Omega^{\frac{1}{8}}} + \frac{x^{\frac{\theta}{4}}}{A_1{}^{\frac{3}{4}\theta}} + \frac{|h|^{\frac{1}{16}}}{A_1{}^{\frac{3}{16}}}\frac{x^{\frac{\theta}{4}}}{A_1{}^{\frac{3}{4}\theta}}\bigg). \tag{5.6}$$

The implied constants depend only on k, the involved functions $w, v_1, \ldots, v_k$ and ε.

When A_1 is so large that it makes sense to average over a_1 alone, we get the first bound (5.4), which is proven in Section 3. The proof essentially boils down to the evaluation of the following sums over arithmetic progressions modulo $b = a_2\cdots a_k$,

$$\sum_{a_1} w\Big(\frac{a_1 b}{x}\Big) v_1(a_1) d(a_1 b + h),$$

for which we can get a non-trivial asymptotic formula as long as $b \ll x^{\frac{2}{3}-\varepsilon}$. Consequently, also the bound (5.4) is non-trivial only for $A_1 \gg x^{\frac{1}{3}+\varepsilon}$.

A further gain in the error term can be achieved here if we average over another variable a_2 as we did very similarly in Chapter 3. The main ingredient is the Kuznetsov formula that enables us to exploit the cancellation between the Kloosterman sums that arise when the Voronoi summation formula is used to evaluate the sums above. We will work this out in Section 4, and the resulting bound (5.5) is useful when $A_1A_2 \gg x^{\frac{1}{2}+\varepsilon}$.

The most difficult case occurs when none of the A_i is particularly large. It is handled in Section 5, and the path we follow there is in some sense dual to the proof of the first bound: Instead of averaging over a_1, we use the Cauchy-Schwarz inequality to merge the variables $a_2,\ldots,a_k$ to one large variable b, so that we can then evaluate the sum over this new variable asymptotically. As mentioned in the introduction, the main difficulty lies in the treatment of the sums

$$\sum_b w\left(\frac{a_1 b}{x}\right) w\left(\frac{\tilde{a}_1 b}{x}\right) d(a_1 b + h) d(\tilde{a}_1 b + h),$$

where a_1 and $\tilde{a}_1$ are of the size $a_1, \tilde{a}_1 \asymp A_1$. In Chapter 6, we will prove an asymptotic formula for these sums, which has (at best) a non-trivial error term as long as $a_1, \tilde{a}_1 \ll x^{\frac{1}{3}-\varepsilon}$, and thus the resulting bound (5.6) is also non-trivial only if $A_1 \ll x^{\frac{1}{3}-\varepsilon}$. Note that this bound is furthermore useful only if $A_1 \gg x^{\varepsilon}$.

Of course, the statement of Lemma 5.2 is symmetric in all the variables. For given $A_1, \ldots, A_k$, the optimal strategy would be to pick the A_i which is the largest, and which is always at least as large as $x^{\frac{1}{k}}$, and then apply either (5.4) or (5.6) with respect to this A_i. This is essentially the path that Bykovskiĭ and Vinogradov [**8**] wanted to take. Unfortunately, this strategy does not go through, as there is a gap at $A_i \asymp x^{\frac{1}{3}}$ where both methods fail to give a non-trivial result – in fact, in the worst case, if for example $A_1 = A_2 = A_3 \asymp x^{\frac{1}{3}}$ and $A_4 = \ldots = A_k \asymp 1$, there is no way to get a non-trivial result from these two bounds alone.

However, we still have another bound at our disposal. In case there exist two $A_{i_1}, A_{i_2} \gg x^{\frac{1}{k}}$, at least one of the estimates (5.5) or (5.6) will always be sufficiently good to get a power saving at the end. If there is only one $A_i \gg x^{\frac{1}{k}}$, we can bridge the gap at $A_i \asymp x^{\frac{1}{3}}$ by using the bound (5.6) with respect to one of the other A_i. More specifically, set

$$X_1 := \frac{x^{\frac{k}{3k-2}}}{\Omega^{\frac{k-1}{2(3k-2)}}}, \quad X_2 := x^{\frac{5}{19}} \quad \text{and} \quad X_3 := \left(\frac{x}{X_1}\right)^{\frac{1}{k-1}} = x^{\frac{2}{3k-2}} \Omega^{\frac{1}{2(3k-2)}}.$$

If one of the A_i is large enough so that $A_i \gg X_1$, we use (5.4) to get the estimate

$$R_{v_1} \ll \frac{x^{1-\frac{1}{3k-2}+\varepsilon}}{\Omega^{\frac{1}{4}+\frac{1}{4(3k-2)}}}.$$

If there are two A_{i_1}, A_{i_2} satisfying $A_{i_1}, A_{i_2} \gg X_2$, we make use of (5.5) and get

$$R_{v_1} \ll x^{\frac{37}{38}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + x^{\frac{\theta}{19}}\right) \left(1 + \left(\frac{|h|}{x^{\frac{18}{19}}}\right)^{\frac{1}{4}}\right).$$

Otherwise, there has to be at least one A_i such that $X_3 \ll A_i \ll X_2$, which means that we can use (5.6), hence getting the bound

$$R_{v_1} \ll \frac{x^{1-\frac{1}{3k-2}+\varepsilon}}{\Omega^{\frac{1}{4}+\frac{1}{4(3k-2)}}} + \frac{x^{\frac{37}{38}+\varepsilon}}{\Omega^{\frac{1}{8}}} + x^{\frac{37}{38}+\frac{\theta}{19}+\varepsilon} \left(1 + \left(\frac{|h|}{x^{\frac{15}{19}}}\right)^{\frac{1}{16}}\right).$$

All in all, this leads to the estimate (5.1).

2. The main term

We first want to describe how to split up the k-th divisor function so that we can conveniently evaluate the main term at the end. Let $u_0 : (0, \infty) \to [0, \infty)$ be a smooth and compactly supported function such that

$$\operatorname{supp} u_0 \subset \left[\frac{1}{2}, 2\right] \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}} u_0\left(\frac{\xi}{2^{\ell}}\right) = 1 \quad \text{for} \quad \xi \in (0, \infty).$$

We set

$$u_\ell(\xi) := u_0\left(\frac{\xi}{2^{\ell}}\right),$$

and

$$h_\ell(\xi) := u_\ell\Big(\frac{\xi}{X_3}\Big) \quad \text{for} \quad \ell \geq 1, \quad \text{and} \quad h_0(\xi) := \sum_{\ell \leq 0} u_\ell\Big(\frac{\xi}{X_3}\Big),$$

and define the sums

$$\Psi^{(j)} := \sum_{a_1,\ldots,a_k} w\Big(\frac{a_1\cdots a_k}{x}\Big) h_{j_1}(a_1)\cdots h_{j_k}(a_k) d(a_1\cdots a_k + h),$$

where $j = (j_1, \ldots, j_k)$ is a k-tuple with elements in $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, so that our main sum can be split up as

$$\Psi = \sum_{\substack{j\in\mathbb{N}^k \\ j\neq(0,\ldots,0)}} \Psi^{(j)}.$$

Given a k-tuple j, there is at least one coordinate $j_i > 0$, so that we can use Lemma 5.2 with respect to the corresponding variable a_i. As it turns out, it does not matter which one we choose – but for the moment we will assume that we can take j_1 to avoid notational complications. We dyadically split all the occurring $h_0(\xi)$ in $\Psi^{(j)}$, apply Lemma 5.2, and then sum everything up again, so that

$$\Psi^{(j)} = M^{(j)} + R^{(j)},$$

where $R^{(j)}$ is bounded by (5.1), and where

$$M^{(j)} := \int w\Big(\frac{\xi}{x}\Big) \sum_d \frac{c_d(h)}{d^2} \lambda_{h,d}(\xi + h) \sum_{\substack{j\in\mathbb{N}^k \\ j\neq(0,\ldots,0)}} M_{\mathrm{a}}^{(j)}(\xi)\, d\xi,$$

with

$$M_{\mathrm{a}}^{(j)}(\xi) := d \sum_{\substack{a_2,\ldots,a_k \\ d|a_2\cdots a_k}} h_{j_1}\Big(\frac{\xi}{a_2 \ldots a_k}\Big) \frac{h_{j_2}(a_2)\cdots h_{j_k}(a_k)}{a_2\cdots a_k}.$$

We use Mellin inversion to write this sum as

$$M_{\mathrm{a}}^{(j)}(\xi) = \frac{1}{2\pi i}\int_{(\sigma)} \hat{h}_{j_1}(s) \frac{d^s}{\xi^s} Z(s,d)\, ds, \quad \sigma < 0,$$

with

$$\hat{h}_{j_1}(s) := \int_0^\infty h_{j_1}(\eta)\eta^{s-1}\, d\eta \quad \text{and} \quad Z(s,d) := d^{1-s} \sum_{\substack{a_2,\ldots,a_k \\ d|a_2\cdots a_k}} \frac{h_{j_2}(a_2)\cdots h_{j_k}(a_k)}{(a_2\cdots a_k)^{1-s}}.$$

For integers $d_2, \ldots, d_k$ such that $d_2\cdots d_k = d$, we define

$$c_2 := \frac{d}{d_2}, \quad c_3 := \frac{d}{d_2 d_3}, \quad \ldots, \quad c_{k-1} := \frac{d}{d_2\cdots d_{k-1}}, \quad c_k := 1,$$

so that we can rewrite the sum appearing in $Z(s,d)$ in the following way,

$$\sum_{\substack{a_2,\ldots,a_k \\ d|a_2\cdots a_k}} (\ldots) = \sum_{d_2\cdots d_k = d} \sum_{(a_2,d)=d_2} \sum_{(a_3,c_2)=d_3} \cdots \sum_{(a_{k-1},c_{k-2})=d_{k-1}} \sum_{d_k|a_k} (\ldots),$$

and as a consequence we can express $Z(s,d)$ as

$$Z(s,d) = \sum_{d_2\cdots d_k = d} \prod_{i=2}^k \Bigg(\sum_{(a,c_i)=1} \frac{h_{j_i}(d_i a)}{a^{1-s}} \Bigg).$$

The sums running over a can be evaluated in the usual way using Mellin inversion and the residue theorem, leading to

$$\sum_{(a,c_i)=1} \frac{h_{j_i}(d_i a)}{a^{1-s}} = \frac{\psi_0(c_i)}{{d_i}^s} H_{j_i}(s) + \mathcal{O}\left(\frac{{d_i}^{1-\varepsilon}}{(2^{j_i}X_3)^{1-\varepsilon}}\right),$$

where the functions $H_j(s)$ are defined as

$$H_0(s) := \zeta(1-s){d_i}^s \frac{\psi_{-s}(c_i)}{\psi_0(c_i)} - \frac{{X_3}^s}{s}\int_1^2 v_0'(\eta)\eta^s \, d\eta,$$

and, for $j_i \geq 1$,

$$H_{j_i}(s) := \left(2^{j_i}X_3\right)^s \int_{\frac{1}{2}}^2 v_0(\eta)\eta^{s-1}\, d\eta,$$

and where

$$\psi_\alpha(n) := \prod_{p|n}\left(1 - \frac{1}{p^{1+\alpha}}\right).$$

Because of

$$\hat{h}_{j_1}(s) = H_{j_1}(s),$$

we can write $M_{\mathrm{a}}^{(j)}(\xi)$ as

$$M_{\mathrm{a}}^{(j)}(\xi) = \frac{1}{2\pi i}\sum_{d_2\cdots d_k = d} \psi_0(c_2)\cdots\psi_0(c_k)\int_{(\sigma)}\prod_{i=1}^k\left(H_{j_i}(s) + \mathcal{O}\left(\frac{{d_i}^{1-\varepsilon}}{(2^{j_i}X_3)^{1-\varepsilon}}\right)\right)\frac{ds}{\xi^s}.$$

Note that this expression is independent of the variable chosen with respect to Lemma 5.2.

At this point, we sum together all the functions $H_{j_i}(s)$ with $j_i \geq 1$, so that

$$\sum_{\substack{j\in\mathbb{N}^k\\ j\neq(0,\ldots,0)}} M_{\mathrm{a}}^{(j)}(\xi) = \sum_{\substack{j\in\{0,1\}^k\\ j\neq(0,\ldots,0)}} M_{\mathrm{b}}^{(j)}(\xi) + \mathcal{O}\left(\frac{d^{1-\varepsilon}}{{X_3}^{1-\varepsilon}}\right),$$

where

$$M_{\mathrm{b}}^{(j)}(\xi) := \frac{1}{2\pi i}\sum_{d_2\cdots d_k = d}\psi_0(c_2)\cdots\psi_0(c_k)\int_{(\sigma)}\frac{1}{\xi^s}\left(\prod_{i=1}^k G_{j_i}(s)\right) ds,$$

with

$$G_1(s) := \frac{{X_3}^s}{s}\int_1^2 v_0'(\eta)\eta^s\, d\eta \quad \text{and} \quad G_0(s) := H_0(s).$$

Next, we move the line of integration to $\sigma = 1-\varepsilon$, and use the residue theorem to extract a main term from the pole at $s=0$. Because of

$$G_1(s) \ll \frac{{X_3}^{\mathrm{Re}(s)}}{|s|^\nu} \quad \text{for} \quad \nu \geq 0, \quad \text{and} \quad \zeta(\varepsilon + it) \ll |t|^{\frac{1}{2}+\varepsilon},$$

we get that

$$M_{\mathrm{b}}^{(j)}(\xi) = P_{k-1,h,d}(\log x, \log\xi) + R_{\mathrm{b}}^{(j)}(\xi) + \mathcal{O}\left(\frac{{X_3}^{k-1}}{x^{1-\varepsilon}}\right),$$

where $P_{k-1,h,d}$ is a polynomial of degree $k-1$, and where

$$R_{\mathrm{b}}^{(j)}(\xi) = (-1)^{k-j_1-\ldots-j_k}\frac{1}{2\pi i}\sum_{d_2\cdots d_k=d}\psi_0(c_2)\cdots\psi_0(c_k)\int_{(1-\varepsilon)}\frac{G_1(s)^k}{\xi^s}\, ds.$$

However, because of the fact that

$$\frac{{X_3}^k}{x} \le \frac{1}{4},$$

and because we can move the line of integration to the right as far as we want, we have at least

$$R_{\mathrm{b}}^{(j)}(\xi) \ll \frac{d^\varepsilon}{x}.$$

All in all, the main term of Ψ is given by

$$\sum_{\substack{j\in\mathbb{N}^k\\ j\neq(0,\ldots,0)}} M^{(j)} = \int w\Big(\frac{\xi}{x}\Big) P_{k,h}(\log x, \log\xi, \log(\xi+h))\, d\xi + \mathcal{O}\bigg(\frac{x^{1+\varepsilon}}{X_3} + x^\varepsilon {X_3}^{k-1}\bigg),$$

where $P_{k,h}$ is a polynomial of degree k. Since the error term here is smaller than in (5.1), this proves the asymptotic evaluation claimed in Lemma 5.1.

