On an analogue of L^2-Betti numbers for finite field coefficients and a question of Atiyah

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Introduction

In 1976 Atiyah introduced an L^2-index theorem for elliptic operators on non-compact manifolds, [**Ati76**], and described the following application. Given a closed Riemannian manifold $X = \tilde{X}/G$, where G is an infinite discrete group and $\tilde{X}$ is a covering of X with free G-action, the Euler characteristic $\chi(X)$ is equal to the index of the operator $D = d + d^*$, where d is the exterior derivative on the differential forms on X.

By Atiyahs L^2-index theorem this is the same as the index of the lifted operator $\tilde{D}$, and with this lifted operator he defined real-valued Betti numbers $b_p^{(2)}\left(\tilde{X}\right)$ of the covering $\tilde{X}$ such that

$$\sum_p (-1)^p \, b_p^{(2)}\left(\tilde{X}\right) = \chi(X).$$

A priori, from their definition using the von Neumann dimension, these Betti numbers are real and so Atiyah asked for examples where they are irrational.

Over the years this question was reformulated to the Atiyah conjecture, which states that L^2-Betti numbers are rational for certain classes of groups. Positive results include free groups and torsionfree elementary amenable groups. For an extensive survey regarding L^2-invariants and the Atiyah conjecture see [**Lüc02b**]. Recent results for irrational L^2-Betti numbers include [**Aus13**], [**PSZ15**] and [**Gra14**], and will be revisited later.

First we recall an equivalent combinatorial approach to L^2-Betti numbers.

Combinatorial L^2-Betti numbers

For a discrete group G recall $l^2(G)$, the Hilbert space of square summable functions on G. It contains the complex group ring $\mathbb{C}[G] \subset l^2(G)$ in a natural way. So if $\gamma \in l^2(G)$ has finite support, then $\gamma \in \mathbb{C}[G] \subset l^2(G)$. Furthermore the inner product on $l^2(G)$ gives also an inner product on $\mathbb{C}[G]$. We sketch the construction shown in [**Eck00**], for more details see also [**Lüc02b**].

Let Y be a connected CW-complex and G be a group acting freely on Y by permuting the cells such that $X = Y/G$ is a finite CW-complex. We call

Y a connected free cocompact G-CW-complex. Now the cellular chain groups $C_i(Y)$ with complex coefficients are finitely generated free $\mathbb{C}[G]$-modules of rank α_i, where α_i is the number of i-cells in X. Let

$$d_i\colon C_i(Y) \to C_{i-1}(Y)$$

be the $\mathbb{C}[G]$-linear boundary map, then the inner product on $\mathbb{C}[G]$ gives the adjoint

$$d_i^*\colon C_{i-1}(Y) \to C_i(Y).$$

Define the combinatorial Laplacian $\Delta_i := d_{i+1}d_{i+1}^* + d_i^* d_i\colon C_i(Y) \to C_i(Y)$. This induces an L^2-Laplacian

$$\Delta_i^{(2)}\colon C_i(Y) \otimes_{\mathbb{C}[G]} l^2(G) \to C_i(Y) \otimes_{\mathbb{C}[G]} l^2(G).$$

Considering the L^2-boundary map $\delta_i = d_i \otimes \mathrm{Id}$, we have the decomposition

$$C_i^{(2)}(Y) = \ker \delta_i \oplus \overline{\operatorname{im} \delta_i^*} = \ker \Delta_i^{(2)} \oplus \overline{\operatorname{im} \delta_{i+1}} \oplus \overline{\operatorname{im} \delta_i^*}.$$

So the i-th L^2-homology of the free cocompact G-CW-complex Y

$$H_i^{(2)}(Y) := \ker \delta_i \Big/ \overline{\operatorname{im} \delta_{i+1}}$$

is isomorphic to the harmonic L^2-chains $\ker \Delta_i^{(2)}$.

Let $V \subset l^2(G)^n$ be a $\mathbb{C}[G]$-submodule. We write $\mathcal{N}(G)$ for the group von Neumann algebra, see [**Lüc02b**]. The von Neumann dimension of V is defined as

$$\dim_{\mathcal{N}(G)}(V) := \sum_{i=1}^{n} \langle P_V e_i, e_i \rangle,$$

where P_V is the orthogonal projection onto the closure of V and (e_i) is the natural basis of $l^2(G)^n$. Via the identification $C_i(Y) \otimes_{\mathbb{C}[G]} l^2(G) \cong l^2(G)^{\alpha_i}$ we have the i-th L^2-Betti number

$$b_i^{(2)}(Y) := \dim_{\mathcal{N}(G)}\left(\ker \Delta_i^{(2)}\right).$$

Now assume we had a dimension function for $\mathbb{K}[G]$-submodules $V \subset \mathbb{K}[G]^n$ which takes the place of the von Neumann dimension for any field $\mathbb{K}$. Then we could consider the combinatorial Laplacian in the construction above with $\mathbb{K}$-coefficients as a $\mathbb{K}[G]$-linear map from $\mathbb{K}[G]^n$ to $\mathbb{K}[G]^n$ and similarly define a $\mathbb{K}[G]$-Betti number of the covering Y, in particular for a finite field $\mathbb{F}_p$. Unfortunately the decomposition used for the identification of harmonic chains and homology only works for Hilbert spaces and fails for positive characteristics. Thus we would need the dimension function to be defined for all $\mathbb{K}[G]$-modules such that we may define the $\mathbb{K}[G]$-Betti numbers as

$$b_i^{\mathbb{K}[G]}(Y) := \dim_{\mathbb{K}[G]}\left(\ker d_i \big/ \operatorname{im} d_{i+1}\right).$$

Results

Let G be a discrete amenable group and $\mathbb{K}$ be a field. Recall that by the Følner condition amenability means that we have almost-translation-invariant finite subsets of G. We can think of these as finite almost-subgroups. By averaging over these sets we establish a dimension function called Følner dimension for submodules of free $\mathbb{K}[G]$-modules.

Theorem 0.1. *Let G be an amenable group with Følner net (F_i), $\mathbb{K}$ be a field, $n \in \mathbb{N}$, for any $\mathbb{K}[G]$-submodule $M \subseteq \mathbb{K}[G]^n$ it holds that*

$$\dim_{\mathbb{K}[G]}(M) := \lim_i \frac{\dim_{\mathbb{K}}\left(\{m \in M \mid \operatorname{supp} m \subseteq F_i\}\right)}{|F_i|}, \tag{0.1}$$

is well-defined and independent of the choice of (F_i).

Remark 0.2. If the group G is finite, then any $\mathbb{K}[G]$-submodule $M \subseteq \mathbb{K}[G]^n$ is a finite dimensional $\mathbb{K}$-vector space. Thus the definition implies

$$\dim_{\mathbb{K}[G]}(M) = \frac{\dim_{\mathbb{K}}(M)}{|G|}. \tag{0.2}$$

This gives a rough idea of what we measure with this dimension function.

We also show desirable properties, so that it behaves similar to the von Neumann dimension for $\mathbb{C}[G]$-submodules $V \subset l^2(G)^n$.

Theorem 0.3. *Let $n \in \mathbb{N}$ and $M, N \subseteq \mathbb{K}[G]^n$ be $\mathbb{K}[G]$-submodule. The Følner dimension $\dim_{\mathbb{K}[G]}$ fulfills the following properties.*

(i) $\dim_{\mathbb{K}[G]}\left(\mathbb{K}[G]^n\right) = n$.
(ii) $\dim_{\mathbb{K}[G]}(M) = 0 \iff M = 0$.
(iii) $\dim_{\mathbb{K}[G]}(M) \leq \dim_{\mathbb{K}[G]}(N)$, *for* $M \subseteq N \subseteq \mathbb{K}[G]^n$.
(iv) $\dim_{\mathbb{K}[G]}(M) = \sup\left\{\dim_{\mathbb{K}[G]}\left(\bar{M}\right) \,\middle|\, \bar{M} \subseteq M \text{ finitely generated}\right\}$.
(v) $\dim_{\mathbb{K}[G]}(M + N) = \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(M \cap N)$.
(vi) $\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(\ker T) + \dim_{\mathbb{K}[G]}(\operatorname{im} T)$, $T\colon M \to N$ *$\mathbb{K}[G]$-linear.*
(vii) $\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[H]}(M_H)$, *where* $M_H \subseteq \mathbb{K}[H]^n$ *for a subgroup* $H \leq G$ *such that* $M = M_H \cdot \mathbb{K}[G]$.

Next we show that inspired by additivity for $\mathbb{K}[G]$-submodules this dimension can be extended to any finitely generated $\mathbb{K}[G]$-module by the following definition, which is shown to be well-defined, monotonous and also satisfies additivity.

Definition 0.4. Let G be a discrete, amenable group, $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. We define

$$\dim_{\mathbb{K}[G]}(M) := n - \dim_{\mathbb{K}[G]}(\ker T). \tag{0.3}$$

Furthermore inspired by property (iv) for $\mathbb{K}[G]$-submodules we can extend this all the way to general $\mathbb{K}[G]$-modules, because any such module is the union of its finitely generated submodules.

Definition 0.5. Let G be a discrete, amenable group, $\mathbb{K}$ be a field. Let M be a $\mathbb{K}[G]$-module. We define

$$\dim_{\mathbb{K}[G]}(M) := \sup\left\{\dim_{\mathbb{K}[G]}\left(\bar{M}\right) \;\middle|\; \bar{M} \subseteq M \text{ is finitely generated}\right\}. \tag{0.4}$$

Thus we may define new Betti numbers as we intended and also have that

$$b_i^{\mathbb{K}[G]}(Y) = \dim_{\mathbb{K}[G]}(\ker d_i) - \dim_{\mathbb{K}[G]}(\operatorname{im} d_{i+1}).$$

Theorem 0.6. *Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let*

$$0 \to N \to M \to P \to 0$$

be an exact sequence of $\mathbb{K}[G]$-modules, then

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(N) + \dim_{\mathbb{K}[G]}(P). \tag{0.5}$$

Further investigation of the dimension function reveals that a group with solvable word-problem produces only computable Følner dimensions. Here computable means that it can be approximated by a Turing machine upto any given error.

Theorem 0.7. *Let G be a finitely generated discrete amenable group and $\mathbb{K}$ be any field. If G has solvable word-problem, then $\dim_{\mathbb{K}[G]}(\ker T)$ and $\dim_{\mathbb{K}[G]}(\operatorname{im} T)$ are computable for any $\mathbb{K}[G]$-linear map*

$$T\colon \mathbb{K}[G]^m \to \mathbb{K}[G]^n.$$

And lastly we describe computational tools and construct finitely generated groups G and elements $T \in \mathbb{K}[G]$ such that their kernels as $\mathbb{K}[G]$-linear maps have irrational Følner dimension.

Theorem 0.8. *For every non-negative real number r there exists a finitely generated amenable group G and $T \in \mathbb{F}_2[G]^{m\times n}$ such that*

$$\dim_{\mathbb{F}_2[G]}(\ker T) = r. \tag{0.6}$$

For L^2-Betti numbers we can connect such purely algebraic results to manifolds by a standard construction shown in [**GLSŻ00**, Theorem 6 and 7] and [**Lüc02a**, Lemma 2.2]. Given a finitely generated group G and an element $T \in \mathbb{Z}[G]$ we find a closed manifold M with fundamental group G, such that the third L^2-Betti number of a normal covering $\tilde{M}$ is the von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker T)$ of the kernel of T as an operator on $l^2(G)$. As is noted in [**PSZ15**], if G is finitely presented then the universal covering of M can be chosen as the normal covering $\tilde{M}$.

In all we get well-defined $\mathbb{K}[G]$-Betti numbers for an amenable group G and a field $\mathbb{K}$. Now additivity shows that the alternating sum of $\mathbb{K}[G]$-Betti numbers coincides with the alternating sum of the number of G-cells in each rank, that is for a free cocompact G-CW-complex Y

$$\sum_i (-1)^i b_i^{\mathbb{K}[G]}(Y) = \chi\left(Y/G\right).$$

Indeed, the proposed $\mathbb{K}[G]$-homology is a homology with local coefficients and therefore satisfies homotopy invariance. Since the Følner dimension is invariant under isomorphisms and we also have homotopy invariance for $b_i^{\mathbb{K}[G]}(Y)$. Further topological properties remain to be investigated.

Outline

Amenable groups

At the beginning of Chapter 1 we recall some topological definitions, namely directed sets and nets. Furthermore we recall the Følner condition [**Føl55**] for amenable groups. Afterwards we introduce the notion of K-boundary of F, where $K, F \subset G$ are subsets of a group. This consists of the elements $g \in G$ for which the translated subset Kg intersects both F and its complement $G \setminus F$.

Thereby we arrive at an equivalent definition of amenability by Følner nets, (1.3.1). We also see that without loss of generality we can demand that such nets are increasing and exhaust the group.

Linear algebra

In Chapter 2 we recall the definition of a module over a ring and define the group ring of a field $\mathbb{K}$ and a group G. Furthermore we define the support of a group ring element which is a finite subset of G. Using this we examine two kinds of finite-dimensional subspaces of a $\mathbb{K}[G]$-submodule $M \subset F[G]$ induced from the $\mathbb{K}$-vector subspace $\mathbb{K}[F] \subset \mathbb{K}[G]$ for a finite subset $F \subset G$.

Namely the subspace of elements which are supported on $F \subset G$, $M \cap \operatorname{im} i_F$, and the subspace of projected elements $p_F(M)$. We find estimates on their sizes which will be used in Chapter 4.

Convergence theorem

In Chapter 3 we reproduce a variation of the quasi-tiling lemma of Ornstein and Weiss [**OW87**] as presented by Krieger, [**Kri07**]. Theorem 3.2.3 shows convergence with regards to Følner nets for subadditive functions on the set of finite subsets of an amenable group.

We prove an adaption to almost-superadditive functions in Theorem 3.2.2, this serves as foundation of the dimension function introduced in Chapter 4.

Følner dimension

Chapter 4 contains the main results of this thesis. In the first part we define a function on submodules of free $\mathbb{K}[G]$-modules which we call the Følner dimension, (4.1.4). For a submodule $M \subset \mathbb{K}[G]^n$ over the group ring of an amenable group G this is

$$\dim_{\mathbb{K}[G]}(M) := \lim_i \frac{\dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_{F_i}\right)}{|F_i|},$$

where (F_i) is a Følner net in G.

This definition is not a new idea. At least similar ones, where the field is the complex numbers $\mathbb{C}$, are used in approximation results for L^2-Betti numbers, [**Ele06**]. We also see for which modules the Følner dimension agrees with similar definitions.

What is new is the well-definition for all submodules of free $\mathbb{K}[G]$-modules extended to any finitely generated $\mathbb{K}[G]$-module and finally any $\mathbb{K}[G]$-module, in combination with the proof of properties similar to the von Neumann dimension. These are proven in the second part of the chapter.

Then in the third part we investigate the connection between different choices of fields for the group ring, as well as the relation to the von Neumann dimension and a rank function introduced by Elek, [**Ele03b**]. The latter allows the reformulation of an approximation result for residually finite groups, [**LLS11**].

Methods of computation

In Chapter 5 we recall our motivation coming from L^2-Betti numbers and the so called Atiyah conjecture. Therefore we investigate which values the Følner dimension takes for kernels of $\mathbb{K}[G]$-matrices, in particular for a field $\mathbb{K}$ of positive characteristic and a finitely generated amenable group G.

The first part concerns finitely generated amenable groups for which the word-problem is solvable. We deduce from the coarse monotony of the sequence which converges to the Følner dimension, that this limit is a computable number.

In the second part we rebuild the computational tool from [**GS14**] for characteristic 2. Then corresponding to L^2-Betti numbers of normal coverings we show, for any real number r, the construction of a finitely generated amenable group and an associated $\mathbb{F}_2[G]$ matrix, whose kernel has Følner dimension r.

Similar, corresponding to L^2-Betti numbers of universal coverings we construct a finitely presented amenable group and an associated $\mathbb{F}_2[G]$-matrix, whose kernel has irrational Følner dimension.

In the final part we use the properties established in Chapter 4 to translate the construction of a transcendental L^2-Betti number, the result of [**PSZ15**], to all positive characteristics except 2.

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CHAPTER 1

Amenable groups

We begin with the well known generalization of sequences from natural numbers to directed sets.

1.1. Nets

Recall the basics of topological spaces, for details see [**Kel75**].

Definition 1.1.1. A binary relation $\geq$ directs a non-empty set I if

(i) for $i, j, k \in I$ such that $i \geq j$ and $j \geq k$ it follows that $i \geq k$,
(ii) for all $i \in I$ holds that $i \geq i$,
(iii) for any $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$.

The pair $(I, \geq)$ is called a directed set.

Definition 1.1.2. (i) A function $f\colon I \to X$, from a directed set $(I, \geq)$ to a set X, is called a net. We write x_i for $f(i), i \in I$ and (x_i) instead of f.
If X is a topological space, a net (x_i) converges to $x \in X$ if and only if for every open neighborhood $U_x \subseteq X$ of x there exists $i_0 \in I$ such that $x_i \in U_i$ for all $i \geq i_0 \in I$. We write $\lim_i x_i = x$ if (x_i) converges to x.
(ii) Let (x_i) and $\left(y_j\right)$ be nets in a topological space X, then $\left(y_j\right)$ with a function $\phi\colon J \to I$ is a subnet of (x_i) if and only if the following holds:
(a) $y_j = x_{\phi(j)}$, for all $j \in J$,
(b) for every $i \in I$ there exists $k \in J$ such that $j \geq k$ implies $\phi(j) \geq i$.
(iii) A point $x \in X$ is a cluster point of a net (x_i) if and only if for every neighborhood U_x of x and $k \in I$ there exists $i \geq k$ such that $x_i \in U_x$.

Theorem 1.1.3. *Let (x_i) be a net in a topological space X then*

(i) $x \in X$ is a cluster point of (x_i) if and only if there exists a subnet of (x_i) that converges to x,
(ii) $C := \bigcap_{k \in I} \overline{\left\{x_i \mid |i \geq k \in I\right\}}$ is the set of all cluster points of (x_i),
(iii) if X is compact then $C \neq \emptyset$.

Definition 1.1.4. Let (x_i) be a net in the real numbers $\mathbb{R}$ such that C is not empty.

(i) The infimum of its clusterpoints is called $\liminf_i x_i$, the limes inferior of (x_i).
(ii) The supremum of its clusterpoints is called $\limsup_i x_i$, the limes superior of (x_i).

Corollary 1.1.5. *Let (x_i) be a net in the real numbers $\mathbb{R}$ such that C is not empty. If $\liminf_i x_i = \limsup_i x_i < \infty$, then there is only one cluster point and (x_i) converges to it.*

1.2. Amenability

The Følner condition for amenability was introduced by [**Føl55**], we recall his original definition and then follow the idea and notation of relative amenability from [**Kri07**]. For more details about amenability see [**Pat88**].

Definition 1.2.1. [**Føl55**] A discrete group G is amenable if and only if for any $0 < \eta < 1$ and any finite subset $L \subseteq G$ there exists a non-empty finite subset $F \subseteq G$ such that

$$|F \cap lF| \geq \eta |F| \tag{1.2.1}$$

for all $l \in L$.

Definition 1.2.2. [**Kri07**] Let G be a discrete group, $F, K \subseteq G$ subsets and $a \in G$, then we define the following subsets of G.

(i) Translates of F,

$$aF := \{af \mid f \in F\},$$
$$Fa := \{fa \mid f \in F\},$$
$$FK^{-1} := \left\{fk^{-1} \mid f \in F, k \in K\right\}.$$

(ii) The K-interior of F,

$$\mathrm{Int}_K(F) := \{g \in G \mid Kg \subseteq F\}.$$

(iii) The K-exterior of F,

$$\mathrm{Ext}_K(F) := \left\{g \in G \mid Kg \subseteq G \setminus F\right\}.$$

(iv) The K-ball of F,

$$B_K(F) := G \setminus \mathrm{Ext}_K(F).$$

(v) The K-boundary of F,

$$\partial_K(F) := G \setminus \big(\operatorname{Ext}_K(F) \cup \operatorname{Int}_K(F)\big)$$

Lemma 1.2.3. *Let G be a discrete group, $F, F', K, K' \subseteq G$ subsets and $a, g \in G$ then the definition above implies*

(i) $\partial_{\{g\}}(F) = \emptyset$,
(ii) $\partial_K(F) = \Big\{ g \in G \;\Big|\; Kg \cap F \neq \emptyset \text{ and } Kg \cap \big(G \setminus F\big) \neq \emptyset \Big\}$,
(iii) $\partial_K(F) = \bigcup_{k_1,k_2 \in K} k_1^{-1}\Big(F \setminus \big(F \cap k_1 k_2^{-1} F\big)\Big)$,
(iv) $\partial_K(F) = \partial_K\big(G \setminus F\big)$,
(v) $\partial_K(F \cup F') \subseteq \partial_K(F) \cup \partial_K(F')$,
(vi) $\partial_K(F) \subseteq \partial_{K'}(F)$, *if* $K \subseteq K' \subseteq G$,
(vii) $\partial_K(Fa) = \partial_K(F)\,a$,
(viii) $\partial_{Ka}(F) = a^{-1}\partial_K(F)$,
(ix) $F \setminus \operatorname{Int}_K(F) \subseteq \partial_K(F)$, *if* $e \in K$,
(x) $KF \subseteq B_K(F)$, *if* $K^{-1} = K$.

Furthermore, if $F, K \subseteq G$ are finite subsets, then $\partial_K(F)$ is finite.

Proof. Let G be a discrete group, $F, F', K, K' \subseteq G$ subsets and $a, g \in G$

(i) then $\operatorname{Int}_{\{g\}}(F) = g^{-1}F$ and $\operatorname{Ext}_{\{g\}}(F) = g^{-1}\big(G \setminus F\big)$. So $\partial_{\{g\}}(F) = G \setminus G = \emptyset$.

(ii) The negation of $\big(Kg \subseteq F$ or $Kg \subseteq G \setminus F\big)$ is equivalent to

$$\Big(Kg \cap F \neq \emptyset \text{ and } Kg \cap \big(G \setminus F\big) \neq \emptyset\Big).$$

(iii) By definition it holds that

$$\begin{aligned} g \in \partial_K(F) &\iff \exists k_1, k_2 \in K\colon k_1 g \in F \text{ and } k_2 g \in G \setminus F \\ &\iff \exists k_1, k_2 \in K\colon g \in k_1^{-1}F \cap \big(G \setminus k_2^{-1}F\big) \\ &\iff \exists k_1, k_2 \in K\colon g \in k_1^{-1}\Big(F \setminus \big(F \cap k_1 k_2^{-1} F\big)\Big). \end{aligned}$$

(iv) Follows from (ii).

(v) Directly follows from $Kg \cap \big(G \setminus (F \cup F')\big) = Kg \cap \big(G \setminus F\big) \cap \big(G \setminus F'\big)$ and the equivalence $Kg \cap (F \cup F') \neq \emptyset \iff Kg \cap F \neq \emptyset$ or $Kg \cap F' \neq \emptyset$.

(vi) Follows from (ii).

(vii) Follows from (iii) with

$$k_1^{-1}\Big(Fa \setminus \big(Fa \cap k_1 k_2^{-1} Fa\big)\Big) = k_1^{-1}\Big(F \setminus \big(F \cap k_1 k_2^{-1} F\big)\Big)\,a.$$

(viii) Follows from (iii) with

$$\bigcup_{k_1,k_2\in Ka} k_1^{-1}\left(F\setminus\left(F\cap k_1k_2^{-1}F\right)\right)=\bigcup_{k_1,k_2\in K} a^{-1}k_1^{-1}\left(F\setminus\left(F\cap k_1k_2^{-1}F\right)\right).$$

(ix) Let $e\in K$, by definition $Kg\subseteq F$ for $g\in \mathrm{Int}_K(F)$, so in particular $eg\in F$. Thus $\mathrm{Int}_K(F)\subseteq F$ and similarly $\mathrm{Ext}_K(F)\subseteq G\setminus F$. The statement follows.

(x) Let $K^{-1}=K$. Assume $kf\in \mathrm{Ext}_K(F), f\in F, k\in K$ then $Kkf\subseteq G\setminus F$. But $k^{-1}\in K$ implies $f\in G\setminus F$ which contradicts $f\in F$. Thus $KF\subseteq B_K(F)$.

□

Lemma 1.2.4. *A discrete group G is amenable if and only if for any $0<\epsilon<1$ and any finite subset $K\subseteq G$ there exists a non-empty finite subset $F\subseteq G$ such that*

$$\frac{|\partial_K(F)|}{|F|}\leq \epsilon. \tag{1.2.2}$$

Proof. Suppose that G is amenable and fix $0<\epsilon<1$ and a finite subset $K\subseteq G$. Let $L:=KK^{-1}$ and $\eta:=1-\epsilon|L|^{-2}$. There exists $F\subseteq G$ finite, such that $|F\cap lF|\geq\eta|F|$ for all $l\in L$. Thus for all $k_1,k_2\in K$ we have

$$|\left(F\setminus\left(F\cap k_1k_2^{-1}F\right)\right)|\leq|F|-\eta|F|=|F|\epsilon|L|^{-2}.$$

From Lemma 1.2.3((iii)) above we deduce that

$$|\partial_K(F)|\leq\sum_{k_1,k_2\in K}|\left(F\setminus\left(F\cap k_1k_2^{-1}F\right)\right)|\leq\epsilon|F|.$$

On the other hand, assume that (1.2.2) holds and fix $0<\eta<1$ and $L\subseteq G$ finite. Let $K:=L^{-1}\cup\{e\}$ and $\epsilon:=1-\eta$. Then Lemma 1.2.3((iii)) shows for all $l\in L$ that $F\setminus(F\cap lF)\subseteq\partial_K(F)$ and there exists $F\subseteq G$ finite with

$$|F\cap lF|=|F|-|\left(F\setminus(F\cap lF)\right)|\geq|F|-|\partial_K(F)|\geq\eta|F|.$$

□

Definition 1.2.5. Let $H\leq G$ be a non-trivial subgroup of a discrete group G. We call a subset $V\subseteq G$ a right transversal (for H) if we get a partition of G,

$$G=\bigsqcup_{v\in V}Hv. \tag{1.2.3}$$

Lemma 1.2.6. *Let $H\leq G$ be finite and $V\subseteq G$ its right transversal.*

(i) For a non-empty subset $E \subseteq H$ we have that

$$\partial_H(E) = \begin{cases} H & \text{if } E \neq H, \\ \emptyset & \text{if } E = H. \end{cases} \tag{1.2.4}$$

(ii) For any non-empty finite subset $F \subseteq G$ we get a partition $F = \bigsqcup_{v \in V} F^v v$ and have that

$$\partial_H(F) = \bigsqcup_{v \in V_F} H \cdot v, \tag{1.2.5}$$

where $V_F = \{v \in V \mid \emptyset \neq F^v \neq H\}$.

Proof. Let $H \leq G$ be finite and $V \subseteq G$ its right transversal.

(i) Since H is a subgroup and $E \subseteq H$ we have

$$\forall g \in G\colon Hg \cap E \neq \emptyset \iff g \in H,$$
$$\forall g \in H\colon Hg \cap (G \setminus E) \neq \emptyset \iff Hg \cap (H \setminus E) \neq \emptyset.$$

So by definition

$$\begin{aligned} \partial_H(E) &= \left\{ g \in G \;\middle|\; Hg \cap E \neq \emptyset \text{ and } Hg \cap (G \setminus E) \neq \emptyset \right\} \\ &= \left\{ g \in H \;\middle|\; Hg \cap (H \setminus E) \neq \emptyset \right\} \\ &= \begin{cases} H & \text{if } E \neq H, \\ \emptyset & \text{if } E = H. \end{cases} \end{aligned}$$

(ii) Let $V^F = \{v \in V \mid F^v \neq \emptyset\}$, then for all $g \in G$ it holds that

$$Hg \cap \left(\bigsqcup_{v \in V^F} F^v v \right) \neq \emptyset \iff g \in \bigsqcup_{v \in V^F} Hv,$$

and thus for all $g \in \bigsqcup_{v \in V^F} Hv$

$$Hg \cap \left(G \setminus \left(\bigsqcup_{v \in V^F} F^v v \right) \right) \neq \emptyset \iff Hg \cap \left(\bigsqcup_{v \in V^F} (H \setminus F^v)\, v \right) \neq \emptyset.$$

Finally this implies

$$\begin{aligned}\partial_H(F) &= \left\{ g \in \bigsqcup_{v \in V^F} Hv \;\middle|\; Hg \cap \left(\bigsqcup_{v \in V^F} \big(H \setminus F^v\big) v \right) \neq \emptyset \right\} \\ &= \bigsqcup_{v \in V_F} Hv,\end{aligned}$$

where $V_F = \big\{ v \in V^F \;\big|\; F^v \neq H \big\}$.

□

1.3. Følner nets

For a discrete group G let $\mathcal{F}(G)$ be the set of non-empty finite subsets of G.

Theorem 1.3.1. *Let G be a discrete group. Then G is amenable if and only if there exists a net $\big(F_i\big)$ in $\mathcal{F}(G)$ such that*

$$\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0 \tag{1.3.1}$$

for all $K \in \mathcal{F}(G)$. Such a net is called Følner net.

Proof. Let G be an amenable group. To construct a Følner net we need a directed set $(I, \geq)$. Let $I := \mathcal{F}(G) \times \mathbb{N}$ be the set of pairs of finite subsets of G and positive integers. We may direct this set as follows:

$$(L, m) \geq (K, n) \iff K \subseteq L \text{ and } n \leq m.$$

By Lemma 1.2.4 there exists a map $f\colon (I, \geq) \to \mathcal{F}(G)$ such that for all $i = (K, n)$ it holds that

$$\frac{|\partial_K\big(f(i)\big)|}{|f(i)|} \leq \frac{1}{n}.$$

This defines a net $(F_i) := f$. Let $U_0 \subseteq (-1, 1)$ be an open neighborhood of 0, fix $n \in \mathbb{N}$ such that $\frac{1}{n} \in U_0$ and let $i_K := (K, n)$ then by Lemma 1.2.3 we have for all $(L, m) = i \geq i_K$ that

$$\frac{|\partial_K(F_i)|}{|F_i|} \leq \frac{|\partial_L(F_i)|}{|F_i|} \leq \frac{1}{m} \leq \frac{1}{n}.$$

This holds for all $K \in \mathcal{F}(G)$ and thus (1.3.1) follows.
On the other hand, suppose that G admits a Følner net (F_i). Let $0 < \epsilon < 1$ and $K \subseteq G$ be a finite subset. Since $\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0$ there exists $j \in I$ such that $\frac{|\partial_K\big(F_j\big)|}{|F_j|} \leq \epsilon$. The theorem follows by Lemma 1.2.4. □

Corollary 1.3.2. *Let G be a countable amenable group then there exists a sequence $(F_i)_{i\in\mathbb{N}}$ in $\mathcal{F}(G)$ such that*

$$\lim_{i\to\infty}\frac{|\partial_K(F_i)|}{|F_i|}=0 \tag{1.3.2}$$

for all $K\in\mathcal{F}(G)$. Such a sequence is called Følner sequence.

Proof. Let G be a countable amenable group and $\left(E_j\right)_{j\in\mathcal{F}(G)\times\mathbb{N}}$ a Følner net as in the proof above. We enumerate the elements of $G=\{g_k\mid k\in\mathbb{N}\}$ and define finite subsets $H_n:=\{g_k\mid 1\le k\le n\}$ for $n\in\mathbb{N}$. Clearly for every $K\in\mathcal{F}(G)$ there exists $n\in\mathbb{N}$ such that $K\subset H_n$. Now let $F_n:=E_{(H_n,n)}$ for $n\in\mathbb{N}$ then $(F_i)_{i\in\mathbb{N}}$ is a Følner sequence by (1.3.1) and Lemma 1.2.3. □

Lemma 1.3.3. *Let $(A_i)_{i\in I},\left(B_j\right)_{j\in J}$ be two Følner nets then there exists a Følner net $(F_k)_{k\in K}$ such that $(A_i),\left(B_j\right)$ are subnets.*

Proof. Let $(A_i)_{i\in I},\left(B_j\right)_{j\in J}$ with directed sets $(I,\ge)$ respective $(J,\ge)$ be Følner nets. We construct a directed set $(K,\ge)$ as follows. Let $\hat{I},\hat{J}$ be copies of I,J and set $K:=I\times J\bigsqcup\hat{I}\times\hat{J}$. $(K,\ge)$ is a directed set by the inherited relation where

$$\begin{aligned}
(i,j)\ge\left(i',j'\right) &\iff i\ge i',j\ge j' \text{ if } (i,j)\in I\times J \text{ and } \left(i',j'\right)\in I\times J,\\
(i,j)\ge\left(\hat{i}',\hat{j}'\right) &\iff i\ge \hat{i}',j\ge \hat{j}' \text{ if } (i,j)\in I\times J \text{ and } \left(\hat{i}',\hat{j}'\right)\in \hat{I}\times \hat{J},\\
\left(\hat{i},\hat{j}\right)\ge\left(i',j'\right) &\iff \hat{i}\ge i',\hat{j}\ge j' \text{ if } \left(\hat{i},\hat{j}\right)\in \hat{I}\times \hat{J} \text{ and } \left(i',j'\right)\in I\times J,\\
\left(\hat{i},\hat{j}\right)\ge\left(\hat{i}',\hat{j}'\right) &\iff \hat{i}\ge \hat{i}',\hat{j}\ge \hat{j}' \text{ if } \left(\hat{i},\hat{j}\right)\in \hat{I}\times \hat{J} \text{ and } \left(\hat{i}',\hat{j}'\right)\in \hat{I}\times \hat{J}.
\end{aligned}$$

We define $(F_k)_{k\in K}$ by

$$F_k:=\begin{cases}A_i & \text{if } k=(i,j),\\ B_j & \text{if } k=\left(\hat{i},\hat{j}\right).\end{cases}$$

Clearly, this is a Følner net, and $(A_i),\left(B_j\right)$ are subnets. □

Lemma 1.3.4. *Let $H\le G$ be a non-trivial finite subgroup of an amenable group G, and (F_i) be a Følner net in $\mathcal{F}(G)$. Furthermore let $V\subseteq G$ be a right transversal for H and define $V_i:=\left\{v\in V\mid F_i\cap Hv\neq\emptyset\right\}$ and $V_i':=\{v\in V_i\mid F_i\cap Hv\neq Hv\}$, then*

$$\lim_i\frac{|V_i'|}{|V_i|}=0 \tag{1.3.3}$$

Proof. Consider the partition $F_i = \bigsqcup_{v\in V_i} F_i^v v$. By Lemma 1.2.6((ii)) we have $\partial_H(F_i) = \bigsqcup_{v\in V_i'} Hv$, and therefore

$$\frac{|V_i'|}{|V_i|} \leq \frac{|V_i'||H|}{|V_i \setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|} = \frac{|\partial_H(F_i)|}{|F_i|} \xrightarrow{\lim i} 0$$

□

This reinforces the intuitive notion that if an amenable group has a non-trivial finite subgroup, any Følner net eventually looks like a union of cosets.