3. Proof of (5.4)

We write (5.2) as

$$\Psi_{v_1,\ldots,v_k} = \sum_{a_2,\ldots,a_k} v_2(a_2)\cdots v_k(a_k)\Phi(a_2\cdots a_k),$$

where

$$\Phi(b) := \sum_{m\equiv h\,(b)} d(m)f(m) = \sum_r d(rb+h)f(rb+h), \tag{5.7}$$

with

$$f(\xi) := w\Big(\frac{\xi-h}{x}\Big)v_1\Big(\frac{\xi-h}{b}\Big).$$

Note that

$$\operatorname{supp} f \asymp x \quad \text{and} \quad f^{(\nu)}(\xi) \ll \frac{1}{(x\Omega)^\nu} \quad \text{for} \quad \nu\ge 0,$$

and

$$\int\Big|f^{(\nu)}(\xi)\Big|\,d\xi \ll \frac{1}{(x\Omega)^{\nu-1}} \quad \text{for} \quad \nu \ge 1.$$

This divisor sum over an arithmetic progression can be treated with Lemma 2.1, and we get

$$\begin{aligned}\Phi(b) = \frac{1}{b}\int \Delta_\delta(\xi)f(\xi)\sum_{d|b}\frac{c_d(h)}{d^{1+\delta}}\,d\xi \\ - \frac{2\pi}{b}\sum_{d|b}\sum_{m=1}^\infty d(m)\frac{S(h,m;d)}{d}\int Y_0\Big(4\pi\frac{\sqrt{m\xi}}{d}\Big)f(\xi)\,d\xi \\ + \frac{4}{b}\sum_{d|b}\sum_{m=1}^\infty d(m)\frac{S(h,-m;d)}{d}\int K_0\Big(4\pi\frac{\sqrt{m\xi}}{d}\Big)f(\xi)\,d\xi,\end{aligned} \tag{5.8}$$

where $\Delta_\delta(\xi)$ is defined in (2.2). From (5.8), it follows easily using Weil's bound for Kloosterman sums and the recurrence relations for Bessel functions, that

$$\Phi(b) = \frac{1}{b}\int w\Big(\frac{\xi}{x}\Big)v_1\Big(\frac{\xi}{b}\Big)\Delta_\delta(\xi+h)\sum_{d|b}\frac{c_d(h)}{d^{1+\delta}}\,d\xi + \mathcal{O}\bigg(x^\varepsilon\frac{b^{\frac12}}{\Omega^{\frac12}}\bigg) \tag{5.9}$$

(we refer to [**4**, Section 2] for a more detailed treatment). This formula holds uniformly in b, and eventually leads to

$$\Psi_{v_1,\dots,v_k} = M_{v_1} + \mathcal{O}\left(\frac{x^{\frac{3}{2}+\varepsilon}}{A_1{}^{\frac{3}{2}}\Omega^{\frac{1}{2}}}\right),$$

with M_{v_1} given as in (5.3).

4. Proof of (5.5)

Now we write (5.2) as

$$\Psi_{v_1,\dots,v_k} = \sum_{a_3,\dots,a_k} v_3(a_3)\cdots v_k(a_k)\Phi_2(a_3\cdots a_k),$$

where

$$\Phi_2(b) := \sum_{a_1,a_2} w\left(\frac{a_1a_2b}{x}\right)v_1(a_1)v_2(a_2)d(a_1a_2b+h).$$

Let $v_0 : \mathbb{R} \to [0,\infty)$ be a smooth and compactly supported function such that

$$\operatorname{supp} v_0 \asymp \frac{x}{A_1A_2} \quad \text{and} \quad v_0^{(\nu)}(\xi) \ll \left(\frac{x}{A_1A_2}\right)^{-\nu} \quad \text{for} \quad \nu \geq 0,$$

and

$$v_0(b) = 1 \quad \text{for} \quad b \in \left[\frac{x}{2A_1A_2}, \frac{x}{A_1A_2}\right].$$

We insert this function into $\Phi_2(b)$, and write it as

$$\Phi_2(b) = \sum_{a_1,a_2} w\left(\frac{ba_2a_1}{x}\right)v(b,a_2,a_1)d(ba_2a_1+h),$$

with

$$v(b,a_2,a_1) := v_0(b)v_2(a_2)v_1(a_1).$$

This sum is just a special case of (3.6) with

$$A := A_3\cdots A_k, \quad B := A_2 \quad \text{and} \quad C := A_1,$$

so that, by Lemma 3.2, we have

$$\Phi_2(b) = M_2(b) + R_2(b),$$

where the main term has the form

$$M_2(b) := \frac{1}{b}\sum_{a_2}\frac{1}{a_2}\int \Delta_\delta(\xi+h)w\left(\frac{\xi}{x}\right)v_1\left(\frac{\xi}{a_2b}\right)v_2(a_2)\sum_{d|a_2b}\frac{c_d(h)}{d^{1+\delta}}\,d\xi,$$

and where $R_2(b)$ is bounded by

$$R_2(b) \ll (b,h)^{\frac{1}{4}}x^{\frac{1}{2}+\varepsilon}\left(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{(A_1A_2)^{2\theta}}{x^\theta}\right)\left(1+\frac{A_2{}^{\frac{1}{2}}}{A_1{}^{\frac{1}{2}}}\right)\left(1+|h|^{\frac{1}{4}}\frac{(A_1A_2)^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right).$$

This immediately leads to (5.5).

5. Proof of (5.6)

We write

$$\Psi_{v_1,\dots,v_k} = \sum_b \delta(b)\Phi(b),$$

where $\Phi(b)$ is defined as in (5.7), and where

$$\delta(b) := \sum_{\substack{a_2,\dots,a_k \\ a_2\cdots a_k = b}} v_2(a_2)\cdots v_k(a_k).$$

Furthermore, set

$$B := A_2 \cdots A_k \asymp \frac{x}{A_1}.$$

If B is too large, it does not make sense to evaluate the divisor sum over arithmetic progressions as in the sections before. Instead, we just insert the main term from (5.8), namely

$$\Phi_0(b) := \frac{1}{b}\int \Delta_\delta(\xi+h) w\left(\frac{\xi}{x}\right) v_1\left(\frac{\xi}{b}\right) \sum_{d|b} \frac{c_d(h)}{d^{1+\delta}}\, d\xi,$$

manually in our sum,

$$\Psi_{v_1,\dots,v_k} = \sum_b \delta(b)\Phi_0(b) - \sum_b \delta(b)(\Phi_0(b) - \Phi(b)).$$

The main term of $\Psi_{v_1,\dots,v_k}$ is then given by the left-most sum. In fact,

$$\sum_b \delta(b)\Phi_0(b) = M_{v_1},$$

with M_{v_1} defined as in (5.3).

It remains to show that the remainder

$$R_{v_1} := \sum_b \delta(b)(\Phi_0(b) - \Phi(b))$$

is small, and as a first step we use Cauchy-Schwarz,

$$R_{v_1} \le \left(\sum_{b \asymp B} |\delta(b)|^2\right)^{\frac{1}{2}} \left(\sum_b |\Phi_0(b) - \Phi(b)|^2\right)^{\frac{1}{2}}.$$

While the first factor can be estimated trivially,

$$\sum_{b\asymp B} |\delta(b)|^2 \ll x^\varepsilon B \ll \frac{x^{1+\varepsilon}}{A_1},$$

the other factor needs more work. We write

$$\sum_b |\Phi_0(b) - \Phi(b)|^2 = \Sigma_1 - 2\Sigma_2 + \Sigma_3,$$

with

$$\Sigma_1 := \sum_b \Phi_0(b)^2, \quad \Sigma_2 := \sum_b \Phi_0(b)\Phi(b) \quad \text{and} \quad \Sigma_3 := \sum_b \Phi(b)^2.$$

In what follows, we will evaluate these sums and show that

$$\Sigma_1 = M_0 + \mathcal{O}\big(x^\varepsilon {A_1}^2\big), \tag{5.10}$$

$$\Sigma_2 = M_0 + \mathcal{O}\left(x^{1+\varepsilon} + \frac{x^{\frac{1}{3}+\varepsilon}{A_1}^2}{\Omega^{\frac{1}{3}}}\right), \tag{5.11}$$

$$\Sigma_3 = M_0 + \mathcal{O}\left(\frac{x^{1+\varepsilon}}{\Omega^{\frac{1}{2}}} + {A_1}^{\frac{7}{4}} x^{\frac{3}{4}+\varepsilon}\left(\frac{1}{\Omega^{\frac{1}{4}}} + \frac{x^{\frac{\theta}{2}}}{{A_1}^{\frac{3}{2}\theta}} + \frac{|h|^{\frac{1}{8}}}{{A_1}^{\frac{3}{8}}}\frac{x^{\frac{\theta}{2}}}{{A_1}^{\frac{3}{2}\theta}}\right)\right), \tag{5.12}$$

where M_0 is defined below in (5.15). Hence

$$R_{v_1} \ll \frac{x^{1+\varepsilon}}{{A_1}^{\frac{1}{2}}\Omega^{\frac14}} + {A_1}^{\frac38} x^{\frac78+\varepsilon}\left(\frac{1}{\Omega^{\frac18}} + \frac{x^{\frac{\theta}{4}}}{{A_1}^{\frac34\theta}} + \frac{|h|^{\frac{1}{16}}}{{A_1}^{\frac{3}{16}}}\frac{x^{\frac{\theta}{4}}}{{A_1}^{\frac34\theta}}\right),$$

thus proving (5.6).

5.1. Evaluation of Σ_1. We have

$$\Sigma_1 = \iint \Delta_{\delta_1}(\xi_1+h)\Delta_{\delta_2}(\xi_2+h) w\left(\frac{\xi_1}{x}\right) w\left(\frac{\xi_2}{x}\right)\Sigma_{1\mathrm{a}}(\xi_1,\xi_2)\, d\xi_1 d\xi_2,$$

where

$$\Sigma_{1\mathrm{a}}(\xi_1,\xi_2) := \sum_b \frac{f_1(b)}{b^2}\sum_{d_1,d_2|b}\frac{c_{d_1}(h)c_{d_2}(h)}{{d_1}^{1+\delta_1}{d_2}^{1+\delta_2}},$$

with

$$f_1(\eta) := v_1\left(\frac{\xi_1}{\eta}\right)v_1\left(\frac{\xi_2}{\eta}\right).$$

We use Mellin inversion to evaluate the sum over b, so that we can write

$$\Sigma_{1\mathrm{a}}(\xi_1,\xi_2) = \frac{1}{2\pi i}\int_{(\sigma)} \hat f_1(s) Z_1(s)\, ds, \quad \sigma > -1, \tag{5.13}$$

where

$$\hat f_1(s) := \int_0^\infty f_1(\eta)\eta^{s-1}\, d\eta \quad \text{and} \quad Z_1(s) := \sum_b \frac{1}{b^{2+s}}\sum_{d_1,d_2|b}\frac{c_{d_1}(h)c_{d_2}(h)}{{d_1}^{1+\delta_1}{d_2}^{1+\delta_2}}.$$

The Dirichlet series $Z_1(s)$ converges absolutely for $\mathrm{Re}(s) > -1$, but it is not hard to find an analytic continuation up to $\mathrm{Re}(s) > -2$, namely

$$Z_1(s) = \zeta(2+s)\sum_{d_1,d_2}\frac{c_{d_1}(h)c_{d_2}(h)(d_1,d_2)^{2+s}}{{d_1}^{3+\delta_1+s}{d_2}^{3+\delta_2+s}}.$$

We move the line of integration in (5.13) to $\sigma = -2+\varepsilon$, and use the residue theorem to extract a main term. The function $Z_1(s)$ has a pole at $s = -1$ with residue

$$\operatorname*{Res}_{s=-1} Z_1(s) = \sum_{d_1,d_2}\frac{c_{d_1}(h)c_{d_2}(h)(d_1,d_2)}{{d_1}^{2+\delta_1}{d_2}^{2+\delta_2}} =: C_{\delta_1,\delta_2}(h).$$

Furthermore, we have the bound

$$Z_1(-2+\varepsilon+it) \ll x^\varepsilon |t|^{\frac12+\varepsilon},$$

which also holds for the derivatives with respect to δ_1 and δ_2, and the following estimate for $\hat f_1$,

$$\hat f_1(s) \ll \frac{B^{\mathrm{Re}(s)}}{1+|s|^2}.$$

It follows that

$$\Sigma_{1\mathrm{a}}(\xi_1,\xi_2) = C_{\delta_1,\delta_2}(h)\int \frac{f_1(\eta)}{\eta^2}\, d\eta + \mathcal{O}\left(\frac{x^\varepsilon}{B^2}\right).$$

One can check that $C_{\delta_1,\delta_2}(h)$ can be written as

$$C_{\delta_1,\delta_2}(h) = C_{\delta_1,\delta_2}\gamma_{\delta_1,\delta_2}(h),$$

where we have set

$$C_{\delta_1,\delta_2} := C_{\delta_1,\delta_2}(1), \tag{5.14}$$

and where $\gamma_{\delta_1,\delta_2}(h)$ is a multiplicative function defined on prime powers by

$$\gamma_{\delta_1,\delta_2}\big(p^\ell\big) := \sum_{i=0}^{\ell} \frac{1}{p^{i+i\delta_1+i\delta_2}} + \sum_{i=0}^{\ell-1}\sum_{j=0}^{i}\bigg(\frac{1}{p^{(i+1)+j\delta_1+(i+1)\delta_2}} + \frac{1}{p^{(i+1)+(i+1)\delta_1+j\delta_2}}\bigg)$$
$$- \frac{p-1}{p^{3+\delta_1+\delta_2}-p^{1+\delta_1}-p^{1+\delta_2}+1}\sum_{i=0}^{\ell-1}\sum_{j=0}^{i}\bigg(\frac{1}{p^{i+j\delta_1+i\delta_2}} + \frac{1}{p^{i+i\delta_1+j\delta_2}}\bigg).$$

Hence

$$\Sigma_1 = M_0 + \mathcal{O}\big(x^\varepsilon {A_1}^2\big),$$

where

$$M_0 := \iiint \Delta_{\delta_1}(\xi_1+h)\Delta_{\delta_2}(\xi_2+h)C_{\delta_1,\delta_2}\gamma_{\delta_1,\delta_2}(h)F(\xi_1,\xi_2,\eta)\,d\xi_1 d\xi_2 d\eta \tag{5.15}$$

with

$$F(\xi_1,\xi_2,\eta) := \frac{1}{\eta^2}v_1\bigg(\frac{\xi_1}{\eta}\bigg)v_1\bigg(\frac{\xi_2}{\eta}\bigg)w\bigg(\frac{\xi_1}{x}\bigg)w\bigg(\frac{\xi_2}{x}\bigg). \tag{5.16}$$

5.2. Evaluation of Σ_2. We have

$$\Sigma_2 = \int \Delta_{\delta_1}(\xi_1+h)w\bigg(\frac{\xi_1}{x}\bigg)\sum_{d_1\le D_1}\frac{c_{d_1}(h)}{{d_1}^{2+\delta_1}}\Sigma_{2\mathrm{a}}(\xi_1;d_1)\,d\xi_1 + \mathcal{O}\bigg(\frac{x^{1+\varepsilon}A_1}{D_1}\bigg),$$

where we have cut the sum over d_1 at D_1, and where we have set

$$\Sigma_{2\mathrm{a}}(\xi_1;d_1) := d_1\sum_r r \sum_{m\equiv h\,(rd_1)} f_2(m-h,r)d(m),$$

with

$$f_2(\xi_2,\eta) := \frac{v_1(\eta)}{\xi_2}w\bigg(\frac{\xi_2}{x}\bigg)v_1\bigg(\frac{\eta\xi_1}{\xi_2}\bigg).$$

We can again use Lemma 2.1 to treat the inner sum, and we get, similarly to (5.9),

$$\Sigma_{2\mathrm{a}}(\xi_1;d_1) = \int \Delta_{\delta_2}(\xi_2+h)\Sigma_{2\mathrm{b}}(\xi_2;d_1)\,d\xi_2 + \mathcal{O}\Bigg(x^\varepsilon\frac{{d_1}^{\frac{3}{2}}{A_1}^{\frac{5}{2}}}{x\Omega^{\frac{1}{2}}}\Bigg),$$

with

$$\Sigma_{2\mathrm{b}}(\xi_2;d_1) := \sum_r f_2(\xi_2,r)\sum_{d_2|rd_1}\frac{c_{d_2}(h)}{{d_2}^{1+\delta_2}}.$$

We can now evaluate the sum over r using Mellin inversion in the same way as in the section before. We have

$$\Sigma_{2\mathrm{b}}(\xi_2;d_1) = \frac{1}{2\pi i}\int_{(\sigma)}\hat{f}_2(\xi_2,s)Z_2(s)\,ds, \quad \sigma>1,$$

where

$$\hat{f}_2(\xi_2,s) := \int f_2(\xi_2,\eta)\eta^{s-1}\,d\eta,$$

and

$$Z_2(s) := \sum_r \frac{1}{r^s}\sum_{d_2|rd_1}\frac{c_{d_2}(h)}{{d_2}^{1+\delta_2}} = \zeta(s)\sum_{d_2}\frac{c_{d_2}(h)(d_1,d_2)^s}{{d_2}^{1+s+\delta_2}}.$$

After moving the line of integration to $\sigma=\varepsilon$ and using the residue theorem, we get

$$\Sigma_{2\mathrm{b}}(\xi_1,\xi_2;d_1) = \sum_{d_2}\frac{c_{d_2}(h)(d_1,d_2)}{{d_2}^{2+\delta_2}}\int f_2(\xi_2,\eta)\,d\eta + \mathcal{O}\bigg(\frac{x^\varepsilon}{x}\bigg),$$

which then leads to

$$\Sigma_{2a}(\xi_1; d_1) = \iint \Delta_{\delta_2}(\xi_2 + h) f_2(\xi_2, \eta) \sum_{d_2} \frac{c_{d_2}(h)(d_1, d_2)}{{d_2}^{2+\delta_2}} \, d\eta d\xi_2 + \mathcal{O}\left(x^{\varepsilon} + x^{\varepsilon} \frac{{d_1}^{\frac{3}{2}} {A_1}^{\frac{5}{2}}}{x \Omega^{\frac{1}{2}}} \right).$$

We complete the sum over d_1 again, and eventually get

$$\Sigma_2 = M_0 + \mathcal{O}\left(x^{1+\varepsilon} + \frac{x^{1+\varepsilon} A_1}{D_1} + x^{\varepsilon} \frac{{D_1}^{\frac{1}{2}} {A_1}^{\frac{5}{2}}}{\Omega^{\frac{1}{2}}} \right).$$

The optimal value for D_1 is

$$D_1 = \frac{x^{\frac{2}{3}} \Omega^{\frac{1}{3}}}{A_1},$$

which gives (5.11).

5.3. Evaluation of Σ_3. We have

$$\Sigma_3 = \sum_{r_1, r_2} v_1(r_1) v_1(r_2) \Sigma_{3a}(r_1, r_2),$$

with

$$\Sigma_{3a}(r_1, r_2) := \sum_b w\left(\frac{r_1 b}{x}\right) w\left(\frac{r_2 b}{x}\right) d(r_1 b + h) d(r_2 b + h).$$

For $r_1 \neq r_2$, this sum is a special case of $D(x_1, x_2, r_1, r_2)$, which we will study in detail in Chapter 6. As stated in Lemma 6.1, we can write $\Sigma_{3a}(r_1, r_2)$ asymptotically as

$$\Sigma_{3a}(r_1, r_2) = M_{3a}(r_1, r_2) + R_{3a}(r_1, r_2), \tag{5.17}$$

with a main term $M_{3a}(r_1, r_2)$ and an error term $R_{3a}(r_1, r_2)$. More precisely, the main term has the form

$$M_{3a}(r_1, r_2) := \int w\Big(\frac{r_1 \eta}{x}\Big) w\Big(\frac{r_2 \eta}{x}\Big) \Delta_{\delta_1}(r_1 \eta + h) \Delta_{\delta_2}(r_2 \eta + h) C_{3a}(r_1, r_2, h) \, d\eta,$$

where

$$C_{3a}(r_1, r_2, h) := \sum_{\substack{u_1^* | (r_1, h) \\ u_2^* | (r_2, h)}} \frac{u_1^{*\delta_1} u_2^{*\delta_2}}{(r_1, h)^{\delta_1} (r_2, h)^{\delta_2}} \psi_{\delta_1}\left(\frac{r_1 u_1^*}{(r_1, h)}\right) \psi_{\delta_2}\left(\frac{r_2 u_2^*}{(r_2, h)}\right) \cdot \sum_{\left(d, \frac{r_1 u_1^* r_2 u_2^*}{(r_1, h)(r_2, h)}\right) = 1} \frac{1}{d^{2+\delta_1+\delta_2}} c_d\left(\frac{r_1 h - r_2 h}{(r_1, h)(r_2, h)}\right),$$

with the arithmetic function $\psi_\alpha(n)$ defined as

$$\psi_\alpha(n) := \prod_{p | n} \left(1 - \frac{1}{p^{1+\alpha}}\right).$$

Concerning the error term, we know from (6.3),

$$R_{3a}(r_1, r_2) \ll (r_1, r_2)^{\star} {A_1}^{\frac{1}{2}} x^{\frac{1}{2}+\varepsilon} \cdot \left(\frac{1}{\Omega^{\frac{1}{2}}} + \left(r_1 r_2, h \frac{r_2 - r_1}{(r_1, r_2)} \right)^{\frac{1}{4}+\theta} \frac{x^{\theta}}{{A_1}^{3\theta}} \left(1 + \frac{|h|^{\frac{1}{4}}}{{A_1}^{\frac{3}{4}}} \right) \right), \tag{5.18}$$

where

$$(r_1, r_2)^{\star} := \min\{(r_1, {r_2}^{\infty}), (r_2, {r_1}^{\infty})\}.$$

Of course, there is also the trivial bound

$$R_{3\mathrm{a}}(r_1,r_2) \ll x^{\varepsilon} B. \tag{5.19}$$

The contribution of the diagonal elements $r_1 = r_2$ is negligible, so that we can bound the respective sums trivially. Otherwise, we use the asymptotic formula (5.17), so that we can write Σ_3 asymptotically as

$$\Sigma_3 = M_3 + R_3,$$

where we have a main term of the form

$$M_3 := \sum_{r_1 \neq r_2} v_1(r_1)v_1(r_2)M_{3\mathrm{a}}(r_1,r_2),$$

and where R_3 is bounded by

$$|R_3| \leq \sum_{r \asymp A_1} |\Sigma_{3\mathrm{a}}(r,r)| + \sum_{\substack{r_1,r_2 \asymp A_1,\ r_1 \neq r_2 \\ (r_1,r_2)^\star > R_0}} |R_{3\mathrm{a}}(r_1,r_2)| + \sum_{\substack{r_1,r_2 \asymp A_1,\ r_1 \neq r_2 \\ (r_1,r_2)^\star \leq R_0}} |R_{3\mathrm{a}}(r_1,r_2)|,$$

with $R_0 \ll A_1$ some constant to be determined at the end. For the first sum, we get

$$\sum_{r \asymp A_1} |\Sigma_{3\mathrm{a}}(r,r)| \ll x^{1+\varepsilon}.$$

For the second sum, we use the trivial bound (5.19),

$$\sum_{\substack{r_1,r_2 \asymp A_1,\ r_1 \neq r_2 \\ (r_1,r_2)^\star > R_0}} |R_{3\mathrm{a}}(r_1,r_2)| \ll x^{\varepsilon} B \sum_{\substack{r_1,r_2 \asymp A_1 \\ (r_1,r_2) > R_0}} 1 \ll x^{\varepsilon} B \sum_{r_0 > R_0} \frac{{A_1}^2}{{r_0}^2} \ll x^{1+\varepsilon} \frac{A_1}{R_0}.$$

Finally, for the third sum, we use (5.18). Note hereby, that

$$\sum_{\substack{r_1,r_2 \asymp A_1 \\ (r_1,r_2)^\star \leq R_0}} (r_1,r_2)^\star \ll \sum_{r_0 \leq R_0} r_0 \sum_{\substack{r_1,r_2 \asymp A_1 \\ (r_1,{r_2}^\infty) = r_0}} 1 \leq \sum_{r_0 \leq R_0} r_0 \sum_{\substack{r_1,r_2 \asymp A_1 \\ r_0 \mid r_1}} 1 \ll R_0 {A_1}^2,$$

and, moreover, that

$$\begin{aligned}
\sum_{\substack{r_1,r_2 \asymp A_1 \\ r_1 \neq r_2 \\ (r_1,r_2)^\star \leq R_0}} (r_1,r_2)^\star \left(r_1 r_2, h\frac{r_2 - r_1}{(r_1,r_2)} \right)^{\frac14 + \theta} &\ll \sum_{r_0 \leq R_0} r_0 \sum_{\substack{r_0 r_1 \asymp A_1 \\ r_2 \asymp A_1 \\ r_0 r_1 \neq r_2}} \left(r_0 r_1 r_2, h\frac{r_2 - r_0 r_1}{(r_0 r_1, r_2)} \right)^{\frac12} \\
&\leq \sum_{r_0 \leq R_0} r_0 \sum_{t \ll A_1} \sum_{\substack{\frac{r_1 r_0}{(r_0,t)} \asymp \frac{A_1}{t} \asymp r_2 \\ \frac{r_0 r_1}{(r_0,t)} \neq r_2 \\ \left(\frac{r_0 r_1}{(r_0,t)}, r_2\right) = 1}} \left(t^2, \left(r_2 - \frac{r_0 r_1}{(r_0,t)} \right) h \right)^{\frac12} \left(\frac{r_0 r_1 r_2}{(r_0,t)}, h \right)^{\frac12} \\
&\leq \sum_{s \leq R_0} \sum_{\substack{r_0 \leq \frac{R_0}{s} \\ t \ll \frac{A_1}{s} \\ (r_0,t)=1}} r_0 s \sum_{\substack{r_1 r_0 \asymp \frac{A_1}{st} \asymp r_2 \\ r_0 r_1 \neq r_2}} (st, (r_2 - r_0 r_1)h)(r_0 r_1 r_2, h)^{\frac12} \\
&\ll \dots,
\end{aligned}$$

and after dividing the ranges of the variables s and t dyadically into ranges $s \asymp S$ and $t \asymp T$,

$$\begin{aligned}
\ldots &\ll \sum_{S,T} S \sum_{r_0 \ll \frac{R_0}{S}} r_0 \sum_{\substack{r_1 r_0 \asymp \frac{A_1}{ST} \asymp r_2 \\ r_0 r_1 \neq r_2}} (r_0 r_1 r_2, h)^{\frac{1}{2}} \sum_{\substack{s \asymp S \\ t \asymp T}} (st, (r_2 - r_0 r_1) h) \\
&\ll |h|^{\varepsilon} {A_1}^{\varepsilon} \sum_{S,T} S^2 T \sum_{r_0 \ll \frac{R_0}{S}} r_0 (r_0, h)^{\frac{1}{2}} \sum_{r_1 \asymp \frac{A_1}{r_0 ST}} (r_1, h)^{\frac{1}{2}} \sum_{r_2 \asymp \frac{A_1}{ST}} (r_2, h)^{\frac{1}{2}} \\
&\ll |h|^{\varepsilon} R_0 {A_1}^{2+\varepsilon}.
\end{aligned}$$

Hence

$$\sum_{\substack{r_1, r_2 \asymp A_1,\ r_1 \neq r_2 \\ (r_1, r_2)^{\star} \leq R_0}} |R_{3\mathrm{a}}(r_1, r_2)| \ll R_0 {A_1}^{\frac{5}{2}} x^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{x^{\theta}}{{A_1}^{3\theta}} \left(1 + \frac{|h|^{\frac{1}{4}}}{{A_1}^{\frac{3}{4}}} \right) \right).$$

We set

$$R_0 = \min \left\{ A_1, \frac{x^{\frac{1}{4}}}{{A_1}^{\frac{3}{4}}} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \frac{x^{\theta}}{{A_1}^{3\theta}} \left(1 + \frac{|h|^{\frac{1}{4}}}{{A_1}^{\frac{3}{4}}} \right) \right)^{-\frac{1}{2}} \right\},$$

which leads to

$$\Sigma_3 = M_3 + \mathcal{O}\left(x^{1+\varepsilon} + {A_1}^{\frac{7}{4}} x^{\frac{3}{4}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{4}}} + \frac{x^{\frac{\theta}{2}}}{{A_1}^{\frac{3}{2}\theta}} + \frac{|h|^{\frac{1}{8}}}{{A_1}^{\frac{3}{8}}} \frac{x^{\frac{\theta}{2}}}{{A_1}^{\frac{3}{2}\theta}} \right) \right).$$

It remains to evaluate the main term M_3.