Lemma 1.3.5. *Let G be an infinite amenable group, let $(F_i)_{i\in I}$ be a Følner net in $\mathcal{F}(G)$ then*

$$\lim_i \frac{1}{|F_i|} = 0. \tag{1.3.4}$$

Proof. Let G be an infinite amenable group and let $F \subset G$ be a non-empty finite subset. Let $f \in F, g \in G \setminus F$ and define $K := \{e, gf^{-1}\}$ then $f \in \partial_K(F)$. Let $(F_i)_{i\in I}$ be a Følner net in $\mathcal{F}(G)$ then for every $i \in I$ there is a finite subset $K_i \subset G$ such that $\partial_{K_i}(F_i) \neq 0$. Now (1.3.1) implies the lemma. □

Lemma 1.3.6. *Let G be an infinite amenable group, let (F_i) be a Følner net in $\mathcal{F}(G)$ and let $A \subseteq G$ be a finite subset of G then $(F_i \cup A)$ is also a Følner net.*

Proof. Let G be an infinite amenable group, let (F_i) be a Følner net in $\mathcal{F}(G)$ and let $A, K \subset G$ be finite subsets of G. By Lemma 1.3.5 it follows that $\lim_i \frac{1}{|F_i|} = 0$, and thus $\lim_i \frac{|\partial_K(A)|}{|F_i|} = 0$. Furthermore we have that

$$\frac{|\partial_K(F_i \cup A)|}{|F_i \cup A|} \leq \frac{|\partial_K(F_i)|}{|F_i|} + \frac{|\partial_K(A)|}{|F_i|},$$

by Lemma 1.2.3 and so $\lim_i \frac{|\partial_K(F_i \cup A)|}{|F_i \cup A|} = 0$. □

Theorem 1.3.7 (Følner exhaustion)**.** *Let G be a discrete amenable group. Then there exists a Følner net (F_i) in $\mathcal{F}(G)$ such that $F_i \supseteq F_j$ for $i \geq j$ and $\bigcup_i F_i = G$.*

Proof. Let G be a discrete amenable group, if G is finite, then the constant net (G) fulfills the properties. So let G be infinite and let $(E_i)_{i\in I}$ be the Følner net constructed in the proof of 1.3.1. We construct $(F_i)_{i\in I}$ recursively. For $(L, m) \in I$ with $|L| = 1$ or $m = 1$ we define

$$F_{(L,m)} := L.$$

Then $\partial_L\left(F_{(L,m)}\right) = \emptyset$ for all $(L,m) \in I$ with $|L| = 1$. For $(K,n) \in I$ with $|K| > 1$ and $n > 1$ we define

$$F_{(K,n)} := F_{(K,n-1)} \bigcup_{L \subsetneq K} F_{(L,n-1)} \cup E_{i_{K,n}},$$

where $i_{K,n} \in I$ such that

$$\frac{\left|\partial_K\left(F_{(K,n-1)} \bigcup_{L \subsetneq K} F_{(L,n-1)} \cup E_{i_{K,n}}\right)\right|}{\left|F_{(K,n-1)} \bigcup_{L \subsetneq K} F_{(L,n-1)} \cup E_{i_{K,n}}\right|} \leq 1/n.$$

We find such an $i_{K,n} \in I$ because of Lemma 1.3.6. Clearly,

$$\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0$$

for all $K \in \mathcal{F}(G)$ and also $F_i \supseteq F_j$ for $i \geq j$ and $\bigcup_i F_i \supseteq \bigcup_{K \in \mathcal{F}(G)} K = G$. □

Lemma 1.3.8. [**Pat88**, p.189] *A discrete group G is amenable if and only if for any $0 < \epsilon < 1$ and any two finite subset $L, A \subseteq G$ there exists a non-empty symmetric finite subset $F \subseteq G$ such that $A \subset F$ and for all $l \in L$ it holds that*

$$|F \cup lF| - |F \cap lF| \leq \epsilon |F|. \tag{1.3.5}$$

Corollary 1.3.9. *[Symmetric Følner sequence] Let G be a countable discrete amenable group. Then there exists a Følner sequence $(F_i)_{i\in\mathbb{N}}$ in $\mathcal{F}(G)$ such that $F_i = F_i^{-1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} F_i = G$.*

Lemma 1.3.10. [**Pat88**, p.14] *The class of amenable groups contains all finite and all abelian groups, and is closed under taking subgroups, forming factor groups, group extensions, and directed unions.*

There is another construction for Følner sequences for amenable, residually finite groups. See [**LLS11**] and [**KKN15**].

Definition 1.3.11. [**KKN15**] Let G be a residually finite group. Let $\left(H_j\right)$ be a chain of finite index subgroups ordered by inclusion. Then $\left(H_j\right)$ is called a Farber chain if and only if

$$\lim_{j\to\infty} \frac{\left|\left\{x \in {}^{G}/_{H_j} \mid gx = x\right\}\right|}{\left[G : H_j\right]} = 0 \tag{1.3.6}$$

for all $g \in G \setminus \{e\}$.

Corollary 1.3.12. [**KKN15**] *Let $(G_i)_{i\in\mathbb{N}}$ be a chain of finite index normal subgroups of G, such that $\bigcap_{i\in\mathbb{N}} G_i = \{e\}$. Then is $(G_i)_{i\in\mathbb{N}}$ a Farber chain.*

Theorem 1.3.13. [**KKN15**, Theorem 7] *Let G be an finitely generated amenable group and $\left(H_j\right)$ a Farber chain in G. Then there exists a Følner sequence $\left(F_j\right)$ such that F_j is a set of coset representatives for H_j in G.*

CHAPTER 2

Linear Algebra

In this chapter we make the linear algebra calculations needed for the main results of this thesis.

2.1. Group ring modules

Recall the definition of $\mathbb{K}$-vector spaces for any field $\mathbb{K}$, [**War90**]. Similarly we define modules.

Definition 2.1.1. [**War90**] Let R be a unitary ring. We call M a (right) R-module if for $\alpha, \beta \in R$ and $m, n \in M$ it holds that

(i) $(m+n)\,\alpha = m\alpha + n\alpha \in M$,
(ii) $m\,(\alpha\beta) = (m\alpha)\,\beta \in M$,
(iii) $m1 = m \in M$.

If instead the ring R acts from the left, it is called a left R-module. Consider R as the free R-module then we see that multiplication from the left by ring elements gives us right R-submodules.

Corollary 2.1.2. *Let R be a ring and let $A, B \in R$ be right R-module homomorphisms $R \to R$ by left multiplication then*

$$\ker\left(A \oplus B\colon\, R \oplus R \to R \oplus R\right) = \ker A \oplus \ker B, \tag{2.1.1}$$

$$\ker\left(\begin{pmatrix} A \\ B \end{pmatrix} : R \to R \oplus R\right) = \ker A \cap \ker B. \tag{2.1.2}$$

For the definition of group rings we follow [**Pas76**].

Definition 2.1.3. (i) Let G be a group and $\mathbb{K}$ any field, then we call the $\mathbb{K}$-vector space with basis G the group ring $\mathbb{K}[G]$. The elements are formal sums $\sum_{x\in G} a_x x$, where only finitely many $a_x \in \mathbb{K}$ are non-zero. It is a ring by multiplication inherited from G,

$$\left(\sum_{x\in G} a_x x\right)\left(\sum_{y\in G} b_y y\right) = \sum_{x,y\in G} \left(a_x b_y\right) xy = \sum_{z\in G} c_z z, \tag{2.1.3}$$

where $c_z = \sum_{x\in G} a_x b_{x^{-1}z}$.

(ii) Let $\alpha = \sum_{x\in G} a_x x \in \mathbb{K}[G]$ be an element of the group ring, we may also consider it as a map $\alpha\colon G \to \mathbb{K}$. Thus we call the finite subset of G for which the coefficients are non-zero its support, $\operatorname{supp}\alpha := \{x \in G \mid a_x \neq 0\}$. On the other hand, we also write $\mathbb{K}^{\oplus G}$ for the $\mathbb{K}$-vector space of maps from G to $\mathbb{K}$ with finite support.

(iii) For $n, m \in \mathbb{N}$ let $\mathbb{K}[G]^n, \mathbb{K}[G]^m$ be the canonical free $\mathbb{K}[G]$-modules of rank n, respective m. Let $(e_i)_{i=1,\dots,m}$ be a basis for $\mathbb{K}[G]^m$ then we have the canonical pairing $\langle\cdot,\cdot\rangle : \mathbb{K}[G]^m \times \mathbb{K}[G]^m \to \mathbb{K}[G]$ defined by

$$\left\langle \alpha e_i, e_j \right\rangle = \begin{cases} \alpha & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Any right $\mathbb{K}[G]$-module homomorphisms $T\colon \mathbb{K}[G]^n \to \mathbb{K}[G]^m$, also called $\mathbb{K}[G]$-linear map, can be written as a matrix $\left(T_{ij}\right) \in \mathbb{K}[G]^{m\times n}$. Where $T_{ij} := \left\langle T(e_i), e_j \right\rangle$ and e_i, e_j are basis elements of the free modules $\mathbb{K}[G]^n$ and $\mathbb{K}[G]^m$. We call

$$L_T := \operatorname{supp} T \cup (\operatorname{supp} T)^{-1} \cup \{e\} \subset G \tag{2.1.4}$$

the symmetric support of T, where

$$\operatorname{supp} T = \bigcup_{i=1}^{m}\bigcup_{j=1}^{n} \operatorname{supp} T_{ij}. \tag{2.1.5}$$

(iv) For a finite subset $F \subseteq G$ denote by $i_F\colon \mathbb{K}[F] \hookrightarrow \mathbb{K}[G]$ and $p_F\colon \mathbb{K}[G] \twoheadrightarrow \mathbb{K}[F]$ the canonical $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. Abusing notation we will also write i_F for the map $\mathbb{K}[F]^n \hookrightarrow \mathbb{K}[G]^n$, as well as p_F for the map $\mathbb{K}[G]^m \twoheadrightarrow \mathbb{K}[F]^m$, and finally $\pi_F := i_F \circ p_F\colon \mathbb{K}[G]^n \to \mathbb{K}[G]^n$.

Corollary 2.1.4. *Let G be a group and $\mathbb{K}$ any field, and let $T \in \mathbb{K}[G]^{m\times n}$ be a $\mathbb{K}[G]$-linear map by left multiplication. Then $\ker T$ and $\operatorname{im} T$ are right $\mathbb{K}[G]$-modules.*

As is well known such group rings are rings with involution and as such make it possible to treat left modules as right modules and vice versa.

Lemma 2.1.5. *Let G be a group and $\mathbb{K}$ any field. For $\alpha = \sum_{x\in G} a_x x \in \mathbb{K}[G]$ we write*

$$\overline{\alpha} := \sum_{x\in G} a_x x^{-1}. \tag{2.1.6}$$

(i) Let M be a left $\mathbb{K}[G]$-module, then M is a right $\mathbb{K}[G]$-module by

$$m\alpha := \bar{\alpha} m, \tag{2.1.7}$$

for $m \in M, \alpha \in \mathbb{K}[G]$.

(ii) Let M be a right $\mathbb{K}[G]$-module, then M is a left $\mathbb{K}[G]$-module by

$$\alpha m := m\bar{\alpha}, \tag{2.1.8}$$

for $m \in M, \alpha \in \mathbb{K}[G]$.

Proof. Let G be a group and $\mathbb{K}$ any field. Let $\alpha = \sum_{x\in G} a_x x \in \mathbb{K}[G]$ and $\beta = \sum_{y\in G} b_y y \in \mathbb{K}[G]$ then

$$\begin{aligned}
\overline{\alpha\beta} &= \overline{\sum_{x,y\in G} \left(a_x b_y\right) xy} \\
&= \sum_{x,y\in G} \left(a_x b_y\right) y^{-1} x^{-1} \\
&= \left(\sum_{y\in G} b_y y^{-1}\right) \left(\sum_{x\in G} a_x x^{-1}\right) \\
&= \overline{\beta}\overline{\alpha}.
\end{aligned}$$

Let M be a left $\mathbb{K}[G]$-module, and let $m \in M, \alpha, \beta \in \mathbb{K}[G]$ then

$$\begin{aligned}
m\left(\alpha\beta\right) &= \overline{\alpha\beta} m \\
&= \left(\overline{\beta}\overline{\alpha}\right) m \\
&= \overline{\beta}\left(\overline{\alpha} m\right) \\
&= \left(m\alpha\right)\beta.
\end{aligned}$$

The second case follows immediately. □

We see that it is enough to treat right $\mathbb{K}[G]$-modules, if a distinction between left and right is necessary.

Definition 2.1.6. Let G be a group and $\mathbb{K}$ any field and let M be a $\mathbb{K}[G]$-module.

(i) We call M finitely generated if there exists a surjective $\mathbb{K}[G]$-linear map

$$\mathbb{K}[G]^n \to M,$$

for some $n \in \mathbb{N}$.

(ii) We call M finitely presented if for some $n \in \mathbb{N}$ there exist an extension of $\mathbb{K}[G]$-modules

$$0 \to N \to \mathbb{K}[G]^n \to M \to 0,$$

such that N is finitely generated.

Corollary 2.1.7. *Let $M \subset \mathbb{K}[G]^m$ be a finitely generated right $\mathbb{K}[G]$-module for some $m \in \mathbb{N}$, then there exists $n \in \mathbb{N}, T \in \mathbb{K}[G]^{m\times n}$ such that $M = \operatorname{im} T$.*

Lemma 2.1.8. *Let $M \subset \mathbb{K}[G]$ be a $\mathbb{K}[G]$-submodule and $F \subset G$ a finite subset, then*

$$\dim_{\mathbb{K}} (M \cap \operatorname{im} i_F) \leq \dim_{\mathbb{K}} p_F(M). \tag{2.1.9}$$

Proof. Let $M \subset \mathbb{K}[G]$ be a $\mathbb{K}[G]$-submodule and $F \subset G$ a finite subset. Let $\alpha \in M \cap \operatorname{im} i_F$ then $\operatorname{supp} \alpha \subset F$ and so $\pi_F(\alpha) = \alpha \in \pi_F(M)$. The lemma follows. □

Next we recall Definition 1.2.2, and find estimates on the propagation of a matrix with $\mathbb{K}[G]$ coefficients in terms of $\mathbb{K}$-vector spaces.

Lemma 2.1.9. *Let $T \in \mathbb{K}[G]^{m\times n}$ be a $\mathbb{K}[G]$-linear map, L_T be its symmetric support and $\alpha \in \mathbb{K}[G]^n$ then*

(i) $\operatorname{supp}(T\alpha) \subseteq (\operatorname{supp} T)(\operatorname{supp} \alpha)$,
(ii) $\operatorname{im}\left(T i_{\operatorname{Int}_{L_T}(F)}\right) \subseteq \operatorname{im} i_F = \left\{\beta \in \mathbb{K}[G]^m \,\middle|\, \operatorname{supp} \beta \subseteq F\right\}$,
(iii) $\operatorname{im}\left(T i_{\operatorname{Ext}_{L_T}(F)}\right) \subseteq \operatorname{im} i_{G\setminus F}$,
(iv) $\operatorname{im}(T i_F) \subseteq \operatorname{im} i_{B_{L_T}(F)}$.

Proof. Let $T \in \mathbb{K}[G]^{m\times n}$ be a $\mathbb{K}[G]$-linear map, L_T be its symmetric support and $\alpha \in \mathbb{K}[G]^n$. Now (i) follows from (2.1.3) and by definitions and Lemma 1.2.3 it holds that

(ii) $(\operatorname{supp} T)\left(\operatorname{Int}_{L_T}(F)\right) \subseteq L_T\left(\operatorname{Int}_{L_T}(F)\right) \subseteq F$,
(iii) $(\operatorname{supp} T)\left(\operatorname{Ext}_{L_T}(F)\right) \subseteq L_T\left(\operatorname{Ext}_{L_T}(F)\right) \subseteq G \setminus F$,
(iv) $(\operatorname{supp} T) F \subseteq L_T F \subseteq B_{L_T}(F)$.

□

2.2. Finite-dimensional subspaces

Note that we will consider $\mathbb{K}[G]$-modules as $\mathbb{K}$-vector spaces when appropriate.

Theorem 2.2.1. *Let G be a group and $\mathbb{K}$ any field. Let $T \in \mathbb{K}[G]^{m\times n}$ be a $\mathbb{K}[G]$-linear map, L_T be its symmetric support and $F \subset G$ be a finite subset. Then*

$$\begin{aligned} \dim_{\mathbb{K}} p_F(\ker T) &\le \dim_{\mathbb{K}} \ker(Ti_F) + n|\partial_{L_T}(F)| \\ &\le \dim_{\mathbb{K}} \ker(p_F T i_F) + n|\partial_{L_T}(F)| \\ &\le \dim_{\mathbb{K}} \ker(Ti_F) + 2n|\partial_{L_T}(F)| \\ &\le \dim_{\mathbb{K}} p_F(\ker T) + 2n|\partial_{L_T}(F)|. \end{aligned} \tag{2.2.1}$$

Proof. Let $T \in \mathbb{K}[G]^{m\times n}$ and let $F \subset G$ be a finite subset. Let L_T be its symmetric support and recall from Lemma 1.2.3 that

$$\begin{aligned} F &\supseteq \mathrm{Int}_{L_T}(F), \\ G \setminus F &\supseteq \mathrm{Ext}_{L_T}(F), \\ G &= \partial_{L_T}(F) \sqcup \mathrm{Int}_{L_T}(F) \sqcup \mathrm{Ext}_{L_T}(F), \\ B_{L_T}(F) &= \partial_{L_T}(F) \sqcup \mathrm{Int}_{L_T}(F). \end{aligned}$$

It is clear that $\ker(Ti_F) \subseteq p_F(\ker T)$ and $\dim_{\mathbb{K}} p_F(\ker T) \le \dim_{\mathbb{K}} p_{B_{L_T}(F)}(\ker T)$.

First we want to show that $\dim_{\mathbb{K}} p_F(\ker T) \le \dim_{\mathbb{K}} \ker(Ti_F) + n|\partial_{L_T}(F)|$. So let

$$P\colon p_{B_{L_T}(F)}(\ker T) \to p_{\partial_{L_T}(F)}(\ker T)$$

be the natural projection. Let $v \in \ker T$ and $p_{\partial_{L_T}(F)}(v) = 0$, then $p_{B_{L_T}(F)}(v) \in \ker P$ and

$$v = \pi_{\mathrm{Int}_{L_T}(F)}(v) + \pi_{\mathrm{Ext}_{L_T}(F)}(v) \in \ker T.$$

By Lemma 2.1.9 it follows that

$$\begin{aligned} T\pi_{\mathrm{Int}_{L_T}(F)}(v) &\in \operatorname{im} i_F \\ T\pi_{\mathrm{Ext}_{L_T}(F)}(v) &\in \operatorname{im} i_{G\setminus F}. \end{aligned}$$

Consequently $T\pi_{\mathrm{Int}_{L_T}(F)}(v) = 0$ and we see that

$$p_F(v) = p_F\left(\pi_{\mathrm{Int}_{L_T}(F)}(v)\right) \in \ker(Ti_F).$$

Thus

$$\begin{aligned}\ker P &= \left\{ p_{B_{L_T}(F)}(v) \,\middle|\, v \in \ker T, p_{\partial_{L_T}(F)}(v) = 0 \right\} \\ &\cong \left\{ p_F(v) \,\middle|\, v \in \ker T, p_{\partial_{L_T}(F)}(v) = 0 \right\} \\ &\subseteq \ker(Ti_F).\end{aligned}$$

By additivity of $\dim_{\mathbb{K}}$, [**War90**], we have that

$$\begin{aligned}\dim_{\mathbb{K}} p_F(\ker T) &\leq \dim_{\mathbb{K}} p_{B_{L_T}(F)}(\ker T) \\ &= \dim_{\mathbb{K}} \ker P + \dim_{\mathbb{K}} \operatorname{im} P \\ &\leq \dim_{\mathbb{K}} \ker(Ti_F) + n|\partial_{L_T}(F)|.\end{aligned}$$

Next we see that

$$\ker(Ti_F) \subseteq \ker(p_F T i_F),$$

and it only remains to show that

$$\dim_{\mathbb{K}} \ker(p_F T i_F) \leq \dim_{\mathbb{K}} \ker(Ti_F) + n|\partial_{L_T}(F)|.$$

Let

$$Q\colon \ker(p_F T i_F) \to \mathbb{K}[F \setminus \operatorname{Int}_{L_T}(F)]^n$$

be the natural projection along $\mathbb{K}[\operatorname{Int}_{L_T}(F)]^n$ and let $v \in \ker Q$, then

$$\begin{aligned}Ti_F(v) &= T\pi_{\operatorname{Int}_{L_T}(F)} i_F(v) \\ &= \pi_F T \pi_{\operatorname{Int}_{L_T}(F)} i_F(v) = 0.\end{aligned}$$

Thus $v \in \ker(Ti_F)$. Recall $F \setminus \operatorname{Int}_{L_T}(F) \subseteq \partial_{L_T}(F)$, now additivity of $\dim_{\mathbb{K}}$ implies

$$\begin{aligned}\dim_{\mathbb{K}} \ker(p_F T i_F) &\leq \dim_{\mathbb{K}} \ker Q + n|\partial_{L_T}(F)| \\ &\leq \dim_{\mathbb{K}} \ker(Ti_F) + n|\partial_{L_T}(F)|.\end{aligned}$$

□

Theorem 2.2.2. *Let G be a group and $\mathbb{K}$ any field. Let $T \in \mathbb{K}[G]^{m\times n}$ be a $\mathbb{K}[G]$-linear map, L_T be its symmetric support and $F \subset G$ be a finite subset.*

Then

$$\begin{aligned}\dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}T\right)\right) &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) + n|\partial_{L_T}\left(F\right)| \\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T i_F\right)\right) + n|\partial_{L_T}\left(F\right)| \\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) + (m+n)|\partial_{L_T}\left(F\right)| \\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}T \cap \operatorname{im} i_F\right) + (m+2n)|\partial_{L_T}\left(F\right)| \\ &\leq \dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}T\right)\right) + (m+2n)|\partial_{L_T}\left(F\right)|.\end{aligned} \tag{2.2.2}$$

Proof. Let $T \in \mathbb{K}[G]^{m\times n}$ and $F \subset G$ be a finite subset. By Lemma 2.1.9(iii) it holds that

$$\operatorname{im}\left(p_F T\right) = p_F\left(\operatorname{im}T\right) = \operatorname{im}\left(p_F T i_{B_{L_T}(F)}\right).$$

Since $B_{L_T}\left(F\right) \setminus F \subseteq \partial_{L_T}\left(F\right)$, we easily see that

$$\dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_{B_{L_T}(F)}\right)\right) \leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) + n|\partial_{L_T}\left(F\right)|.$$

By Lemma 2.1.9(iv) it holds that $\operatorname{im}\left(T i_F\right) = \operatorname{im}\left(\pi_{B_{L_T}(F)} T i_F\right)$ and so

$$\dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) \leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T i_F\right)\right) \leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) + m|\partial_{L_T}\left(F\right)|.$$

By Lemma 2.1.9(ii) it holds that

$$\operatorname{im}\left(\pi_F T i_{\operatorname{Int}_{L_T}(F)}\right) = \operatorname{im}\left(T i_{\operatorname{Int}_{L_T}(F)}\right) \subseteq \operatorname{im}T \cap \operatorname{im} i_F.$$

Therefore

$$\begin{aligned}\dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_F\right)\right) &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T i_{\operatorname{Int}_{L_T}(F)}\right)\right) + n|\partial_{L_T}\left(F\right)| \\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}T \cap \operatorname{im} i_F\right) + n|\partial_{L_T}\left(F\right)|.\end{aligned}$$

The statement follows with Lemma 2.1.8. □

Lemma 2.2.3. *Let $T \in \mathbb{K}[G]^{m\times n}$ and $S \in \mathbb{K}[G]^{s\times r}$ be $\mathbb{K}[G]$-linear maps, L_T, L_S as before and let $F \subset G$ be a finite subset.*

(i) For $r = n$ we have the following dimension formula,

$$\begin{aligned}&\dim_{\mathbb{K}}\left(p_F\left(\ker T + \ker S\right)\right) \\ &= \dim_{\mathbb{K}}\left(p_F\left(\ker T\right)\right) + \dim_{\mathbb{K}}\left(p_F\left(\ker S\right)\right) - \dim_{\mathbb{K}}\left(p_F\left(\ker T\right) \cap p_F\left(\ker S\right)\right).\end{aligned}$$

(ii) We estimate dimensions for inner sums.

(a) For $r = n$ we have

$$\begin{aligned}&\dim_{\mathbb{K}}\big(p_F(\ker T + \ker S)\big)\\ &\le \dim_{\mathbb{K}}\big((\ker T + \ker S)\cap \operatorname{im} i_F\big) + 3n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\ker T + \ker S)\big) + 3n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

(b) For $s = m$ we have

$$\begin{aligned}&\dim_{\mathbb{K}}\big(p_F(\operatorname{im} T + \operatorname{im} S)\big)\\ &\le \dim_{\mathbb{K}}\big((\operatorname{im} T + \operatorname{im} S)\cap \operatorname{im} i_F\big) + 3m\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\operatorname{im} T + \operatorname{im} S)\big) + 3m\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

(c) For $s = n$ we have

$$\begin{aligned}&\dim_{\mathbb{K}}\big(p_F(\ker T + \operatorname{im} S)\big)\\ &\le \dim_{\mathbb{K}}\big((\ker T + \operatorname{im} S)\cap \operatorname{im} i_F\big) + 3n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\ker T + \operatorname{im} S)\big) + 3n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

(iii) We estimate dimensions for sections.

(a) For $r = n$ we have

$$\begin{aligned}&\dim_{\mathbb{K}}\big(p_F(\ker T)\cap p_F(\ker S)\big)\\ &\le \dim_{\mathbb{K}}(\ker T \cap \ker S \cap \operatorname{im} i_F) + n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\ker T \cap \ker S)\big) + n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\ker T)\cap p_F(\ker S)\big) + n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

(b) For $s = m$ we have

$$\begin{aligned}&\dim_{\mathbb{K}}\big(p_F(\operatorname{im} T)\cap p_F(\operatorname{im} S)\big)\\ &\le \dim_{\mathbb{K}}(\operatorname{im} T \cap \operatorname{im} S \cap \operatorname{im} i_F) + m\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\operatorname{im} T \cap \operatorname{im} S)\big) + m\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right)\\ &\le \dim_{\mathbb{K}}\big(p_F(\operatorname{im} T)\cap p_F(\operatorname{im} S)\big) + m\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

(c) For $s = n$ we have

$$\begin{aligned}
&\dim_{\mathbb{K}}\left(p_F(\ker T)\cap p_F(\operatorname{im} S)\right)\\
&\leq \dim_{\mathbb{K}}(\ker T\cap \operatorname{im} S\cap \operatorname{im} i_F)+n\left(|\partial_{L_T}(F)|+|\partial_{L_S}(F)|\right)\\
&\leq \dim_{\mathbb{K}}\left(p_F(\ker T\cap \operatorname{im} S)\right)+n\left(|\partial_{L_T}(F)|+|\partial_{L_S}(F)|\right)\\
&\leq \dim_{\mathbb{K}}\left(p_F(\ker T)\cap p_F(\operatorname{im} S)\right)+n\left(|\partial_{L_T}(F)|+|\partial_{L_S}(F)|\right).
\end{aligned}$$

Proof. Let $T \in \mathbb{K}[G]^{m\times n}$ and $S \in \mathbb{K}[G]^{s\times r}$ be $\mathbb{K}[G]$-linear maps, L_T, L_S as before and let $F \subset G$ be a finite subset. Since p_F is a $\mathbb{K}$-linear map, we have that

$$p_F(\ker T+\ker S)=p_F(\ker T)+p_F(\ker S).$$

(i) As we know that $p_F(\ker T+\ker S) = p_F(\ker T)+p_F(\ker S)$. The statement follows from the dimension formula for $\dim_{\mathbb{K}}$, [**War90**]. That is for any two finite subspaces A, B of a $\mathbb{K}$-vector space C it holds that

$$\dim_{\mathbb{K}}(A+B)+\dim_{\mathbb{K}}(A\cap B)=\dim_{\mathbb{K}}(A)+\dim_{\mathbb{K}}(B).$$

(ii) For inner sums let $M, N \subset \mathbb{K}[G]$ be $\mathbb{K}[G]$-submodules, then

$$M\cap \operatorname{im} i_F+N\cap \operatorname{im} i_F\subseteq (M+N)\cap \operatorname{im} i_F.$$

Now the statements follow from Theorems 2.2.1 and 2.2.2.

(iii) For sections we have to consider kernels and images seperately.

(a) Let $r = n$ and $\alpha \in \ker T\cap\ker S$ then clearly $p_F(\alpha) \in p_F(\ker T)\cap p_F(\ker S)$ and so,

$$p_F(\ker T\cap\ker S)\subseteq p_F(\ker T)\cap p_F(\ker S). \tag{2.2.3}$$

Furthermore, let $\beta \in \ker(Ti_F)\cap\ker(Si_F)$ then $i_F\beta \in \ker T\cap\ker S$ and $p_F(i_F\beta) = \beta$, therefore

$$\ker(Ti_F)\cap\ker(Si_F)\subseteq p_F(\ker T\cap\ker S). \tag{2.2.4}$$

and $\ker(Ti_F)\cap\ker(Si_F) = p_F(\ker T\cap\ker S\cap \operatorname{im} i_F)$. As consequence of Theorem 2.2.1 there exist $\mathbb{K}$-vector spaces $W_T \cong \mathbb{K}[\partial_{L_T}(F)]^n$ and $W_S \cong \mathbb{K}[\partial_{L_S}(F)]^n$ such that

$$p_F(\ker T)\cap p_F(\ker S)\subseteq \left(\ker(Ti_F)+W_T\right)\cap\left(\ker(Si_F)+W_S\right).$$

We write $V := \ker(Si_F) + W_S$. Now let $v_1, v_2 \in \ker(Ti_F)$ and $w \in W_T$ such that $v_1 + w, v_2 + w \in \big(\ker(Ti_F) + W_T\big) \cap V$ and write

$$\underline{v_1 + w}, \underline{v_2 + w} \in \frac{\big(\ker(Ti_F) + W_T\big) \cap V}{\ker Ti_F \cap V}.$$

Then $v_1 + w - v_2 - w = v_1 - v_2 \in \ker(Ti_F) \cap V$ and thus $\underline{v_1 + w} = \underline{v_2 + w}$. This implies

$$\dim_{\mathbb{K}}\Big(\big(\ker(Ti_F) + W_T\big) \cap V\Big) \leq \dim_{\mathbb{K}}\big(\ker(Ti_F \cap V)\big) + n|\partial_{L_T}(F)|. \tag{2.2.5}$$

Repeating the same argument for $\dim_{\mathbb{K}}\Big(\big(\ker(Si_F) + W_S\big) \cap \ker(Ti_F)\Big)$ we arrive at

$$\begin{aligned}\dim_{\mathbb{K}}\big(p_F(\ker T) \cap p_F(\ker S)\big) \leq{}& \dim_{\mathbb{K}}\big(\ker(Ti_F) \cap \ker(Si_F)\big)\\ &+ n\left(|\partial_{L_T}(F)| + |\partial_{L_S}(F)|\right).\end{aligned}$$

Thus together with (2.2.4) and (2.2.3) this implies the second statement.

(b) Next we look at the images of T and S. Let $s = m$ and so we deduce from Lemma 2.1.9 that

$$\begin{aligned}\operatorname{im}\left(Ti_{\operatorname{Int}_{L_T}(F)}\right) \cap \operatorname{im}\left(Si_{\operatorname{Int}_{L_S}(F)}\right) &\subseteq \operatorname{im} T \cap \operatorname{im} S \cap \operatorname{im} i_F\\ &\subseteq \pi_F(\operatorname{im} T \cap \operatorname{im} S)\\ &\subseteq \pi_F(\operatorname{im} T) \cap \pi_F(\operatorname{im} S)\end{aligned}$$

and by the argument that gives estimate (2.2.5) it holds for any $V \subset \mathbb{K}[F]^n$ that

$$\begin{aligned}\dim_{\mathbb{K}}\big(\pi_F(\operatorname{im} T) \cap V\big) &= \dim_{\mathbb{K}}\left(\operatorname{im}\left(\pi_F T i_{B_{L_T}(F)}\right) \cap V\right)\\ &= \dim_{\mathbb{K}}\left(\left(\operatorname{im}\left(\pi_F T i_{\operatorname{Int}_{L_T}(F)}\right) + \operatorname{im}\left(\pi_F T i_{\partial_{L_T}(F)}\right)\right) \cap V\right)\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T i_{\operatorname{Int}_{L_T}(F)}\right) \cap V\right) + n|\partial_{L_T}(F)|.\end{aligned}$$

The equivalent holds for $\pi_F(\operatorname{im} S) \cap V$ and so translating all inclusions to $\dim_{\mathbb{K}}$ gives the statement.

(c) And finally let $s = n$, then we find that

$$\begin{aligned}\operatorname{im}\left(p_F T i_{\operatorname{Int}_{L_T}(F)}\right) \cap \ker\left(S i_F\right) &\subseteq p_F\left(\operatorname{im} T \cap \ker S \cap \operatorname{im} i_F\right)\\ &\subseteq p_F\left(\operatorname{im} T \cap \ker S\right)\\ &\subseteq p_F\left(\operatorname{im} T\right) \cap p_F\left(\ker S\right),\end{aligned}$$

and as seen above

$$\dim_{\mathbb{K}}\left(p_F\left(\operatorname{im} T\right) \cap p_F\left(\ker S\right)\right) \leq \dim_{\mathbb{K}}\left(p_F\left(\operatorname{im} T\right) \cap \ker\left(S i_F\right)\right) + n|\partial_{L_S}(F)|.$$

Since

$$\begin{aligned}p_F\left(\operatorname{im} T\right) \cap \ker\left(S i_F\right) &= \operatorname{im}\left(p_F T i_{B_{L_T}(F)}\right) \cap \ker\left(S i_F\right)\\ &= \left(\operatorname{im}\left(p_F T i_{\operatorname{Int}_{L_T}(F)}\right) + \operatorname{im}\left(p_F T i_{\partial_{L_T}(F)}\right)\right) \cap \ker\left(S i_F\right),\end{aligned}$$

we deduce, as for estimate (2.2.5), that

$$\begin{aligned}&\dim_{\mathbb{K}}\left(p_F\left(\operatorname{im} T\right) \cap \ker\left(S i_F\right)\right)\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(p_F T i_{\operatorname{Int}_{L_T}(F)}\right) \cap \ker\left(S i_F\right)\right) + n|\partial_{L_T}(F)|.\end{aligned}$$

This concludes the proof. □

2.3. Functions on the set of finite subsets

Definition 2.3.1. Let G be a discrete group and let $\mathcal{F}(G)$ be the set of non-empty finite subsets of G. Let $M \subseteq \mathbb{K}[G]^n$ be a $\mathbb{K}[G]$-submodule, for any field $\mathbb{K}$. We consider two maps

(i) $\overline{h}\colon \mathcal{F}(G) \to \mathbb{N}$ with

$$\overline{h}(F) := \dim_{\mathbb{K}}\left(p_F M\right), \tag{2.3.1}$$

(ii) $\underline{h}\colon \mathcal{F}(G) \to \mathbb{N}$ with

$$\underline{h}(F) := \dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_F\right). \tag{2.3.2}$$

Lemma 2.3.2. *We have the following properties for the maps above.*

(i) $\underline{h}(F) \leq \overline{h}(F)$ *for all* $F \in \mathcal{F}(G)$.
(ii) $\underline{h}(F), \overline{h}(F) \leq n|F|$ *for all* $F \in \mathcal{F}(G)$.
(iii) right invariance: $\overline{h}(Fa) = \overline{h}(F)$ *and* $\underline{h}(Fa) = \underline{h}(F)$.
(iv) subadditivity: $\overline{h}(A \cup B) \leq \overline{h}(A) + \overline{h}(B)$ *for any* $A, B \in \mathcal{F}(G)$.

(v) almost superadditivity: For any finite subset $\{A_1, \ldots, A_n\} \subset \mathcal{F}(G)$

$$\underline{h}\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \underline{h}(A_i) - n\left(\sum_{i=1}^{n} |A_i| - |\bigcup_{i=1}^{n} A_i|\right).$$

Proof. Let $M \subseteq \mathbb{K}[G]^n$ be a $\mathbb{K}[G]$-submodule.

(i) Follows from Lemma 2.1.8.

(ii) Follows from the definition.

(iii) Let $F \subset G$ be a finite subset and $a \in G$, then

$$\dim_{\mathbb{K}}(p_{Fa}M) = \dim_{\mathbb{K}}\left(p_F\left(Ma^{-1}\right)\right) = \dim_{\mathbb{K}}(p_F M),$$
$$\dim_{\mathbb{K}}(M \cap \operatorname{im} i_{Fa}) = \dim_{\mathbb{K}}\left(\left(Ma^{-1}\right) \cap \operatorname{im} i_F\right) = \dim_{\mathbb{K}}(M \cap \operatorname{im} i_F).$$

(iv) Let $A, B \in \mathcal{F}(G), C := B \setminus (A \cap B)$ and $\alpha \in M$, then

$$\pi_{A\cup B}(\alpha) = \pi_A(\alpha) + \pi_C(\alpha).$$

So $\pi_{A\cup B}M \subseteq \pi_A(M) + \pi_C(M)$ and since $\dim_{\mathbb{K}}(p_C M) \leq \dim_{\mathbb{K}}(p_B M)$ the statement follows.

(v) Let $A, B \in \mathcal{F}(G)$ and $C := B \setminus (A \cap B)$ then

$$M \cap \operatorname{im} i_{A\cup B} \supseteq M \cap \operatorname{im} i_A \oplus M \cap \operatorname{im} i_C.$$

Furthermore define

$$P\colon M \cap \operatorname{im} i_B \to \mathbb{K}[A \cap B]^n$$

by $p_{A\cap B}$. Let $v \in \ker P$ then $v = \pi_C(v) \in M \cap \operatorname{im} i_C$ and by additivity it follows that

$$\dim_{\mathbb{K}}(M \cap \operatorname{im} i_C) \geq \dim_{\mathbb{K}}(M \cap \operatorname{im} i_B) - n|A \cap B|.$$

The statement follows by iteration and

$$\sum_{i=2}^{n} |\left(\bigcup_{j=1}^{i-1} A_j\right) \cap A_i| = \sum_{i=2}^{n}\left(|\bigcup_{j=1}^{i-1} A_j| + |A_i| - |\bigcup_{j=1}^{i} A_j|\right)$$
$$= \sum_{i=1}^{n} |A_i| - |\bigcup_{i=1}^{n} A_i|$$

□

CHAPTER 3

Convergence Theorem

Now we want to find a convergence result for functionals as introduced in the last section of the previous chapter. This will be the foundation of the dimension function we are looking for.

3.1. Fillings of finite subsets

First we cite this section from [**Kri07**]. Let G be a discrete group and let $\mathcal{F}(G)$ be the set of non-empty finite subsets of G.

Definition 3.1.1. A family of finite subsets (A_i) is called ϵ-disjoint, if there is a disjoint family $(B_i \subset A_i)$ such that

$$\frac{|B_i|}{|A_i|} \geq 1 - \epsilon, \tag{3.1.1}$$

for all i.

Lemma 3.1.2. *Let (A_i) be a ϵ-disjoint family, then*

$$(1-\epsilon)\sum_i |A_i| \leq |\bigcup_i A_i| \tag{3.1.2}$$

Lemma 3.1.3. *Let G be a discrete group, $K \subset G$ finite, $0 < \epsilon < 1$. Let $A_1, A_2, \ldots, A_n$ be an ϵ-disjoint family of non-empty finite subsets of G and let $\eta > 0$ such that $\frac{|\partial_K(A_i)|}{|A_i|} \leq \eta$ for all i. Then it follows that*

$$\frac{|\partial_K\left(\bigcup_{i=1}^n A_i\right)|}{|\bigcup_{i=1}^n A_i|} \leq \frac{\eta}{1-\epsilon}. \tag{3.1.3}$$

Lemma 3.1.4. *Let G be a discrete group and let K, A and Ω be finite subsets of G such that $\emptyset \neq A \subset \Omega$. Suppose that there exists $\epsilon > 0$ such that $|\Omega \setminus A| \geq \epsilon|\Omega|$. Then*

$$\frac{|\partial_K\left(\Omega \setminus A\right)|}{|\Omega \setminus A|} \leq \epsilon^{-1}\left(\frac{|\partial_K(\Omega)|}{|\Omega|} + \frac{|\partial_K(A)|}{|A|}\right). \tag{3.1.4}$$

Lemma 3.1.5. *Let G be a discrete group and let A and B be two finite subsets of G. Then one has*

$$\sum_{g \in G} |Ag \cap B| = |A||B|. \tag{3.1.5}$$

Definition 3.1.6. Let G be a discrete group. Let K and Ω be finite subsets of G and $\epsilon > 0$. A subset $R \subset G$ is called an (ϵ, K)-filling of Ω if the following conditions are satisfied:

(i) $R \subseteq \mathrm{Int}_K(\Omega)$;
(ii) the family $(Kg)_{g \in R}$ is ϵ-disjoint.

Lemma 3.1.7. [**Kri07**, Lemma 3.5] *Let Ω and K be non-empty finite subsets of a group G. For all $0 < \epsilon \leq 1$ there exists a finite subset $R \subset G$ such that:*

(i) R is an (ϵ, K)-filling of Ω;
(ii) $|KR| \geq \epsilon(1 - \alpha_0)|\Omega|$, where $\alpha_0 := \frac{|\partial_K(\Omega)|}{|\Omega|}$.

3.2. Variation of the Ornstein-Weiss lemma

We will prove a variation of the Ornstein-Weiss lemma [**OW87**] based on [**Kri07**] and [**Kri10**]. The following lemma is extracted from [**Kri10**]. Again we denote the set of non-empty finite subsets of a discrete group G by $\mathcal{F}(G)$.

Lemma 3.2.1. *Let G be a discrete group and let $0 < \epsilon \leq \frac{1}{2}$ and*

$$n = n_\epsilon := \lceil \frac{\log \epsilon}{\log(2 - \epsilon) - \log 2} \rceil.$$

Let $K_1, \ldots, K_n$ be a family of finite subsets of G such that

$$\frac{|\partial_{K_i}(K_j)|}{|K_j|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i < j \leq n. \tag{3.2.1}$$

Then for any finite subset $D \subset G$ with

$$\frac{|\partial_{K_i}(D)|}{|D|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i \leq n, \tag{3.2.2}$$

we have a decomposition into disjoint subsets

$$D = \bigsqcup_{j=1}^{n} K_j R_j \sqcup D_n, \tag{3.2.3}$$

such that $|D_n| \leq \epsilon |D|$. Where $R_j \subset G$ are finite, possible empty, and $(K_j g)_{g \in R_j}$ are ϵ-disjoint families.

Proof. Let G be a discrete group and let $0 < \epsilon \le \frac{1}{2}$ and $n := n_\epsilon = \lceil \frac{\log \epsilon}{\log(2-\epsilon) - \log 2} \rceil$. Let $K_1, \dots, K_n$ be a family of finite subsets of G such that

$$\frac{|\partial_{K_i}(K_j)|}{|K_j|} \le \epsilon^{2n} \quad \text{for } 1 \le i < j \le n.$$

Let $D \subset G$ be a finite subset with

$$\frac{|\partial_{K_i}(D)|}{|D|} \le \epsilon^{2n} \quad \text{for } 1 \le i \le n.$$

We will find a decomposition into disjoint subsets

$$D = \bigsqcup_{j=1}^{n} K_j R_j \sqcup D_n$$

where $R_j \subset G$ are finite, but possible empty, and $(K_j g)_{g \in R_j}$ are ϵ-disjoint families such that $|D_n| \le \epsilon |D|$.

We define D_j for $1 \le j \le n$ recursively, and set $D_0 := D$. We use Lemma 3.1.7 with $\Omega = D_{j-1}$ and $K = K_{n-j+1}$ to find $R_{n-j+1} \in \mathcal{F}(G)$, an (ϵ, K_n)-filling of D_{j-1} such that

$$\frac{|K_{n-j+1} R_{n-j+1}|}{|D_{j-1}|} \ge \epsilon \left(1 - \frac{|\partial_{K_{n-j+1}}(D_{j-1})|}{|D_{j-1}|} \right).$$

So define $D_j := D_{j-1} \setminus K_{n-j+1} R_{n-j+1}$ and see that

$$|D_j| \le |D_{j-1}| \left(1 - \epsilon \left(1 - \frac{|\partial_{K_{n-j+1}}(D_{j-1})|}{|D_{j-1}|} \right) \right).$$

If for any $1 \le j < n$: $|D_j| \le \epsilon |D_{j-1}|$ then $|D_n| \le \epsilon |D|$, since $|D_j| \le |D_{j-1}|$ for all $1 \le j \le n$, and we are done.

So assume $|D_j| > \epsilon |D_{j-1}|$ for all $1 \le j < n$, and recall that $\frac{|\partial_{K_i}(K_j)|}{|K_j|} \le \epsilon^{2n}$, then it follows from Lemma 3.1.3 and Lemma 1.2.3 that

$$\begin{aligned} \frac{|\partial_{K_k}(K_{n-j+1} R_{n-j+1})|}{|K_{n-j+1} R_{n-j+1}|} &\le \frac{|\partial_{K_k}\left(\bigcup_{g \in R_{n-j+1}} K_{n-j+1} g\right)|}{|K_{n-j+1}|} \\ &\le \frac{\epsilon^{2n}}{1-\epsilon}. \end{aligned}$$

Now Lemma 3.1.4 shows that

$$\frac{|\partial_{K_k}(D_j)|}{|D_j|} \leq \epsilon^{-1}\left(\frac{|\partial_{K_k}(D_{j-1})|}{|D_{j-1}|} + \frac{|\partial_{K_k}(K_{n-j+1}R_{n-j+1})|}{|K_{n-j+1}R_{n-j+1}|}\right)$$
$$\leq \epsilon^{-1}\left(\frac{|\partial_{K_k}(D_{j-1})|}{|D_{j-1}|} + \frac{\epsilon^{2n}}{1-\epsilon}\right),$$

for $1 \leq k \leq n$ and all $1 \leq j < n$.

Then $\frac{|\partial_{K_i}(D)|}{|D|} \leq \epsilon^{2n}$ together with $0 < \epsilon \leq \frac{1}{2}$ implies

$$\frac{|\partial_{K_k}(D_j)|}{|D_j|} \leq \epsilon^{-1}\left(\epsilon^{-1}\left(\frac{|\partial_{K_k}(D_{j-2})|}{|D_{j-2}|} + \frac{\epsilon^{2n}}{1-\epsilon}\right) + \frac{\epsilon^{2n}}{1-\epsilon}\right)$$
$$\leq \epsilon^{2n-j} + \sum_{i=1}^{j} \frac{\epsilon^{2n-i}}{1-\epsilon} \leq (2j+1)\,\epsilon^{2n-j},$$

for all $1 \leq k \leq n$. We get for $0 \leq j < n$

$$|D_{j+1}| \leq |D_j|\left(1 - \epsilon\left(1 - (2j+1)\,\epsilon^{2n-j}\right)\right).$$

and thus

$$|D_n| \leq |D| \prod_{j=0}^{n-1}\left(1 - \epsilon\left(1 - (2j+1)\,\epsilon^{2n-j}\right)\right)$$
$$\leq |D|\left(1 - \epsilon\left(1 - (2n-1)\,\epsilon^{n+1}\right)\right)^n$$
$$\leq |D|\left(1 - \frac{\epsilon}{2}\right)^n$$
$$\leq \epsilon|D|.$$

Where we used that for $k \in \mathbb{N}$, $\epsilon \leq \frac{1}{2}$ and $n = \lceil \frac{\log \epsilon}{\log(2-\epsilon) - \log 2} \rceil$ it holds that

$$(2k-1)\,\epsilon^{k+1} \leq 2k\epsilon^{k+1} \leq k\left(\frac{1}{2}\right)^k \leq \frac{1}{2},$$

as well as $\left(1 - \frac{\epsilon}{2}\right)^n \leq \epsilon$. $\square$

We come to the convergence theorems, wherein we adapt the main result of [**Kri07**], respectively [**Kri10**], and its proof to a slightly different setting.

Theorem 3.2.2. *Let G be an amenable discrete group and let $\underline{h}\colon \mathcal{F}(G) \to \mathbb{R}$ be a map satisfying the following conditions:*

(i) $\underline{h}$ is right-invariant, $\underline{h}(Fa) = \underline{h}(F)$ for all $F \in \mathcal{F}(G), a \in G$;

(ii) $\underline{h}$ *is almost superadditive, there exists* $C > 0$ *such that* $\underline{h}\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \underline{h}(A_i) - C\left(\sum_{i=1}^n |A_i| - |\bigcup_{i=1}^n A_i|\right)$ *for any finite subset* $\{A_1, \dots, A_n\} \subset \mathcal{F}(G)$*;*

(iii) $\underline{h}$ *is relatively bounded, there exists* $C > 0$ *such that* $0 \leq \underline{h}(F) \leq C|F|$ *for all* $F \in \mathcal{F}(G)$*.*

Then for every Følner net (F_i) *in* $\mathcal{F}(G)$ *the limit*

$$\lim_i \frac{\underline{h}(F_i)}{|F_i|}$$

exists, is finite and independent of the choice of Følner net.

Proof. Let G be an amenable group and $\underline{h}\colon \mathcal{F}(G) \to \mathbb{R}$ a map satisfying the conditions above. Furthermore let (F_i) be a Følner net in $\mathcal{F}(G)$ and fix $0 < \epsilon \leq \frac{1}{2}$. Let $x_i := \frac{\underline{h}(F_i)}{|F_i|}$ and since (x_i) is bounded we have a largest cluster point

$$\lambda := \limsup_i x_i.$$

By Theorem 1.1.3 and Definition 1.1.2 there exists a subnet $\left(x_{\phi(j)}\right)$ such that $x_{\phi(j)} \geq \lambda - \epsilon$ for all j. Note that $\left(F_{\phi(j)}\right)$ is also a Følner net. Now let $n = n_\epsilon := \lceil \frac{\log \epsilon}{\log(2-\epsilon) - \log 2} \rceil$ and let $\mathcal{E} \subset \mathcal{F}(G)$ be a finite family of finite subsets of G then there exists $F_{\phi(j_\mathcal{E})}$ such that

$$\forall A \in \mathcal{E}\colon \frac{|\partial_A\left(F_{\phi(j_\mathcal{E})}\right)|}{|F_{\phi(j_\mathcal{E})}|} \leq \epsilon^{2n}.$$

Therefore there exists a finite sequence $K_1, \dots, K_n$ from (F_i) such that

$$\frac{\underline{h}(K_i)}{|K_i|} \geq \lambda - \epsilon \quad \text{for } 1 \leq i \leq n, \tag{3.2.4}$$

$$\frac{|\partial_{K_i}\left(K_j\right)|}{|K_j|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i < j \leq n. \tag{3.2.5}$$

Let $D \subset G$ be a finite subset with

$$\frac{|\partial_{K_i}\left(D\right)|}{|D|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i \leq n. \tag{3.2.6}$$

By Lemma 3.2.1 we find a decomposition into disjoint subsets

$$D = \bigsqcup_{j=1}^n K_j R_j \sqcup D_n$$

where $|D_n| \leq \epsilon |D|$, $R_j \subset G$ are finite, but possible empty, and $(K_j g)_{g \in R_j}$ are ϵ-disjoint families.

It follows that

$$\frac{\underline{h}(D)}{|D|} \geq \frac{\underline{h}(\bigsqcup_{j=1}^{n} K_j R_j)}{|D|} + \frac{\underline{h}(D_n)}{|D|} \geq \frac{\sum_{j=1}^{n} \underline{h}(K_j R_j)}{|D|}.$$

Since $(K_j g)_{g\in R_j}$ are ϵ-disjoint families that do not intersect, the combination of those is again an ϵ-disjoint family and Lemma 3.1.2 gives

(3.2.7)

$$\begin{aligned}\frac{\underline{h}(\bigsqcup_{j=1}^{n} K_j R_j)}{|D|} &\geq \sum_{j=1}^{n}\sum_{g\in R_j} \frac{\underline{h}(K_j)}{|K_j|}\frac{|K_j g|}{|D|} - C\frac{\sum_{j=1}^{n}\sum_{g\in R_j}|K_j g| - |\bigsqcup_{j=1}^{n} K_j R_j|}{|D|} \\ &\geq \sum_{j=1}^{n}\sum_{g\in R_j} \frac{\underline{h}(K_j)}{|K_j|}\frac{|K_j g|}{|D|} - C\frac{|D|/(1-\epsilon) - (1-\epsilon)\,|D|}{|D|}.\end{aligned}$$

Using (3.2.4) this shows,

$$\begin{aligned}\frac{\underline{h}(D)}{|D|} &\geq (\lambda-\epsilon)\sum_{j=1}^{n}\sum_{g\in R_j}\frac{|K_j g|}{|D|} - C\frac{2\epsilon-\epsilon^2}{1-\epsilon} \\ &\geq (\lambda-\epsilon)(1-\epsilon) - 4\epsilon C \\ &\geq \lambda - \epsilon\,(\lambda + 4C + 1)\,.\end{aligned}$$

Since (F_i) is Følner, we get a lower bound $i_0 \in I$ such thatF_i meets condition (3.2.6) for all $i \geq i_0$. We consider the subnet (F_k) which converges to the limit inferior μ and the corresponding lower bound k_0. Then for $k > k_0$ we have

$$\mu = \lim_k \frac{\underline{h}(F_k)}{|F_k|} \geq \lambda - \epsilon\,(\lambda + 5C)\,.$$

All the above holds for any $0 < \epsilon \leq \frac{1}{2}$, so we may take the limit of ϵ tending to 0 and get

$$\liminf_i x_i = \mu \geq \lambda = \limsup_i x_i.$$

It remains to be seen, that the limit is independent of the choice of Følner net. Consider any two Følner nets $(A_i)\,,\big(B_j\big)$ then by Lemma 1.3.3 there exists a Følner net (F_k) such that $(A_i)\,,\big(B_j\big)$ are subnets. Therefore $\left(\frac{\underline{h}(A_i)}{|A_i|}\right), \left(\frac{\underline{h}(B_j)}{|B_j|}\right)$ are subnets of $\left(\frac{\underline{h}(F_k)}{|F_k|}\right)$ and converge to the same cluster point. This gives independence from the choice of Følner net. □

The following theorem is a reproduction of [**Kri10**, Theorem 1.1].

Theorem 3.2.3. *Let G be an amenable discrete group and let $\overline{h}\colon \mathcal{F}(G) \to \mathbb{R}$ be a map satisfying the following conditions:*

(i) $\overline{h}$ *is right-invariant,* $\overline{h}(Fa) = \overline{h}(F)$ *for all* $F \in \mathcal{F}(G), a \in G$*;*

(ii) $\overline{h}$ *is subadditive, that is* $\overline{h}(A \cup B) \leq \overline{h}(A) + \overline{h}(B)$ *for any two finite subsets* $A, B \in \mathcal{F}(G)$*.*

Then for every Følner net (F_i) *in* $\mathcal{F}(G)$ *the limit*

$$\lim_i \frac{\overline{h}(F_i)}{|F_i|}$$

exists, is finite and independent of the choice of Følner net.

Proof. Let G be an amenable group and $\overline{h} \colon \mathcal{F}(G) \to \mathbb{R}$ a map satisfying the conditions above. Furthermore let (F_i) be a Følner net in $\mathcal{F}(G)$ and fix $0 < \epsilon \leq \frac{1}{2}$. The conditions imply that $\overline{h}(A) \leq \overline{h}\left(\{e\}\right)|A|$ for all $A \in \mathcal{F}(G)$. Let $x_i := \frac{\underline{h}(F_i)}{|F_i|}$ and since (x_i) is bounded we have a least cluster point

$$\lambda := \liminf_i x_i.$$

Therefore there exists a finite sequence $K_1, \dots, K_n$ from (F_i) such that

$$\frac{\overline{h}(K_i)}{|K_i|} \leq \lambda + \epsilon \quad \text{for } 1 \leq i \leq n, \tag{3.2.8}$$

$$\frac{|\partial_{K_i}(K_j)|}{|K_j|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i < j \leq n. \tag{3.2.9}$$

Let $D \subset G$ be a finite subset with

$$\frac{|\partial_{K_i}(D)|}{|D|} \leq \epsilon^{2n} \quad \text{for } 1 \leq i \leq n. \tag{3.2.10}$$

By Lemma 3.2.1 we find a decomposition into disjoint subsets

$$D = \bigsqcup_{j=1}^{n} K_j R_j \sqcup D_n$$

where $|D_n| \leq \epsilon|D|$, $R_j \subset G$ are finite, but possible empty, and $(K_j g)_{g \in R_j}$ are ϵ-disjoint families.

It follows that

$$\frac{\overline{h}(D)}{|D|} \leq \frac{\overline{h}(\bigsqcup_{j=1}^{n} K_j R_j)}{|D|} + \frac{\overline{h}(D_n)}{|D|} \leq \frac{\sum_{j=1}^{n} \overline{h}(K_j R_j)}{|D|} + \overline{h}\left(\{e\}\right)|D_n|.$$

$$\frac{\overline{h}(\bigsqcup_{j=1}^{n} K_j R_j)}{|D|} \leq \sum_{j=1}^{n} \sum_{g \in R_j} \frac{\overline{h}(K_j)}{|K_j|} \frac{|K_j g|}{|D|}$$

Using (3.2.8) this shows,

$$\frac{\overline{h}(\bigsqcup_{j=1}^{n} K_j R_j)}{|D|} \leq (\lambda+\epsilon) \sum_{j=1}^{n} \sum_{g \in R_j} \frac{|K_j g|}{|D|}$$

Since $(K_j g)_{g \in R_j}$ are ϵ-disjoint families that do not intersect, the combination of those is again an ϵ-disjoint family and Lemma 3.1.2 gives

$$\begin{aligned} \frac{\overline{h}(D)}{|D|} &\leq (\lambda+\epsilon) \sum_{j=1}^{n} \sum_{g \in R_j} \frac{|K_j g|}{|D|} + \epsilon \overline{h}\left(\{e\}\right) \\ &\leq \frac{\lambda+\epsilon}{1-\epsilon} + \epsilon \overline{h}\left(\{e\}\right). \end{aligned}$$

Since (F_i) is Følner, we get a lower bound $i_0 \in I$ such that F_i meets condition (3.2.10) for all $i \geq i_0$. We consider the subnet (F_k) which converges to the limit superior μ and the corresponding lower bound k_0. Then for $k > k_0$ we have

$$\mu = \lim_k \frac{h(F_k)}{|F_k|} \leq \frac{\lambda+\epsilon}{1-\epsilon} + \epsilon \overline{h}\left(\{e\}\right).$$

All the above holds for any $0 < \epsilon \leq \frac{1}{2}$, so we may take the limit of ϵ tending to 0 and get

$$\limsup_i x_i = \mu \leq \lambda = \liminf_i x_i.$$

□

CHAPTER 4

Følner dimension

In this chapter we will define the Følner dimension $\dim_{\mathbb{K}[G]}(M)$ of a $\mathbb{K}[G]$-submodule M for any field $\mathbb{K}$ and a discrete, amenable group G. Recall that by Lemma 2.1.5 we may consider left $\mathbb{K}[G]$-modules as right $\mathbb{K}[G]$-modules and thus we write $\mathbb{K}[G]$-modules to mean right $\mathbb{K}[G]$-modules. Furthermore we take matrices $T \in \mathbb{K}[G]^{m\times n}$ as $\mathbb{K}[G]$-linear maps by left multiplication.

4.1. Definition

Definition 4.1.1. Let G be a group and $\mathbb{K}$ a field. We define the sets of $\mathbb{K}[G]$-modules which come from $\mathbb{K}[G]$-matrices,

$$\begin{aligned} \mathcal{K}_n(G) &:= \left\{\ker T \;\middle|\; T \in \mathbb{K}[G]^{m\times n}, m \in \mathbb{N}\right\}, \\ \mathcal{R}_n(G) &:= \left\{\operatorname{im} T \;\middle|\; T \in \mathbb{K}[G]^{n\times m}, m \in \mathbb{N}\right\}, \end{aligned} \tag{4.1.1}$$

for $n \in \mathbb{N}$.

Remark 4.1.2. (i) Note that

$$\begin{aligned} &\left\{M \;\middle|\; \exists n \in \mathbb{N}\colon M \subseteq \mathbb{K}[G]^n \text{ is a finitely generated } \mathbb{K}[G]\text{-submodule}\right\} \\ &\subset \left\{N_1 + N_2, N_1 \cap N_2 \;\middle|\; N_1, N_2 \in \mathcal{K}_n(G) \cup \mathcal{R}_n(G), n \in \mathbb{N}\right\} \\ &\subset \left\{M \;\middle|\; \exists n \in \mathbb{N}\colon M \subseteq \mathbb{K}[G]^n \text{ is a } \mathbb{K}[G]\text{-submodule}\right\}. \end{aligned} \tag{4.1.2}$$

(ii) $\mathcal{K}_n(G)$ is closed under finite sections, see Corollary 2.1.2.

(iii) $\mathcal{R}_n(G)$ is closed under finite inner sums, since

$$\operatorname{im} T + \operatorname{im} S = \operatorname{im}\begin{pmatrix} T & S \end{pmatrix}. \tag{4.1.3}$$

Theorem 4.1.3 (Følner dimension)**.** *Let G be an amenable group with Følner net (F_i), $\mathbb{K}$ be a field and $n \in \mathbb{N}$, then for any submodule $M \subseteq \mathbb{K}[G]^n$ it holds that*

$$\dim_{\mathbb{K}[G]}(M) := \lim_i \frac{\dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_{F_i}\right)}{|F_i|}, \tag{4.1.4}$$

$$\overline{\dim_{\mathbb{K}[G]}}(M) := \lim_i \frac{\dim_{\mathbb{K}}\left(p_{F_i}(M)\right)}{|F_i|}, \tag{4.1.5}$$

are well-defined and independent of the choice of (F_i).

Proof. This follows from Theorems 3.2.3 and 3.2.2 with Lemma 2.3.2. □

Remark 4.1.4. These definitions make use of the canonical $\mathbb{K}[G]$-module structure of $\mathbb{K}[G]^n$ in how the support of an element is defined. By Corollary 1.3.9 we can require all Følner sets to be symmetric. Thus for a $\mathbb{K}[G]$-bimodule that inherits the canonical $\mathbb{K}[G]$-module structure of $\mathbb{K}[G]^n$ it makes no difference whether we consider its natural right $\mathbb{K}[G]$-module structure, or the one gained by Lemma 2.1.5 from its left $\mathbb{K}[G]$-module structure.

Nevertheless, we assume that for every $\mathbb{K}[G]$-module a right $\mathbb{K}[G]$-module structure was chosen beforehand.

Theorem 4.1.5 (Properties). *The Følner dimension* $\dim_{\mathbb{K}[G]}$ *fulfills the following properties for all* $n \in \mathbb{N}$.

(i) $\dim_{\mathbb{K}[G]}\left(\mathbb{K}[G]^n\right) = n$.

(ii) $\dim_{\mathbb{K}[G]}(M) = 0 \iff M = 0$ *for any* $\mathbb{K}[G]$*-submodule* $M \subseteq \mathbb{K}[G]^n$.

(iii) For $N_1, N_2 \in \mathcal{K}_n(G) \cup \mathcal{R}_n(G)$ *we have*

$$\dim_{\mathbb{K}[G]}(N_1 + N_2) = \overline{\dim_{\mathbb{K}[G]}}(N_1 + N_2),$$
$$\dim_{\mathbb{K}[G]}(N_1 \cap N_2) = \overline{\dim_{\mathbb{K}[G]}}(N_1 \cap N_2).$$

(iv) Monotony.

(v) Continuity from below.

(vi) Dimension formula.

(vii) Additivity.

(viii) Induction from subgroups.

Proof. We will prove all properties except the first one in separate lemmas in the next section.

(i) Let $F \subset G$ be a finite subset, then

$$\dim_{\mathbb{K}}\left(\mathbb{K}[G]^n \cap \operatorname{im} i_F\right) = \dim_{\mathbb{K}}\left(\mathbb{K}[F]^n\right) = n|F|.$$

This holds for all Følner sets and so the statement follows.

(ii) Lemma 4.2.1

(iii) Lemma 4.2.3.

(iv) Lemma 4.2.4.

(v) Lemma 4.2.5.

(vi) Lemma 4.2.6.

(vii) Lemma 4.2.7.
(viii) Lemma 4.2.8.

□

Next we extend the previous definition to all finitely generated $\mathbb{K}[G]$-modules. But first note a consequence of additivity.