5.4. The main term of Σ_3. The main term of Σ_3 is given by

$$M_3 = \int \Delta_{\delta_1}(1) \Delta_{\delta_2}(1) \frac{1}{h^{\delta_1+\delta_2}} \sum_{\substack{u_1, u_2 | h \\ u_1^* | \frac{h}{u_1},\ u_2^* | \frac{h}{u_2}}} {u_1}^{\delta_1} {u_2}^{\delta_2} {u_1^*}^{\delta_1} {u_2^*}^{\delta_2} \cdot \sum_{\substack{d \leq D \\ (d, u_1^* u_2^*) = 1}} \frac{M_{3\mathrm{b}}(d)}{d^{2+\delta_1+\delta_2}} \, d\eta + \mathcal{O}\left(\frac{A_1 x^{1+\varepsilon}}{D} \right),$$

where we have cut the sum over d at $D \ll A_1$, and where

$$M_{3\mathrm{b}}(d) := \sum_{\substack{r_1, r_2 \\ (r_1, u_1 d) = 1 \\ (r_2, u_2 d) = 1}} f_{3,1}\left(\frac{h r_1}{u_1} \right) f_{3,2}\left(\frac{h r_2}{u_2} \right) \psi_{\delta_1}(r_1 u_1^*) \psi_{\delta_2}(r_2 u_2^*) c_d(r_1 u_2 - r_2 u_1),$$

with

$$f_{3,i}(\xi) := (\xi \eta + h)^{\frac{\delta_i}{2}} v_1(\xi) w\left(\frac{\xi \eta}{x} \right).$$

We open the Ramanujan sum, so that

$$M_{3\mathrm{b}}(d) = \sum_{\substack{y\ (d) \\ (y,d)=1}} \sum_{(r_1, u_1 d) = 1} f_{3,1}\left(\frac{h r_1}{u_1} \right) \psi_{\delta_1}(r_1 u_1^*) e\left(\frac{y r_1 h_2}{d_2} \right) \cdot \sum_{(r_2, u_2 d) = 1} f_{3,2}\left(\frac{h r_2}{u_2} \right) \psi_{\delta_1}(r_2 u_2^*) e\left(-\frac{y r_2 h_1}{d_1} \right),$$

where we have set

$$d_1 := \frac{d}{(d, u_1)}, \quad h_1 := \frac{u_1}{(d, u_1)} \quad \text{and} \quad d_2 := \frac{d}{(d, u_2)}, \quad h_2 := \frac{u_2}{(d, u_2)}.$$

In order to evaluate these sums, we encode the additive twists by means of Dirichlet characters,

$$e\left(\frac{yr_1h_2}{d_2}\right) = \frac{1}{\varphi(d_2)} \sum_{\chi_2 \bmod d_2} \overline{\chi_2}(yr_1h_2)\tau(\chi_2),$$

so that we can write $M_{3\mathrm{b}}(d)$ as

$$M_{3\mathrm{b}}(d) = \frac{\varphi(d)}{\varphi(d_1)\varphi(d_2)} \sum_{\substack{\chi_1 \bmod d_1 \\ \chi_2 \bmod d_2 \\ \chi_2 \equiv \overline{\chi_1} \bmod d}} \overline{\chi_1}(-h_1)\overline{\chi_2}(h_2)\tau(\chi_1)\tau(\chi_2)W_{2,1}W_{1,2},$$

where

$$W_{i,j} := \sum_{(r,u_id)=1} f_{3,i}\left(\frac{hr}{u_i}\right)\psi_{\delta_i}(ru_i^*)\overline{\chi_j}(r).$$

Now we use Mellin inversion to write these sums as follows,

$$W_{i,j} = \frac{1}{2\pi i}\int_{(\sigma)} \frac{{u_i}^s}{h^s}\hat{f}_{3,i}(s)Z_3(s)\,ds, \quad \sigma > 1,$$

where

$$\hat{f}_{3,i}(s) := \int f_{3,i}(\xi)\xi^{s-1}\,d\xi \quad \text{and} \quad Z_3(s) := \sum_{\left(r,\frac{hd}{u_i}\right)=1} \frac{\psi_{\delta_i}(ru_i^*)\overline{\chi_j}(r)}{r^s}.$$

The Dirichlet series $Z_3(s)$ converges absolutely for $\mathrm{Re}(s) > 1$, but it is easy to check that an analytic continuation is given by

$$Z_3(s) = \psi_{\delta_i}(u_i^*)\prod_{p \mid u_id}\left(1 - \frac{\overline{\chi_j}(p)}{p^s}\right)\prod_{p \mid u_iu_i^*d}\left(1 - \frac{\overline{\chi_j}(p)}{p^{1+s+\delta_i}}\right)^{-1}\frac{L(s,\overline{\chi_j})}{L(1+s+\delta_i,\overline{\chi_j})}.$$

We move the line of integration to $\sigma = \varepsilon$, and the only pole we need to take care of lies at $s = 1$ and appears only when χ_j is the principal character, in which case

$$\operatorname*{Res}_{s=1} Z_3(s) = \frac{1}{\zeta(2+\delta_i)}\frac{\psi_{\delta_i}(u_i^*)\psi_0(u_id)}{\psi_{1+\delta_i}(u_iu_i^*d)}.$$

Furthermore we have the following bound for $\hat{f}_{3,i}(s)$,

$$\hat{f}_{3,i}(s) \ll {A_1}^{\mathrm{Re}(s)}\min\left\{1, \frac{1}{|s|}, \frac{1}{\Omega|s||s+1|}\right\},$$

which also holds for its derivative with respect to δ_i, and the following bounds for the involved L-functions,

$$\zeta(s) \ll |t|^{\frac{1-\sigma}{2}+\varepsilon} \quad \text{and} \quad L(s,\overline{\chi_j}) \ll (|t|d_j)^{\frac{1-\sigma}{2}+\varepsilon}$$

(see [**29**, (3)] for the latter). This way, we get the following asymptotic formula for $W_{i,j}$ when χ_j is principal,

$$W_{i,j} = \frac{\hat{f}_{3,i}(1)}{\zeta(2+\delta_i)}\frac{u_i}{h}\frac{\psi_{\delta_i}(u_i^*)\psi_0(u_id)}{\psi_{1+\delta_i}(u_iu_i^*d)} + \mathcal{O}\left(\frac{x^\varepsilon}{\Omega^{\frac{1}{2}}}\right),$$

while otherwise we get the following upper bound,

$$W_{i,j} \ll x^\varepsilon \frac{{d_j}^{\frac{1}{2}}}{\Omega^{\frac{1}{2}}}.$$

Eventually, this leads to

$$M_3 = \iiint \Delta_{\delta_1}(\xi_1+h)\Delta_{\delta_2}(\xi_2+h)F(\xi_1,\xi_2,\eta)C_3(h)\,d\eta d\xi_1 d\xi_2\,d\eta \\ + \mathcal{O}\biggl(\frac{x^{1+\varepsilon}}{\Omega^{\frac12}} + \frac{A_1 x^{1+\varepsilon}}{D} + \frac{x^{1+\varepsilon}D}{A_1\Omega}\biggr),$$

with $F(\xi_1,\xi_2,\eta)$ as defined in (5.16), and with

$$C_3(h) := \frac{1}{\zeta(2+\delta_1)\zeta(2+\delta_2)} \sum_{\substack{h_1,h_2|h\\ u_1|h_1,\ u_2|h_2}} \frac{{h_1}^{\delta_1}{h_2}^{\delta_2}u_1u_2}{h^{2+\delta_1+\delta_2}}\psi_{\delta_1}\Bigl(\frac{h_1}{u_1}\Bigr)\psi_{\delta_1}\Bigl(\frac{h_2}{u_2}\Bigr) \\ \cdot \sum_{\left(d,\frac{h_1h_2}{u_1u_2}\right)=1} \frac{1}{d^{1+\delta_1+\delta_2}} \frac{\mu\Bigl(\frac{d}{(d,u_1)}\Bigr)\mu\Bigl(\frac{d}{(d,u_2)}\Bigr)}{\varphi\Bigl(\frac{d}{(d,u_1)}\Bigr)\varphi\Bigl(\frac{d}{(d,u_2)}\Bigr)} \frac{\psi_0(d)\psi_0(u_1d)\psi_0(u_2d)}{\psi_{1+\delta_1}(h_1d)\psi_{1+\delta_2}(h_2d)}.$$

One can easily check that $C_3(1) = C_{\delta_1,\delta_2}$, with C_{δ_1,δ_2} as defined in (5.14), and that the arithmetic function

$$\gamma_3(h) := \frac{C_3(h)}{C_3(1)}$$

is multiplicative in h. A much more tedious calculation then shows that $\gamma_3(h)$ and $\gamma_{\delta_1,\delta_2}(h)$ indeed agree on prime powers, and hence must be the same function. As a consequence, our main term has the form

$$M_3 = M_0 + \mathcal{O}\biggl(\frac{x^{1+\varepsilon}}{\Omega^{\frac12}} + \frac{A_1 x^{1+\varepsilon}}{D} + \frac{x^{1+\varepsilon}D}{A_1\Omega}\biggr).$$

Clearly, the optimal value for D is

$$D = A_1\Omega^{\frac12},$$

and we finally get (5.12).

CHAPTER 6

Proof of Theorems 1.9 and 1.10

In this chapter we will work out an asymptotic formula for $D(x_1, x_2, r_1, r_2)$, and prove Theorems 1.9 and 1.10

Let $w_1, w_2 : [1/2, 1] \to \mathbb{R}$ be smooth and compactly supported functions satisfying

$$w_i^{(\nu)}(\xi) \ll \frac{1}{\Omega^\nu} \quad \text{for} \quad \nu \geq 0, \quad \text{and} \quad \int \left| w_i^{(\nu)}(\xi) \right| d\xi \ll \frac{1}{\Omega^{\nu-1}} \quad \text{for} \quad \nu \geq 1,$$

for some $\Omega \leq 1$. We will look at the sum

$$\Psi := \sum_n w_1\left(\frac{r_1 n + f_1}{x_1}\right) w_2\left(\frac{r_2 n + f_2}{x_2}\right) d(r_1 n + f_1) d(r_2 n + f_2),$$

and, assuming that

$$f_1 \ll {x_1}^{1-\varepsilon}, \quad f_2 \ll {x_2}^{1-\varepsilon} \quad \text{and} \quad h \ll r_2 {x_1}^{1-\varepsilon} \Omega^2, \tag{6.1}$$

our aim is to prove the following asymptotic formula for Ψ.

Lemma 6.1. *The sum Ψ can be written asymptotically as*

$$\Psi = M + R,$$

where M denotes the main term, which has the form

$$M := \int w_1\left(\frac{r_1 \xi + f_1}{x_1}\right) w_2\left(\frac{r_2 \xi + f_2}{x_2}\right) P_2(\log(r_1 \xi + f_1), \log(r_2 \xi + f_2))\, d\xi$$

with a quadratic polynomial $P_2(\xi_1, \xi_2)$ depending on r_1, r_2, f_1 and f_2, and where the error term R can be bounded by

$$R \ll r_0 (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{|h|^\theta}{\Omega^{\frac{1}{2}}} + (r_2 x_1)^\theta \right), \tag{6.2}$$

$$R \ll r_0 (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \left(\frac{(r_0 r_1 r_2, h) x_1}{r_0 {r_1}^2 r_2} \right)^\theta \left(1 + \frac{(r_0 r_1 r_2, h)^{\frac{1}{4}} |h|^{\frac{1}{4}}}{{r_0}^{\frac{1}{4}} (r_1 r_2)^{\frac{1}{2}}} \right) \right), \tag{6.3}$$

$$R \ll r_0 (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \left(\frac{r_2 x_1}{|h|} \right)^\theta \left(1 + \frac{(r_0 r_1 r_2, h)^{\frac{1}{4}} |h|^{\frac{1}{4}}}{{r_0}^{\frac{1}{4}} (r_1 r_2)^{\frac{1}{2}}} \right) \right). \tag{6.4}$$

Recall that we had set

$$r_0 := \min\{(r_1, {r_2}^\infty), (r_2, {r_1}^\infty)\},$$

and that θ was defined in (2.8). Note that Lemma 6.1 gives a non-trivial result only when

$${r_0}^2 {r_1}^2 r_2 \ll x_1,$$

which we will implicitly assume from now on. Furthermore, from the first two bounds in (6.1) and the size of the supports of w_1 and w_2, it follows that

$$r_2 x_1 \asymp r_1 x_2.$$

Theorem 1.9 follows immediately from Lemma 6.1 by using the first bound (6.2) for the error term and choosing $\Omega = 1$. In order to prove Theorem 1.10 from Lemma 6.1, we choose

$$\Omega = \frac{{r_0}^{\frac{2}{3}} {r_1}^{\frac{2}{3}} {r_2}^{\frac{1}{3}}}{{x_1}^{\frac{1}{3}}},$$

and use the second bound (6.3) to show that

$$R \ll (r_0 r_1 r_2, h)^{\theta} {r_0}^{\frac{2}{3}+\theta} (r_1 r_2)^{\frac{1}{3}} \left(\frac{x_1}{r_1}\right)^{\frac{2}{3}+\varepsilon}, \tag{6.5}$$

for h satisfying

$$(r_0 r_1 r_2, h) h \ll \frac{(r_1 r_2)^{\frac{4}{3}}}{{r_0}^{\frac{1}{3}}} \left(\frac{x_1}{r_1}\right)^{\frac{2}{3}} \left(\frac{{r_0}^2 {r_1}^2 r_2}{x_1}\right)^{4\theta}.$$

Unfortunately, due to the presence of θ, the possible range for h is weakened considerably in its size, even if we take the currently best value for this constant. We can improve this slightly, however, by making use of the third bound (6.4) in Lemma 6.1 to show that the bound (6.5) also holds in the range

$$\frac{(r_1 r_2)^{\frac{4}{3}}}{{r_0}^{\frac{1}{3}}} \left(\frac{x_1}{r_1}\right)^{\frac{2}{3}} \left(\frac{{r_0}^2 {r_1}^2 r_2}{x_1}\right)^{4\theta} \ll (r_0 r_1 r_2, h) h \ll {r_0}^{\frac{1}{3}} {r_1}^{\frac{4}{3}} {r_2}^{\frac{5}{3}} {x_1}^{\frac{1}{3}-\varepsilon}.$$

Theorem 1.10 follows by setting $x_1 = r_1 x$, $x_2 = r_2 x$ and using suitable weight functions.