Corollary 4.1.6. *Let $M \subset \mathbb{K}[G]^m$ be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear, then*

$$\dim_{\mathbb{K}[G]}(M) = n - \dim_{\mathbb{K}[G]}(\ker T). \tag{4.1.6}$$

Proof. Apply Lemma 4.2.7 to $T\colon \mathbb{K}[G]^n \to M \subset \mathbb{K}[G]^m$. □

Definition 4.1.7 (Finitely generated $\mathbb{K}[G]$-modules)**.** Let G be a discrete, amenable group, $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. We define

$$\dim_{\mathbb{K}[G]}(M) := n - \dim_{\mathbb{K}[G]}(\ker T). \tag{4.1.7}$$

Theorem 4.1.8 (Well-definition)**.** *Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let M and N be finitely generated $\mathbb{K}[G]$-modules with a $\mathbb{K}[G]$-linear isomorphism $f\colon N \to M$. Then*

$$\dim_{\mathbb{K}[G]}(N) = \dim_{\mathbb{K}[G]}(M). \tag{4.1.8}$$

Proof. Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let M and N be finitely generated $\mathbb{K}[G]$-modules with an $\mathbb{K}[G]$-linear isomorphism $f\colon N \to M$. Since M and N are finitely generated there exist surjective and $\mathbb{K}[G]$-linear maps $T\colon \mathbb{K}[G]^n \to N$ and $S\colon \mathbb{K}[G]^m \to M$ for some $n, m \in \mathbb{N}$. We may lift f as follows. Let $(b_1, \dots, b_n)$ be a basis for $\mathbb{K}[G]^n$ and consider $f\big(T(b_i)\big) \in M$. Then by surjectivity of S there exist $c_i \in \mathbb{K}[G]^m$ such that $S(c_i) = f\big(T(b_i)\big)$ for $1 \le i \le n$. We define the lift

$$\tilde{f} : \mathbb{K}[G]^n \to \mathbb{K}[G]^m$$

by $\tilde{f}(b_i) := c_i$ for $1 \le i \le n$ and $\mathbb{K}[G]$-linearity, by this definition $S \circ \tilde{f} = f \circ T$. Let $K := \ker T$ and $L := \ker S$ then $S\Big(\tilde{f}(K)\Big) = f\big(T(K)\big) = 0$, so $\tilde{f}(K) \subseteq L$. Furthermore let $\alpha \in \mathbb{K}[G]^m$ then $S(\alpha)$ has a preimage $\beta \in \mathbb{K}[G]^n$ under $f \circ T$, by surjectivity of f and T. Thus $\alpha - \tilde{f}(\beta) \in L$ and

$$\operatorname{im} \tilde{f} + L = \mathbb{K}[G]^m.$$

We want to show that $n - \dim_{\mathbb{K}[G]}(K) = m - \dim_{\mathbb{K}[G]}(L)$. We need another step before we can use additivity. Let $\beta \in \ker \tilde{f}$ and $\alpha \in \operatorname{im} \tilde{f} \cap L$ then

$$S\left(\tilde{f}(\beta)\right) = 0 = S(\alpha)$$

and $\beta \in K$ as well as $\alpha \in \tilde{f}(K)$ by injectivity of f. So in fact, $\ker \tilde{f} \subset K$ and $\tilde{f}(K) = \operatorname{im} \tilde{f} \cap L$. Now additivity from Lemma 4.2.7 implies

$$\begin{aligned} n &= \dim_{\mathbb{K}[G]}\left(\ker \tilde{f}\right) + \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right), \\ \dim_{\mathbb{K}[G]}(K) &= \dim_{\mathbb{K}[G]}\left(\ker \tilde{f}\right) + \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right), \\ n - \dim_{\mathbb{K}[G]}(K) &= \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right) - \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right). \end{aligned}$$

On the other hand the dimension formula in Lemma 4.2.6 says that

$$\begin{aligned} m = \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} + L\right) &= \dim_{\mathbb{K}[G]}(L) + \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right) - \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right), \\ m - \dim_{\mathbb{K}[G]}(L) &= \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right) - \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right). \end{aligned}$$

□

Corollary 4.1.9. *Let M be a finitely generated $\mathbb{K}[G]$-module and $N \subseteq M$ be a finitely generated $\mathbb{K}[G]$-submodule, then*

$$\dim_{\mathbb{K}[G]}(N) \leq \dim_{\mathbb{K}[G]}(M). \tag{4.1.9}$$

Proof. Recall the proof of well-definition above. The lack of surjectivity gives

$$m \geq \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} + L\right)$$

while the rest remains the same. □

The diagram chases used before also produce additivity for finitely generated $\mathbb{K}[G]$-modules.

Theorem 4.1.10. *Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let*

$$0 \to N \to M \to P \to 0$$

be an exact sequence of finitely generated $\mathbb{K}[G]$-modules, then

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(N) + \dim_{\mathbb{K}[G]}(P). \tag{4.1.10}$$

Proof. Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let $0 \to N \to M \to P \to 0$ be an exact sequence of finitely generated $\mathbb{K}[G]$-modules, then there exists the following commutative diagram, where the columns and the bottom row are exact.

$$
\begin{array}{ccccc}
J & & L & & K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{K}[G]^m & \overset{\tilde{f}}{\dashrightarrow} & \mathbb{K}[G]^p & \overset{\tilde{g}}{\dashrightarrow} & \mathbb{K}[G]^n \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
N & \overset{f}{\hookrightarrow} & M & \overset{g}{\twoheadrightarrow} & P
\end{array}
$$

Where $\tilde{f} \colon \mathbb{K}[G]^m \to \mathbb{K}[G]^p$ and $\tilde{g} \colon \mathbb{K}[G]^p \to \mathbb{K}[G]^n$ are lifts defined as in the proof of well-definition. Then exactness and commutativity in the diagram give $\ker \tilde{f} \subset J$ and $\tilde{f}\,(J) = \operatorname{im} \tilde{f} \cap L$ and so, by additivity,

$$\dim_{\mathbb{K}[G]}(N) = m - \dim_{\mathbb{K}[G]}(J) = \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right) - \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right).$$

Similarly, we see that $\tilde{g}\,(L) \subset K$ and $\operatorname{im} \tilde{g} + K = \mathbb{K}[G]^n$ and so, by the dimension formula,

$$\dim_{\mathbb{K}[G]}(P) = \dim_{\mathbb{K}[G]}(\operatorname{im} \tilde{g}) - \dim_{\mathbb{K}[G]}(\operatorname{im} \tilde{g} \cap K).$$

Let $\alpha \in \mathbb{K}[G]^p$ be such that it maps to $f\,(N) \subset M$, then $\alpha \in \operatorname{im} \tilde{f} + L$. By the exactness of the bottom row we furthermore deduce $\operatorname{im} \tilde{g} \cap K = \tilde{g}\left(\operatorname{im} \tilde{f} + L\right)$ and $\ker \tilde{g} \subset \operatorname{im} \tilde{f} + L$. Therefore we have that

$$\dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} + L\right) = \dim_{\mathbb{K}[G]}(\ker \tilde{g}) + \dim_{\mathbb{K}[G]}(\operatorname{im} \tilde{g} \cap K),$$
$$\dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} + L\right) = \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f}\right) + \dim_{\mathbb{K}[G]}(L) - \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} \cap L\right).$$

Recall that

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(\operatorname{im} \tilde{g}) + \dim_{\mathbb{K}[G]}(\ker \tilde{g}) - \dim_{\mathbb{K}[G]}(L).$$

Now all the above combines to

$$\begin{aligned}\dim_{\mathbb{K}[G]}(M) &= \dim_{\mathbb{K}[G]}(P) + \dim_{\mathbb{K}[G]}\left(\operatorname{im} \tilde{f} + L\right) - \dim_{\mathbb{K}[G]}(L) \\ &= \dim_{\mathbb{K}[G]}(P) + \dim_{\mathbb{K}[G]}(N).\end{aligned}$$

□

Corollary 4.1.11. *Let M be a finitely generated $\mathbb{K}[G]$-module such that there exists a surjective $\mathbb{K}[G]$-linear map $\mathbb{K}[G]^n \to M$. Then*

$$\dim_{\mathbb{K}[G]}(M) = n \iff M \cong \mathbb{K}[G]^n. \tag{4.1.11}$$

Proof. Let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Let $\dim_{\mathbb{K}[G]}(M) = n$, then by definition $\dim_{\mathbb{K}[G]}(\ker T) = 0$. This implies $\ker T = 0$ by Lemma 4.2.1 and thus

$$M \cong \mathbb{K}[G]^n.$$

□

Finally we use the fact that any module is the union of its finitely generated submodules to further extend the definition to general $\mathbb{K}[G]$-modules.

Definition 4.1.12 (General $\mathbb{K}[G]$-modules)**.** Let G be a discrete, amenable group, $\mathbb{K}$ be a field. Let M be a $\mathbb{K}[G]$-module. We define

$$\dim_{\mathbb{K}[G]}(M) := \sup\left\{\dim_{\mathbb{K}[G]}\left(\bar{M}\right) \;\middle|\; \bar{M} \subseteq M \text{ is finitely generated}\right\}. \tag{4.1.12}$$

Corollary 4.1.13. *Let $M \cong N$ be isomorphic $\mathbb{K}[G]$-modules, then*

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(N). \tag{4.1.13}$$

Proof. Let $\phi\colon M \to N$ be a $\mathbb{K}[G]$-module isomorphism. Let $\bar{M} \subseteq M$ be a finitely generated submodule, then $\phi\left(\bar{M}\right) \subseteq N$ is a finitely generated submodule and

$$\dim_{\mathbb{K}[G]}\left(\bar{M}\right) = \dim_{\mathbb{K}[G]}\left(\phi\left(\bar{M}\right)\right).$$

Thus $\dim_{\mathbb{K}[G]}(M) \leq \dim_{\mathbb{K}[G]}(N)$. On the other hand let $\bar{N} \subseteq N$ be a finitely generated submodule, then $\phi^{-1}\left(\bar{N}\right) \subseteq M$ is a finitely generated submodule and

$$\dim_{\mathbb{K}[G]}\left(\bar{N}\right) = \dim_{\mathbb{K}[G]}\left(\phi^{-1}\left(\bar{N}\right)\right).$$

This implies $\dim_{\mathbb{K}[G]}(N) \leq \dim_{\mathbb{K}[G]}(M)$ and so (4.1.13) holds. □

We show additivity for arbitrary $\mathbb{K}[G]$-modules in two steps.

Lemma 4.1.14. *Let G be a discrete, amenable group, $\mathbb{K}$ be a field. Let M be a finitely generated $\mathbb{K}[G]$-module and let $N \subseteq M$ be a submodule then*

$$\dim_{\mathbb{K}[G]}\left(M/N\right) = \dim_{\mathbb{K}[G]}(M) - \dim_{\mathbb{K}[G]}(N). \tag{4.1.14}$$

Proof. Let G be a discrete, amenable group, $\mathbb{K}$ be a field. Let $N \subseteq M$ be $\mathbb{K}[G]$-modules such that M is finitely generated then M/N is also finitely generated.

By definition there exists a surjective $\mathbb{K}[G]$-linear map $T\colon \mathbb{K}[G]^n \to M$ for some $n \in \mathbb{N}$. Define $\bar{N} := T^{-1}(N) \subseteq \mathbb{K}[G]^n$, then

$$\bar{N} = \ker\left(\mathbb{K}[G]^n \to M \to M/N\right) \subseteq \mathbb{K}[G]^n$$

and by definition $\dim_{\mathbb{K}[G]}\left(M/N\right) = n - \dim_{\mathbb{K}[G]}\left(\bar{N}\right)$. Now let $\epsilon > 0$, clearly any finitely generated submodule of $\bar{N}$ maps to a finitely generated submodule of N and so there exist finitely generated submodules $\bar{N}_\epsilon \subseteq \bar{N}$ and $N_\epsilon := T\left(\bar{N}_\epsilon\right) \subseteq N$ such that

$$\dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(N_\epsilon) \leq \epsilon,$$
$$\dim_{\mathbb{K}[G]}\left(\bar{N}\right) - \dim_{\mathbb{K}[G]}\left(\bar{N}_\epsilon\right) \leq \epsilon.$$

By monotony $\dim_{\mathbb{K}[G]}\left(\bar{N}\right) - \dim_{\mathbb{K}[G]}\left(T^{-1}(N_\epsilon)\right) \leq \epsilon$ and therefore by definition of $\dim_{\mathbb{K}[G]}$ for finitely generated $\mathbb{K}[G]$-modules

$$\dim_{\mathbb{K}[G]}\left(M/N_\epsilon\right) - \dim_{\mathbb{K}[G]}\left(M/N\right) \leq \epsilon.$$

So by additivity for finitely generated $\mathbb{K}[G]$-modules, Theorem 4.1.10, we have

$$\dim_{\mathbb{K}[G]}\left(M/N_\epsilon\right) - \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N_\epsilon) = 0.$$

Therefore we conclude for every $\epsilon > 0$ that

$$\begin{aligned}
&\left|\dim_{\mathbb{K}[G]}\left(M/N\right) - \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N)\right| \\
&\leq \left|\dim_{\mathbb{K}[G]}\left(M/N\right) - \dim_{\mathbb{K}[G]}\left(M/N_\epsilon\right)\right| + \left|\dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(N_\epsilon)\right| \\
&\leq 2\epsilon.
\end{aligned}$$

□

Theorem 4.1.15. *Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let*

$$0 \to N \to M \to P \to 0$$

be an exact sequence of $\mathbb{K}[G]$-modules, then

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(N) + \dim_{\mathbb{K}[G]}(P). \tag{4.1.15}$$

Proof. Let G be a discrete, amenable group and $\mathbb{K}$ be a field. Let

$$0 \to N \xrightarrow{\phi} M \xrightarrow{\psi} P \to 0$$

be an exact sequence of $\mathbb{K}[G]$-modules and $\epsilon > 0$, then by monotony, see Corollary 4.1.9 there exist finitely generated $\mathbb{K}[G]$-modules $N_\epsilon \subset N, M_\epsilon \subset M, P_\epsilon \subset P$ such that

$$\begin{aligned} \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(N_\epsilon) &\leq \epsilon \\ \dim_{\mathbb{K}[G]}(M) - \dim_{\mathbb{K}[G]}(M_\epsilon) &\leq \epsilon \\ \dim_{\mathbb{K}[G]}(P) - \dim_{\mathbb{K}[G]}(P_\epsilon) &\leq \epsilon \end{aligned}$$

and

$$\begin{aligned} \phi(N_\epsilon) \subseteq M_\epsilon \cap \phi(N) =: \bar{N}_\epsilon, \\ \psi(M_\epsilon) = P_\epsilon. \end{aligned}$$

It follows that $M_\epsilon / \bar{N}_\epsilon \cong P_\epsilon$ and by Lemma 4.1.14

$$\dim_{\mathbb{K}[G]}(P_\epsilon) = \dim_{\mathbb{K}[G]}(M_\epsilon) - \dim_{\mathbb{K}[G]}\left(\bar{N}_\epsilon\right).$$

Together with $\dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}\left(\bar{N}_\epsilon\right) \leq \epsilon$ by monotony this implies

$$\begin{aligned} &\left|\dim_{\mathbb{K}[G]}(P) - \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N)\right| \\ \leq &\left|\dim_{\mathbb{K}[G]}(P) - \dim_{\mathbb{K}[G]}(P_\epsilon)\right| \\ &+ \left|\left(\dim_{\mathbb{K}[G]}(M) - \dim_{\mathbb{K}[G]}(M_\epsilon)\right)\right| \\ &+ \left|\dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}\left(\bar{N}_\epsilon\right)\right| \\ \leq &\, 3\epsilon. \end{aligned}$$

□

4.2. Proof of properties

We come back to the postponed proofs of Theorem 4.1.5. Let G be a discrete, amenable group and $\mathbb{K}$ a field.

Lemma 4.2.1. *Let $M \subseteq \mathbb{K}[G]^n$ be a $\mathbb{K}[G]$-submodule, then*

$$\dim_{\mathbb{K}[G]}(M) = 0 \iff M = 0. \tag{4.2.1}$$

Proof. By definition it is clear that $\dim_{\mathbb{K}[G]}(0) = 0$.

On the other hand let $M \subseteq \mathbb{K}[G]^n$ be a $\mathbb{K}[G]$-submodule such that $M \neq 0$. Now let $0 \neq m \in M$. We will show that $\dim_{\mathbb{K}[G]}(M) > 0$. Let $L := \operatorname{supp} m \cup \{e\} \subset G$ and let $F \subset G$ be a finite subset such that $L \subseteq F$, recall the definition of $\operatorname{Int}_L(F)$ and consider the subset $\mathbb{K}[\operatorname{Int}_L(F)] \subset \mathbb{K}[G]$. Then

(4.2.2)

$$m \cdot \mathbb{K}[\operatorname{Int}_L(F)] = \left(m \cdot \mathbb{K}[\operatorname{Int}_L(F)]\right) \cap \operatorname{im} i_F \subseteq M \cap \operatorname{im} i_F \subseteq \operatorname{im} i_F \cong \mathbb{K}[F]^n,$$

as $\mathbb{K}$-vector spaces. Thus left multiplication by m induces a $\mathbb{K}$-linear map

$$\mathbb{K}[\mathrm{Int}_L(F)] \to \mathbb{K}[F]^n,$$

which we may identify with a matrix $T \in \mathbb{K}^{r \times s}$, where $r = n|F|$ and $s = |\mathrm{Int}_L(F)|$, such that $m \cdot \mathbb{K}[\mathrm{Int}_L(F)] \cong \mathrm{im}\, T$ as $\mathbb{K}$-vector spaces.

We write $l := |\mathrm{supp}\, m|$, from the definition of group ring multiplication we know that each row of T has at most l non-zero entries and each column is a permutation of $m \in \mathbb{K}[F]^n$ as a $\mathbb{K}$-vector, in particular each column has the same number of non-zero entries. Recall from linear algebra ([**Hef15**]) that the number of linearly independent columns of T is the same as $\dim_F(\mathrm{im}\, T)$ and that deleting a row or column from a matrix can only reduce this.

We want to see that T has at least $\frac{s}{l}$ linearly independent columns. We reduce the matrix as follows.

- Pick a column with the fewest non-zero entries, then pick a non-zero entry in this column.
- Delete all other columns which have a non-zero entry in the corresponding row, these are at most $l - 1$.
- Delete all other rows for which the chosen column has a non-zero entry. Now the chosen column is linearly independent of all other remaining columns. All other remaining columns have at least one non-zero entry.
- Repeat until all remaining columns are linearly independent.

In each iteration we reduce at most l columns to one and do not touch its non-zero entry afterwards. So this leaves at least $\frac{s}{l}$ linearly independent reduced columns.

This implies with (4.2.2) that

$$\dim_{\mathbb{K}}(M \cap \mathrm{im}\, i_F) \geq \dim_{\mathbb{K}}\big(m \cdot \mathbb{K}[\mathrm{Int}_L(F)]\big) \geq \frac{s}{l} = \frac{|\mathrm{Int}_L(F)|}{|\mathrm{supp}\, m|}.$$

Furthermore by Lemma 1.2.3 we know that

$$|\mathrm{Int}_L(F)| \geq |F| - \partial_L(F).$$

Let (F_i) be a Følner exhaustion for G according to Theorem 1.3.7 then all the above implies

$$\begin{aligned}\dim_{\mathbb{K}[G]}(M) &= \lim_i \frac{\dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_{F_i}\right)}{|F_i|} \\ &\geq \lim_i \frac{1}{|\operatorname{supp} m|}\frac{|F_i| - \partial_L(F_i)}{|F_i|} \\ &= \frac{1}{|\operatorname{supp} m|} - \lim_i \frac{1}{|\operatorname{supp} m|}\frac{\partial_L(F_i)}{|F_i|} \\ &= \frac{1}{|\operatorname{supp} m|} > 0.\end{aligned}$$

□

Corollary 4.2.2. *Let $M \subseteq \mathbb{K}[G]$ be a non-trivial $\mathbb{K}[G]$-submodule. We denote the size of the minimal support of M by $s := \min\left\{|\operatorname{supp} m| \;\middle|\; 0 \neq m \in M\right\} \in \mathbb{N}$. Then*

$$\frac{1}{s} \leq \dim_{\mathbb{K}[G]}(M) \leq 1. \tag{4.2.3}$$

Lemma 4.2.3. *Let G be an amenable group, $\mathbb{K}$ be a field, then for any submodule $M \in \left\{N_1 + N_2, N_1 \cap N_2 \;\middle|\; N_1, N_2 \in \mathcal{K}_n(G) \cup \mathcal{R}_n(G), n \in \mathbb{N}\right\}$ it holds that*

$$\dim_{\mathbb{K}[G]}(M) = \overline{\dim_{\mathbb{K}[G]}}(M). \tag{4.2.4}$$

Proof. Let $M \in \mathcal{K}_n(G) \cup \mathcal{R}_n(G)$, then (4.2.4) follows from Theorems 2.2.1 and 2.2.2 due to the property of Følner nets (1.3.1). Furthermore let $N \in \mathcal{K}_n(G) \cup \mathcal{R}_n(G)$ then Lemma 2.2.3 implies that,

$$\begin{aligned}\dim_{\mathbb{K}[G]}(M + N) &= \overline{\dim_{\mathbb{K}[G]}}(M + N), \\ \dim_{\mathbb{K}[G]}(M \cap N) &= \overline{\dim_{\mathbb{K}[G]}}(M \cap N).\end{aligned}$$

□

Lemma 4.2.4 (Monotony)**.** *Let $n \in \mathbb{N}$ and let $M \subseteq N \subseteq \mathbb{K}[G]^n$ be two submodules, then*

$$\dim_{\mathbb{K}[G]}(M) \leq \dim_{\mathbb{K}[G]}(N), \tag{4.2.5}$$

$$\overline{\dim_{\mathbb{K}[G]}(M)} \leq \overline{\dim_{\mathbb{K}[G]}(N)}. \tag{4.2.6}$$

Proof. Let $M \subseteq N \subseteq \mathbb{K}[G]$ be two submodules, then for all $F \in \mathcal{F}(G)$ it holds that $M \cap \operatorname{im} i_F \subseteq N \cap \operatorname{im} i_F$ and $p_F(M) \subseteq p_F(N)$ therefore the claim follows. □

Lemma 4.2.5 (Continuity)**.** *Let $M \subseteq \mathbb{K}[G]^m$, $m \in \mathbb{N}$, be a submodule and $\{M_\alpha | \alpha \in A\}$ be an exhausting directed system of submodules, i.e. $\bigcup_{\alpha \in A} M_\alpha = M$ and for any two indices α, β there is an index $\gamma \in A$ such that $M_\alpha \subset M_\gamma$ and $M_\beta \subset M_\gamma$. Then it holds that*

$$\dim_{\mathbb{K}[G]}(M) = \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha). \tag{4.2.7}$$

Proof. Let $M \subseteq \mathbb{K}[G]^m$ be a submodule, $m \in \mathbb{N}$. Because of Theorem 1.3.7 we may consider a Følner net $(F_i)_{i \in I}$ such that $F_i \subseteq F_j$ for $i \leq j \in I$. We want to show a coarse monotonicity of

$$I \to \mathbb{R} : i \mapsto \frac{\dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_{F_i}\right)}{|F_i|}.$$

Recall the proof of Theorem 3.2.2. Fix $\epsilon \in]0, 1/2]$ and with it $n_\epsilon := \lceil \frac{\log \epsilon}{\log(2-\epsilon) - log 2} \rceil$. Let $J^\epsilon \subset I$ be a directed subset such that $\left(F_j\right)_{j \in J^\epsilon}$ is a subnet where for all $k > j \in J^\epsilon$ holds that $F_j \subset F_k$ and

$$\frac{|\partial_{F_j}(F_k)|}{|F_k|} < \epsilon^{2n_\epsilon}. \tag{4.2.8}$$

Let $\underline{h} \colon \mathcal{F}(G) \to \mathbb{N}$ be as in Definition 2.3.1 and we define

$$j_0 := \operatorname*{arg\,min}_{j_1 \leq j_i \leq j_{n_\epsilon}} \frac{h(F_{j_i})}{|F_{j_i}|},$$

for a finite sequence $j_1 < j_2 < j_3 < \ldots < j_{n_\epsilon}$ of length n_ϵ. By Lemma 3.2.1 we have a decomposition of F_k, $k > j_{n_\epsilon}$ into disjoint subsets

$$F_k = \bigsqcup_{l=1}^{n_\epsilon} F_{j_l} R_l \sqcup D_{n_\epsilon},$$

such that $|D_{n_\epsilon}| \leq \epsilon|F_k|$. Where $R_l \subset G$ are finite, possible empty, and $(F_{j_l}g)_{g\in R_l}$ are ϵ-disjoint families. Now as in the proof of Theorem 3.2.2,

$$\begin{aligned}
\frac{\underline{h}(F_k)}{|F_k|} &\geq \sum_{l=1}^{n_\epsilon}\sum_{g\in R_l}\frac{\underline{h}(F_{j_l})}{|F_{j_l}|}\frac{|F_{j_l}g|}{|F_k|} - m\frac{2\epsilon-\epsilon^2}{1-\epsilon} \\
&\geq \frac{\underline{h}(F_{j_0})}{|F_{j_0}|}\sum_{j=1}^{n_\epsilon}\sum_{g\in R_j}\frac{|F_{j_l}g|}{|F_k|} - m\frac{2\epsilon-\epsilon^2}{1-\epsilon} \\
&\geq \frac{\underline{h}(F_{j_0})}{|F_{j_0}|}(1-\epsilon) - m\frac{2\epsilon-\epsilon^2}{1-\epsilon} \qquad (4.2.9)\\
&\geq \frac{\underline{h}(F_{j_0})}{|F_{j_0}|} - m\left(\epsilon + \frac{2\epsilon-\epsilon^2}{1-\epsilon}\right) \\
&\geq \frac{\underline{h}(F_{j_0})}{|F_{j_0}|} - 5m\epsilon.
\end{aligned}$$

So $k \mapsto \frac{\dim_{\mathbb{K}}\left(M\cap \operatorname{im} i_{F_k}\right)}{|F_k|}$ is almost monotone increasing. Finally, we may consider a submodule $M \subseteq \mathbb{K}[G]^m$, $m \in \mathbb{N}$, and an exhausting directed system of submodules $\{M_\alpha|\alpha \in A\}$, i.e. $\bigcup_{\alpha\in A} M_\alpha = M$ and for any two indices α, β there is an index $\gamma \in A$ such that $M_\alpha \subset M_\gamma$ and $M_\beta \subset M_\gamma$. Then the following holds

$$\begin{aligned}
\dim_{\mathbb{K}[G]}(M) &= \lim_i \frac{\dim_{\mathbb{K}}\left(M\cap \operatorname{im} i_{F_i}\right)}{|F_i|} \\
&= \lim_i \frac{\sup_\alpha \dim_{\mathbb{K}}\left(M_\alpha\cap \operatorname{im} i_{F_i}\right)}{|F_i|} \qquad (4.2.10)\\
&\geq \sup_\alpha \lim_i \frac{\dim_{\mathbb{K}}\left(M_\alpha\cap \operatorname{im} i_{F_i}\right)}{|F_i|} \\
&= \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha).
\end{aligned}$$

We write $h_\alpha(F_i) := \dim_{\mathbb{K}}\left(M_\alpha \cap \operatorname{im} i_{F_i}\right)$, $h_\infty(F_i) := \dim_{\mathbb{K}}\left(M \cap \operatorname{im} i_{F_i}\right)$ and assume (4.2.10) is a strict inequality. Then there exists $\epsilon \in]0, 1/2]$ such that

$$\lim_i \frac{h_\infty(F_i)}{|F_i|} = \lim_i \sup_\alpha \frac{h_\alpha(F_i)}{|F_i|} \geq \sup_\alpha \lim_i \frac{h_\alpha(F_i)}{|F_i|} + 10m\epsilon. \qquad (4.2.11)$$

For this ϵ we get n_ϵ and a Følner subsequence $\left(F_j\right)_{j\in J^\epsilon}$ as in (4.2.8). Since the limits converge independent of the choice of Følner sequence we find a lower bound $j_0 \in J^\epsilon$ such that

$$\forall j \geq j_0 : \frac{h_\infty(F_j)}{|F_j|} \geq \dim_{\mathbb{K}[G]}(M) - m\epsilon.$$

Fix a finite sequence $j_1 < j_2 < j_3 < \ldots < j_{n_\epsilon}$ of length n_ϵ with $j_0 < j_1$. Since F_{j_k} is a finite set we have that $\sup_\alpha h_\alpha(F_{j_k})$ stabilizes at some α_k. We note the maximum $\alpha_0 := \max_{1\le k\le n_\epsilon} \alpha_k$, then (4.2.9) gives us that

$$\frac{h_\alpha(F_j)}{|F_j|} \geq \min_{1\le k\le n_\epsilon} \frac{h_{\alpha_0}(F_{j_k})}{|F_{j_k}|} - 5m\epsilon \geq \dim_{\mathbb{K}[G]}(M) - 6m\epsilon,$$

for all $\alpha > \alpha_0$ and $j > j_{n_\epsilon}$. But then

$$\sup_\alpha \lim_j \frac{h_\alpha(F_j)}{|F_j|} \geq \lim_j \frac{h_\infty(F_j)}{|F_j|} - 6m\epsilon,$$

which contradicts (4.2.11) and we see that (4.2.10) is indeed an equality

$$\dim_{\mathbb{K}[G]}(M) = \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha).$$

□

Lemma 4.2.6 (Dimension Formula)**.** *Let $M, N \subset \mathbb{K}[G]^n, n \in \mathbb{N}$, be $\mathbb{K}[G]$-submodules then*

$$\dim_{\mathbb{K}[G]}(M+N) = \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(M\cap N). \tag{4.2.12}$$

Proof. We start with the case $M, N \in \mathcal{R}_n(G), n \in \mathbb{N}$. Now Lemma 2.2.3 implies that

$$\overline{\dim_{\mathbb{K}[G]}}(M+N) = \overline{\dim_{\mathbb{K}[G]}}(M) + \overline{\dim_{\mathbb{K}[G]}}(N) - \overline{\dim_{\mathbb{K}[G]}}(M\cap N).$$

Thus by Lemma 4.2.3

$$\dim_{\mathbb{K}[G]}(M+N) = \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(M\cap N).$$

Now consider an elementary consequence of monotony. Let $A_1 \subset A_2 \subset \mathbb{K}[G]$ be two submodules, $B \subset \mathbb{K}[G]$ another one and $F \subset G$ a finite subset. Then linear algebra shows that

$$\begin{aligned} &\dim_{\mathbb{K}}(A_1 \cap \operatorname{im} i_F) - \dim_{\mathbb{K}}(A_1 \cap B \cap \operatorname{im} i_F) \\ \le &\dim_{\mathbb{K}}(A_2 \cap \operatorname{im} i_F) - \dim_{\mathbb{K}}(A_2 \cap B \cap \operatorname{im} i_F), \end{aligned}$$

and therefore

$$\dim_{\mathbb{K}[G]}(A_1) - \dim_{\mathbb{K}[G]}(A_1\cap B) \leq \dim_{\mathbb{K}[G]}(A_2) - \dim_{\mathbb{K}[G]}(A_2\cap B). \tag{4.2.13}$$

Now let $M, N \subset \mathbb{K}[G]^n$ be arbitrary submodules. We find that

$$\left\{ M_\alpha \middle| M_\alpha \subset M \text{ finitely generated submodule} \right\}$$

and

$$\left\{N_\alpha \middle| N_\alpha \subset N \text{ f. g. submodule}\right\}$$

are exhausting directed systems of M respective N. These induce further exhausting directed systems,

$$\begin{aligned} M \cap N &= \bigcup_\alpha (M_\alpha \cap N) = \bigcup_\alpha \bigcup_\beta \left(M_\alpha \cap N_\beta\right) \\ &= \bigcup_\beta \left(M \cap N_\beta\right) = \bigcup_{(\alpha,\beta)} \left(M_\alpha \cap N_\beta\right), \\ M + N &= \left(\bigcup_\alpha M_\alpha\right) + \left(\bigcup_\beta N_\beta\right) = \bigcup_{(\alpha,\beta)} \left(M_\alpha + N_\beta\right). \end{aligned}$$

By our previous results we know that

$$\begin{aligned} D_1 &:= \sup_{(\alpha,\beta)} \Big(\dim_{\mathbb{K}[G]}(M_\alpha) + \dim_{\mathbb{K}[G]}\left(N_\beta\right) - \dim_{\mathbb{K}[G]}\left(M_\alpha \cap N_\beta\right)\Big) \\ &= \sup_{(\alpha,\beta)} \dim_{\mathbb{K}[G]}\left(M_\alpha + N_\beta\right) \\ &= \dim_{\mathbb{K}[G]}(M+N), \\ D_2 &:= \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha) + \sup_\beta \dim_{\mathbb{K}[G]}\left(N_\beta\right) - \sup_{(\alpha,\beta)} \dim_{\mathbb{K}[G]}\left(M_\alpha \cap N_\beta\right) \\ &= \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(M \cap N). \end{aligned}$$

We assume there exists $\epsilon > 0$ such that $|D_1 - D_2| > \epsilon$. By definition of the supremum we find indices $(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2)$ such that

$$\begin{aligned} \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha) - \dim_{\mathbb{K}[G]}(M_{\alpha_1}) &\le \frac{\epsilon}{4}, \\ \sup_\beta \dim_{\mathbb{K}[G]}\left(N_\beta\right) - \dim_{\mathbb{K}[G]}\left(N_{\beta_1}\right) &\le \frac{\epsilon}{4}, \\ \sup_{(\alpha,\beta)} \dim_{\mathbb{K}[G]}\left(M_\alpha \cap N_\beta\right) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_2} \cap N_{\beta_2}\right) &\le \frac{\epsilon}{4} \end{aligned}$$

and

$$D_1 - \Big(\dim_{\mathbb{K}[G]}(M_{\alpha_0}) + \dim_{\mathbb{K}[G]}\left(N_{\beta_0}\right) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_0} \cap N_{\beta_0}\right)\Big) \le \frac{\epsilon}{4}.$$

By property of directed systems we find (α_3, β_3) such that

$$M_{\alpha_0}, M_{\alpha_1}, M_{\alpha_2} \subset M_{\alpha_3} \text{ and } N_{\beta_0}, N_{\beta_1}, N_{\beta_2} \subset N_{\beta_3},$$

and we call $D_3 := \dim_{\mathbb{K}[G]}(M_{\alpha_3}) + \dim_{\mathbb{K}[G]}\left(N_{\beta_3}\right) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_3} \cap N_{\beta_3}\right).$

Now monotony gives us that $|D_3 - D_2| \leq \frac{2\epsilon}{4}$ and with (4.2.13) we see

$$\dim_{\mathbb{K}[G]}(M_{\alpha_0}) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_0} \cap N_{\beta_0}\right) \leq \dim_{\mathbb{K}[G]}(M_{\alpha_3}) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_3} \cap N_{\beta_0}\right),$$
$$\dim_{\mathbb{K}[G]}\left(N_{\beta_0}\right) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_3} \cap N_{\beta_0}\right) \leq \dim_{\mathbb{K}[G]}\left(N_{\beta_3}\right) - \dim_{\mathbb{K}[G]}\left(M_{\alpha_3} \cap N_{\beta_3}\right).$$

Which implies that $|D_1 - D_3| \leq \frac{\epsilon}{4}$ and therefore by triangle inequality

$$|D_1 - D_2| \leq \frac{3\epsilon}{4}.$$

This contradicts our assumption and thus

$$\dim_{\mathbb{K}[G]}(M + N) = \dim_{\mathbb{K}[G]}(M) + \dim_{\mathbb{K}[G]}(N) - \dim_{\mathbb{K}[G]}(M \cap N).$$

□

Lemma 4.2.7 (Additivity)**.** *Let $T\colon M \to N$ be a $\mathbb{K}[G]$-linear map with domain $M \subset \mathbb{K}[G]^m$ and codomain $N \subset \mathbb{K}[G]^n$ $\mathbb{K}[G]$-submodules then*

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(\ker T) + \dim_{\mathbb{K}[G]}(\operatorname{im} T). \tag{4.2.14}$$

Proof. Let $T\colon M \to N$ be a $\mathbb{K}[G]$-linear map with domain $M \subset \mathbb{K}[G]^m$ and codomain $N \subset \mathbb{K}[G]^n$ $\mathbb{K}[G]$-submodules. We start with the case of $M = \operatorname{im} S$ with $S\colon \mathbb{K}[G]^l \to \mathbb{K}[G]^m$ a $\mathbb{K}[G]$-linear map. Then $T \circ S\colon \mathbb{K}[G]^l \to \mathbb{K}[G]^n$ and we may consider it as a matrix with $\mathbb{K}[G]$ entries and therefore its support.

Let $F \subset G$ be any finite subset and as before let $L_S = \operatorname{supp} S \cup (\operatorname{supp} S)^{-1} \cup \{e\}$ and $L_{T\circ S} = \operatorname{supp}(T \circ S) \cup \left(\operatorname{supp}(T \circ S)\right)^{-1} \cup \{e\}$, we set $K := L_{T\circ S} L_S$ then K is finite and $L_S, L_{T\circ S} \subseteq K$. This implies by definition that $\operatorname{Int}_K(F) \subseteq \operatorname{Int}_{L_S}(F)$ and $B_{L_S}(F), B_{L_{T\circ S}}(F) \subset B_K(F)$. With Theorem 2.2.2 we see the following estimates,

$$\begin{aligned}\dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}(T\circ S)\right)\right) &= \dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}\left(T\circ S\circ i_{B_K(F)}\right)\right)\right)\\ &\leq \dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}\left(T\circ S\circ i_{\operatorname{Int}_K(F)}\right)\right)\right)+l|\partial_K(F)|\\ &= \dim_{\mathbb{K}}\left(\operatorname{im}\left(T\circ S\circ i_{\operatorname{Int}_K(F)}\right)\right)+l|\partial_K(F)|\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T\circ S\circ i_{\operatorname{Int}_{L_S}(F)}\right)\right)+l|\partial_K(F)|\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T|_{\operatorname{im}S\cap\operatorname{im}i_F}\right)\right)+l|\partial_K(F)|\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T\circ S\circ i_{B_K(F)}\right)\right)+l|\partial_K(F)|\\ &\leq \dim_{\mathbb{K}}\left(\operatorname{im}\left(T\circ S\circ i_{\operatorname{Int}_K(F)}\right)\right)+2l|\partial_K(F)|\\ &\leq \dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}(T\circ S)\right)\right)+2l|\partial_K(F)|.\end{aligned}$$

By definition of Følner nets this implies that

$$\lim_i \frac{\dim_{\mathbb{K}}\left(p_{F_i}\left(\operatorname{im}(T\circ S)\right)\right)}{|F_i|} = \lim_i \frac{\dim_{\mathbb{K}}\left(\operatorname{im}\left(T|_{\operatorname{im}S\cap\operatorname{im}i_{F_i}}\right)\right)}{|F_i|}$$

for any Følner net (F_i).

Thus by Lemma 4.2.3 and $\operatorname{im}T = \operatorname{im}(T\circ S)$ we have

$$\dim_{\mathbb{K}[G]}(\operatorname{im}T) = \lim_i \frac{\dim_{\mathbb{K}}\left(\operatorname{im}\left(T|_{\operatorname{im}S\cap\operatorname{im}i_{F_i}}\right)\right)}{|F_i|}, \tag{4.2.15}$$

for any Følner net (F_i).

Also Lemma 2.1.9 implies

$$\operatorname{im}\left(T|_{\operatorname{im}S\cap\operatorname{im}i_F}\right) \subseteq \operatorname{im}\left(T\circ S\circ i_{B_{L_S}(F)}\right) \subseteq \operatorname{im}(T\circ S)\cap\operatorname{im}i_{B_K\left(B_{L_S}(F)\right)}.$$

So consider now

$$T|_{M\cap\operatorname{im}i_F}\colon\ (M\cap\operatorname{im}i_F)\to\left(\mathbb{K}[G]^n\cap\operatorname{im}i_{B_K\left(B_{L_S}(F)\right)}\right)$$

as a $\mathbb{K}$-linear map, then $\ker T|_{M\cap\operatorname{im}i_F} = \ker T\cap\operatorname{im}i_F$. So additivity for $\mathbb{K}$-linear maps [**War90**] tells us

$$\dim_{\mathbb{K}}(M\cap\operatorname{im}i_F) = \dim_{\mathbb{K}}(\ker T\cap\operatorname{im}i_F)+\dim_{\mathbb{K}}\left(\operatorname{im}T|_{M\cap\operatorname{im}i_F}\right).$$

Together with (4.2.15) this gives us additivity for $M = \operatorname{im} S$:

$$\dim_{\mathbb{K}[G]}(M) = \dim_{\mathbb{K}[G]}(\ker T) + \dim_{\mathbb{K}[G]}(\operatorname{im} T).$$

Next let $M \subset \mathbb{K}[G]$ be any submodule and consider the exhausting directed system

$$\{M_\alpha \mid M_\alpha \subset M \text{ finitely generated submodule}\}$$

then $T(M_\alpha) \subset T(M_\beta)$ for $M_\alpha \subset M_\beta$ and

$$\bigcup_\alpha T(M_\alpha) = T(M).$$

With the considerations above the additivity result from before extends,

$$\begin{aligned}\dim_{\mathbb{K}[G]}(M) &= \sup_\alpha \dim_{\mathbb{K}[G]}(M_\alpha)\\ &= \sup_\alpha \Big(\dim_{\mathbb{K}[G]}(\ker T \cap M_\alpha) + \dim_{\mathbb{K}[G]}\big(T(M_\alpha)\big)\Big)\\ &= \sup_\alpha \dim_{\mathbb{K}[G]}(\ker T \cap M_\alpha) + \sup_\alpha \dim_{\mathbb{K}[G]}\big(T(M_\alpha)\big)\\ &= \dim_{\mathbb{K}[G]}(\ker T) + \dim_{\mathbb{K}[G]}\big(T(M)\big).\end{aligned}$$

□

Lemma 4.2.8 (Induction from subgroups)**.** *Let $H \leq G$ be a subgroup of a discrete, amenable group G. Let $M \subseteq \mathbb{K}[H]^n, n \in \mathbb{N}$ be a $\mathbb{K}[H]$-submodule, with induced submodule $M^G := M \cdot \mathbb{K}[G] \subseteq \mathbb{K}[G]^n$, then*

$$\dim_{\mathbb{K}[H]}(M) = \dim_{\mathbb{K}[G]}(M^G). \tag{4.2.16}$$

For $H \leq G$ a finite subgroup it also holds that

$$\overline{\dim_{\mathbb{K}[H]}(M)} = \overline{\dim_{\mathbb{K}[G]}(M^G)}. \tag{4.2.17}$$

Proof. Let $H \leq G$ be a subgroup of a discrete, amenable group G and let $\big(F_i\big)_{i\in I}$ be a Følner net of G. Furthermore, let $V \subset G$ be a transversal, such that

$$G = \bigsqcup_{v\in V} H \cdot v$$

is a disjoint union of conjugacy classes. Then $F_i = \bigsqcup_{v\in V} F_i^v \cdot v$ is a disjoint union, where $F_i^v := (F_i \cap H \cdot v) \cdot v^{-1} \subseteq H$ and $V_i := \big\{v \in V \bigm| F_i^v \neq \emptyset\big\} \subset V$ is a finite subset.

We prove the cases of H finite and infinite separately. Let H be a finite subgroup then we see by Lemma 1.2.6 that

$$\partial_H(F_i) = \bigsqcup_{v\in V_i'} H\cdot v,$$

where $V_i' = \{v \in V_i \mid F_i^v \neq H\}$. So by Lemma 1.3.4 we have

$$\lim_i \frac{|V_i'|}{|V_i|} = 0$$

By partition of G into conjugacy classes we get that $\mathbb{K}[G] = \bigoplus_{v\in V} \mathbb{K}[H]\cdot v$. Let $M \subset \mathbb{K}[H]$ be a $\mathbb{K}[H]$-submodule, and consider

$$M^G := M\cdot \mathbb{K}[G] = \bigoplus_{v\in V} M\cdot v$$

the induction to G.

Then

$$\begin{aligned} p_{F_i}\left(M^G\right) = p_{F_i}\left(\bigoplus_{v\in V} M\cdot v\right) &= \bigoplus_{v\in V_i\setminus V_i'} p_{F_i}(M\cdot v) \oplus \bigoplus_{v\in V_i'} p_{F_i}(M\cdot v) \\ &= \bigoplus_{v\in V_i\setminus V_i'} M\cdot v \oplus \bigoplus_{v\in V_i'} p_{F_i^v\cdot v}(M\cdot v) \\ &= \bigoplus_{v\in V_i\setminus V_i'} M\cdot v \oplus p_{\bigsqcup_{v\in V_i'} F_i^v\cdot v}\left(M^G\right). \end{aligned}$$

Thus calculating the Følner dimension we arrive at

$$\begin{aligned} \frac{\dim_{\mathbb{K}} p_{F_i}\left(M^G\right)}{|F_i|} &= \frac{|V_i\setminus V_i'|\dim_{\mathbb{K}} M + \dim_{\mathbb{K}} p_{\bigsqcup_{v\in V_i'} F_i^v\cdot v}\left(M^G\right)}{|V_i\setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|} \\ &= \left(\frac{|V_i\setminus V_i'|\dim_{\mathbb{K}} M}{|V_i\setminus V_i'||H|}\right)\left(\frac{|V_i\setminus V_i'||H|}{|V_i\setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|}\right) \\ &\quad + \left(\frac{\sum_{v\in V_i'}|F_i^v|}{|V_i\setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|}\right)\left(\epsilon_i + \frac{\dim_{\mathbb{K}} M}{|H|}\right) \\ &= \frac{\dim_{\mathbb{K}} M}{|H|} + \epsilon_i\left(\frac{\sum_{v\in V_i'}|F_i^v|}{|V_i\setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|}\right), \end{aligned}$$

where

$$\epsilon_i := \frac{\dim_{\mathbb{K}} p_{\bigsqcup_{v\in V_i'} F_i^v\cdot v}\left(M^G\right)}{|\bigsqcup_{v\in V_i'} F_i^v\cdot v|} - \frac{\dim_{\mathbb{K}} M}{|H|}$$

and $|\epsilon_i| \leq 1$ for all $i \in I$.

Finally we may estimate

$$\begin{aligned}
\left|\frac{\dim_{\mathbb{K}} p_{F_i}\left(M^G\right)}{|F_i|} - \frac{\dim_{\mathbb{K}} M}{|H|}\right| &\leq \frac{\sum_{v\in V_i'}|F_i^v|}{|V_i \setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|} \\
&\leq \frac{|V_i'||H|}{|V_i \setminus V_i'||H| + \sum_{v\in V_i'}|F_i^v|} = \frac{|\partial_H(F_i)|}{|F_i|}.
\end{aligned}$$

It follows that

$$\overline{\dim_{\mathbb{K}[H]}}(M) = \frac{\dim_{\mathbb{K}} M}{|H|} = \overline{\dim_{\mathbb{K}[G]}}\left(M^G\right).$$

Now consider the exhausting directed system

$$\{M_\alpha \mid M_\alpha \subset M \text{ finitely generated submodule}\}$$

then

$$\left\{M_\alpha^G \,\middle|\, M_\alpha \subset M \text{ finitely generated submodule}\right\}$$

is an exhausting directed system of the induced submodule M^G and so

$$\begin{aligned}
\dim_{\mathbb{K}[H]}(M) &= \sup_\alpha \dim_{\mathbb{K}[H]}(M_\alpha) = \sup_\alpha \overline{\dim_{\mathbb{K}[H]}}(M_\alpha) \\
&= \sup_\alpha \overline{\dim_{\mathbb{K}[G]}}\left(M_\alpha^G\right) = \sup_\alpha \dim_{\mathbb{K}[G]}\left(M_\alpha^G\right) \\
&= \dim_{\mathbb{K}[G]}\left(M^G\right).
\end{aligned}$$

This concludes the finite subgroup case and leaves the case of an infinite subgroup.

Next let $H < G$ be an infinite subgroup, then we find $h_v \in H$ such that

$$H \supset F_i' := \bigsqcup_{v\in V_i} F_i^v \cdot h_v$$

is a disjoint union. Now let $E \in \mathcal{F}(H)$ be a finite subset of H. We want to see that $\partial_E(F_i)$ is a disjoint union of $\partial_E(F_i^v \cdot v)$. For any $g \in G$ we have that $Eg \subset Hg = Hv$ for some $v \in V$. Clearly $Hv \cap F_i^{v'}v' \neq \emptyset$ implies $v = v'$ for any two $v, v' \in V$. So,

$$\begin{aligned}
\partial_E(F_i^v v) &= \left\{g \in G \middle| Eg \cap F_i^v v \neq \emptyset,\ Eg \cap \left(G \setminus F_i^v v\right) \neq \emptyset\right\} \\
&= \left\{g \in G \middle| Eg \cap F_i^v v \neq \emptyset,\ Eg \cap \left(Hv \setminus F_i^v v\right) \neq \emptyset\right\} \\
&\subset \left\{g \in G \middle| Eg \cap F_i \neq \emptyset,\ Eg \cap \left(G \setminus F_i\right) \neq \emptyset\right\} \\
&= \partial_E(F_i).
\end{aligned}$$

Take any $g \in \partial_E(F_i)$, then $Eg \subset Hv$ for precisely one $v \in V$ and therefore $Eg \cap F_i^v v \neq \emptyset$. Furthermore

$$Eg \cap \big(G \setminus F_i\big) = Eg \cap \big(G \setminus F_i\big) \cap Hv = Eg \cap \big(Hv \setminus F_i^v v\big) \neq \emptyset,$$

that is $g \in \partial_E(F_i^v v)$. Thus

$$\partial_E(F_i) = \bigsqcup_{v \in V_i} \partial_E(F_i^v v).$$

Since $\partial_E(Fg) = \partial_E(F)g$ for any $g \in G$ and finite subset $F \subset G$, we have that $|\partial_E(F_i^v v)| = |\partial_E(F_i^v h_v)|$. This means that

$$\frac{|\partial_E(F_i')|}{|F_i'|} \leq \frac{\sum_{v \in V_i} |\partial_E(F_i^v h_v)|}{\sum_{v \in V_i} |F_i^v h_v|} = \frac{\sum_{v \in V_i} |\partial_E(F_i^v v)|}{\sum_{v \in V_i} |F_i^v v|} = \frac{|\partial_E(F_i)|}{|F_i|}, \tag{4.2.18}$$

and $\lim_i \frac{|\partial_E(F_i')|}{|F_i'|} = 0$, thus $\big(F_i'\big)_{i \in I}$ is a Følner net of H.

Let $\tau \in \mathbb{K}[H]$ define a submodule

$$M := \tau \cdot \mathbb{K}[H] \subset \mathbb{K}[H]$$

then the induced module is

$$M^G := \tau \cdot \mathbb{K}[H] \cdot \mathbb{K}[G] = \tau \cdot \mathbb{K}[G] \subset \mathbb{K}[G].$$

As before let $L := \operatorname{supp} \tau \cup (\operatorname{supp} \tau)^{-1} \cup \{e\} \in \mathcal{F}(H)$. For the induced module we have $M^G = \bigoplus_{v \in V} \tau \cdot \mathbb{K}[H] \cdot v$ and

$$\begin{aligned} p_{F_i}\left(M^G\right) &= \bigoplus_{v \in V_i} p_{F_i}\big(\tau \cdot \mathbb{K}[H] \cdot v\big) \\ &= \bigoplus_{v \in V_i} p_{F_i^v v}\big(\tau \cdot \mathbb{K}[H] \cdot v\big) \\ &\cong \bigoplus_{v \in V_i} p_{F_i^v h_v}\big(\tau \cdot \mathbb{K}[H] \cdot h_v\big) \\ &= \bigoplus_{v \in V_i} p_{F_i^v h_v}\big(\tau \cdot \mathbb{K}[H]\big) \\ &\supseteq p_{\left(\bigoplus_{v \in V_i} F_i^v h_v\right)}(M) \\ &\supseteq \operatorname{im} i_{F_i'} \cap M \\ &\supseteq \bigoplus_{v \in V_i} \operatorname{im} i_{F_i^v h_v} \cap M. \end{aligned}$$

So the inclusion above and Theorem 2.2.2 show that

$$\begin{aligned}\dim_{\mathbb{K}}\left(p_{F_i}\left(M^G\right)\right) &\geq \dim_{\mathbb{K}}\left(\operatorname{im} i_{F_i'} \cap M\right)\\ &\geq \dim_{\mathbb{K}}\left(\bigoplus_{v\in V_i} \operatorname{im} i_{F_i^v h_v} \cap M\right)\\ &= \sum_{v\in V_i} \dim_{\mathbb{K}}\left(\operatorname{im} i_{F_i^v h_v} \cap M\right)\\ &\geq \sum_{v\in V_i}\left(\dim_{\mathbb{K}}\left(p_{F_i^v h_v}\left(\tau\cdot\mathbb{K}[H]\right)\right) - 3|\partial_L\left(F_i^v h_v\right)|\right)\\ &= \dim_{\mathbb{K}}\left(\bigoplus_{v\in V_i} p_{F_i^v h_v}\left(\tau\cdot\mathbb{K}[H]\right)\right) - \sum_{v\in V_i} 3|\partial_L\left(F_i^v v\right)|\\ &= \dim_{\mathbb{K}}\left(p_{F_i}\left(M^G\right)\right) - 3|\partial_L\left(F_i\right)|.\end{aligned}$$

Since $\left(F_i'\right)_{i\in I}$ is a Følner net of H and $|F_i| = |F_i'|$, we get that

$$\dim_{\mathbb{K}[H]}(M) = \overline{\dim_{\mathbb{K}[G]}}\left(M^G\right) = \dim_{\mathbb{K}[G]}\left(M^G\right).$$

Clearly this extends to

$$\dim_{\mathbb{K}[H]}(M) = \dim_{\mathbb{K}[G]}\left(M^G\right),$$

for all $M \in \left\{\operatorname{im} T \;\middle|\; T \in \mathbb{K}[H]^{n\times m}, m\in\mathbb{N}\right\}, n\in\mathbb{N}$.

Again for any submodule $M \subset \mathbb{K}[H]^n$ consider the exhausting directed system

$$\{M_\alpha \mid M_\alpha \subset M \text{ finitely generated submodule}\}$$

then

$$\left\{M_\alpha^G \;\middle|\; M_\alpha \subset M \text{ finitely generated submodule}\right\}$$

is an exhausting directed system of the induced submodule M^G, and so

$$\begin{aligned}\dim_{\mathbb{K}[H]}(M) &= \sup_\alpha \dim_{\mathbb{K}[H]}(M_\alpha) = \sup_\alpha \dim_{\mathbb{K}[G]}\left(M_\alpha^G\right)\\ &= \dim_{\mathbb{K}[G]}\left(M^G\right).\end{aligned}$$

$\square$

4.3. Example: Module over the lamplighter group

Let $G := \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ be the lamplighter group. For $(\alpha, z) \in G$ we can identify α with a map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ of finite support. Denote by τ_l the element that maps l to 1 and all else to 0 then $(\tau_l, 0)^2 = (\tau_l + \tau_l, 0) = (\mathbf{0}, 0) = 1$ is the neutral element of G. The inverse of $(\alpha, 0) \cdot (\mathbf{0}, z) = (\alpha, z)$ is $(\mathbf{0}, -z) \cdot (\alpha, 0) = (\alpha^{-z}, -z)$ where $\alpha^{-z}(s) = \alpha(s+z)$.

A presentation of G is $\langle a, t \big| a^2 = 1, [t^n a t^{-n}, a] = 1 \;\; \forall n \in \mathbb{Z} \rangle$. In this presentation we write $(\alpha, z) = \left(\prod_{k\in\mathbb{Z}} t^k a^{\alpha(k)} t^{-k}\right) t^z$, in particular we have that $(\tau_l, 0) = t^l a t^{-l}$.

Let $\mathbb{K}$ be a field of characteristic $\neq 2$. We define a Følner net by the finite subsets

$$F_i := \left\{(\alpha, z) \in G \;\middle|\; \operatorname{supp} \alpha \subseteq [-i, i] \ni z\right\}^{-1}, \quad i \in \mathbb{N}.$$

Indeed, let $K \subset G$ be a finite subset, then there exists $j \in \mathbb{N}$ such that $KK^{-1} \subset F_j^{-1}$, and thus also $KK^{-1} \subset F_j$. Note that for $i > j$ we have that $t^{i-j}\left(t^{-j} a t^j t^j a t^{-j}\right) t^j = \left(t^{i-2j} a t^{-i+2j} t^i a t^{-i}\right) t^i \in F_i^{-1}$ and so $\left\{(\alpha, z) \in F_i^{-1} \;\middle|\; -i+j \le z \le i-j\right\} F_j^{-1} \subseteq F_i^{-1}$ and therefore

(4.3.1)
$$\left\{(\alpha, z)^{-1} \in F_i \;\middle|\; -i+j \le z \le i-j\right\} \subseteq \left\{(\alpha, z)^{-1} \in F_i \;\middle|\; F_j \cdot (\alpha, z)^{-1} \subset F_i\right\}.$$

Thus

$$\begin{aligned}|F_i \cap k_1 k_2^{-1} F_i| &\ge \left|\left\{(\alpha, z)^{-1} \in F_i \;\middle|\; -i+j \le z \le i-j\right\}\right| \\ &= |F_i| \left(1 - \frac{2j}{2i+1}\right),\end{aligned}$$

for all $k_1, k_2 \in K$.

Now Lemma 1.2.3 implies

$$\frac{\partial_K (F_i)}{|F_i|} \le \frac{|K|^2 \left(|F_i| - |F_i|\left(1 - \frac{2j}{2i+1}\right)\right)}{|F_i|} = |K|^2 \frac{2j}{2i+1} \xrightarrow{i\to\infty} 0.$$

Note that if $l \notin [-2i, 2i]$ then $(\tau_l, 0) \cdot (\alpha^{-z}, -z) \notin F_i$ for all $(\alpha^{-z}, -z) \in F_i$. For all $i \in \mathbb{N}$ we make a choice of $H_i \subset \mathbb{Z} \setminus [-2i, 2i]$ with $|H_i| = i+1$.

Now define idempotents $x_i := \prod_{l\in H_i} \left(\frac{1-(\tau_l, 0)}{2}\right) \in \mathbb{K}[G]$ and furthermore $X_i := x_i \cdot \mathbb{K}[G]$, $M_m := \sum_{i=1}^m X_i$ and $M := \bigcup_m M_m$. Then for all $i \in \mathbb{N}$ we have $p_{F_i} X_i = \mathbb{K}[F_i]$ and therefore $p_{F_i} M = \mathbb{K}[F_i]$, because for $l \in H_i$ and $(\alpha^{-z}, -z) \in F_i$

it holds that

$$p_{F_i}\Big(\big(1-(\tau_l,0)\big)\cdot(\alpha^{-z},-z)\Big)=p_{F_i}\left((\alpha^{-z},-z)-(\tau_l,0)\cdot(\alpha^{-z},-z)\right)=(\alpha^{-z},-z).$$

On the other hand for every H_i there is $j\in\mathbb{N}$ such that $H_i\subset[-j,j]$, and thus $\left(\sum_{l\in H'}\tau_l,0\right)\in F_j$ for every subset $H'\subseteq H_i$. So for all $n>j$ we define the subset

$$U_n^i:=\left\{g\in F_n\;\middle|\;\forall H'\subseteq H_i\colon\left(\sum_{l\in H'}\tau_l,0\right)\cdot g\in F_n\right\}\subseteq F_n.$$

Let $l\in H_i$ and $H'\subseteq H_i$, then $\left(\sum_{l'\in H'}\tau_{l'},0\right)\cdot(\tau_l,0)=\left(\sum_{l'\in\tilde{H}}\tau_{l'},0\right)$, with $\tilde{H}\subseteq H_i$. Thus $g\in U_n^i$ implies $(\tau_l,0)\,g\in U_n^i$ for all $l\in H_i$. Then for $l\in H_i$ this implies $\left|\left\{(\alpha^{-z},-z)\in U_n^i\;\middle|\;\alpha^{-z}(l)=0\right\}\right|=\frac{1}{2}|U_n^i|$ and for any $y\colon H_i\to\{0,1\}$,

$$\begin{aligned}(\tau_l,0)\cdot&\left\{(\alpha^{-z},-z)\in U_n^i\;\middle|\;\forall k\in H_i\setminus\{l\}:\alpha^{-z}(k)=y(k),\alpha^{-z}(l)=0\right\}\\&=\left\{(\alpha^{-z},-z)\in U_n^i\;\middle|\;\forall k\in H_i\setminus\{l\}:\alpha^{-z}(k)=y(k),\alpha^{-z}(l)=1\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}\dim_{\mathbb{K}}\left(\left(\frac{1-(\tau_l,0)}{2}\right)\cdot\mathbb{K}[U_n^i]\right)&=\frac{1}{2}|U_n^i|,\\\dim_{\mathbb{K}}\left(x_i\cdot\mathbb{K}[U_n^i]\right)&=2^{-|H_i|}|U_n^i|.\end{aligned}$$

By (4.3.1) we see that

$$1-\frac{2j}{2n+1}\leq\frac{|U_n^i|}{|F_n|}\leq 1.$$

and consequently,

$$\begin{aligned}\dim_{\mathbb{K}}\left(x_i\cdot\mathbb{K}[F_n]\right)&\leq\dim_{\mathbb{K}}\left(x_i\cdot\mathbb{K}[U_n^i]\right)+\dim_{\mathbb{K}}\left(x_i\cdot\mathbb{K}[F_n\setminus U_n^i]\right)\\&\leq 2^{-|H_i|}|U_n^i|+\left(|F_n|-|U_n^i|\right)\\&\leq 2^{-|H_i|}|F_n|+\frac{2j}{2n+1}|F_n|.\end{aligned}$$

Corollary 4.2.2 and Theorem 2.2.2 now imply

$$2^{-|H_i|}\leq\dim_{\mathbb{K}[G]}(X_i)=\overline{\dim_{\mathbb{K}[G]}}(X_i)\leq 2^{-|H_i|}=\frac{1}{2}2^{-i}.$$

So by dimension formula we have for all $m\in\mathbb{N}$ that

$$\dim_{\mathbb{K}[G]}(M_m)=\overline{\dim_{\mathbb{K}[G]}}(M_m)\leq\frac{1}{2}\sum_{i=1}^{m}2^{-i}\leq\frac{1}{2}\sum_{i=1}^{\infty}2^{-i}=\frac{1}{2}.$$

Thus $\frac{1}{4} \leq \dim_{\mathbb{K}[G]}(M) = \sup_m \dim_{\mathbb{K}[G]}(M_m) \leq \frac{1}{2}$, while $\overline{\dim_{\mathbb{K}[G]}}(M) = 1$.

In the end we see that $\dim_{\mathbb{K}[G]} \neq \overline{\dim_{\mathbb{K}[G]}}$ and as we can prove more nice properties for it, $\dim_{\mathbb{K}[G]}$ is preferable as a dimension function.

4.4. Relation to the von Neumann dimension

In the introduction we saw that we want to establish a dimension function similar to the von Neumann dimension for the group von Neumann algebra. We will work towards this connection and begin by looking at how the Følner dimension changes when we change the field.

Lemma 4.4.1 (Change of field)**.** *Let G be a discrete, amenable group and let $\mathbb{F}_p$ be the canonical field of p elements, $p, q \in \mathbb{N}$, where p is a prime number and $p \nmid q$. We consider an element $\tau \in \mathbb{Z}[\frac{1}{q}][G] \subset \mathbb{Q}[G]$ and its image $\overline{\tau} \in \mathbb{F}_p[G]$ under the natural projection $\mathbb{Z}[\frac{1}{q}][G] \twoheadrightarrow \mathbb{F}_p[G]$. Then*

$$\dim_{\mathbb{F}_p[G]}\left(\overline{\tau} \cdot \mathbb{F}_p[G]\right) \leq \dim_{\mathbb{Q}[G]}\left(\tau \cdot \mathbb{Q}[G]\right). \tag{4.4.1}$$

Proof. Let G be a discrete, amenable group and let $\mathbb{F}_p$ be the field of p elements, $p, q \in \mathbb{N}$, where p is a prime number and $p \nmid q$. Let $\tau \in \mathbb{Z}[\frac{1}{q}]G \subset \mathbb{Q}[G]$ and its image $\overline{\tau} \in \mathbb{F}_p[G]$ under the natural projection $\mathbb{Z}[\frac{1}{q}]G \twoheadrightarrow \mathbb{F}_p[G]$. Slightly abusing notation we may write $\tau \cdot \mathbb{F}_p[G]$ for $\overline{\tau} \cdot \mathbb{F}_p[G]$.

Like before, we define the finite subset $L := \operatorname{supp} \tau \cup (\operatorname{supp} \tau)^{-1} \cup \{e\} \subset G$ and let $F \subset G$ be any finite subset. As we have seen before, it holds that

$$\begin{aligned}\dim_{\mathbb{K}}\left(p_F\left(\operatorname{im}\left(\mathbb{K}[G] \xrightarrow{\tau} \mathbb{K}[G]\right)\right)\right) &= \dim_{\mathbb{K}}\left(\operatorname{im}\left(\mathbb{K}[B_L(F)] \xrightarrow{p_F \circ \tau} \mathbb{K}[F]\right)\right) \\ &= \dim_{\mathbb{K}}\left(\operatorname{im}\left(\mathbb{K}^n \xrightarrow{A} \mathbb{K}^m\right)\right),\end{aligned}$$

where $\mathbb{K} \in \{\mathbb{Q}, \mathbb{F}_p\}$, $n = |B_L(F)|$, $m = |F|$ and $A \in \mathbb{Z}[\frac{1}{q}]^{m \times n}$ is the matrix corresponding to the left multiplication with τ. Therefore it would suffice to show for any matrix $A \in \mathbb{Z}[\frac{1}{q}]^{m \times n}$ that

$$\operatorname{rk}_{\mathbb{F}_p}(A) \leq \operatorname{rk}_{\mathbb{Q}}(A).$$

Let A_i be the i-th column vector of A. We define the index sets of families of linear independent column vectors

$$\mathcal{I}_{\mathbb{K}}(A) := \left\{ I \subset \{1, \ldots, n\} \middle| \langle \left\{ A_i \middle| i \in \left(I \setminus \{j\}\right) \right\} \rangle \subsetneq \langle \{A_i | i \in I\} \rangle \subset \mathbb{K}^m \ \forall j \in I \right\},$$

then

$$\mathrm{rk}_{\mathbb{K}}(A) = \max_{I \in \mathcal{I}_{\mathbb{K}}(A)} |I|.$$

Now let $I \in \mathcal{I}_{\mathbb{F}_p}(A)$ and $r := |I|$ and consider the matrix produced by column vectors $A_I := (A_i)_{i \in I}$ and let $A_I^\dagger$ be the transpose of A_I. Then we know by linear algebra, see [**Hef15**], that $\mathrm{rk}_{\mathbb{K}}(A_I) = \mathrm{rk}_{\mathbb{K}}\left(A_I^\dagger\right)$ and there exists $J \in \mathcal{I}_{\mathbb{F}_p}\left(A_I^\dagger\right)$ such that $|J| = r$ and $\det\left(\left(A_I^\dagger\right)_J\right) \not\equiv 0 \mod p$. But then we already have that

$$\det\left(\left(A_I^\dagger\right)_J\right) \neq 0$$

and therefore $J \in \mathcal{I}_{\mathbb{Q}}\left(A_I^\dagger\right)$. This in turn implies that $I \in \mathcal{I}_{\mathbb{Q}}(A)$.