1. A decomposition of the divisor function

Before diving into the proof of Lemma 6.1, we want to describe first the decomposition we will use for the divisor function. Let $u_0 : \mathbb{R} \to [0, \infty)$ be a smooth and compactly supported function such that

$$u_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1, \quad \text{and} \quad u_0(\xi) = 0 \quad \text{for} \quad |\xi| \geq 2,$$

and set

$$u_1(\xi) := u_0\left(\frac{\xi}{\sqrt{x_2}}\right) \quad \text{and} \quad h(a,b) := u_1(a)(2 - u_1(b)).$$

For $ab \leq x_2$, we have

$$(u_1(a) - 1)(u_1(b) - 1) = 0,$$

so that we can write $d(n)$, for $n \leq x_2$, as

$$d(n) = \sum_{ab=n} u_1(a)(2 - u_1(b)) = \sum_{ab=n} h(a,b).$$

This construction was used already by Meurman [**33**] to treat the binary additive divisor problem (and originally goes back to Heath-Brown).

It will furthermore be helpful to dyadically divide the supports of the two involved variables a and b. In order to do so, we choose smooth and compactly supported functions $u_X : (0, \infty) \to [0, \infty)$ such that

$$\operatorname{supp} u_X \subset \left[\frac{X}{2}, 2X\right] \quad \text{and} \quad u_X^{(\nu)}(\xi) \ll \frac{1}{X^{\nu}} \quad \text{for} \quad \nu \geq 0,$$

and

$$\sum_X u_X(\xi) = 1 \quad \text{for} \quad \xi \in (0, \infty),$$

where the last sum runs over powers of 2. Then we set

$$h_{AB}(a,b) := h(a,b) u_A(a) u_B(b).$$

Back to our sum – we split the second divisor function and use the dyadic decomposition described above, so that

$$\Psi = \sum_{A,B} \Psi_{AB},$$

where

$$\begin{aligned}\Psi_{AB} &:= \sum_n w_1\left(\frac{r_1 n + f_1}{x_1}\right) w_2\left(\frac{r_2 n + f_2}{x_2}\right) d(r_1 n + f_1) \sum_{\substack{a,b \\ ab = r_2 n + f_2}} h_{AB}(a,b) \\ &= \sum_{\substack{a,b \\ ab \equiv f_2\,(r_2)}} \tilde{f}(a,b) d\left(\frac{r_1}{r_2}(ab - f_2) + f_1\right),\end{aligned}$$

and

$$\tilde{f}(a,b) := w_1\left(\frac{\frac{r_1}{r_2}(ab - f_2) + f_1}{x_1}\right) w_2\left(\frac{ab}{x_2}\right) h_{AB}(a,b).$$

Note that the variables A and B, which run over powers of 2, satisfy

$$AB \asymp x_2, \quad A \ll B \quad \text{and} \quad A \ll x_2^{\frac{1}{2}}.$$

In the following, we have to pay a lot of attention to possible common divisors between the different parameters, and it will be helpful to define, for $i = 1, 2$,

$$u_i := (r_i, f_i), \quad s_i := \frac{r_i}{u_i}, \quad g_i := \frac{f_i}{u_i}, \quad \text{and} \quad h := r_1 f_2 - f_1 r_2, \quad h_0 := \frac{h}{u_1 u_2}.$$

Now, since the product ab in the above sum must be divisible by u_2, we can write

$$\Psi_{AB} = \sum_{u_2^* | u_2} \sum_{\substack{a \\ (a, s_2 u_2^*) = 1}} \sum_{\substack{b \\ ab \equiv g_2\,(s_2)}} \tilde{f}\left(\frac{u_2}{u_2^*} a, u_2^* b\right) d\left(\frac{r_1}{s_2}(ab - g_2) + f_1\right).$$

Choose $\tilde{a}$ and $\tilde{s}_2$ such that

$$a\tilde{a} + s_2 \tilde{s}_2 = 1,$$

so that b in the above sum has the form

$$b = \tilde{a} g_2 + s_2 n \quad \text{with} \quad n \in \mathbb{Z},$$

and hence

$$\begin{aligned}\Psi_{AB} &= \sum_{\substack{u_2^* | u_2 \\ a,n \\ (a, s_2 u_2^*) = 1}} \tilde{f}\left(\frac{u_2}{u_2^*} a, \frac{u_2^*}{u_2 a}(r_2(an - g_2\tilde{s}_2) + f_2)\right) d(r_1(an - g_2 \tilde{s}_2) + f_1) \\ &= \sum_{\substack{u_2^* | u_2 \\ a \\ (a, s_2 u_2^*) = 1}} \sum_{m \equiv f_1 - g_2 r_1 \overline{s_2}\,(r_1 a)} d(m) f(m; a),\end{aligned}$$

with

$$f(\xi; a) := w_1\left(\frac{\xi}{x_1}\right) w_2\left(\frac{\frac{r_2}{r_1}(\xi - f_1) + f_2}{x_2}\right) h_{AB}\left(\frac{u_2}{u_2^*} a, \frac{u_2^*}{u_2 a}\left(\frac{r_2}{r_1}(\xi - f_1) + f_2\right)\right).$$

Note that the modular inverse $\overline{s_2}$, which occurs in the congruence condition, is understood to be mod a. Also note that the support of $f(\xi; a)$ is given by

$$\operatorname{supp} f(\bullet; a) \asymp x_1 \quad \text{and} \quad \operatorname{supp} f(\xi; \bullet) \asymp \frac{u_2^*}{u_2} A,$$

and that its derivatives can be bounded by

$$\frac{\partial^{\nu_1+\nu_2}}{\partial\xi^{\nu_1}a^{\nu_2}}f(\xi;a) \ll \frac{1}{(x_1\Omega)^{\nu_1}}\left(\frac{u_2}{u_2^*A}\right)^{\nu_2} \quad \text{for} \quad \nu_1,\nu_2 \geq 0,$$

while also satisfying

$$\int\left|\frac{\partial^{\nu_1+\nu_2}}{\partial\xi^{\nu_1}a^{\nu_2}}f(\xi;a)\right|d\xi \ll \frac{1}{(x_1\Omega)^{\nu_1-1}}\left(\frac{u_2}{u_2^*A}\right)^{\nu_2} \quad \text{for} \quad \nu_1 \geq 1,\, \nu_2 \geq 0.$$

2. Use of the Voronoi summation formula

We use Theorem 2.1 to treat the divisor sums in arithmetic progressions appearing in Ψ_{AB}. This way we are led to

$$\Psi_{AB} = M_{AB} - 2\pi\Sigma_{AB}^+ + 4\Sigma_{AB}^-,$$

where

$$M_{AB} := \frac{1}{r_1}\sum_{u_2^*|u_2}\sum_{(a,s_2u_2^*)=1}\frac{1}{a}\int\lambda_{f_1-g_2r_1\overline{s_2},r_1a}(\xi)f(\xi;a)\,d\xi,$$

with $\lambda_{f_1-g_2r_1\overline{s_2},r_1a}(\xi)$ defined as in (2.1), and where

$$\Sigma_{AB}^{\pm} := \frac{1}{r_1}\sum_{u_2^*|u_2}\sum_{(a,s_2u_2^*)=1}\sum_{c|r_1a}\frac{c}{a}\sum_{m=1}^{\infty}d(m)\frac{S(f_1-g_2r_1\overline{s_2},\pm m;c)}{c^2}$$
$$\cdot\int B^{\pm}\left(4\pi\frac{\sqrt{m\xi}}{c}\right)f(\xi;a)\,d\xi,$$

with

$$B^+(\xi) := Y_0(\xi) \quad \text{and} \quad B^-(\xi) := K_0(\xi).$$

The main term will be extracted from M_{AB}, but we will postpone this until the end, and first take care of $\Sigma_{AB}^{\pm}$.

We reshape these sums a little bit,

$$\Sigma_{AB}^{\pm} = \frac{1}{r_1}\sum_{u_2^*|u_2}\sum_{\substack{a,c\\ c|r_1a\\ (a,s_2u_2^*)=1}}(\dots) = \frac{1}{r_1}\sum_{\substack{u_2^*|u_2\\ r_1^*|r_1}}\sum_{\substack{d\\ (d,r_1)=r_1^*}}\sum_{\substack{a,c\\ dc=r_1a\\ (a,s_2u_2^*)=1}}(\dots)$$
$$= \frac{1}{r_1}\sum_{\substack{u_2^*|u_2\\ r_1^*|r_1}}\sum_{\substack{d\\ (d,r_1^*s_2u_2^*)=1}}\sum_{\substack{c\\ (c,s_2u_2^*)=1}}(\dots),$$

where we have to replace c by r_1^*c and a by dc, so that

$$\Sigma_{AB}^{\pm} = \sum_{\substack{u_2^*|u_2\\ r_1^*|r_1}}\sum_{\substack{d\\ (d,r_1^*s_2u_2^*)=1}}\frac{R_{AB}^{\pm}}{d},$$

with

$$R_{AB}^{\pm} := \sum_{\substack{c\\ (c,s_2u_2^*)=1}}\sum_{m=1}^{\infty}d(m)\frac{S(f_1-g_2r_1\overline{s_2},\pm m;r_1^*c)}{r_1^*c}F^{\pm}(r_1^*c;dc,m),$$

and

$$F^{\pm}(\eta;a,m) := \frac{r_1^*}{\eta r_1}\int B^{\pm}\left(4\pi\frac{\sqrt{m\xi}}{\eta}\right)f(\xi;a)\,d\xi.$$

As a reminder, the modular inverse $\overline{s_2}$ occuring in the Kloosterman sum is now understood to be mod dc.

Let

$$M_0^- := \frac{{x_1}^\varepsilon}{x_1} A^{*2}, \quad M_0^+ := \frac{{x_1}^\varepsilon}{x_1 \Omega^2} A^{*2} \quad \text{and} \quad A^* := \frac{u_2^*}{u_2} \frac{r_1^* A}{d}.$$

Regarding $F^\pm(r_1^* c; dc, n)$, we have the bounds

$$F^+(r_1^* d; dc, m) \ll \frac{(x_1\Omega)^{\frac12}}{m^{\frac12}} \left(\frac{A^*}{\sqrt{m x_1 \Omega}} \right)^{\nu - \frac12},$$

$$F^-(r_1^* d; dc, m) \ll \frac{{x_1}^{\frac12}}{m^{\frac12}} \left(\frac{A^*}{\sqrt{m x_1}} \right)^{\nu - \frac12},$$

which can be proven using (2.4). With the help of these bounds, it is not hard to see that the sum over m in $R_{AB}^\pm$ can be cut at $M_0^\pm$. After dyadically dividing the remaining sum, we are left with

$$R_{AB}^\pm(M) := \sum_{\substack{c \\ (c, s_2 u_2^*)=1}} \sum_{M < m \le 2M} d(m) \frac{S(f_1 - g_2 r_1 \overline{s_2}, \pm m; r_1^* c)}{r_1^* c} F^\pm(r_1^* c; dc, m).$$

3. Treatment of the Kloosterman sums

Not surprisingly, we would like to treat the sum of Kloosterman sums occuring in $R_{AB}^\pm(M)$ with the Kuznetsov formula. However, in our situation this does not seem to be possible directly. To deal with this difficulty, we factor out the part of the variable r_1^* which has the same prime factors as $s_2 u_2^*$,

$$v := (r_1^*, (s_2 u_2^*)^\infty), \quad t_1 := \frac{r_1^*}{v},$$

and use the twisted multiplicativity of Kloosterman sums,

$$\frac{S(f_1 - g_2 r_1 \overline{s_2}, \pm m; r_1^* c)}{r_1^* c} = \frac{S\big(f_1 \overline{c t_1}, \pm m \overline{c t_1}; v\big)}{v} \frac{S\big(h_0 u_1, \pm m \overline{v}^2 \overline{s_2}; c t_1\big)}{c t_1}.$$

Here, all the modular inverses are finally understood to be modulo the respective modulus of the Kloosterman sum. Obviously, the first factor still depends on c, but here we follow an idea of Blomer and Milićević [**7**] and use Dirichlet characters to separate this variable. We define

$$\hat{S}_v(\chi; m) := \sum_{\substack{y\,(v) \\ (y,v)=1}} \overline{\chi}(y) \frac{S(f_1 \overline{y}, \pm m \overline{y}; v)}{v},$$

where χ is a Dirichlet character modulo v, so that by the orthogonality relations of Dirichlet characters it follows that

$$\frac{S\big(f_1 \overline{c t_1}, \pm m \overline{c t_1}; v\big)}{v} = \frac{1}{\varphi(v)} \sum_{\chi \bmod v} \overline{\chi}(c t_1) \hat{S}_v(\overline{\chi}; m),$$

where the sum runs over all Dirichlet characters modulo v. Hence

$$R_{AB}^\pm(M) = \frac{1}{\varphi(v)} \sum_{\chi \bmod v} \overline{\chi}(t_1) R_{AB}^\pm(M; \chi),$$

with

$$R_{AB}^\pm(M; \chi) := \sum_{M < m \le 2M} d(m) \hat{S}_v(\overline{\chi}; m) K_{AB}^\pm(\chi; m),$$

and

$$K_{AB}^\pm(m; \chi) := \sum_{\substack{c \\ (c, s_2 u_2^*)=1}} \frac{S\Big(h_0 u_1 u_2^*, \pm m \overline{s_2 u_2^* v^2}; t_1 c\Big)}{t_1 c} \overline{\chi}(c) F^\pm(r_1^* c; dc, m).$$

Of course it is important to have good bounds for $\hat{S}_v(\chi;m)$. Directly using Weil's bound for Kloosterman sums, we get

$$\hat{S}_v(\chi;m) \ll (f_1,m,v)^{\frac{1}{2}} v^{\frac{1}{2}+\varepsilon},$$

but this can be improved with a little bit of effort, and the remainder of this section will be concerned with proving the following improved bound,

$$\hat{S}_v(\chi;m) \ll \left(f_1, m, \frac{v}{\mathrm{cond}(\chi)}\right) v^{\varepsilon}, \tag{6.6}$$

where $\mathrm{cond}(\chi)$ is the conducter of χ. The sum actually vanishes in a lot of cases, in particular when f_1, m and v have certain common factors, but this result will be sufficient for our purposes. At this point, we also want to mention that

$$\frac{1}{\varphi(v)} \sum_{\chi \bmod v} \frac{v}{\mathrm{cond}(\chi)} = \frac{v}{\varphi(v)} \sum_{v^*|v} \frac{1}{v^*} \sum_{\substack{\chi \bmod v\\ \mathrm{cond}(\chi)=v^*}} 1 \leq \frac{v}{\varphi(v)} d(v) \ll v^{\varepsilon}, \tag{6.7}$$

which will be useful later.