So $\mathcal{I}_{\mathbb{F}_p}(A) \subseteq \mathcal{I}_{\mathbb{Q}}(A)$ and

$$\mathrm{rk}_{\mathbb{F}_p}(A) \leq \mathrm{rk}_{\mathbb{Q}}(A).$$

We see that this implies $\dim_{\mathbb{F}_p[G]}\left(\overline{\tau} \cdot \mathbb{F}_p[G]\right) \leq \dim_{\mathbb{Q}[G]}\left(\tau \cdot \mathbb{Q}[G]\right)$. □

Corollary 4.4.2. *(i) Let $T \in \mathbb{Z}[\frac{1}{q}]G^{m \times n}$ be a matrix over the group ring $\mathbb{Z}[\frac{1}{q}]G$ then*

$$\dim_{\mathbb{F}_p[G]}\left(\mathrm{im}\, \overline{T}\right) \leq \dim_{\mathbb{Q}[G]}(\mathrm{im}\, T), \tag{4.4.2}$$

and by additivity

$$\dim_{\mathbb{F}_p[G]}\left(\ker \overline{T}\right) \geq \dim_{\mathbb{Q}[G]}(\ker T). \tag{4.4.3}$$

In case of $T = T^2$ we have that $T(\mathbb{1} - T) = 0 = (\mathbb{1} - T)T$, so $\ker T = \mathrm{im}(\mathbb{1} - T)$ and $\mathrm{im}\, T = \ker(\mathbb{1} - T)$. Then the same holds for $\overline{T}$ and these estimates become equations.

(ii) Let $A \in \mathbb{K}^{n \times n}$ be a quadratic matrix with coefficients in $\mathbb{K}$. Let $\bar{\mathbb{K}}$ be a field extension of $\mathbb{K}$, then $A \in \bar{\mathbb{K}}^{n \times n}$ and by definition of the determinant we see that

$$\det_{\mathbb{K}}(A) \neq 0 \iff \det_{\bar{\mathbb{K}}}(A) \neq 0.$$

So the estimates above become an equality for fields of the same characteristic. In particular we see for $T \in \mathbb{Q}[G]^{m \times n}$ that

$$\dim_{\mathbb{Q}[G]}(\ker T) = \dim_{\mathbb{C}[G]}(\ker T), \tag{4.4.4}$$

$$\dim_{\mathbb{Q}[G]}(\mathrm{im}\, T) = \dim_{\mathbb{C}[G]}(\mathrm{im}\, T). \tag{4.4.5}$$

We follow [**Ele03a**] to see the relation to the von Neumann dimension for $\mathcal{N}(G)$, the group von Neumann algebra of a discrete group. For a more general discussion of the von Neumann dimension see [**Lüc02b**].

Definition 4.4.3. [**Ele03a**] Let G be a discrete group and let

$$l^2(G) := \left\{ \sum_{x\in G} a_x x \;\middle|\; \forall x \in G\colon a_x \in \mathbb{C}, \sum_{x\in G} |a_x|^2 < \infty \right\} \tag{4.4.6}$$

be the Hilbert space of square summable functions on G. Note that if $\gamma \in l^2(G)$ has finite support, then $\gamma \in \mathbb{C}[G]$. Let $V \subset l^2(G)$ be a $\mathbb{C}[G]$-submodule.

(i) The von Neumann dimension of V is defined as

$$\dim_{\mathcal{N}(G)}(V) = \langle P_V e, e \rangle , \tag{4.4.7}$$

where P_V is the orthogonal projection onto the closure of V and $e \in l^2(G)$ is the sum which contains only the neutral element of G with coefficient $1 \in \mathbb{C}$.

(ii) For arbitrary linear subspaces $W \subset l^2(G)$ and finite subsets $F \subset G$ we define

$$\dim_F(W) = \frac{\sum_{g\in F} \langle P_W g, g\rangle}{|F|}. \tag{4.4.8}$$

Corollary 4.4.4. *Let $V \subset l^2(G)$ be a $\mathbb{C}[G]$-submodule then*

$$\dim_F(V) = \dim_{\mathcal{N}(G)}(V).$$

On the other hand note that for a finite dimensional subspace $W \subset l^2(G)$ which is supported on F we have that

$$\dim_F(W) = \frac{\dim_{\mathbb{C}}(W)}{|F|}.$$

Proof. Let $V \subset l^2(G)$ be a $\mathbb{C}[G]$-submodule, and P_V be the orthogonal projection onto the closure of V. Then $\langle P_V g, g\rangle = \langle P_V e, e\rangle$ for all $g \in G$. On the other hand it is well known in linear algebra that the dimension of a finite dimensional subspace is equal to the trace of its orthogonal projection. So let $W \subset l^2(G)$ be a finite dimensional subspace which is supported on $F \in \mathcal{F}(G)$ then we have that

$$\dim_F(W) = \frac{\sum_{g\in F} \langle P_W g, g\rangle}{|F|} = \frac{\sum_{g\in G} \langle P_W g, g\rangle}{|F|} = \frac{\dim_{\mathbb{C}}(W)}{|F|}.$$

□

Theorem 4.4.5. *Let G be a discrete, amenable group. Let $\tau \in \mathbb{C}[G]$ and $M_\tau \colon \mathbb{C}[G] \to \mathbb{C}[G]$, $M_\tau^{(2)} \colon l^2(G) \to l^2(G)$ be the left multiplication by τ.*

$$\dim_{\mathbb{C}[G]} (\ker M_\tau) = \dim_{\mathcal{N}(G)} \left(\ker M_\tau^{(2)}\right), \tag{4.4.9}$$

$$\dim_{\mathbb{C}[G]} (\operatorname{im} M_\tau) = \dim_{\mathcal{N}(G)} \left(\operatorname{im} M_\tau^{(2)}\right). \tag{4.4.10}$$

Proof. Let $\tau \in \mathbb{C}[G]$ and $M_\tau \colon \mathbb{C}[G] \to \mathbb{C}[G]$, $M_\tau^{(2)} \colon l^2(G) \to l^2(G)$ be the left multiplication by τ. For any finite subset $F \subset G$, we have that

$$\ker M_\tau^{(2)} \cap \operatorname{im} i_F \subset \mathbb{C}[G].$$

So $\ker M_\tau^{(2)} \cap \operatorname{im} i_F = \ker M_\tau \cap \operatorname{im} i_F$. Then by monotony for subspaces we have that

$$\begin{aligned} \dim_F (\ker M_\tau \cap \operatorname{im} i_F) &\le \dim_F \left(\ker M_\tau^{(2)}\right) = \dim_{\mathcal{N}(G)} \left(\ker M_\tau^{(2)}\right), \\ \dim_F (\operatorname{im} M_\tau \cap \operatorname{im} i_F) &\le \dim_F \left(\operatorname{im} M_\tau^{(2)}\right) = \dim_{\mathcal{N}(G)} \left(\operatorname{im} M_\tau^{(2)}\right). \end{aligned} \tag{4.4.11}$$

Now by additivity and $\dim_{\mathbb{C}[G]} \left(\mathbb{C}[G]\right) = 1 = \dim_{\mathcal{N}(G)} \left(l^2(G)\right)$ we see that

$$\begin{aligned} \dim_{\mathbb{C}[G]} (\ker M_\tau) + \dim_{\mathbb{C}[G]} (\operatorname{im} M_\tau) &= 1, \\ \dim_{\mathcal{N}(G)} \left(\ker M_\tau^{(2)}\right) + \dim_{\mathcal{N}(G)} \left(\operatorname{im} M_\tau^{(2)}\right) &= 1, \end{aligned}$$

which together with (4.4.11) implies

$$\begin{aligned} \dim_{\mathbb{C}[G]} (\ker M_\tau) &= \dim_{\mathcal{N}(G)} (\ker M_\tau), \\ \dim_{\mathbb{C}[G]} (\operatorname{im} M_\tau) &= \dim_{\mathcal{N}(G)} (\operatorname{im} M_\tau). \end{aligned}$$

□

4.5. Comparison to Elek's rank function

In [**Ele03b**] a rank function for finitely generated modules over the group ring of a finitely generated amenable group is introduced. It also has nice properties like additivity, so naturally we want to see if it agrees with the previously defined Følner dimension. To that end we will reproduce Elek's definition.

Definition 4.5.1. Let G be a discrete group, $\mathbb{K}$ be a field and $n \in \mathbb{N}$. We call $(\mathbb{K}^n)^{\oplus G}$ the maps from G to $\mathbb{K}^n$ with finite support, and $(\mathbb{K}^n)^G$ general maps from G to $\mathbb{K}^n$. We define

$$\langle \cdot, \cdot \rangle : \ (\mathbb{K}^n)^G \times (\mathbb{K}^n)^{\oplus G} \to \mathbb{K} \tag{4.5.1}$$

by $\langle f,\alpha\rangle := \sum_{g\in G}\big(f(g),\alpha(g)\big)$ for $f\in(\mathbb{K}^n)^G, \alpha\in(\mathbb{K}^n)^{\oplus G}$, where $(\cdot,\cdot)$ is the canonical bilinear form in $\mathbb{K}^n$

Lemma 4.5.2. *Let G be a discrete group, $\mathbb{K}$ be a field and $n\in\mathbb{N}$. There exists a canonical isomorphism $\Phi\colon \mathbb{K}[G]^n\to(\mathbb{K}^n)^{\oplus G}$.*

Proof. Let G be a discrete group, $\mathbb{K}$ be a field and $n\in\mathbb{N}$. Let

$$\alpha=\left(\sum_{x\in G}a_x^1x,\sum_{x\in G}a_x^2x,\dots,\sum_{x\in G}a_x^nx\right)\in\mathbb{K}[G]^n$$

then we define $\Phi(\alpha)(g) := \left(a_g^1,a_g^2,\dots,a_g^n\right)\in\mathbb{K}^n$. Note that $\Phi(\alpha)$ has finite support and Φ is clearly bijective and $\mathbb{K}$-linear. □

Lemma 4.5.3. *Let G be a discrete group and $\mathbb{K}$ be a field, we see that*

$$\big(\mathbb{K}[G]^n\big)^* := \hom_{\mathbb{K}}\big(\mathbb{K}[G]^n,\mathbb{K}\big)\cong(\mathbb{K}^n)^G \tag{4.5.2}$$

and $(\mathbb{K}^n)^G$ is a $\mathbb{K}$-vector space and there is a natural right G-action, $(f.h)(g) := f\left(gh^{-1}\right)$ for $f\in(\mathbb{K}^n)^G$ and $g,h\in G$.

Proof. Let G be a discrete group and $\mathbb{K}$ be a field, by the previous lemma we have $\hom_{\mathbb{K}}\big(\mathbb{K}[G]^n,\mathbb{K}\big)\cong\hom_{\mathbb{K}}\left((\mathbb{K}^n)^{\oplus G},\mathbb{K}\right)$. Now let $f\in(\mathbb{K}^n)^G$ then $\langle f,\cdot\rangle\in\hom_{\mathbb{K}}\left((\mathbb{K}^n)^{\oplus G},\mathbb{K}\right)$.

On the other hand let $1_{g,i}\in(\mathbb{K}^n)^{\oplus G}$ be the map that maps g to the i-th basis vector of $\mathbb{K}^n$ and is zero elsewhere, then $\left(1_{g,i}\right)_{g\in G,i=1,\dots,n}$ is a basis for $(\mathbb{K}^n)^{\oplus G}$. Now we define

$$\Psi\colon \hom_{\mathbb{K}}\left((\mathbb{K}^n)^{\oplus G},\mathbb{K}\right)\to(\mathbb{K}^n)^G$$

by $\phi\mapsto\left(g\mapsto\sum_{i=1}^n\phi(1_{g,i})1_{g,i}(g)\right)$. It is easy to see that $\sum_{i=1}^n\left\langle f,1_{g,i}\right\rangle 1_{g,i}(g)=f(g)$ for all $g\in G$ and

$$\begin{aligned}\left\langle\Psi(\phi),1_{h,j}\right\rangle&=\sum_{g\in G}\left(\sum_{i=1}^n\phi(1_{g,i})1_{g,i}(g),1_{h,j}(g)\right)\\&=\big(\phi(1_{h,j})1_{h,j}(h),1_{h,j}(h)\big)=\phi(1_{h,j}),\end{aligned}$$

for all $\phi(1_{h,j})\in(\mathbb{K}^n)^{\oplus G}$. This shows that $f\mapsto\langle f,\cdot\rangle$ and Ψ are inverse to each other. □

Lemma 4.5.4. *Let G be a discrete group, $\mathbb{K}$ be a field and let M be a finitely generated right $\mathbb{K}[G]$-module with $T\colon\mathbb{K}[G]^n\to M$ surjective and $\mathbb{K}[G]$-linear for* *some $n \in \mathbb{N}$. Then we have $T^*\colon \hom_{\mathbb{K}}(M,\mathbb{K}) \to \hom_{\mathbb{K}}\big(\mathbb{K}[G]^n,\mathbb{K}\big)$ by precomposition and T^* is injective and $\mathbb{K}[G]$-linear. So we have*

$$M^* \cong (\ker T)^{\perp} := \Big\{ f \in (\mathbb{K}^n)^G \;\Big|\; \big\langle f, \Phi(\alpha)\big\rangle = 0 \;\; \forall \alpha \in \ker T \Big\} \subset (\mathbb{K}^n)^G, \tag{4.5.3}$$

and it is invariant under right G-action.

Proof. Let G be a group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely generated right $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Let $f, f' \in M^*$ such that $T^*(f) = T^*(f')$ then

$$f\big(T(\alpha)\big) = T^*(f)(\alpha) = T^*(f')(\alpha) = f'\big(T(\alpha)\big)$$

for all $\alpha \in \mathbb{K}[G]^n$. Since T is surjective it follows that $f(m) = f'(m)$ for all $m \in M$ and thus $f = f' \in M^*$. Furthermore for all $\alpha \in \mathbb{K}[G]^n$, $\sum_{x\in G} a_x x \in \mathbb{K}[G]$ and $f \in M^*$ it holds that

$$\begin{aligned}\left((T^*f)\cdot\left(\sum_{x\in G} a_x x\right)\right)(\alpha) &= \sum_{x\in G} a_x\,(T^*f)\left(\alpha x^{-1}\right) = \sum_{x\in G} a_x f\left(T\left(\alpha x^{-1}\right)\right)\\ &= \sum_{x\in G} a_x f\left(T(\alpha)\,x^{-1}\right) = \sum_{x\in G} a_x\,(fx)\big(T(\alpha)\big)\\ &= \left(T^*\left(f\cdot\left(\sum_{x\in G} a_x x\right)\right)\right)(\alpha),\end{aligned}$$

so T^* is $\mathbb{K}[G]$-linear. Finally let $f \in M^*$ then $T^*f(x) = f\big(T(x)\big) = f(0) = 0$ for all $x \in \ker T$. On the other hand, let $f \in (\ker T)^{\perp}$ then it provides

$$\bar{f} \in \left(\frac{\mathbb{K}[G]^n}{\ker T}\right)^* \cong (M)^*$$

by $\bar{f}\left([\alpha]\right) := \big\langle f, \Phi(\alpha)\big\rangle$. □

Definition 4.5.5. [**Ele03b**] Let G be a discrete, amenable group, $\mathbb{K}$ be a field and (F_i) a Følner net of G. Let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear.

Then we have $T^*\colon \hom_{\mathbb{K}}(M,\mathbb{K}) \to \hom_{\mathbb{K}}\big(\mathbb{K}[G]^n,\mathbb{K}\big)$ by precomposition and T^* is injective. So consider $M^* \cong (\ker T)^{\perp} \subset \big(\mathbb{K}[G]^n\big)^G$ and define

$$\operatorname{rk}_{\mathrm{Elek}}(M) := \lim_i \frac{\dim_{\mathbb{K}}\left((\ker T)^{\perp}_{F_i}\right)}{|F_i|}, \tag{4.5.4}$$

where

$$(\ker T)^{\perp}_{F_i} := \left\{ f \in (\mathbb{K}^n)^G \;\middle|\; \operatorname{supp} f \subseteq F_i,\ \exists f' \in (\ker T)^{\perp} \colon f(g) = f'(g)\ \ \forall g \in F_i \right\}.$$

Theorem 4.5.6. [**Ele03b**]*Let G be a finitely generated, discrete and amenable group, $\mathbb{K}$ be a field and let M, N, P be finitely generated $\mathbb{K}[G]$-modules. Then the following statements hold.*

(i) If $M \cong N$, then $\operatorname{rk}_{Elek}(M) = \operatorname{rk}_{Elek}(N)$.

(ii) If $0 \to M \to P \to N \to 0$ is an exact sequence, then

$$\operatorname{rk}_{Elek}(M) + \operatorname{rk}_{Elek}(N) = \operatorname{rk}_{Elek}(P).$$

(iii) $\operatorname{rk}_{Elek}\big(\mathbb{K}[G]\big) = 1$.

Our next goal is to approximate this rank by the previously defined dimensions. For this purpose we need to compare the segments the normalized average is taken over.

Lemma 4.5.7. *Let G be a group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely generated right $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. For $F \in \mathcal{F}(G)$ let $i_F\colon \mathbb{K}[F]^n \hookrightarrow \mathbb{K}[G]^n$ and $p_F\colon \mathbb{K}[G]^n \twoheadrightarrow \mathbb{K}[F]^n$ be the canonical $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. Then*

$$(\ker T)^{\perp}_F \cong \left\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \exists f \in (\ker T)^{\perp} : \forall \alpha \in \mathbb{K}[G]^n \colon \\ \phi\big(p_F(\alpha)\big) = \sum_{g \in F} \Big(f(g), \big(\Phi(\alpha)\big)(g) \Big) \end{array} \right\}, \tag{4.5.5}$$

and also

$$\frac{\mathbb{K}[F]^n}{p_F(\ker T)} \cong \left\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \phi\big(p_F(\alpha)\big) = 0\ \ \forall \alpha \in \ker T \right\}, \tag{4.5.6}$$

$$\frac{\mathbb{K}[F]^n}{\ker T \cap \operatorname{im} i_F} \cong \left\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \forall \alpha \in \ker T \cap \operatorname{im} i_F \colon \\ \phi\big(p_F(\alpha)\big) = 0 \end{array} \right\}. \tag{4.5.7}$$

Proof. Let G be a group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely generated right $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Let $F \in \mathcal{F}(G)$ and let $i_F\colon \mathbb{K}[F]^n \hookrightarrow \mathbb{K}[G]^n$ and $p_F\colon \mathbb{K}[G]^n \twoheadrightarrow \mathbb{K}[F]^n$ be the canonical $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. We define $\pi_F := i_F \circ p_F$. Let $f \in (\mathbb{K}^n)^G$ then by Definition 4.5.1 and Lemma 4.5.2 we have that $\operatorname{supp} f \subseteq F$ if and only if

$$\forall \alpha \in \mathbb{K}[G]^n \colon \big\langle f, \Phi(\alpha) \big\rangle = \sum_{g \in F} \Big(f(g), \big(\Phi(\alpha)\big)(g) \Big) = \big\langle f, \Phi(\pi_F \alpha) \big\rangle.$$

With Lemmas 4.5.3 and 4.5.2 we get that $(\ker T)_F^{\perp}$ is isomorphic to

$$\left\{ \langle f, \cdot \rangle \;\middle|\; \begin{array}{c} f \in (\mathbb{K}^n)^G, \exists f' \in (\ker T)^{\perp} : \forall \alpha \in \mathbb{K}[G]^n : \\ \big\langle f, \Phi(\alpha) \big\rangle = \sum_{g \in F} \Big(f(g), \big(\Phi(\alpha)\big)(g) \Big) = \sum_{g \in F} \Big(f'(g), \big(\Phi(\alpha)\big)(g) \Big) \end{array} \right\}$$

$$\cong \left\{ \phi\colon (\mathbb{K}^n)^{\oplus G} \to \mathbb{K} \;\middle|\; \begin{array}{c} \exists f \in (\ker T)^{\perp} : \forall \alpha \in \mathbb{K}[G]^n : \\ \phi\big(\Phi(\alpha)\big) = \phi\big(\Phi(\pi_F \alpha)\big) = \sum_{g \in F} \Big(f(g), \big(\Phi(\alpha)\big)(g) \Big) \end{array} \right\}$$

$$\cong \left\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \exists f \in (\ker T)^{\perp} : \forall \alpha \in \mathbb{K}[G]^n : \\ \phi\big(p_F(\alpha)\big) = \sum_{g \in F} \Big(f(g), \big(\Phi(\alpha)\big)(g) \Big) \end{array} \right\}.$$

As $\mathbb{K}[F]^n$ is a finite dimensional $\mathbb{K}$-vector space, we have that $\dim_{\mathbb{K}}\big(\mathbb{K}[F]^n\big) = \dim_{\mathbb{K}}\Big(\big(\mathbb{K}[F]^n\big)^*\Big)$. The same holds for any quotient space.

Furthermore let $\phi\colon \mathbb{K}[F]^n \to \mathbb{K}$ and let $V \subset \mathbb{K}[F]^n$ be a $\mathbb{K}$-subspace, then $V \subset \ker \phi$ if and only if ϕ factors through the quotient space $\frac{\mathbb{K}[F]^n}{V}$. Thus,

$$\frac{\mathbb{K}[F]^n}{p_F(\ker T)} \cong \Big\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\Big|\; \phi\big(p_F(\alpha)\big) = 0 \;\; \forall \alpha \in \ker T \Big\},$$

$$\frac{\mathbb{K}[F]^n}{\ker T \cap \operatorname{im} i_F} \cong \left\{ \phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \forall \alpha \in \ker T \cap \operatorname{im} i_F : \\ \phi\big(p_F(\alpha)\big) = 0 \end{array} \right\}.$$

□

We see that Elek's rank lies between the "upper" and "lower" Følner dimension in some sense.

Theorem 4.5.8. *Let G be a discrete, amenable group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Then*

$$n - \overline{\dim}_{\mathbb{K}[G]}(\ker T) \le \operatorname{rk}_{Elek}(M) \le n - \dim_{\mathbb{K}[G]}(\ker T) = \dim_{\mathbb{K}[G]}(M). \tag{4.5.8}$$

Proof. Let G be a discrete, amenable group and (F_i) a Følner net of G. Let $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely generated $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. For $F \in \mathcal{F}(G)$ let $i_F\colon \mathbb{K}[F]^n \hookrightarrow \mathbb{K}[G]^n$ and $p_F\colon \mathbb{K}[G]^n \twoheadrightarrow \mathbb{K}[F]^n$ be the canonical $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces.

Let $\phi\colon \mathbb{K}[F]^n \to \mathbb{K}$ be a $\mathbb{K}$-linear map such that for all $\alpha \in \ker T$ we have $\phi\big(p_F(\alpha)\big) = 0$ and let $f \in (\mathbb{K}^n)^G$ be the image of ϕ precomposed by $p_F \circ \Phi^{-1}$

under the map Ψ, then $f \in (\ker T)^{\perp}$. Indeed by the proof of Lemma 4.5.3,

$$\left\langle f, \Phi\left(\alpha\right)\right\rangle = \phi\left(p_F \circ \Phi^{-1}\left(\Phi\left(\alpha\right)\right)\right) = \phi\left(p_F\left(\alpha\right)\right) = 0$$

for all $\alpha \in \ker T$.

On the other hand let $\phi\colon \mathbb{K}[F]^n \to \mathbb{K}$ be a $\mathbb{K}$-linear map and $f \in (\ker T)^{\perp}$ such that $\phi\left(p_F\left(\alpha\right)\right) = \sum_{g\in F}\left(f(g), \left(\Phi(\alpha)\right)(g)\right)$ for all $\alpha \in \ker T$. Now let $\alpha \in \ker T \cap \operatorname{im} i_F$ then

$$0 = \left\langle f, \Phi\left(\alpha\right)\right\rangle = \sum_{g\in F}\left(f(g), \left(\Phi(\alpha)\right)(g)\right) = \phi\left(p_F\left(\alpha\right)\right).$$

We have shown the following inclusions,

$$\begin{aligned}
&\left\{\phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \phi\left(p_F\left(\alpha\right)\right) = 0 \;\; \forall \alpha \in \ker T\right\}\\
\subseteq &\left\{\phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \exists f \in (\ker T)^{\perp} : \forall \alpha \in \mathbb{K}[G]^n : \\ \phi\left(p_F\left(\alpha\right)\right) = \sum_{g\in F}\left(f(g), \left(\Phi(\alpha)\right)(g)\right) \end{array}\right\}\\
\subseteq &\left\{\phi\colon \mathbb{K}[F]^n \to \mathbb{K} \;\middle|\; \begin{array}{c} \forall \alpha \in \ker T \cap \operatorname{im} i_F : \\ \phi\left(p_F\left(\alpha\right)\right) = 0 \end{array}\right\}.
\end{aligned}$$

By Lemma 4.5.7 we get for all $F \in \mathcal{F}(G)$ that

$$\begin{aligned}
n|F| - \dim_{\mathbb{K}}\left(p_F\left(\ker T\right)\right) &\le \dim_{\mathbb{K}}\left(\left(\ker T\right)^{\perp}_F\right)\\
&\le n|F| - \dim_{\mathbb{K}}\left(\ker T \cap \operatorname{im} i_F\right).
\end{aligned}$$

Now the statement follows from the definitions of $\overline{\dim_{\mathbb{K}[G]}}$, $\mathrm{rk}_{\mathrm{Elek}}$ and $\dim_{\mathbb{K}[G]}$. □

Corollary 4.5.9. *Let G be a discrete, amenable group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely presented $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Then*

$$\dim_{\mathbb{K}[G]}\left(M\right) = \mathrm{rk}_{Elek}\left(M\right). \tag{4.5.9}$$

Proof. Let G be a discrete, amenable group, $\mathbb{K}$ be a field, $n \in N$ and let M be a finitely presented $\mathbb{K}[G]$-module with $T\colon \mathbb{K}[G]^n \to M$ surjective and $\mathbb{K}[G]$-linear. Then by definition of finitely presented modules $\ker T$ is finitely generated. Now Lemma 4.2.3 implies the corollary. □

Remark 4.5.10. Let M be a finitely generated $\mathbb{K}[G]$-module with a surjective $\mathbb{K}[G]$-linear map $T\colon \mathbb{K}[G]^n \to M$ and let $\{K_\alpha | K_\alpha \subset \ker T \text{ finitely generated}\}$ be an exhausting directed system of submodules of $\ker T$. We have corresponding finitely presented $M_\alpha := \mathbb{K}[G]^n \big/ K_\alpha$, such that by the corollary above and continuity from below, Lemma 4.2.5, we see that

$$\begin{aligned} \inf_\alpha \left(\mathrm{rk}_{\mathrm{Elek}}\left(M_\alpha\right)\right) &= \inf_\alpha \left(\dim_{\mathbb{K}[G]}\left(M_\alpha\right)\right) \\ &= n - \sup_\alpha \left(\dim_{\mathbb{K}[G]}\left(K_\alpha\right)\right) \\ &= n - \dim_{\mathbb{K}[G]}\left(\ker T\right) \\ &= \dim_{\mathbb{K}[G]}\left(M\right). \end{aligned} \tag{4.5.10}$$

Thus if this kind of continuity from above holds for Elek's rank function,

$$\mathrm{rk}_{\mathrm{Elek}}\left(M\right) = \inf_\alpha \left(\mathrm{rk}_{\mathrm{Elek}}\left(M_\alpha\right)\right), \tag{4.5.11}$$

then $\dim_{\mathbb{K}[G]} = \mathrm{rk}_{\mathrm{Elek}}$ for all finitely generated $\mathbb{K}[G]$-modules. On the other hand if it does not hold, then $\dim_{\mathbb{K}[G]}$ seems preferable.

4.6. Residually finite groups

Again recall Lemma 2.1.5, which says that we can consider a left $\mathbb{K}[G]$-module as a right $\mathbb{K}[G]$-module. Now we reformulate the approximation result from [**LLS11**] which uses Elek's rank.

Theorem 4.6.1. *Let G be a finitely generated amenable group and $(G_i)_{i\in\mathbb{N}}$ be a chain of finite index normal subgroups of G, such that $\bigcap_{i\in\mathbb{N}} G_i = \{e\}$. Let $\mathbb{K}$ be a field and let M be a finitely presented left $\mathbb{K}[G]$-module. Then*

$$\dim_{\mathbb{K}[G]}\left(M\right) = \lim_{i\to\infty} \frac{\dim_{\mathbb{K}}\left(\mathbb{K}\otimes_{\mathbb{K}[G_i]} M\right)}{[G:G_i]}. \tag{4.6.1}$$

Proof. With $\dim_{\mathbb{K}[G]} = \mathrm{rk}_{\mathrm{Elek}}$ (Corollary 4.5.9) this follows directly from [**LLS11**, Theorem 2.1]. □

Remark 4.6.2. As is done in [**LLS11**, p.10] it is possible to drop the condition finitely generated from the group G.

If we recall Definition 1.3.11 and Theorem 1.3.13, the same result for Farber chains follows from the proof of [**LLS11**, Theorem 2.1].

Corollary 4.6.3. *Let G be a finitely generated amenable group and $(G_i)_{i\in\mathbb{N}}$ a Farber chain in G. Let $\mathbb{K}$ be a field and let M be a finitely presented left $\mathbb{K}[G]$-module. Then*

$$\dim_{\mathbb{K}[G]}\left(M\right) = \lim_{i\to\infty} \frac{\dim_{\mathbb{K}}\left(\mathbb{K}\otimes_{\mathbb{K}[G_i]} M\right)}{[G:G_i]}. \tag{4.6.2}$$

CHAPTER 5

Methods of computation

In this chapter we want to investigate which numbers lie in the range of the introduced Følner dimension. Recall our motivation from L^2-Betti numbers in the introduction. In accordance with the definition of L^2-Betti numbers we limit our investigation to the Følner dimension of kernels of $\mathbb{K}[G]$-matrices, where $\mathbb{K}[G]$ is the group ring of a finitely generated amenable group G over a field $\mathbb{K}$. Our main interest lies with fields of positive characteristics.

5.1. Computable numbers

Definition 5.1.1. [**VS03**] A function $r\colon \mathbb{Q}_{>0} \to \mathbb{Q}$ is called *computable*, if there exists an algorithm which halts (in finite time) on input $q \in \mathbb{Q}_{>0}$ and prints $r(q)$.

Remark 5.1.2. Vereshchagin and Shen mean algorithm in the sense of any programming language and remark that any algorithm computed by a mathematician with pen and paper can be build as a Turing machine as the requirements of finite memory and finite alphabet are met.

Definition 5.1.3. [**Mil04**] A real number $x \in \mathbb{R}$ is *computable* if there exists a computable function $r\colon \mathbb{Q}_{>0} \to \mathbb{Q}$ such that for all $q \in \mathbb{Q}_{>0}$ we have

$$|r(q) - x| < q. \tag{5.1.1}$$

Lemma 5.1.4. *Let G be a finitely generated discrete amenable group with solvable word-problem. We fix a generating set S and for $k \in \mathbb{N}$ we write B_k for the set of all words of length k or less. Then there exists an algorithm which halts on input $k \in \mathbb{N}$ and prints $F_k \subset G$ finite, such that*

$$\frac{|\partial_{B_k}(F_k)|}{|F_k|} \leq \frac{1}{k}, \tag{5.1.2}$$

and $F_{k-1} \subset F_k$.

Proof. Let G be a finitely generated discrete amenable group with solvable word-problem. We fix a generating set S and for $k \in \mathbb{N}$ we write B_k for the set of all words of length k or less. Enumerate a symmetric generating set of G, and thus fix an enumeration of G by word length, $g_1 = e, g_2, g_3, \ldots \in G$. We find an enumeration $L_i, i \in \mathbb{N}$, of finite subsets of G in the following way. Let $L_1 := \emptyset$ and $L_2 := \{g_1\}$, then define $L_3 := L_1 \cup \{g_2\}$ and $L_4 := L_2 \cup \{g_2\}$. Let $L_5 := L_1 \cup \{g_3\}$ and so on.

We describe a recursive algorithm for which the claim holds. Note that we make use of a word-problem algorithm throughout. On input $k \in \mathbb{N}$:

(1) If $k = 1$, set $F_0 := \{e\}$. Otherwise call the algorithm with input $k-1$ and get F_{k-1}.
(2) Calculate $F_{k-1} \cap B_s$ for $s \in \mathbb{N}$ until $F_{k-1} \cap B_s = F_{k-1}$ and therefore $F_{k-1} \subseteq B_s$.
(3) For $i > 0$ check $B_s \subset L_i$ and if so calculate $\frac{|\partial_{B_k}(L_i)|}{|L_i|}$ until
$$\frac{|\partial_{B_k}(L_i)|}{|L_i|} \leq \frac{1}{k}.$$
By Theorem 1.3.7 this halts because G is amenable.
(4) Set $F_k := L_i$.

We have $F_{k-1} \subseteq B_s \subset F_k$ and the claim holds. □

Corollary 5.1.5. *The sequence $(F_k)_{k\in\mathbb{N}}$ defined by an algorithm as in the lemma above is a Følner sequence.*

Proof. Let $(F_k)_{k\in\mathbb{N}}$ be of sequence of finite subsets of G such that
$$\frac{|\partial_{B_k}(F_k)|}{|F_k|} \leq \frac{1}{k},$$
and $F_{k-1} \subset F_k$. Now let $L \subset G$ be a finite subset, then there exists $k_L \in \mathbb{N}$ such that $L \subseteq B_k$ for all $k \geq k_L$. Then
$$\frac{|\partial_L(F_k)|}{|F_k|} \leq \frac{|\partial_{B_k}(F_k)|}{|F_k|} \leq \frac{1}{k},$$
for all $k \geq k_L$. Thus $\lim_{k\to\infty} \frac{|\partial_L(F_k)|}{|F_k|} = 0$. □

Theorem 5.1.6. *Let G be a finitely generated discrete amenable group and $\mathbb{K}$ be any field. If G has solvable word-problem, then $\dim_{\mathbb{K}[G]}(\ker T)$ and $\dim_{\mathbb{K}[G]}(\operatorname{im} T)$ are computable for any $\mathbb{K}[G]$-linear map*
$$T\colon \mathbb{K}[G]^m \to \mathbb{K}[G]^n.$$

Proof. Let $\mathbb{K}$ be any field and G be a finitely generated discrete amenable group such that G has solvable word-problem. Since G is finitely generated it is countable and all nets used in definitions of previous chapters can be reduced to sequences. For any $\mathbb{K}[G]$-linear map $T\colon \mathbb{K}[G]^m \to \mathbb{K}[G]^n$ we need a computable function $r\colon \mathbb{Q}_{>0} \to \mathbb{Q}$ such that

$$|r(q) - \dim_{\mathbb{K}[G]}(\ker T)| < q.$$

We will describe the algorithm for r in two parts, the first part finds a certain Følner set F, and the second part is the computation

$$F \mapsto \dim_{\mathbb{K}}(\ker T \cap \operatorname{im} i_F) = \dim_{\mathbb{K}}\left(\ker\left(\mathbb{K}[F]^m \xrightarrow{T} \mathbb{K}[B_L(F)]^n\right)\right).$$

Since G has solvable word-problem, the second part halts on any input $F \subset G$ finite. Without loss of generality we may assume that $0 < q \leq 1 \in \mathbb{Q}_{>0}$ and set $\epsilon := \frac{q}{10m}$. Fix the finite subset $L := \operatorname{supp} T \cup \operatorname{supp} T^{-1} \cup \{e\} \subset G$. We fix a symmetric generating set S of G and for $k \in \mathbb{N}$ we write B_k for the set of all words of length s or less.

Find $s \in \mathbb{N}$ such that $L \subset B_s$ and $\frac{1}{s} < \epsilon$. Now the algorithm from Lemma 5.1.4 produces Følner sets F_i for $i \geq s$, such that

$$\frac{|\partial_L(F_i)|}{|F_i|} \leq \frac{|\partial_{B_s}(F_i)|}{|F_i|} \leq \frac{1}{s} < \epsilon.$$

Calculate $n_\epsilon := \lceil \frac{\log \epsilon}{\log(2-\epsilon) - \log 2} \rceil$. Now we use the algorithm to find a family of Følner sets $\left(F_{k_1}, \ldots, F_{k_{n_\epsilon}}\right)$ with $k_j \geq s$ and $k_j < k_{j+1}$ such that

$$\frac{|\partial_{F_{k_j}}(F_{k_l})|}{|F_{k_l}|} < \epsilon^{2n_\epsilon}, \tag{5.1.3}$$

and

$$\left|\left(\frac{\dim_{\mathbb{K}}\left(\ker T \cap \operatorname{im} i_{F_{k_j}}\right)}{|F_{k_j}|}\right) - \left(\frac{\dim_{\mathbb{K}}\left(\ker T \cap \operatorname{im} i_{F_{k_l}}\right)}{|F_{k_l}|}\right)\right| < 2m\epsilon, \tag{5.1.4}$$

for all $1 \leq j < l \leq n_\epsilon$. To do this we may start with $k_1 := s$. Then find the next k_l under the restriction (5.1.3) and (5.1.4) for $k_j = k_{l-1}$. Then check (5.1.4) for all $k_j \leq k_l$ found so far, and if it fails, discard k_1 and shift all labels until it holds. We repeat until the family is found.

We can find such a family in finite time, because the first condition reduces to $\frac{|\partial_{F_{k_j}}\left(F_{k_{j+1}}\right)|}{|F_{k_{j+1}}|} < \epsilon^{2n_\epsilon}$ by $F_{k_j} \subset F_{k_{j+1}}$ and furthermore $\underline{h}\left(k_j\right) := \frac{\dim_{\mathbb{K}}\left(\ker T \cap \operatorname{im} i_{F_{k_j}}\right)}{|F_{k_j}|}$

is an element of the same convergent sequence for all $k_j \geq s$. On the other hand define $\overline{h}\left(k_j\right) := \frac{\dim_{\mathbb{K}}\left(p_{F_{k_j}}(\ker T)\right)}{|F_{k_j}|}$, then by Theorem 2.2.1 we have

$$\begin{aligned}\overline{h}\left(k_j\right) &\leq \underline{h}\left(k_j\right) + m\frac{|\partial_L\left(F_{k_j}\right)|}{|F_{k_j}|}\\ &\leq \underline{h}\left(k_j\right) + m\epsilon,\end{aligned}$$

and thus

$$\max_{1\leq j\leq n_\epsilon} \overline{h}\left(k_j\right) \leq \min_{1\leq j\leq n_\epsilon} \underline{h}\left(k_j\right) + 3m\epsilon.$$

Recall that $\overline{h}\left(k_j\right), \underline{h}\left(k_j\right) \leq m$ for all $1 \leq j < l \leq n_\epsilon$. Now by the estimate (4.2.9) for all Følner sets F_i with $i > k_{n_\epsilon}$ we know that

$$\min_{1\leq j\leq n_\epsilon} \underline{h}\left(k_j\right) \leq \dim_{\mathbb{K}[G]}(\ker T) + 5m\epsilon.$$

For $i > k_{n_\epsilon}$ such that

$$\frac{|\partial_{F_{k_{n_\epsilon}}}(F_i)|}{|F_i|} < \epsilon^{2n_\epsilon}$$

we get by decomposition of F_i according to Lemma 3.2.1 and the estimates seen in the proof of Theorem 3.2.3 that

$$\begin{aligned}\max_{1\leq j\leq n_\epsilon} \overline{h}\left(k_j\right) &\geq \left(\overline{\dim_{\mathbb{K}[G]}}(\ker T) - \epsilon\dim_{\mathbb{K}}\left(p_{\{e\}}(\ker T)\right)\right)(1-\epsilon)\\ &\geq \left(\overline{\dim_{\mathbb{K}[G]}}(\ker T) - m\epsilon\right) - \left(m\epsilon + m\epsilon^2\right)\\ &\geq \dim_{\mathbb{K}[G]}(\ker T) - 2m\epsilon.\end{aligned}$$

Finally we can estimate that

$$-5m\epsilon \leq \underline{h}\left(k_1\right) - \dim_{\mathbb{K}[G]}(\ker T) \leq 7m\epsilon,$$

and so we define $r\colon \mathbb{Q}_{>0} \to \mathbb{Q}$ by $r(q) := \underline{h}\left(k_1\right)$. Thus

$$|r(q) - \dim_{\mathbb{K}[G]}(\ker T)| \leq 7m\epsilon < q.$$

So $r\colon \mathbb{Q}_{>0} \to \mathbb{Q}$ is computable.

This shows the case of $\ker T$, for $\operatorname{im} T$ we merely need to consider

$$\dim_{\mathbb{K}}(\operatorname{im} T \cap \operatorname{im} i_F) = \dim_{\mathbb{K}}\left(\operatorname{im}\left(\mathbb{K}[B_L(F)]^m \xrightarrow{T} \mathbb{K}[B_L\left(B_L(F)\right)]^n\right) \cap \operatorname{im} i_F\right),$$

and Theorem 2.2.2. If we replace m by $(2m+n)$ the proof works just the same.

□

5.2. Graphical representation

To showcase some calculations of the Følner dimension previously defined, we amend the results of [**GS14**] with the case of $\mathbb{F}_2$, the field of 2 elements.

Theorem 5.2.1. *For every non-negative real number r there exists a finitely generated amenable group Λ and $T \in \mathbb{F}_2[\Lambda]^{m\times n}$ such that*

$$\dim_{\mathbb{F}_2[\Lambda]} (\ker T) = r. \tag{5.2.1}$$

Theorem 5.2.2. *There exists a finitely presented amenable group Λ and $T \in \mathbb{F}_2[\Lambda]^{2\times 1}$ such that*

$$\dim_{\mathbb{F}_2[\Lambda]} (\ker T) = \frac{2^8}{3^{10}} \cdot \sum_{k\in\mathbb{N}} \frac{1}{3^{k^2+3k}}, \tag{5.2.2}$$

which is irrational.

Remark 5.2.3. By Corollary 4.2.2 we can deduce from $\dim_{\mathbb{F}_2[\Lambda]} (\ker T)$ the minimal number of summands every element in $\ker T$ must have.

Because $\frac{2^8}{3^{10}} \cdot \sum_{k\in\mathbb{N}} \frac{1}{3^{k^2+3k}} \le \frac{2^8}{3^{10}} \cdot \frac{81}{80}$ we see that every element in $\ker T$ has atleast 228 summands.

5.2.1. Computational tool

This section is a reproduction of parts of [**GS14**] with some major modifications.

Let Λ be an amenable group, which is discrete and countable, and $\mathbb{K}$ be any field. Recall Definition 2.1.3. So for a finite subset $F \subset \Lambda$ let $i_F\colon \mathbb{K}[F] \hookrightarrow \mathbb{K}[\Lambda]$ be the inclusion of the finite-dimensional $\mathbb{K}$-vector subspace spanned by F and for any matrix $T \in \mathbb{K}[\Lambda]^{m\times n}$ denote the compression of T to F by the natural projection $p_F\colon \mathbb{K}[\Lambda] \twoheadrightarrow \mathbb{K}[F]$,

$$T_F := p_F T i_F\colon \mathbb{K}[F]^n \to \mathbb{K}[\Lambda]^m \to \mathbb{K}[F]^m.$$

Now we have by Theorem 2.2.1 and Corollary 1.3.2 that

$$\overline{\dim_{\mathbb{K}[\Lambda]}} (\ker T) = \dim_{\mathbb{K}[\Lambda]} (\ker T) = \lim_{i\to\infty} \frac{\dim_{\mathbb{K}} \left(\ker T_{F_i}\right)}{|F_i|},$$

for any Følner sequence $(F_i)_{i\in\mathbb{N}}$ of Λ.

Let G be a discrete, finitely generated amenable group and let $\mathbb{F}_2$, $\mathbb{F}_4 = \mathbb{F}_2[z]$ with $z^2 = 1 + z$, be the fields of two respectively four elements. Then there is a natural embedding of group rings $\mathbb{F}_2[G] \subset \mathbb{F}_4[G]$. Let A be a discrete, countable, abelian group, where each element has order 3, then all homomorphisms $A \to \mathbb{T}$ factor through $\left\{1, \exp\left(\frac{2}{3}\pi i\right), \exp\left(\frac{4}{3}\pi i\right)\right\}$. As we have an isomorphism between $\left\{1, \exp\left(\frac{2}{3}\pi i\right), \exp\left(\frac{4}{3}\pi i\right)\right\} \subset \mathbb{T}$ and the multiplicative group $\mathbb{F}_4^*$, let

$$X := \widehat{A} = \operatorname{Hom}(A, \mathbb{T}) \cong \operatorname{Hom}(A, \mathbb{F}_4^*)$$

be its compact Pontryagin dual and μ the normalized Haar measure on X.

Now by Pontryagin Duality there is an embedding $\mathbb{F}_4[A] \widehat{\to} \mathbb{F}_4^X$ given by

$$\sum_{c_i \in \mathbb{F}_4} c_i a_i \mapsto \left(x \mapsto \sum_{c_i \in \mathbb{F}_4} c_i x(a_i) \right).$$

So the image of $p \in \mathbb{F}_4[A]$ in $\mathbb{F}_4^X$ is written as $\widehat{p}$, while the preimage of $\chi \in \widehat{\mathbb{F}_4[A]}$ is written as $\check{\chi} \in \mathbb{F}_4[A]$.

Now let $e \in A$ be the neutral element, $E \subset A$ be a finite subset and $\Psi \colon E \to \mathbb{F}_4^*$, for these we define cylinder sets in X by

$$C_\Psi := \left\{ x \in X \;\middle|\; \forall a \in E \colon x(a) = \Psi(a) \right\}.$$

Then

$$p_\Psi := \prod_{a \in E} \left(e + \Psi(a)^2 a + \Psi(a) a^2 \right)$$

maps to the indicator function of C_Ψ, $\widehat{p_\Psi} = 1_{C_\Psi}$. To check this, let $x \in C_\Psi$ then

$$\begin{aligned} \widehat{p_\Psi}(x) &= \prod_{a \in E} \left(x(e) + \Psi(a)^2 x(a) + \Psi(a) x\left(a^2\right) \right) \\ &= \prod_{a \in E} (1 + 1 + 1) = 1. \end{aligned}$$

On the other hand, let $x \notin C_\Psi$, then there exists $a \in E$ such that $x(a) \neq \Psi(a)$. Thus $1 = x(a)^3 \neq \Psi(a) x(a)^2$, $\Psi(a)^2 x(a) \neq \Psi(a)^3 = 1$ and $\Psi(a) x(a)^2 \neq \Psi(a)^2 x(a)$. So we see that

$$x(e) + \Psi(a)^2 x(a) + \Psi(a) x\left(a^2\right) = 1 + \Psi(a)^2 x(a) + \Psi(a) x(a)^2 = 0,$$

and thus $\widehat{p_\Psi}(x) = 0$. This means the image of the duality map $\widehat{\mathbb{F}_4[A]}$ contains all $f \in \mathbb{F}_4^X$, which are linear combinations of such indicator functions.

Let $G \curvearrowright X$ be an action by automorphisms, then G acts on $\mathbb{F}_4^X$ by precomposition, $g.\chi(x) = \chi\left(g^{-1}.x\right)$, and $G \curvearrowright A$ such that the map above is equivariant.

Then we may define the semidirect product $G \ltimes A$ and for $g \in G, a \in A$ we denote the group multiplication by $g \cdot a = g.a \cdot g$ while the action is $g.a$.

Definition 5.2.4. Let T be the sum $\sum_{i=1}^{m} g_i \cdot f_i$ where $g_i \in G'$ for a symmetric subset $G' \subseteq G$ with $e \in G'$ and $f_i \in \mathbb{F}_4[A]$. Then $T \in \mathbb{F}_4[G \ltimes A]$ and we define a good basic graph with respect to G' for T as a set $\Gamma = \{p_1, \dots, p_n\} \subset \mathbb{F}_4[A]$ such that

(i) $\widehat{p_i}$ are indicator functions of pairwise disjoint subsets of X which are all of the same measure;
(ii) we have either $\operatorname{supp}\left(\widehat{p_j}\right) \subseteq \operatorname{supp}\left(\widehat{f_k}\right)$ or $\operatorname{supp}\left(\widehat{p_j}\right) \cap \operatorname{supp}\left(\widehat{f_k}\right) = \emptyset$ for all $j \leq n, k \leq m$;
(iii) if $\operatorname{supp}\left(\widehat{p_j}\right) \subseteq \operatorname{supp}\left(\widehat{f_k}\right)$ then $g_k.p_j \in \Gamma$ and $\widehat{f_k}$ is constant on $\operatorname{supp}\left(\widehat{p_j}\right)$;
(iv) for each pair $p_j, p_l \in \Gamma$ there exists exactly one $g \in G'$ such that $g.p_j = p_l$ and for $g, g', \tilde{g} \in G'$ and $p \in \Gamma$,

$$g.p = g'\tilde{g}.p \Rightarrow g = g'\tilde{g}.$$

Definition 5.2.5. Let Γ be a good basic graph with respect to G' for $T = \sum_{i=1}^{m} g_i \cdot f_i \in \mathbb{F}_4[G \ltimes A]$. For $1 \leq i \leq m$ and $p_j \in \Gamma$ let

$$t_{p_j,g_i} := \begin{cases} 1 & \text{if } \operatorname{supp}\left(\widehat{p_j}\right) \subset \operatorname{supp}\left(\widehat{f_i}\right), \\ 0 & \text{otherwise;} \end{cases} \tag{5.2.3}$$

and

$$\bar{f}_i(p_j) := \begin{cases} \widehat{f_i}\left(\operatorname{supp}\left(\widehat{p_j}\right)\right) \in \mathbb{F}_4 & \text{if } \operatorname{supp}\left(\widehat{p_j}\right) \subset \operatorname{supp}\left(\widehat{f_i}\right), \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.4}$$

Let $\mathcal{S}(\Gamma, T)$ be the directed edge-labelled graph whose vertices are $p_j \in \Gamma$ and for $1 \leq i \leq m$ and $p_j \in \Gamma$ such that $t_{p_j,g_i} = 1$ there is an edge from p_j to $g_i.p_j$ labeled with $\bar{f}_i(p_j)$. Let T_Γ be the directed adjacency operator on $\mathcal{S}(\Gamma, T)$, that is $T_\Gamma \colon \mathbb{F}_4[\Gamma] \to \mathbb{F}_4[\Gamma]$ is given by

$$T_\Gamma(p) := \sum_{i=1}^{m} \bar{f}_i(p) g_i.p.$$

Also we write $\mathcal{G}(\Gamma, T)$ for the underlying directed graph of $\mathcal{S}(\Gamma, T)$.

Remark 5.2.6. Let $\alpha := \left(p_{j_1}, g_{i_1}^{\epsilon_1}, p_{j_2}, g_{i_2}^{\epsilon_2}, \dots, g_{i_b}^{\epsilon_b}, p_{j_{b+1}}\right)$ be a path in $\mathcal{S}(\Gamma, T)$, where $\epsilon_1 = -1$ means that the path goes along the edge generated by $t_{p_{j_2},g_{i_1}}$ in the reverse direction, then we write $g(\alpha) := \prod_{l=1}^{b} g_{i_l}^{\epsilon_l}$. Note that if $\mathcal{G}(\Gamma, T)$ is a tree, i.e. disregarding the direction of the edges it is connected and cycle-free, then $\{g(\alpha) \mid \alpha \text{ is an undirected path in } \mathcal{S}(\Gamma, T)\}$ does the job for G' in condition (iv).

Definition 5.2.7. Let $p \in \Gamma$ then we define $G(p) := \{g \in G' \mid \exists q \in \Gamma \colon g.p = q\}$ and $G(\Gamma) := \bigcup_{p \in \Gamma} G(p)$. Note that $G(\Gamma) \subset G'$ is symmetric and contains the neutral element. Also $|G(p)| = |\Gamma|$ for all $p \in \Gamma$.

Lemma 5.2.8. *Let Γ be a good basic graph with respect to G' for $T = \sum_{i=1}^{m} g_i \cdot f_i \in \mathbb{F}_4[G \ltimes A]$ and fix $p \in \Gamma$ then* $\operatorname{span}\{g \cdot p \mid g \in G(p)\} \subset \mathbb{F}_4[G \ltimes A]$ *is T-invariant and T is isomorphic to T_Γ on it.*

Proof. Let Γ be a good basic graph with respect to G' for $T = \sum_{i=1}^{m} g_i \cdot f_i \in \mathbb{F}_4[G \ltimes A]$ and fix $p \in \Gamma$ and $g \in G(p)$. By definition

$$\widehat{f_i \cdot g.p} = \widehat{f_i} \cdot \widehat{g.p} = \bar{f}_i(g.p)\widehat{g.p},$$

and thus

$$\begin{aligned} T \cdot g \cdot p &= \sum_{i=1}^{m} g_i \cdot f_i \cdot g \cdot p \\ &= \sum_{i=1}^{m} g_i \cdot f_i \cdot g.p \cdot g \\ &= \sum_{i=1}^{m} g_i \cdot \bar{f}_i(g.p) g.p \cdot g \\ &= \sum_{i=1}^{m} \bar{f}_i(g.p) g_i \cdot (g \cdot p), \end{aligned}$$

where $g_i g \in G(p)$ if $\bar{f}_i(g.p) \neq 0$ by property of good basic graphs. For each pair $p_j, p_l \in \Gamma$ there exists exactly one $g \in G'$ such that $g.p_j = p_l$, thus the map

$$\operatorname{span}\{g \cdot p \mid g \in G(p)\} \to \mathbb{F}_4[\Gamma]$$

induced by $g \cdot p \mapsto g.p$ is an isomorphism. The lemma then follows from the calculation above. □

Definition 5.2.9. Let Γ be a good basic graph with respect to G' for $T \in \mathbb{F}_4[G \ltimes A]$ then

(i) $\operatorname{supp}(\Gamma) := \bigcup_{p \in \Gamma} \operatorname{supp}(\widehat{p})$.
(ii) Two good basic graphs for T are disjoint if their supports are disjoint.

(iii) We call a sequence of good basic graphs $\Gamma_1, \Gamma_2, \ldots$ exhausting if $\sum_{i=1}^{\infty} \mu\left(\operatorname{supp}(\Gamma_i)\right) = 1$ and Γ_i, Γ_j are pairwise disjoint.

(iv) A set of idempotent elements $K \subset \mathbb{F}_4[A]$ is called a refinement of Γ if $\bigcup_{p \in K} \operatorname{supp}(\widehat{p}) = \operatorname{supp}(\Gamma)$ and for each $p \in K$ there exists $q \in \Gamma$ such that $\operatorname{supp}(\widehat{p}) \subset \operatorname{supp}(\widehat{q})$.

Theorem 5.2.10. *Let G be a discrete, finitely generated amenable group acting on A, an elementary abelian 3-group, by automorphisms. If $\Gamma_1, \Gamma_2, \ldots$ is an exhausting sequence of good basic graphs with respect to G' for a family of operators $T^1, \ldots, T^l \in \mathbb{F}_4[G \ltimes A]$, which are represented by sums as before, then*

$$\dim_{\mathbb{F}_4[G \ltimes A]} \left(\bigcap_j \ker T^j \right) = \sum_{i=1}^{\infty} \frac{\mu\left(\operatorname{supp}(\Gamma_i)\right)}{|\Gamma_i|} \dim_{\mathbb{F}_4} \left(\bigcap_j \ker T^j_{\Gamma_i} \right).$$

Proof. Since the left hand side of the equation is calculated via the vector spaces spanned by Følner sets, the idea of the proof is to find a Følner sequence such that we can decompose these vector spaces upto a small remainder into T-invariant subspaces of known size. This leads to the right hand side.

First we will prove the case of only one operator. Let $\Gamma_1, \Gamma_2, \ldots$ be an exhausting sequence of good basic graphs with respect to G' for $T = \sum_{i=1}^{m} g_i \cdot f_i \in \mathbb{F}_4[G \ltimes A]$ and a subset $G' \subset G$. For $n \in \mathbb{N}$ let $N \in \mathbb{N}$ be such that $\sum_{i=1}^{N} \mu\left(\operatorname{supp}(\Gamma_i)\right) > 1 - \frac{1}{n}$. We note that by definition $p = p \cdot p \in \bigcup_{i=1}^{N} \Gamma_i$ and any two different $p \neq p' \in \bigcup_{i=1}^{N} \Gamma_i$ have disjoint support. Thus we define $p_\infty := 1 - \sum_{p \in \bigcup_i^N \Gamma_i} p$.

Let Σ' be a symmetric generating set for G and define

$$\Sigma := \bigcup_{i=1}^{K} G(\Gamma_i) \cup \Sigma'.$$

Note that Σ is symmetric and contains the neutral element of G. Recall Corollary 1.3.9 and let $\Phi_n \subset G$ be a symmetric Følner set such that

$$\frac{|\partial_\Sigma(\Phi_n)|}{|\Phi_n|} \leq \frac{1}{n},$$

fix $n \in \mathbb{N}$ and write $\Phi := \Phi_n$.

We will construct a special Følner set $F \subset A$ such that for every $p \in \bigcup_{i=1}^{N} \Gamma_i$ we find a decomposition of $\operatorname{supp}(\widehat{p}) = \bigsqcup U$ where

(i) the indicator function of U can be written using elements from F,

(ii) for any $\varphi \in \Phi$ we have that $\varphi.U \cap \left(\bigcup_{p\in\bigcup_{i=1}^N \Gamma_i} \operatorname{supp}(\widehat{p})\right) = \emptyset$ or $\varphi.U \subseteq \operatorname{supp}(\widehat{p})$ for some $p \in \bigcup_{i=1}^N \Gamma_i$.

Lemma 5.2.11. *There exist refinements $K_1, K_2, \ldots, K_N$ for $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ such that for every $p \in \bigcup_{i=1}^N K_i, \varphi \in \Phi, q \in \bigcup_{i=1}^N \Gamma_i$ we have that*

$$\operatorname{supp}(\varphi.\widehat{p}) \cap \operatorname{supp}(\widehat{q}) \neq \emptyset$$

implies $\operatorname{supp}(\varphi.\widehat{p}) \subseteq \operatorname{supp}(\widehat{q})$.

Proof. We fix an enumeration $\varphi_1, \varphi_2, \ldots \in \Phi$ and for $1 \leq j \leq |\Phi|$ we define recursively refinements Γ_i^j of Γ_i. With $\Gamma_i^0 := \Gamma_i$, we define

$$\Gamma_i^j := \left\{ p \cdot \left(\varphi_j^{-1}.q\right) \;\middle|\; \begin{array}{c} p \in \Gamma_i^{j-1}, q \in \left(\bigcup_{i=1}^N \Gamma_i \cup \{p_\infty\}\right), \\ \varphi_j.\operatorname{supp}(\widehat{p}) \cap \operatorname{supp}(\widehat{q}) \neq \emptyset \end{array} \right\}.$$

This is a refinement, because $\operatorname{supp}\left(\bigcup_{i=1}^N \Gamma_i \cup \{p_\infty\}\right) = X$. Now let $p \in \Gamma_i^j$ and fix $k \leq j$. Recall that A is abelian, then

$$\varphi_k.p = \varphi_k.p' \cdot q_k \cdot \left(\prod_{l=k+1}^{j} \left(\varphi_k \varphi_l^{-1}.q_l\right)\right),$$

where $p' \in \Gamma_i^{k-1}$ and $q_l \in \left(\bigcup_{i=1}^N \Gamma_i \cup \{p_\infty\}\right)$. Let $q \in \bigcup_{i=1}^N \Gamma_i$, if $q = q_k$ then

$$\operatorname{supp}(\varphi.\widehat{p}) \cap \operatorname{supp}(\widehat{q}) \neq \emptyset$$

and $\operatorname{supp}(\varphi.\widehat{p}) \subseteq \operatorname{supp}(\widehat{q})$. On the other hand if $q \neq q_k$ then $\operatorname{supp}(\varphi.\widehat{p}) \cap \operatorname{supp}(\widehat{q}) = \emptyset$, because the supports of different elements in an exhaustion of good basic graphs are disjoint. The lemma follows with $K_i := \Gamma_i^{|\Phi|}$. □

Now let $S < A$ be the finite subgroup generated by the finite subset

$$\bigcup_{p\in\bigcup_{i=1}^N K_i} \operatorname{supp}(p) \subset A.$$

Let F_j be an ascending sequence of finite overgroups of S such that $\bigcup_j F_j = A$ and $\Phi_j.F_{j-1} \subset F_j$.

Lemma 5.2.12. $\Phi_j \cdot F_j \subset G \ltimes A$ *is a Følner sequence in* $G \ltimes A$.

Proof. Let $a_{j-1} \in F_{j-1}, a_j \in F_j$ and $\varphi_j \in \Phi_j$ then by symmetry of Φ_j and $\Phi_j.F_{j-1} \subset F_j$ we have that

$$a_{j-1} \cdot \left(\varphi_j \cdot a_j\right) = \varphi_j \cdot \left(\left(\varphi_j^{-1}.a_{j-1}\right) \cdot a_j\right) \in \Phi_j \cdot F_j.$$

Let $L \subset G \ltimes A$ be finite and $0 < \epsilon < 1$ then there exists some $k \in \mathbb{N}$ such that $L \subset \Sigma^k \cdot F_k$. Let $j > k \in \mathbb{N}$ be such that $|\Phi_j \cap l\Phi_j| \geq (1-\epsilon)|\Phi_j|$ for all $l \in \Sigma^k$, we find this by Lemma 1.2.4. Recall that F_j is a group, then for any $g \cdot f \in L \subset \Sigma^k \cdot F_k$

$$\begin{aligned} |\Phi_j \cdot F_j \cap g \cdot f \cdot \Phi_j \cdot F_j| &= |\Phi_j \cdot F_j \cap g \cdot \Phi_j \cdot F_j| \\ &\geq |\left(\Phi_j \cap g \cdot \Phi_j\right) \cdot F_j| \\ &= |\Phi_j \cap g \cdot \Phi_j||F_j| \\ &\geq (1-\epsilon)|\Phi_j||F_j| = (1-\epsilon)|\Phi_j \cdot F_j|. \end{aligned}$$

□

Let $F := F_n$ for previously fixed $n \in \mathbb{N}$ and $\bar{\Phi} := \Phi \cup \partial_\Sigma(\Phi)$. Note that F is a Følner set as it is described above Lemma 5.2.11. We want to compute $\dim_{\mathbb{F}_4}(\ker T_{\bar{\Phi}\cdot F})$.

Lemma 5.2.13. *Let* $p \in \bigcup_{i=1}^N K_i$ *then*

$$\dim_{\mathbb{F}_4} \operatorname{span}\{p \cdot a \mid a \in F\} = |F| \cdot \mu\left(\operatorname{supp}(\widehat{p})\right). \tag{5.2.5}$$

Proof. Let X_F be the Pontryagin dual of $F \lhd A$, since F is finite abelian, we have that $|F| = |X_F|$. Let $f \in \mathbb{F}_4^{X_F}$, since X_F is finite, we have that $f = \sum_{x \in X_F} f(x) 1_{C_x}$, where the cylinder sets

$$C_x = \left\{x' \in X_F \;\middle|\; \forall a \in F \colon x'(a) = x(a)\right\} = \{x\}$$

are singletons. Then the $\mathbb{F}_4$-algebra embedding $\mathbb{F}_4[F] \xrightarrow{\hat{}} \mathbb{F}_4^{X_F}$ is surjective and we denote the left multiplication with $\hat{p}$ by $M_{\hat{p}} \colon \mathbb{F}_4^{X_F} \to \mathbb{F}_4^{X_F}$.

Let ν be the normalized Haar measure on X_F and recall that $\hat{p}$ is the indicator function for $\operatorname{supp}(\hat{p})$, then

$$|\left\{x \in X_F \;\middle|\; \hat{p}(x) = 1\right\}| = |F| \cdot \nu\left(\operatorname{supp}(\hat{p})\right).$$

Thus $\dim_{\mathbb{F}_4} \operatorname{im}\left(M_{\hat{p}}\right) = |F| \cdot \nu\left(\operatorname{supp}(\hat{p})\right)$. Denote by $M_p \colon \mathbb{F}_4[F] \to \mathbb{F}_4[F]$ the left multiplication with p then by the $\mathbb{F}_4$-algebra isomorphism we have that

$$\begin{aligned}\dim_{\mathbb{F}_4} \operatorname{span}\{p \cdot a \mid a \in F\} &= \dim_{\mathbb{F}_4} \operatorname{im}\left(M_p\right) = \dim_{\mathbb{F}_4} \operatorname{im}\left(M_{\hat{p}}\right)\\ &= |F| \cdot \nu\left(\operatorname{supp}(\hat{p})\right).\end{aligned}$$

It remains to be seen, that $\nu\left(\operatorname{supp}(\hat{p})\right) = \mu\left(\operatorname{supp}(\widehat{p})\right)$.

Let $\tilde{x} \in \operatorname{supp}(\hat{p}) \subset X_F$, that is $\tilde{x} \colon F \to \mathbb{F}_4^*$, $a \in F \setminus \{e\}$ and consider the cylinder sets $C'_{\tilde{x},a} := \left\{x \in X_F \,\middle|\, x(a) = \tilde{x}(a)\right\}$ and $C_{\tilde{x},a} := \left\{x \in X \,\middle|\, x(a) = \tilde{x}(a)\right\}$. Then $\nu\left(C'_{\tilde{x},a}\right) = \frac{1}{3} = \mu\left(C_{\tilde{x},a}\right)$. Finally let $E \subset F \setminus \{e\}$ be a minimal generating set, then $\frac{1}{3}^{|E|} = \nu\left(\bigcap_{a \in E} C'_{\tilde{x},a}\right) = \nu\left(\{\tilde{x}\}\right) = \frac{1}{|F|}$ and $\bigcap_{a \in E} C_{\tilde{x},a} = C_{\tilde{x}}$. Finally $\mu\left(\operatorname{supp}(\widehat{p})\right) = \sum_{\tilde{x} \in \operatorname{supp}(\hat{p}) \subset X_F} \frac{1}{|F|} = \nu\left(\operatorname{supp}(\hat{p})\right)$. $\square$

Now for each $p \in \bigcup_{i=1}^N K_i$ let $F(p) \subseteq F$ be the minimal subset such that

$$\dim_{\mathbb{F}_4} \operatorname{span}\left\{p \cdot a \,\middle|\, a \in F(p)\right\} = |F| \cdot \mu\left(\operatorname{supp}(\widehat{p})\right).$$

We say that a pair $\varphi \in \Phi, p \in \bigcup_{i=1}^N K_i$ is *lovely* if there is $i \leq N$ such that for some $q \in \Gamma_i$ we have $\operatorname{supp}(\varphi.\widehat{p}) \subseteq \operatorname{supp}(\widehat{q})$. Then we define

$$\begin{aligned}\Gamma(\varphi, p) &:= \Gamma_i,\\ G(\varphi, p) &:= G(q) \subseteq G(\Gamma_i),\end{aligned}$$

and $Q(\varphi, p) := q$.

We call a triple $\varphi \in \Phi, p \in \bigcup_{i=1}^N K_i, a \in F$ a lovely triple, if (φ, p) is lovely and $a \in F(p)$.

Corollary 5.2.14. *Let (φ, p, a) be such a lovely triple, then*

$$Y(\varphi, p, a) := \operatorname{span}\left\{g \cdot \varphi \cdot p \cdot a \,\middle|\, g \in G(\varphi, p)\right\} \subseteq \mathbb{F}_4[\bar{\Phi} \cdot F]. \tag{5.2.6}$$

Proof. Since Σ is symmetric and contains the neutral element of G, we have by Lemma 1.2.3 that $\Sigma \cdot \Phi \subseteq \Phi \cup \partial_\Sigma(\Phi) = \bar{\Phi}$. Let (φ, p, a) be a lovely triple and $g \in G(\varphi, p)$, then $\operatorname{supp}(p \cdot a) \subseteq F$ and $g \cdot \varphi \in \Sigma \cdot \Phi \subseteq \bar{\Phi}$. Thus

$$\operatorname{supp}(g \cdot \varphi \cdot p \cdot a) \in \bar{\Phi} \cdot F.$$

$\square$

Next we shall investigate how T acts on such $Y(\varphi, p, a)$.