In order to prove (6.6), note first that $\hat{S}_v(\chi;m)$ is quasi-multiplicative in the sense that, if $v=v_1v_2$ with coprime v_1 and v_2, and $\chi=\chi_1\chi_2$ with the corresponding Dirichlet characters $\chi_1 \pmod{v_1}$ and $\chi_2 \pmod{v_2}$, then

$$\hat{S}_v(\chi;m) = \chi_1(v_2)\chi_2(v_1)\hat{S}_{v_1}(\chi_1;m)\hat{S}_{v_2}(\chi_2;m).$$

It is therefore enough to look at the case where v is a prime power $v=p^\ell$.

Assume first that $\chi=\chi_0$ is the principal character. For $v=p$, we have

$$\hat{S}_p(\chi;m) = \frac{1}{p} \sum_{\substack{x,y\,(p)\\ (x,p)=1}} e\left(\frac{y(f_1x \pm m\overline{x})}{p}\right) - \frac{\varphi(p)}{p} = \sum_{\substack{x\,(p)\\ f_1x\pm m\overline{x}\equiv 0\,(p)}} 1 - \frac{\varphi(p)}{p} \ll (f_1,m,p),$$

and, for prime powers $v=p^\ell$, $\ell \geq 2$, we have

$$\begin{aligned}
\hat{S}_{p^\ell}(\chi;m) &= \frac{1}{p^\ell} \sum_{\substack{x,y\,(p^\ell)\\ (x,p)=1}} e\left(\frac{y(f_1x \pm m\overline{x})}{p^\ell}\right) - \frac{1}{p^\ell} \sum_{\substack{x\,(p^\ell),y\,(p^{\ell-1})\\ (x,p)=1}} e\left(\frac{y(f_1x \pm m\overline{x})}{p^{\ell-1}}\right)\\
&= \#\big\{x\,(p^\ell)\big|f_1x \pm m\overline{x} \equiv 0\,(p^\ell)\big\} - \frac{1}{p}\#\big\{x\,(p^\ell)\big|f_1x \pm m\overline{x} \equiv 0\,(p^{\ell-1})\big\}\\
&\ll \big(f_1,m,p^\ell\big).
\end{aligned}$$

In the following, we can now assume that χ is non-principal. For $v=p$ prime, this means that χ is primitive and hence

$$\begin{aligned}
\hat{S}_p(\chi;m) &= \frac{1}{p} \sum_{\substack{x,y\,(p)\\ (xy,p)=1}} \chi(y) e\left(\frac{y(f_1x \pm m\overline{x})}{p}\right)\\
&= \frac{1}{p} \sum_{\substack{x,y\,(p)\\ f_1x\pm m\overline{x}\not\equiv 0\,(p)}} \chi(y)\overline{\chi}(f_1x \pm m\overline{x}) e\left(\frac{y}{p}\right) - \frac{1}{p} \sum_{\substack{x,y\,(p)\\ f_1x\pm m\overline{x}\equiv 0\,(p)}} \chi(y)\\
&= \frac{\tau(\chi)}{p} \sum_{\substack{x\,(p)\\ (x,p)=1}} \overline{\chi}(f_1x \pm m\overline{x})\\
&\ll 1,
\end{aligned}$$

where we have used the fact that both the Gauß sum $\tau(\chi)$ and the character sum on the right are bounded by $\mathcal{O}\big(\sqrt{p}\big)$, which is well-known for the former and follows from Weil's work for the latter (see e.g. [**24**, Theorem 11.23] or [**30**, Chapter 6, Theorem 3]).

It remains to look at the case of χ having modulus $v = p^\ell$, $\ell \geq 2$, which is slightly more complicated. Let χ be induced by the primitive character χ^* of modulus $v^* = p^{\ell^*}$, and set $v^\circ := p^{\ell-\ell^*}$. In our sum

$$\hat{S}_{p^\ell}(\chi; m) = \frac{1}{p^\ell} \sum_{\substack{x\,(p^\ell) \\ (x,p^\ell)=1}} \sum_{y\,(p^\ell)} \chi(y) e\left(\frac{y(f_1 x \pm m\overline{x})}{p^\ell}\right),$$

we parametrize y by

$$y = y_1 + v^* y_2, \quad \text{with} \quad y_1 \bmod v^* \quad \text{and} \quad y_2 \bmod v^\circ.$$

Then

$$\begin{aligned}
\hat{S}_{p^\ell}(\chi; m) &= \frac{1}{v} \sum_{\substack{x\,(v) \\ (x,v)=1}} \sum_{y_1\,(v^*)} \chi^*(y_1) e\left(\frac{y_1(f_1 x \pm m\overline{x})}{v}\right) \sum_{y_2\,(v^\circ)} e\left(\frac{y_2(f_1 x \pm m\overline{x})}{v^\circ}\right) \\
&= \frac{1}{v^*} \sum_{\substack{x\,(v) \\ (x,v)=1 \\ f_1 x \pm m\overline{x} \equiv 0\,(v^\circ)}} \sum_{y_1\,(v^*)} \chi^*(y_1) e\left(\frac{y_1(f_1 x \pm m\overline{x})}{v}\right) \\
&= \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x\,(v) \\ (x,v)=1 \\ f_1 x \pm m\overline{x} \equiv 0\,(v^\circ)}} \overline{\chi^*}\left(\frac{f_1 x \pm m\overline{x}}{v^\circ}\right).
\end{aligned}$$

We set

$$\tilde{v}^\circ := \frac{v^\circ}{(f_1, m, v^\circ)}, \quad \tilde{v} := v^* \tilde{v}^\circ, \quad \tilde{f}_1 := \frac{f_1}{(f_1, m, v^\circ)} \quad \text{and} \quad \tilde{m} := \frac{m}{(f_1, m, v^\circ)},$$

and the sum becomes

$$\hat{S}_{p^\ell}(\chi; m) = (f_1, m, v^\circ) \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x\,(\tilde{v}) \\ \tilde{f}_1 x \pm \tilde{m}\overline{x} \equiv 0\,(\tilde{v}^\circ)}} \overline{\chi^*}\left(\frac{\tilde{f}_1 x \pm \tilde{m}\overline{x}}{\tilde{v}^\circ}\right).$$

If $\tilde{v}^\circ = 1$, we have square-root cancellation for the character sum on the right (see e.g. [**46**, Theorem 2]), so that $\hat{S}_{p^\ell}(\chi; m) \ll (f_1, m, v^\circ)$.

Otherwise note that both $\tilde{f}_1$ and $\tilde{m}$ have to be coprime with p, as otherwise the sum is empty. We parametrize x by

$$x = x_1(1 + \tilde{v}^\circ x_2), \quad \text{with} \quad x_1 \bmod \tilde{v}^\circ, \quad (x_1, \tilde{v}^\circ) = 1 \quad \text{and} \quad x_2 \bmod v^*.$$

In this case we can write $\overline{x} \bmod \tilde{v}$ in the following way

$$\overline{x} \equiv \overline{x_1}\left(1 - \tilde{v}^\circ x_2 \overline{(1 + \tilde{v}^\circ x_2)}\right) \bmod \tilde{v},$$

and after putting this in our sum, we have

$$\hat{S}_{p^\ell}(\chi; m) = (f_1, m, v^\circ) \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x_1\,(\tilde{v}^\circ) \\ \tilde{f}_1 x_1 \pm \tilde{m}\overline{x_1} \equiv 0\,(\tilde{v}^\circ)}} \sum_{x_2\,(v^*)} \overline{\chi^*}(P(x_2)),$$

where $P(X)$ is the rational function

$$P(X) := \frac{\tilde{f}_1 x_1 \tilde{v}^\circ X^2 + 2\tilde{f}_1 x_1 X + \frac{\tilde{f}_1 x_1 \pm \tilde{m}\overline{x_1}}{\tilde{v}^\circ}}{\tilde{v}^\circ X + 1}.$$

If $p \geq 3$, we can use [**9**, Theorem 1.1] to get the bound

$$\sum_{x_2\ (v^*)} \overline{\chi^*}(P(x_2)) \ll 1.$$

If $p = 2$ and $\tilde{v}^\circ \geq 8$, we rewrite this sum

$$\sum_{x_2\ (v^*)} \overline{\chi^*}(P(x_2)) = \sum_{x_2\ (2v^*)} \overline{\chi^*}\Big(P\Big(\frac{x_2}{2}\Big)\Big) = 2 \sum_{x_2\ (v^*)} \overline{\chi^*}\Big(P\Big(\frac{x_2}{2}\Big)\Big),$$

so that we can again apply [**9**, Theorem 1.1] and show that this sum is $\mathcal{O}(1)$. Finally, for the remaining cases $\tilde{v}^\circ = 2$ and $\tilde{v}^\circ = 4$, we can use [**9**, Theorem 2.1] to show square-root cancellation. This concludes the proof of (6.6).

4. Auxiliary estimates

We want to use the Kuznetsov formula in the form (2.10) with

$$\tilde{q} := t_1 s_2 u_2^* v^2, \quad \tilde{r} := s_2 u_2^* v^2, \quad \tilde{s} := t_1 \quad \tilde{q}_0 := v, \quad \tilde{m} := h_0 u_1 u_2^*, \quad \tilde{n} := m.$$

However, before we can do so, some technical arrangements have to be made. Set

$$\tilde{F}^\pm(c;m) := h(m)\frac{r_1^*}{r_1}\frac{v\sqrt{s_2 u_2^*}}{4\pi}\sqrt{\frac{r_2}{m|h|}}\int cB^\pm\left(c\sqrt{\xi\frac{r_2}{|h|}}\right) f\left(\xi; 4\pi\frac{d\sqrt{m}}{r_1^* c}\sqrt{\frac{|h|}{r_2}}\right) d\xi,$$

where h is a smooth and compactly supported bump function such that

$$\operatorname{supp} h \asymp M \quad \text{and} \quad h^{(\nu)}(m) \ll \frac{1}{M^\nu} \quad \text{for} \quad \nu \geq 0,$$

and

$$h(m) = 1 \quad \text{for} \quad m \in [M, 2M].$$

We have defined $\tilde{F}^\pm(c;m)$ in such a way that

$$F^\pm(r_1^* c; dc, m) = \frac{1}{\sqrt{\tilde{r}}}\tilde{F}^\pm\left(\frac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{\sqrt{\tilde{r}}\tilde{s}c}; m\right) \quad \text{for} \quad m \in [M, 2M].$$

Note that

$$\operatorname{supp} \tilde{F}^\pm(\bullet; m) \asymp C := \frac{1}{A^*}\sqrt{\frac{M|h|}{r_2}} \quad \text{and} \quad \tilde{F}^\pm(c;m) \ll v\sqrt{s_2 u_2^*}\frac{r_1^*}{A^* r_1} {x_1}^{1+\varepsilon}.$$

We furthermore need to separate the variable m to be able to use the large sieve inequalities later, and to this end we make use of Fourier inversion,

$$\tilde{F}^\pm(c;m) = \int G_0(\lambda) G_\lambda^\pm(c) e(\lambda m)\, d\lambda, \quad G_\lambda^\pm(c) := \frac{1}{G_0(\lambda)}\int \tilde{F}^\pm(c;m) e(-\lambda m)\, dm,$$

where

$$G_0(\lambda) := v\sqrt{s_2 u_2^*}\frac{r_1^*}{A^* r_1} {x_1}^{1+\varepsilon} \min\left(M, \frac{1}{M\lambda^2}\right).$$

Eventually, our sum of Kloosterman sums looks like

$$K_{AB}^\pm(\chi;m) = \int G_0(\lambda) e(\lambda m) \sum_{\substack{c \\ (c,\tilde{r})=1}} \overline{\chi}(c) \frac{S(\tilde{m}, \pm\tilde{n}\bar{\tilde{r}}; \tilde{s}c)}{c\tilde{s}\sqrt{\tilde{r}}} \tilde{G}_\lambda^\pm\left(4\pi\frac{\sqrt{|\tilde{m}\tilde{n}|}}{c\tilde{s}\sqrt{\tilde{r}}}\right) d\lambda.$$

Next, we need to find good estimates for the Bessel transforms occuring in the Kuznetsov formula. For convenience set

$$W := \frac{1}{A^*}\sqrt{\frac{|h|M}{r_2}} \quad \text{and} \quad Z := \frac{1}{A^*}\sqrt{x_1 M}.$$

Note that due to the assumptions made in (6.1), it holds that $W \ll 1$.

LEMMA 6.2. *If $M \ll M_0^-$, we have*

$$\begin{aligned} \tilde{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) &\ll W^{-2t} && \text{for} \quad 0 \le t < \frac{1}{4}, \\ \frac{\tilde{G}_\lambda^\pm(t)}{(1+t)^\kappa}, \check{G}_\lambda^\pm(t), \dot{G}_\lambda^\pm(t) &\ll \frac{x_1{}^\varepsilon}{1+t^{\frac{5}{2}}} && \text{for} \quad t \ge 0. \end{aligned}$$

If $M_0^- \ll M \ll M_0^+$, we have, for any $\nu \ge 0$,

$$\begin{aligned} \tilde{G}_\lambda^\pm(it), \check{G}_\lambda^\pm(it) &\ll x_1{}^{-\nu} && \text{for} \quad 0 \le t < \frac{1}{4}, \\ \frac{\tilde{G}_\lambda^\pm(t)}{(1+t)^\kappa}, \check{G}_\lambda^\pm(t), \dot{G}_\lambda^\pm(t) &\ll \frac{x_1{}^\varepsilon}{Z^{\frac{5}{2}}} \left(\frac{Z}{t}\right)^\nu && \text{for} \quad t \ge 0. \end{aligned}$$

PROOF. Except for obvious modifications, these bounds can be proven the same way as Lemma 3.3. □

5. Use of the Kuznetsov formula

Here we will only look at $K_{AB}^+(\chi; m)$ and we will assume that $h > 0$, since all other cases can be treated in essentially the same way.

We use the Kuznetsov formula as explained above and get

$$R_{AB}^+(M; \chi) = \int G_0(\lambda)(\Xi_1(M) + \Xi_2(M) + \Xi_3(M))\, d\lambda,$$

where

$$\begin{aligned} \Xi_1(M) &:= \sum_{j=1}^\infty \frac{\tilde{G}_\lambda^+(t_j)}{(1+|t_j|)^\kappa} \left(\frac{(1+|t_j|)^{\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t_j)}} \rho_j(\tilde{m}, \infty) \sqrt{\tilde{m}} \right) \Sigma_j^{(1)}(M), \\ \Xi_2(M) &:= \sum_{\mathfrak{c} \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^\infty \frac{\tilde{G}_\lambda^+(r)}{(1+|t|)^\kappa} \left(\frac{(1+|t|)^{\frac{\kappa}{2}}}{\sqrt{\cosh(\pi r)}} \varphi_{\mathfrak{c},r}(\tilde{m}, \infty) \sqrt{\tilde{m}} \right) \Sigma_{\mathfrak{c},r}^{(2)}(M)\, dr, \\ \Xi_3(M) &:= \sum_{\substack{k \equiv \kappa\ (2),\ k > \kappa \\ 1 \le j \le \theta_k(q,\chi)}} \dot{G}_\lambda^+(k) \Big(\sqrt{(k-1)!}\, \psi_{j,k}(\tilde{m}, \infty) \sqrt{\tilde{m}} \Big) \Sigma_{j,k}^{(3)}(M), \end{aligned}$$

with

$$\begin{aligned} \Sigma_j^{(1)}(M) &:= \frac{(1+|t_j|)^{\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t_j)}} \sum_{M < m \le 2M} d(m) \hat{S}_v(\overline{\chi}; m) e\left(\lambda m - m \frac{\overline{\tilde{s}}}{\tilde{r}}\right) \overline{\rho_j}\left(m, \frac{1}{\tilde{s}}\right) \sqrt{m}, \\ \Sigma_{\mathfrak{c},r}^{(2)}(M) &:= \frac{(1+|t|)^{\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t)}} \sum_{M < m \le 2M} d(m) \hat{S}_v(\overline{\chi}; m) e\left(\lambda m - m \frac{\overline{\tilde{s}}}{\tilde{r}}\right) \overline{\varphi_{\mathfrak{c},r}}\left(m, \frac{1}{\tilde{s}}\right) \sqrt{m}, \\ \Sigma_{j,k}^{(3)}(M) &:= \sqrt{(k-1)!} \sum_{M < m \le 2M} d(m) \hat{S}_v(\overline{\chi}; m) e\left(\lambda m - m \frac{\overline{\tilde{s}}}{\tilde{r}}\right) \overline{\psi_{j,k}}\left(m, \frac{1}{\tilde{s}}\right) \sqrt{m}. \end{aligned}$$

Assume first that $M \ll M_0^-$. We divide $\Xi_1(M)$ into three parts:

$$\Xi_1(M) = \sum_{t_j \le 1} (\dots) + \sum_{t_j > 1} (\dots) + \sum_{t_j \text{ exc.}} (\dots) =: \Xi_{1\mathrm{a}}(M) + \Xi_{1\mathrm{b}}(M) + \Xi_{1\mathrm{c}}(M).$$

We use Cauchy-Schwarz on $\Xi_{1\mathrm{a}}(M)$, and then make use of Lemma 6.2, Theorem 2.8 and Lemma 2.10 to bound the different factors, so that we get

$$\begin{aligned}\Xi_{1\mathrm{a}}(M) &\leq \max_{t_j\leq 1}\left|\frac{\tilde{G}_\lambda^+(t_j)}{(1+t_j)^\kappa}\right|\left(\sum_{t_j\leq 1}\frac{(1+t_j)^\kappa}{\cosh(\pi t_j)}|\rho_j(\tilde{m},\infty)|^2\tilde{m}\right)^{\frac{1}{2}}\left(\sum_{t_j\leq 1}\left|\Sigma_j^{(1)}(M)\right|^2\right)^{\frac{1}{2}}\\
&\ll {x_1}^\varepsilon\tilde{m}^\theta\left(1+\tilde{q}_0^{\frac{1}{2}}\frac{M}{\tilde{q}}\right)^{\frac{1}{2}}\left(M^\varepsilon v^\varepsilon\sum_{M<m\leq 2M}(f_1,m,v^\circ)^2\right)^{\frac{1}{2}}\\
&\ll {v^\circ}^{\frac{1}{2}}{x_1}^\varepsilon h^\theta\frac{A^*}{x_1}\left({x_1}^{\frac{1}{2}}+\frac{A^*}{v^{\frac{1}{4}}(r_1^*s_2u_2^*)^{\frac{1}{2}}}\right),\end{aligned}$$

where we have set

$$v^\circ := \frac{v}{\mathrm{cond}(\overline{\chi})}.$$

We split up $\Xi_{1\mathrm{b}}(M)$ into dyadic segments

$$\Xi_{1\mathrm{b}}(M,T) := \sum_{T<t_j\leq 2T}\frac{\tilde{G}_\lambda^+(t_j)}{(1+|t_j|)^\kappa}\left(\frac{(1+|t_j|)^{\frac{\kappa}{2}}}{\sqrt{\cosh(\pi t_j)}}\rho_j(\tilde{m},\infty)\sqrt{\tilde{m}}\right)\Sigma_j^{(1)}(M),$$

and in the same way as above we can show that

$$\Xi_{1\mathrm{b}}(M,T) \ll {v^\circ}^{\frac{1}{2}}{x_1}^\varepsilon h^\theta\frac{A^*}{T^{\frac{1}{2}}x_1}\left({x_1}^{\frac{1}{2}}+\frac{A^*}{v^{\frac{1}{4}}(r_1^*s_2u_2^*)^{\frac{1}{2}}}\right),$$

which leads to the same bound for $\Xi_{1\mathrm{b}}(M)$ as for $\Xi_{1\mathrm{a}}(M)$. Finally, for $\Xi_{1\mathrm{c}}(M)$, we get

$$\Xi_{1\mathrm{c}}(M) \ll {v^\circ}^{\frac{1}{2}}{x_1}^\varepsilon(r_2x_1)^\theta\frac{A^*}{x_1}\left({x_1}^{\frac{1}{2}}+\frac{A^*}{v^{\frac{1}{4}}(r_1^*s_2u_2^*)^{\frac{1}{2}}}\right),$$

and all in all, we end up with

$$\int G_0(\lambda)\Xi_1(M)\,d\lambda \ll {v^\circ}^{\frac{1}{2}}v(r_2x_1)^{\frac{1}{2}+\theta+\varepsilon}. \tag{6.8}$$

In exactly the same manner, but using Lemma 2.9 instead of Lemma 2.10, we can also get the bounds

$$\Xi_{1\mathrm{a}}(M),\,\Xi_{1\mathrm{b}}(M) \ll {v^\circ}^{\frac{1}{2}}{x_1}^\varepsilon\frac{A^*}{x_1}\left({x_1}^{\frac{1}{2}}+\frac{A^*}{v^{\frac{1}{4}}(r_1^*s_2u_2^*)^{\frac{1}{2}}}\right)\left(1+\frac{(r_1^*r_2v,h)^{\frac{1}{4}}h^{\frac{1}{4}}}{(r_1^*r_2)^{\frac{1}{2}}v^{\frac{1}{4}}}\right),$$

and

$$\Xi_{1\mathrm{c}}(M) \ll {v^\circ}^{\frac{1}{2}}{x_1}^\varepsilon\left(\frac{r_2x_1}{h}\right)^\theta\frac{A^*}{x_1}\left({x_1}^{\frac{1}{2}}+\frac{A^*}{v^{\frac{1}{4}}(r_1^*s_2u_2^*)^{\frac{1}{2}}}\right)\left(1+\frac{(r_1^*r_2v,h)^{\frac{1}{4}}h^{\frac{1}{4}}}{(r_1^*r_2)^{\frac{1}{2}}v^{\frac{1}{4}}}\right),$$

so that

$$\int G_0(\lambda)\Xi_1(M)\,d\lambda \ll {v^\circ}^{\frac{1}{2}}v(r_2x_1)^{\frac{1}{2}+\varepsilon}\left(\frac{r_2x_1}{h}\right)^\theta\left(1+\frac{(r_1r_2v,h)^{\frac{1}{4}}h^{\frac{1}{4}}}{(r_1r_2)^{\frac{1}{2}}v^{\frac{1}{4}}}\right). \tag{6.9}$$

Furthermore, since

$$\frac{{x_1}^\varepsilon}{C} \gg \frac{{x_1}^{\frac{1}{2}}{r_2}^{\frac{1}{2}}}{h^{\frac{1}{2}}} \gg \frac{r_1^*r_2v^{\frac{1}{2}}}{(r_1^*r_2v,h)^{\frac{1}{2}}h^{\frac{1}{2}}} = \frac{\tilde{q}}{(\tilde{q},\tilde{m})^{\frac{1}{2}}\tilde{q}_0^{\frac{1}{2}}\tilde{m}^{\frac{1}{2}}},$$

we can make use of Lemma 2.11 here, which means that concerning $\Xi_{1\mathrm{c}}(M)$ we also have the bound

$$\Xi_{1\mathrm{c}}(M) \ll \left(\sum_{t_j \le 1} \frac{|\rho_j(\tilde{m},\infty)|^2 \tilde{m}}{\cosh(\pi t_j)} \left(\frac{{x_1}^{\varepsilon}}{C} \right)^{4it_j} \right)^{\frac{1}{2}} \left(\sum_{t_j \le 1} \left| \Sigma_j^{(1)}(M) \right|^2 \right)^{\frac{1}{2}}$$
$$\ll v^{\circ \frac{1}{2}} {x_1}^{\theta+\varepsilon} \frac{(r_1^* r_2 v, h)^{\theta}}{({r_1^*}^2 r_2 v)^{\theta}} \frac{A^*}{x_1} \left({x_1}^{\frac{1}{2}} + \frac{A^*}{v^{\frac{1}{4}} (r_1^* s_2 u_2^*)^{\frac{1}{2}}} \right) \left(1 + \frac{(r_1^* r_2 v, h)^{\frac{1}{4}} h^{\frac{1}{4}}}{(r_1^* r_2)^{\frac{1}{2}} v^{\frac{1}{4}}} \right),$$

and hence

$$\int G_0(\lambda) \Xi_1(M)\, d\lambda \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(x_1 \frac{(r_1 r_2 v, h)}{{r_1}^2 r_2 v} \right)^{\theta} \left(1 + \frac{(r_1 r_2 v, h)^{\frac{1}{4}} h^{\frac{1}{4}}}{(r_1 r_2)^{\frac{1}{2}} v^{\frac{1}{4}}} \right). \tag{6.10}$$

Next, assume $M_0^- \ll M \ll M_0^+$. We split $\Xi_1(M)$ into three parts as follows,

$$\Xi_1(M) = \sum_{t_j \le Z} (\ldots) + \sum_{t_j > Z} (\ldots) + \sum_{t_j \text{ exc.}} (\ldots).$$

The sum over the exceptional eigenvalues causes no problems in this case, as the respective Bessel transforms are very small. The rest can be treated in the same way as above, and we get the bounds

$$\int G_0(\lambda) \Xi_1(M)\, d\lambda \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \frac{h^{\theta}}{\Omega^{\frac{1}{2}}}, \tag{6.11}$$

$$\int G_0(\lambda) \Xi_1(M)\, d\lambda \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \frac{1}{\Omega^{\frac{1}{2}}} \left(1 + \Omega^{\frac{1}{2}} \frac{(r_1 r_2 v, h)^{\frac{1}{4}} h^{\frac{1}{4}}}{(r_1 r_2)^{\frac{1}{2}} v^{\frac{1}{4}}} \right). \tag{6.12}$$

The same reasoning applies similarly to $\Xi_2(M)$ and $\Xi_3(M)$, the main difference being that we do not have to worry about exceptional eigenvalues at all. In the end we get from (6.8) and (6.11),

$$R_{AB}^{+}(M;\chi) \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{h^{\theta}}{\Omega^{\frac{1}{2}}} + (r_2 x_1)^{\theta} \right),$$

from (6.10) and (6.12),

$$R_{AB}^{+}(M;\chi) \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \left(\frac{x_1 (r_1 r_2 v, h)}{{r_1}^2 r_2 v} \right)^{\theta} \left(1 + \frac{(r_1 r_2 v, h)^{\frac{1}{4}} h^{\frac{1}{4}}}{(r_1 r_2)^{\frac{1}{2}} v^{\frac{1}{4}}} \right) \right),$$

and from (6.9) and (6.12),

$$R_{AB}^{+}(M;\chi) \ll v^{\circ \frac{1}{2}} v (r_2 x_1)^{\frac{1}{2}+\varepsilon} \left(\frac{1}{\Omega^{\frac{1}{2}}} + \left(\frac{r_2 x_1}{h} \right)^{\theta} \left(1 + \frac{(r_1 r_2 v, h)^{\frac{1}{4}} h^{\frac{1}{4}}}{(r_1 r_2)^{\frac{1}{2}} v^{\frac{1}{4}}} \right) \right).$$

With the help of (6.7), these bounds eventually lead to (6.2), (6.3) and (6.4).

6. The main term

The only thing left to do in order to prove Lemma 6.1 is the evaluation of the main term. After summing over all A and B, it has the form

$$M_0 := \frac{1}{r_1} \sum_{u_2^* | u_2} \sum_{\substack{a \\ (a, s_2 u_2^*)=1}} \frac{1}{a} \int \lambda_{f_1 - r_1 g_2 \overline{s_1}, r_1 a}(\xi) f(\xi; a)\, d\xi$$
$$= \int w_1\left(\frac{r_1 \xi + f_1}{x_1} \right) w_2\left(\frac{r_2 \xi + f_2}{x_2} \right) \left(\sum_{u_2^* | u_2} M_{\mathrm{a}}(\xi, u_2^*) \right) d\xi, \tag{6.13}$$

with

$$M_{\mathrm{a}}(\xi,u_2^*) := \sum_{\substack{a\\(a,s_2u_2^*)=1}} \frac{\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1)}{a} h\bigg(\frac{u_2a}{u_2^*},\frac{u_2^*}{u_2a}(r_2\xi+f_2)\bigg).$$

Using Mellin inversion, we can write the last sum as

$$M_{\mathrm{a}}(\xi,u_2^*) = \frac{1}{2\pi i}\int_{(\sigma)} \hat{h}(s;\xi)Z(s;\xi)\,ds, \quad \sigma>0, \tag{6.14}$$

where $\hat{h}(s;\xi)$ is the Mellin transform

$$\hat{h}(s;\xi) := \int_0^\infty h\bigg(\frac{u_2a}{u_2^*},\frac{u_2^*}{u_2a}(r_2\xi+f_2)\bigg)a^{s-1}\,da, \quad \mathrm{Re}(s)>0,$$

and where the function $Z(s;\xi)$ is defined as the Dirichlet series

$$Z(s;\xi) := \sum_{\substack{a\\(a,s_2u_2^*)=1}} \frac{\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1)}{a^{1+s}}, \quad \mathrm{Re}(s)>0.$$

Our plan is to move the line of integration in (6.14) to $\sigma=-1+\varepsilon$, so that we can use the residue theorem to extract a main term. Using partial integration, a meromorphic continuation of $\hat{h}(s;\xi)$ can easily be found, but for $Z(s;\xi)$ the situation is not quite as obvious.

We write

$$\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1) = \Delta_\delta(r_1\xi+f_1)\sum_{d|r_1a}\frac{c_d(f_1-r_1g_2\overline{s_2})}{d^{1+\delta}},$$

with $\Delta_\delta(r_1\xi+f_1)$ as defined in (2.2). Now we separate the part of r_1 which shares common factors with $s_2u_2^*$ from the rest by setting

$$v := (r_1,(s_2u_2^*)^\infty), \quad t_1 := \frac{r_1}{v},$$

so that

$$\sum_{d|r_1a}\frac{c_d(f_1-r_1g_2\overline{s_2})}{d^{1+\delta}} = \bigg(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\bigg)\bigg(\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}}\bigg),$$

and hence

$$Z(s;\xi) = \Delta_\delta(r_1\xi+f_1)\bigg(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\bigg)\sum_{\substack{a\\(a,s_2u_2^*)=1}}\frac{1}{a^{1+s}}\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}}.$$

The two right-most sums can be transformed to

$$\sum_{\substack{a\\(a,s_2u_2^*)=1}}\frac{1}{a^{1+s}}\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}} = \sum_{\substack{d\\(d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)}{d^{2+\delta}}\tilde{Z}(s;d),$$

with

$$\tilde{Z}(s;d) := \zeta(1+s)\frac{(d,t_1)^s}{d^s}\prod_{p|s_2u_2^*}\bigg(1-\frac{1}{p^{1+s}}\bigg).$$

This is a meromorphic function, defined on the whole complex plane, which means that the desired meromorphic continuation for $Z(s;\xi)$ can be given by

$$Z(s;\xi) = \Delta_\delta(r_1\xi+f_1)\bigg(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\bigg)\Bigg(\sum_{\substack{d\\(d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)}{d^{2+\delta}}\tilde{Z}(s;d)\Bigg).$$

Hence

$$M_{\mathrm{a}}(\xi,u_2^*)=\Delta_\delta(r_1\xi+f_1)\Bigg(\sum_{d\mid v}\frac{c_d(f_1)}{d^{1+\delta}}\Bigg)\Bigg(\sum_{\substack{d\\(d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)}{d^{2+\delta}}\tilde{I}^0(\xi,d)\Bigg),$$

with

$$\tilde{I}^0(\xi,d):=\frac{1}{2\pi i}\int_{(\sigma)}\hat{h}(s;\xi)\tilde{Z}(s;d)\,ds.$$

The Mellin transform $\hat{h}(s;\xi)$ has, at $s=0$, the Taylor expansion

$$\hat{h}(s;\xi)=\frac{2}{s}+\log(r_2\xi+f_2)+2\log\frac{u_2^*}{u_2}+\mathcal{O}(s),$$

while that of $\tilde{Z}(s;d)$ is given by

$$\tilde{Z}(s;d)=\bigg(\frac{1}{s}+\gamma+\frac{\partial}{\partial\rho}\bigg)\bigg|_{\rho=0}\frac{(d,t_1)^\rho}{d^\rho}\prod_{p\mid s_2u_2^*}\bigg(1-\frac{1}{p^{1+\rho}}\bigg)+\mathcal{O}(s).$$

All in all, the residue of their product at $s=0$ is

$$\operatorname*{Res}_{s=0}\Big(\hat{h}(s;\xi)\tilde{Z}(s;d)\Big)=\Delta_\rho(r_2\xi+f_2)\bigg(\frac{u_2^*}{u_2}\bigg)^\rho\frac{(d,t_1)^\rho}{d^\rho}\prod_{p\mid s_2u_2^*}\bigg(1-\frac{1}{p^{1+\rho}}\bigg).$$

We now move the line of integration to $\sigma=-1+\varepsilon$,

$$\tilde{I}^0(\xi,d)=\Delta_\rho(r_2\xi+f_2)\bigg(\frac{u_2^*}{u_2}\bigg)^\rho\frac{(d,t_1)^\rho}{d^\rho}\prod_{p\mid s_2u_2^*}\bigg(1-\frac{1}{p^{1+\rho}}\bigg)+\mathcal{O}\bigg(\frac{d^{1-\varepsilon}}{{x_2}^{\frac12-\varepsilon}}\bigg),$$

which leads to

$$M_{\mathrm{a}}(\xi,u_2^*)=\Delta_\delta(r_1\xi+f_1)\Delta_\rho(r_2\xi+f_2)M_{\mathrm{b}}(\xi,u_2^*)+\mathcal{O}\bigg(\frac{{x_2}^\varepsilon}{{x_2}^{\frac12}}\bigg),$$

with

$$M_{\mathrm{b}}(\xi,u_2^*):=\bigg(\frac{u_2^*}{u_2}\bigg)^\rho\Bigg(\sum_{d\mid v}\frac{c_d(f_1)}{d^{1+\delta}}\Bigg)\prod_{p\mid s_2u_2^*}\bigg(1-\frac{1}{p^{1+\rho}}\bigg)\cdot\Bigg(\sum_{\substack{d\\(d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)^{1+\rho}}{d^{2+\delta+\rho}}\Bigg).$$

An elementary but quite tedious calculation shows that this product can be transformed in such a way that we can write

$$\sum_{u_2^*\mid u_2}M_{\mathrm{b}}(\xi,u_2^*)=C_{\delta,\rho}(r_1,r_2,f_1,f_2),$$

where

$$C_{\delta,\rho}(r_1,r_2,f_1,f_2):=\sum_{\substack{u_1^*\mid u_1\\u_2^*\mid u_2}}\bigg(\frac{u_1^*}{u_1}\bigg)^\delta\bigg(\frac{u_2^*}{u_2}\bigg)^\rho\psi_\delta(s_1u_1^*)\psi_\rho(s_2u_2^*)\gamma_{\delta+\rho}(s_1u_1^*s_2u_2^*),$$

with

$$\psi_\alpha(n):=\prod_{p\mid n}\bigg(1-\frac{1}{p^{1+\alpha}}\bigg)\quad\text{and}\quad\gamma_\alpha(n):=\sum_{(d,n)=1}\frac{c_d(h_0)}{d^{2+\alpha}}.$$

After a look back at (6.13), we finally see that M_0 has the form

$$M_0=M+\mathcal{O}\bigg(\frac{{x_2}^{\frac12+\varepsilon}}{r_2}\bigg),$$

with

$$M := \int w_1\left(\frac{r_1\xi + f_1}{x_1}\right) w_2\left(\frac{r_2\xi + f_2}{x_2}\right) P_2(\log(r_1\xi + f_1), \log(r_2\xi + f_2))\, d\xi,$$

where $P_2(\xi_1, \xi_2)$ is the quadratic polynomial defined by

$$P_2(\log \xi_1, \log \xi_2) := \Delta_\delta(\xi_1)\Delta_\rho(\xi_2)C_{\delta,\rho}(r_1, r_2, f_1, f_2).$$

This concludes the proof of Lemma 6.1.

Bibliography

1. F. Aryan, *On binary and quadratic divisor problems*, preprint (2015), arXiv:1502.06067 [math.NT].
2. V. Blomer, *Shifted convolution sums and subconvexity bounds for automorphic L-functions*, Int. Math. Res. Not. **2004** (2004), no. 73, 3905–3926.
3. V. Blomer, *Sums of Hecke eigenvalues over values of quadratic polynomials*, Int. Math. Res. Not. **2008** (2008).
4. V. Blomer, *The average value of divisor sums in arithmetic progressions*, Q. J. Math. **59** (2008), no. 3, 275–286.
5. V. Blomer, G. Harcos, and Ph. Michel, *Bounds for modular L-functions in the level aspect*, Ann. Sci. Éc. Norm. Supér. (4) **40** (2007), no. 5, 697–740.
6. V. Blomer, G. Harcos, and Ph. Michel, *A Burgess-like subconvex bound for twisted L-functions*, Forum Math. **19** (2007), no. 1, 61–105, Appendix 2 by Z. Mao.
7. V. Blomer and D. Milićević, *Kloosterman sums in residue classes*, J. Eur. Math. Soc. **17** (2015), no. 1, 51–69.
8. V. A. Bykovskiĭ and A. I. Vinogradov, *Neodnorodnye svertki [Inhomogeneous convolutions]*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **160** (1987), 16–30 (Russian).
9. T. Cochrane, C. Liu, and Z. Zheng, *Upper bounds on character sums with rational function entries*, Acta Math. Sin. (Engl. Ser.) **19** (2003), no. 2, 327–338.
10. J. B. Conrey and S. M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107** (2001), no. 3, 577–604.
11. J.-M. Deshouillers, *Majorations en moyenne de sommes de Kloosterman [Upper bounds in mean of Kloosterman sums]*, Séminaire de Théorie des Nombres de Bordeaux **11** (1981–1982), 1–6 (French).
12. J.-M. Deshouillers and H. Iwaniec, *An additive divisor problem*, J. Lond. Math. Soc. (2) **26** (1982), no. 1, 1–14.
13. J.-M. Deshouillers and H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. **70** (1982), no. 2, 219–288.
14. S. Drappeau, *Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method*, preprint (2015), arXiv:1504.05549v3 [math.NT].
15. W. Duke, J. B. Friedlander, and H. Iwaniec, *A quadratic divisor problem*, Invent. Math. **115** (1994), no. 2, 209–217.
16. W. Duke, J. B. Friedlander, and H. Iwaniec, *The subconvexity problem for Artin L-functions*, Invent. Math. **149** (2002), no. 3, 489–577.
17. É. Fouvry and G. Tenenbaum, *Sur la corrélation des fonctions de Piltz [On the correlation of Piltz functions]*, Rev. Mat. Iberoam. **1** (1985), no. 3, 43–54 (French).
18. J. B. Friedlander and H. Iwaniec, *Incomplete Kloosterman sums and a divisor problem*, Ann. of Math. (2) **121** (1985), no. 2, 319–350, Appendix by B. J. Birch and E. Bombieri.
19. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, seventh ed., Elsevier/Academic Press, Amsterdam, 2007.
20. D. R. Heath-Brown, *The divisor function $d_3(n)$ in arithmetic progressions*, Acta Arith. **47** (1986), no. 1, 29–56.
21. C. Hooley, *An asymptotic formula in the theory of numbers*, Proc. Lond. Math. Soc. (3) **7** (1957), 396–413.
22. A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function.*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser., vol. 237, Cambridge University Press, Cambridge, 1997, pp. 205–243.
23. H. Iwaniec, *Spectral methods of automorphic forms*, second ed., Grad. Stud. Math., vol. 53, American Mathematical Society, Providence; Revista Matemática Iberoamericana, Madrid, 2002.

24. H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc. Colloq. Publ., vol. 53, American Mathematical Society, Providence, 2004.
25. M. Jutila, *Lectures on a method in the theory of exponential sums*, Tata Inst. Fund. Res. Lectures on Math. and Phys., vol. 80, Springer-Verlag, Berlin, 1987.
26. M. Jutila, *Convolutions of Fourier coefficients of cusp forms*, Publ. Inst. Math. (Beograd) (N.S.) **65(79)** (1999), 31–51.
27. H. H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2*, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, Appendix 1 by D. Ramakrishnan, Appendix 2 by H. H. Kim and P. Sarnak.
28. A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Mem. Amer. Math. Soc. **224** (2013), no. 1055.
29. G. Kolesnik, *On the order of Dirichlet L-functions*, Pacific J. Math. **82** (1979), no. 2, 479–484.
30. W. C. W. Li, *Number theory with applications*, Series on University Mathematics, vol. 7, World Scientific Publishing, Singapore, 1996.
31. Yu. V. Linnik, *Dispersionnyĭ metod v binarnykh additivnykh zadachakh [The dispersion method in binary additive problems]*, Izdatelstvo Leningradskogo Universiteta, Leningrad, 1961 (Russian).
32. L. Matthiesen, *Correlations of the divisor function*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 4, 827–858.
33. T. Meurman, *On the binary additive divisor problem*, Number theory (Turku, 1999), de Gruyter, Berlin, 2001, pp. 223–246.
34. Ph. Michel, *The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points*, Ann. of Math. (2) **160** (2004), no. 1, 185–236.
35. Y. Motohashi, *An asymptotic series for an additive divisor problem*, Math. Z. **170** (1980), no. 1, 43–63.
36. Y. Motohashi, *The binary additive divisor problem*, Ann. Sci. Éc. Norm. Supér. (4) **27** (1994), no. 5, 529–572.
37. Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge Tracts in Math., vol. 127, Cambridge University Press, Cambridge, 1997.
38. R. Munshi, *Shifted convolution of divisor function d_3 and Ramanujan τ function*, The legacy of Srinivasa Ramanujan (Delhi, 2012), Ramanujan Math. Soc. Lect. Notes Ser., vol. 20, Ramanujan Mathematical Society, Mysore, 2013, pp. 251–260.
39. N. J. E. Pitt, *On shifted convolutions of $\zeta^3(s)$ with automorphic L-functions*, Duke Math. J. **77** (1995), no. 2, 383–406.
40. N. J. E. Pitt, *On an analogue of Titchmarsh's divisor problem for holomorphic cusp forms*, J. Amer. Math. Soc. **26** (2013), no. 3, 735–776.
41. N. V. Proskurin, *Ob obshchikh summakh Klostermana [On the general Kloosterman sums]*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. **302** (2003), 107–134 (Russian).
42. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., Clarendon Press, Oxford, 1986.
43. B. Topacogullari, *On a certain additive divisor problem*, preprint (2015), arXiv:1512.05770 [math.NT].
44. B. Topacogullari, *The shifted convolution of divisor functions*, Q. J. Math. **67** (2016), no. 2, 331–363.
45. B. Topacogullari, *The shifted convolution of generalized divisor functions*, preprint (2016), arXiv:1605.02364 [math.NT].
46. J. Yang and Z. Zheng, *Bounds for certain character sums*, Tsinghua Sci. Technol. **8** (2003), no. 5, 553–556.