Lemma 5.2.15. *Let $(\varphi, p, a), (\varphi_1, p_1, a_1)$ and (φ_2, p_2, a_2) be lovely triples.*

(i) $Y(\varphi,p,a)$ *is* T*-invariant and the restriction of* T *to* $Y(\varphi,p,a)$ *is isomorphic to* $T_{\Gamma(\varphi,p)}$.

(ii) $Y(\varphi_1,p_1,a_1)=Y(\varphi_2,p_2,a_2)$ *if and only if* $p_1=p_2$, $a_1=a_2$ *and for some* $g\in G(\varphi_1,p_1)$ *we have* $g\varphi_1=\varphi_2$.

(iii) The subspace of $\mathbb{F}_4[\bar{\Phi}\cdot F]$ *generated by all the spaces* $Y(\varphi,p,a)$*, is a direct sum of all the different such spaces.*

Proof. (i) Let $q:=Q(\varphi,p)$, then by definition we have $\mathrm{supp}(\varphi.\widehat{p})\subseteq\mathrm{supp}(\widehat{q})$ and also $\mathrm{supp}(g\varphi.\widehat{p})\subseteq\mathrm{supp}(g.\widehat{q})$ for $g\in G(\varphi,p)$. Recall that $\widehat{q}$ is an indicator function, thus $(g\varphi.\widehat{p})\cdot(g.\widehat{q})=(g\varphi.\widehat{p})$ and also $(g.q)\cdot(g\varphi.p)=(g\varphi.p)$. As is Lemma 5.2.8 we see that

$$\begin{aligned}T(g\cdot\varphi\cdot p\cdot a)&=\sum_{i=1}^{m}g_if_i\cdot(g\varphi).p\cdot(g\varphi)\cdot a\\&=\sum_{i=1}^{m}\big(g_if_i\cdot(g.q)\big)\cdot(g\varphi).p\cdot(g\varphi)\cdot a\\&=\sum_{i=1}^{m}\bar{f}_i(g.q)\big(g_i\cdot(g.q)\big)\cdot(g\varphi).p\cdot(g\varphi)\cdot a\\&=\sum_{i=1}^{m}\bar{f}_i(g.q)\,g_ig\cdot\varphi\cdot p\cdot a.\end{aligned}$$

So $Y(\varphi,p,a)$ is T-invariant because $\bar{f}_i(g.q)\neq 0$ implies $g_i\in G(g.q)$ and thus $g_ig\in G(\varphi,p)$. Since for each pair $q,q'\in\Gamma(\varphi,p)$ there exists at most one $g\in G$ such that $g.q=q'$, the map

$$Y(\varphi,p,a)\to\mathbb{F}_4[\Gamma(\varphi,p)]$$

induced by $g\cdot\varphi\cdot p\cdot a\mapsto g.q$ is an isomorphism. The statement follows from the definition of $T_{\Gamma(\varphi,p)}$.

(ii) Let $p_1=p_2$, $a_1=a_2, q_1:=Q(\varphi_1,p_1), q_2=Q(\varphi_2,p_2)$, and $\bar{g}\in G(\varphi_1,p_1)$ such that $\bar{g}\varphi_1=\varphi_2$. Then $\bar{g}\varphi_1\cdot p_1=\varphi_2\cdot p_2$. and

$$\emptyset\neq\mathrm{supp}(\bar{g}\varphi_1.\widehat{p_1})=\mathrm{supp}(\varphi_2.\widehat{p_2})\subseteq\mathrm{supp}(\bar{g}.\widehat{q_1})\cap\mathrm{supp}(\widehat{q_2}),$$

so $\bar{g}.q_1=q_2$ and therefore $\Gamma(\varphi_1,p_1)=\Gamma(\varphi_2,p_2)$ as well as $G(\varphi_1,p_1)=G(\varphi_2,p_2)\,\bar{g}$. Finally, for $g\in G(\varphi_2,p_2)$ we have $g\cdot\varphi_2\cdot p_2\cdot a_2=g\bar{g}\cdot\varphi_1\cdot p_1\cdot a_1$.

On the other hand let $Y(\varphi_1,p_1,a_1)=Y(\varphi_2,p_2,a_2)$, that means that for every $g\in G(\varphi_2,p_2)$ we have $g\cdot\varphi_2\cdot p_2\cdot a_2=\sum_k s_kg_k\cdot\varphi_1\cdot p_1\cdot a_1$, with $s_k\in\mathbb{F}_4, g_k\in G(\varphi_1,p_1)$.

But the group multiplication in $G \ltimes A$ then implies that $g_k \cdot \varphi_1 = g \cdot \varphi_2$ for all k and $\sum_k s_k = 1$, because $\widehat{p_1}$ and $\widehat{p_2}$ are indicator functions. In particular $\varphi_2 = \bar{g} \cdot \varphi_1$ for some $\bar{g} \in G(\varphi_1, p_1)$, $\varphi_2 \cdot p_2 \cdot a_2 = \bar{g} \cdot \varphi_1 \cdot p_1 \cdot a_1$ and so $p_2 \cdot a_2 = p_1 \cdot a_1$. Recall that $\widehat{p_1}$ and $\widehat{p_2}$ have disjoint support if $p_1 \neq p_2$ and that $\widehat{a}(x) \neq 0$ for all $x \in X, a \in A$ thus $\operatorname{supp}\left(\widehat{p_1 \cdot a_1}\right) = \operatorname{supp}(\widehat{p_1} \cdot \widehat{a}) = \operatorname{supp}(\widehat{p_1})$, so $p_1 = p_2$ and in turn $a_1 = a_2$, since $F(p_1)$ was chosen to be minimal.

(iii) Consider

$$0 = \sum_{j \in J} s_j \left(g_j \varphi_j \cdot p_j \cdot a_j\right) \in \mathbb{F}_4[\bar{\Phi} \cdot F],$$

for some $0 \neq s_j \in \mathbb{F}_4$, lovely triples $\left(\varphi_j, p_j, a_j\right)$ and $g_j \in G\left(\varphi_j, p_j\right)$. Then the summands where $g_j \varphi_j$ is some fixed element in $\bar{\Phi}$ also add up to 0. So we may assume that $g_j \varphi_j = g_1 \varphi_1$ for all $j \in j$.

Now $\sum_{j \in J} s_j \left(p_j \cdot a_j\right) = 0$, again recall that $\widehat{p_j}$ and $\widehat{p_{j'}}$ with $j \neq j'$ have disjoint support and as above $\operatorname{supp}\left(\widehat{p_j}\right) = \operatorname{supp}\left(\widehat{p_j \cdot a_j}\right)$. Again we may assume $p_j = p_1$ for all $j \in J$ and linear independence of $p_1 \cdot a$ for $a \in F(p_1)$ implies $a_j = a_1$.

But then $g_j \varphi_j . p_j = g_1 \varphi_1 . p_1$ for all $j \in J$ and so $g_j . q_j = g_1 . q_1$ for $q_j := Q\left(\varphi_j, p_j\right)$, because $g_j . q_j \neq g_1 . q_1$ implies disjoint support. Since the graphs Γ_i are pairwise disjoint we have that $q_j \in \Gamma(\varphi_1, p_1)$ and so $g_j^{-1} g_1 \in G(\varphi_1, p_1)$ for all $j \in J$. It follows that $g_j^{-1} g_1 \varphi_1 \cdot p_1 = \varphi_j \cdot p_j$ and therefore $Y(\varphi_1, p_1, a_1) = Y\left(\varphi_j, p_j, a_j\right)$.

□

Our aim is to calculate the dimension of the kernel of T compressed to $\mathbb{F}_4[\Phi F]$, for that we investigate the subspace where T behaves well-understood, as seen in the previous lemma. We define $Y \subseteq \mathbb{F}_4[\bar{\Phi} F]$ to be the span of all $Y(\varphi, p, a)$ where (φ, p, a) is a lovely triple and T_Y to be the restriction of T to Y. Furthermore let Y^c be its complement in $\mathbb{F}_4[\bar{\Phi} F]$, i.e. $Y \oplus Y^c = \mathbb{F}_4[\bar{\Phi} F]$.

To estimate the size of the complement Y^c, let $Y_\infty \subseteq \mathbb{F}_4[\Phi \cdot F]$ be

$$Y_\infty := \operatorname{span}\left\{ \varphi \cdot p \cdot a \;\middle|\; \varphi \in \Phi, p \in \bigcup_{i=1}^{N} K_i, a \in F(p), (\varphi, p) \text{ is not lovely} \right\}.$$

Recall that $\widehat{p_\infty} = 1 - \sum_{q \in \bigcup_{i=1}^{N} \Gamma_i} \widehat{q}$ and $\sum_{i=1}^{N} \mu\left(\operatorname{supp}(\Gamma_i)\right) > 1 - \frac{1}{n}$.

Lemma 5.2.16. *For Y_∞ as above we have* $\dim_{\mathbb{F}_4}(Y_\infty) \leq \frac{1}{n}|\Phi||F|$.

Proof. We consider the subspace of Y_∞ for a fixed φ first. For all $p \in \bigcup_{i=1}^N K_i$ such that (φ, p) is not lovely we have $\operatorname{supp}(\varphi.\widehat{p}) \subseteq \operatorname{supp}\left(\widehat{p_\infty}\right)$ by Lemma 5.2.11. Since $\mu\left(\operatorname{supp}\left(\widehat{p_\infty}\right)\right) \leq \frac{1}{n}$ and the measure on X is preserved by automorphisms we see

$$\sum_{(\varphi,p) \text{ not lovely}} \mu\left(\operatorname{supp}\widehat{p}\right) \leq \frac{1}{n}.$$

Then we see by Lemma 5.2.13 that the dimension of the subspace of Y_∞ for a fixed φ is bounded by $\frac{1}{n}|F|$. As Y_∞ lies in the sum of all such subspaces for $\varphi \in \Phi$ the statement follows. $\square$

Corollary 5.2.17. $\dim_{\mathbb{F}_4}(Y) \geq \left(1 - \frac{2}{n}\right)|\bar{\Phi}||F|$.

Proof. We see that $Y^c \subseteq \mathbb{F}_4[\partial_\Sigma(\Phi) \cdot F] + Y_\infty$ and recall that Φ is a $\frac{1}{n}$-Følner set for Σ, therefore

$$\dim_{\mathbb{F}_4} Y^c \leq \frac{|\partial_\Sigma(\Phi)|}{|\Phi|}|\Phi||F| + \frac{1}{n}|\Phi||F| \leq \frac{2}{n}|\Phi||F| \leq \frac{2}{n}|\bar{\Phi}||F|.$$

$\square$

Now we want to estimate $\dim_{\mathbb{F}_4}(\ker T_{\bar{\Phi}\cdot F})$ and thus note that

$$\dim_{\mathbb{F}_4}\left(\mathbb{F}_4[\bar{\Phi} \cdot F]\right) = |\bar{\Phi}||F| \geq \dim_{\mathbb{F}_4}(Y) \geq \left(1 - \frac{2}{n}\right)|\bar{\Phi}||F|.$$

As Y is T-invariant, we have that

(5.2.7)

$$\dim_{\mathbb{F}_4}(\ker T_{\bar{\Phi}\cdot F}) - \dim_{\mathbb{F}_4}(\ker T_Y) \leq \frac{2}{n}|\bar{\Phi}||F| \leq \frac{2}{n}\left(1 + \frac{1}{n}\right)|\Phi||F| \leq \frac{4}{n}|\Phi||F|.$$

Furthermore, with Theorem 2.2.1 we have that

$$|\dim_{\mathbb{F}_4}(\ker T_{\bar{\Phi}\cdot F}) - \dim_{\mathbb{F}_4}(\ker T_{\Phi\cdot F})| \leq \frac{3}{n}|\Phi||F|. \tag{5.2.8}$$

Next we estimate $\dim_{\mathbb{F}_4}(\ker T_Y)$ in multiple steps. Let Y_{Γ_i} be the subspace of Y where T acts like T_{Γ_i}. Define for all $\varphi \in \bar{\Phi}$ the set of projections p which are shifted into Γ_i,

$$K_{\varphi,\Gamma_i} := \left\{ p \in \bigcup_{i=1}^N K_i \;\middle|\; \exists q \in \Gamma_i \colon \operatorname{supp}(\varphi.\widehat{p}) \subseteq \operatorname{supp}(\widehat{q}) \right\},$$

then by the Lemma 5.2.15

$$Y_{\Gamma_i} = \sum_{\varphi\in\Phi} \bigoplus_{p\in K_{\varphi,\Gamma_i}} \bigoplus_{a\in F(p)} Y(\varphi, p, a).$$

On the other hand we define for all $\varphi \in \bar{\Phi}$ the following subspace corresponding to Γ_i,

$$\begin{aligned} Y_{\varphi,\Gamma_i} &:= \mathrm{span}\left\{\varphi \cdot p \cdot a \,\middle|\, p \in K_{\varphi,\Gamma_i}, a \in F(p)\right\} \\ &= \bigoplus_{p\in K_{\varphi,\Gamma_i}} \mathrm{span}\left\{\varphi \cdot p \cdot a \,\middle|\, a \in F(p)\right\}. \end{aligned}$$

Note that for $\varphi \in \Phi$ this is the vector space spanned by $\varphi \cdot p \cdot a$ for lovely triples (φ, p, a) such that $\Gamma(\varphi, p) = \Gamma_i$.

Recall that different p have disjoint support and $\bigcup_{i=1}^{N} \mathrm{supp}(K_i) \sqcup \mathrm{supp}\left(\widehat{p_\infty}\right) = X$, so by Lemma 5.2.11

$$\bigcup_{p\in K_{\varphi,\Gamma_i}} \mathrm{supp}(\varphi.\widehat{p}) = \mathrm{supp}(\Gamma_i) \setminus \mathrm{supp}\left(\varphi.\widehat{p_\infty}\right).$$

This implies by Lemma 5.2.13

Corollary 5.2.18.

$$\begin{aligned} \dim_{\mathbb{F}_4}\left(Y_{\varphi,\Gamma_i}\right) &= \mu\left(\mathrm{supp}(\Gamma_i) \setminus \mathrm{supp}\left(\varphi.\widehat{p_\infty}\right)\right)|F| \\ &= |F|\left(\mu\left(\mathrm{supp}(\Gamma_i)\right) - \mu\left(\mathrm{supp}(\Gamma_i) \cap \mathrm{supp}\left(\varphi.\widehat{p_\infty}\right)\right)\right), \end{aligned} \tag{5.2.9}$$

and also

$$\sum_{i=1}^{N} \mu\left(\mathrm{supp}(\Gamma_i) \cap \mathrm{supp}\left(\varphi.\widehat{p_\infty}\right)\right) \leq \frac{1}{n}.$$

Next we introduce an equivalence relation $\sim_{\Gamma_i}$ on $\bigoplus_{\varphi\in\bar{\Phi}} Y_{\varphi,\Gamma_i}$ generated by

$$\varphi \cdot p \cdot a \sim_{\Gamma_i} \varphi' \cdot p' \cdot a'$$

if (φ, p) is lovely, $p = p', a = a'$ and $\exists g \in G(\varphi, p) : g\varphi = \varphi'$.

Note that for $\varphi, \varphi' \in \Phi$ we have

$$\varphi \cdot p \cdot a \sim_{\Gamma_i} \varphi' \cdot p \cdot a \iff Y(\varphi, p, a) = Y\left(\varphi', p, a\right),$$

and recall that for every lovely pair (φ, p)

$$|G(\varphi, p)| = |\Gamma(\varphi, p)|.$$

Lemma 5.2.19. *With Y_{Γ_i} as before and $\sim_{\Gamma_i}$ restricted to $Y_{\Phi,\Gamma_i} := \bigoplus_{\varphi\in\Phi} Y_{\varphi,\Gamma_i}$ we have that*

$$Y_{\Gamma_i} \cong {}^{Y_{\Phi,\Gamma_i}}/{\sim_{\Gamma_i}} \otimes \mathbb{F}_4[\Gamma_i]. \tag{5.2.10}$$

Proof. We define the isomorphism on basis elements $g\cdot\varphi\cdot p\cdot a \in Y_{\Gamma_i}$ by

$$g\cdot\varphi\cdot p\cdot a \mapsto [\varphi\cdot p\cdot a]\otimes g.Q(\varphi,p).$$

We need to show injectivity and surjectivity for basis elements, then we get the isomorphism by linear extension.

(i) Let $[\varphi\cdot p\cdot a]\otimes g.Q(\varphi,p) = [\varphi'\cdot p'\cdot a']\otimes g'.Q(\varphi',p')$, since $\varphi,\varphi'\in\Phi$ this implies $p=p'$, $a=a'$ and there exists $\tilde{g}\in G(\varphi,p)$ such that $\tilde{g}\varphi=\varphi'$. Furthermore $g.Q(\varphi,p) = g'.Q(\varphi',p) = g'\tilde{g}.Q(\varphi,p)$, so by property of good basic graphs (iv), $g=g'\tilde{g}$. Therefore,

$$g\cdot\varphi\cdot p\cdot a = g'\tilde{g}\cdot\varphi\cdot p\cdot a = g'\cdot\varphi'\cdot p'\cdot a',$$

shows injectivity.

(ii) Let

$$[\varphi\cdot p\cdot a]\otimes q \in {}^{Y_{\Phi,\Gamma_i}}/{\sim_{\Gamma_i}}\otimes\mathbb{F}_4[\Gamma_i],$$

since $\varphi\in\Phi$ we know that (φ,p,a) is a lovely triple. By definition of good basic graphs, there exists $g\in G(\varphi,p)$ such that $g.Q(\varphi,p)=q$. Then

$$g\cdot\varphi\cdot p\cdot a \mapsto [\varphi\cdot p\cdot a]\otimes g.Q(\varphi,p) = [\varphi\cdot p\cdot a]\otimes q,$$

shows surjectivity.

□

Corollary 5.2.20. *$T_{Y_{\Gamma_i}}$, the restriction of T to Y_{Γ_i}, is isomorphic to* $\mathrm{Id}\otimes T_{\Gamma_i}$ *on* ${}^{Y_{\Phi,\Gamma_i}}/{\sim_{\Gamma_i}}\otimes\mathbb{F}_4[\Gamma_i]$.

Proof. Let $g\cdot\varphi\cdot p\cdot a\in Y_{\Gamma_i}$ with $q := Q(\varphi,p)\in\Gamma_i$ and $g\in G(\varphi,p)$, then as in the proof of Lemma 5.2.15, we have

$$T(g\cdot\varphi\cdot p\cdot a) = \sum_{i=1}^{m} \bar{f}_i(g.q)\, g_i g\cdot\varphi\cdot p\cdot a,$$

and $g_i g\in G(\varphi,p)$.

On the other hand the Lemma above shows that

$$g\cdot\varphi\cdot p\cdot a\mapsto[\varphi\cdot p\cdot a]\otimes g.q,$$
$$\sum_{i=1}^{m}\bar{f}_i\left(g.q\right)g_ig\cdot\varphi\cdot p\cdot a\mapsto[\varphi\cdot p\cdot a]\otimes\sum_{i=1}^{m}\bar{f}_i\left(g.q\right)g_ig.q$$

This agrees with the definition of $\operatorname{Id}\otimes T_{\Gamma_i}$. □

As Y_{Γ_i} is T-invariant for all $1\leq i\leq N$, we can use this isomorphism to describe the kernel of T_Y,

$$\ker T_Y=\bigoplus_{i=1}^{N}\ker T_{Y_{\Gamma_i}}\cong\bigoplus_{i=1}^{N}{}^{Y_{\Phi,\Gamma_i}}\!/\!\sim_{\Gamma_i}\otimes\ker T_{\Gamma_i}.$$

This shows that we need to estimate the dimension of ${}^{Y_{\Phi,\Gamma_i}}\!/\!\sim_{\Gamma_i}$.

Lemma 5.2.21. *For every $1\leq i\leq n$ we have that*

$$\begin{aligned}&|\left(\dim_{\mathbb{F}_4}\left({}^{Y_{\Phi,\Gamma_i}}\!/\!\sim_{\Gamma_i}\right)-\frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}|\Phi||F|\right)|\\ &\leq\frac{|F|}{|\Gamma_i|}\left(\frac{|\Phi|}{n}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)+\sum_{\varphi\in\Phi}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\cap\operatorname{supp}\left(\varphi.\widehat{p_\infty}\right)\right)\right).\end{aligned}\tag{5.2.11}$$

Proof. First we find the lower bound for $\dim_{\mathbb{F}_4}\left({}^{Y_{\Phi,\Gamma_i}}\!/\!\sim_{\Gamma_i}\right)$. Note that for $\sim_{\Gamma_i}$ at most $|\Gamma_i|$ basis elements lie in the same equivalence class. Therefore, in combination with Corollary 5.2.18,

$$\begin{aligned}&\dim_{\mathbb{F}_4}\left({}^{Y_{\Phi,\Gamma_i}}\!/\!\sim_{\Gamma_i}\right)\\ &\geq\frac{1}{|\Gamma_i|}\sum_{\varphi\in\Phi}\dim_{\mathbb{F}_4}\left(Y_{\varphi,\Gamma_i}\right)\\ &=\frac{1}{|\Gamma_i|}\sum_{\varphi\in\Phi}\left(|F|\left(\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)-\mu\left(\operatorname{supp}\left(\Gamma_i\right)\cap\operatorname{supp}\left(\varphi.\widehat{p_\infty}\right)\right)\right)\right)\\ &=\frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}|\Phi||F|-\frac{|F|}{|\Gamma_i|}\sum_{\varphi\in\Phi}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\cap\operatorname{supp}\left(\varphi.\widehat{p_\infty}\right)\right).\end{aligned}$$

For the upper bound note that $Y\subseteq\bigoplus_{\varphi\in\bar{\Phi}}Y_{\varphi,\Gamma_i}$ and $Y_{\Phi,\Gamma_i}\subseteq\bigoplus_{\varphi\in\bar{\Phi}}Y_{\varphi,\Gamma_i}\cap Y$. Let $g\cdot\varphi\cdot p\cdot a\in\bigoplus_{\varphi\in\bar{\Phi}}Y_{\varphi,\Gamma_i}\cap Y$ then

$$\varphi\cdot p\cdot a\sim_{\Gamma_i}g'\cdot\varphi\cdot p\cdot a$$

for all $g'\in G\left(\varphi,p\right)$. So from the definition of $\sim_{\Gamma_i}$ it is apparent that every equivalence class on $\bigoplus_{\varphi\in\bar{\Phi}}Y_{\varphi,\Gamma_i}\cap Y$ contains exactly $|\Gamma_i|$ basis elements. Therefore, again with Corollary 5.2.18,

$$\begin{aligned}
\dim_{\mathbb{F}_4}\left({}^{Y_{\Phi,\Gamma_i}}/\sim_{\Gamma_i}\right) &\leq \dim_{\mathbb{F}_4}\left(\bigoplus_{\varphi\in\bar{\Phi}} Y_{\varphi,\Gamma_i}\cap Y \Big/ \sim_{\Gamma_i}\right)\\
&= \frac{1}{|\Gamma_i|}\dim_{\mathbb{F}_4}\left(\bigoplus_{\varphi\in\bar{\Phi}} Y_{\varphi,\Gamma_i}\cap Y\right)\\
&\leq \frac{1}{|\Gamma_i|}\sum_{\varphi\in\bar{\Phi}}\dim_{\mathbb{F}_4}\left(Y_{\varphi,\Gamma_i}\right)\\
&\leq \frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}|\bar{\Phi}||F|\\
&\leq \frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}\left(1+\frac{1}{n}\right)|\Phi||F|
\end{aligned}$$

□

With this lemma we may now estimate

(5.2.12)

$$\begin{aligned}
&|\dim_{\mathbb{F}_4}\ker T_Y - \left(\sum_{i=1}^{N}\frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}|\Phi||F|\right)\dim_{\mathbb{F}_4}\ker\left(T_{\Gamma_i}\right)|\\
&= |\left(\sum_{i=1}^{N}\dim_{\mathbb{F}_4}\left({}^{Y_{\Phi,\Gamma_i}}/\sim_{\Gamma_i}\right)\right) - \left(\sum_{i=1}^{N}\frac{\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)}{|\Gamma_i|}|\Phi||F|\right)| \cdot \dim_{\mathbb{F}_4}\ker\left(T_{\Gamma_i}\right)\\
&\leq \sum_{i=1}^{N}\left(\frac{|F|}{|\Gamma_i|}\left(\frac{|\Phi|}{n}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)+\sum_{\varphi\in\Phi}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\cap\operatorname{supp}\left(\varphi.\widehat{p_\infty}\right)\right)\right)\right)\\
&\qquad\qquad \cdot\dim_{\mathbb{F}_4}\ker\left(T_{\Gamma_i}\right)\\
&\leq \sum_{i=1}^{N}|F|\left(\frac{|\Phi|}{n}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\right)+\sum_{\varphi\in\Phi}\mu\left(\operatorname{supp}\left(\Gamma_i\right)\cap\operatorname{supp}\left(\varphi.\widehat{p_\infty}\right)\right)\right)\\
&\leq |F|\left(\frac{|\Phi|}{n}+\sum_{\varphi\in\Phi}\frac{1}{n}\right)\\
&= \frac{2}{n}|\Phi||F|.
\end{aligned}$$

Combine the previous estimates (5.2.8),(5.2.7) and (5.2.12), while recalling that $\Phi = \Phi_n$, $F = F_n$ and $N = N_n$, and so the estimates hold for every $n \in \mathbb{N}$.

We conclude

$$|\frac{\dim_{\mathbb{F}_4}(\ker T_{\Phi_n\cdot F_n})}{|\Phi_n||F_n|} - \sum_{i=1}^{N_n} \frac{\mu\left(\operatorname{supp}(\Gamma_i)\right)}{|\Gamma_i|} \dim_{\mathbb{F}_4} \ker\left(T_{\Gamma_i}\right)| \leq \frac{9}{n}.$$

This shows the statement for one operator.

The general case with any number of operators $T^1, \dots, T^l \in \mathbb{F}_4[G' \ltimes A]$, follows directly, because the construction of Y and its subspaces is the same for all T^j and so

$$\dim_{\mathbb{F}_4}\left(\bigcap_j \ker T^j_{\bar{\Phi}\cdot F}\right) - \dim_{\mathbb{F}_4}\left(\bigcap_j \ker T^j_Y\right) \leq \frac{4}{n}|\Phi||F|,$$

$$|\dim_{\mathbb{F}_4}\left(\bigcap_j \ker T^j_{\bar{\Phi}\cdot F}\right) - \dim_{\mathbb{F}_4}\left(\bigcap_j \ker T^j_{\Phi\cdot F}\right)| \leq \frac{3}{n}|\Phi||F|,$$

$$\bigcap_j \ker T^j_Y \cong \bigoplus_{i=1}^{N} Y_{\Phi,\Gamma_i}/\sim_{\Gamma_i} \otimes \bigcap_j \ker T^j_{\Gamma_i}.$$

□

The theorem we have just proven allows us to compute the dimension of the kernel by way of adjacency operators on certain labeled graphs. In the calculation for the main theorems of this section, we need one class of labeled graphs, simply called nice graphs following the idea in [**GS14**].

Definition 5.2.22 (Nice graph)**.** Let $k \in \mathbb{N}_0, l \in \{0,1\}$. A *nice graph* of type (k,l) $\mathcal{G}$, is a finite rooted tree $\mathcal{G}_\alpha$, where all edges are directed towards the root and there are loops at all vertices with exception of k leafs, together with a subgraph $\mathcal{G}_\beta$ which has all the same vertices and no edges but the loop at the root if $l = 1$.

Let $\mathcal{G} = \left(\mathcal{G}_\alpha, \mathcal{G}_\beta\right)$ be a nice graph of type (k,l). We denote the adjacency matrix of $\mathcal{G}_\alpha$ by T_α and the one for $\mathcal{G}_\beta$ by T_β.

Example 5.2.23. We draw a nice graph $\mathcal{G}$ of type $(2,1)$, that is a finite rooted tree $\mathcal{G}_\alpha$ (Figure 5.1) together with its subgraph $\mathcal{G}_\beta$ (Figure 5.2).

Lemma 5.2.24. *Let $\mathcal{G}$ be a nice graph of type (k,l) and $\mathbb{K}$ any field then*

$$\dim_{\mathbb{K}}\left(\ker T_\alpha \cap \ker T_\beta\right) = \begin{cases} 0 & \text{if } k = 0, \\ k - l & \text{else.} \end{cases} \tag{5.2.13}$$

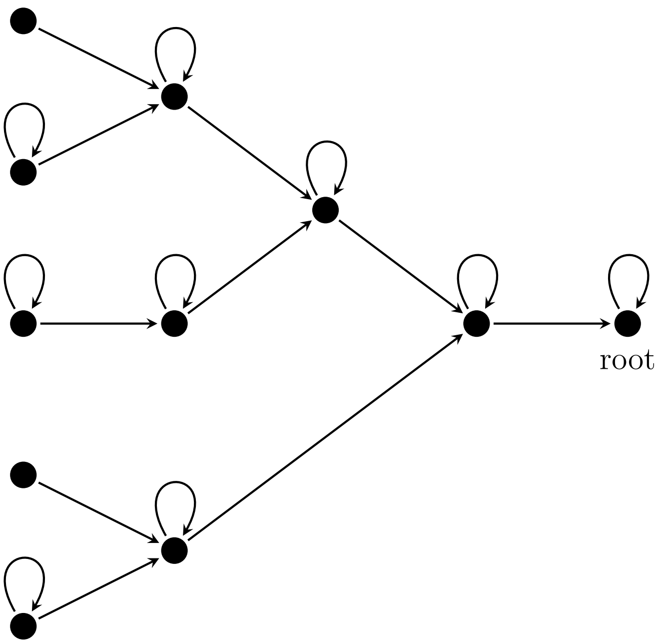

Figure 5.1. $\mathcal{G}_\alpha$ of a nice graph $\mathcal{G}$ of type $(2,1)$

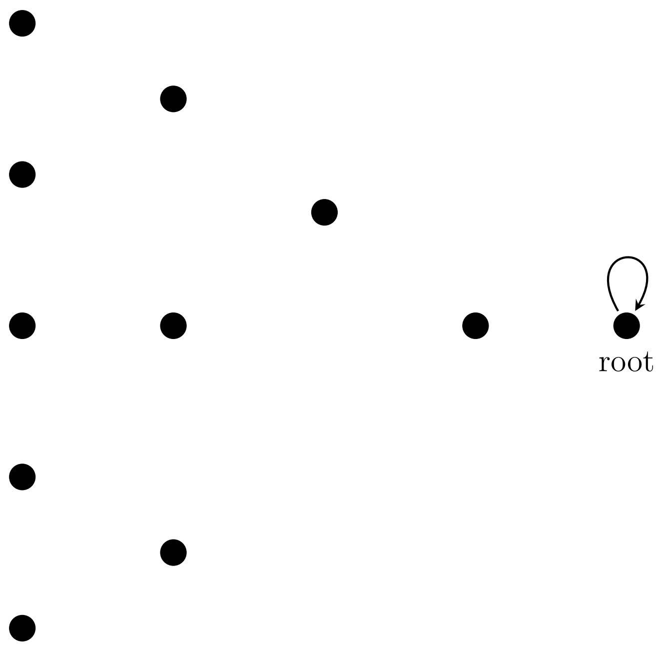

Figure 5.2. $\mathcal{G}_\beta$ of a nice graph $\mathcal{G}$ of type $(2,1)$

Proof. Let $\mathcal{G}$ be a nice graph of type (k,l) and $\mathbb{K}$ any field, where $1 \in \mathbb{K}$ is the neutral element of the group of units. Let $V := V(\mathcal{G})$ be the set of vertices of this tree, $m = |V|$ and $r \in V$ the root of the tree.

Let $(\zeta_i)_{i=1,\ldots,m}$ be a basis for $\mathbb{K}[V]$ such that the first k elements correspond to the leafs without loops, the last element corresponds to the root and the partial order induced by the tree is respected. Then for this basis $T_\alpha \in \mathbb{K}^{m\times m}$ is a lower triangle matrix such that

$$T_\alpha = \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline C & B & 0 \\ \hline \underline{C} & \underline{B} & 1 \end{array}\right),$$

where A is an $k \times k$ zero-matrix, $\binom{B}{\underline{B}}$ is an $(m-k)\times(m-k-1)$ matrix with 1's on the diagonal and one additional 1 in every column and $\binom{C}{\underline{C}}$ is an $(m-k)\times a$ matrix with a single 1 in every column. Using elementary matrix operations, see [**Hef15**], we may diagonalize from the right starting with column $m-1$, which has 1's in the two lowest entries and 0's elsewhere. This does not change the rank but pushes the off-diagonal non-zero entries to the bottom row. Thus we arrive at the following matrix

$$\left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & I & 0 \\ \hline D & \underline{D} & 1 \end{array}\right),$$

where I is the identity matrix, all the entries of D and $\underline{D}$ are ± 1 and the rank of this matrix is $m-k$.

Now we treat the two cases of T_β. If $l = 0$ then T_β is the zero operator, and

$$\dim_{\mathbb{K}}\left(\ker T_\alpha \cap \ker T_\beta\right) = \dim_{\mathbb{K}}\left(\ker T_\alpha\right) = m - \mathrm{rk}_{\mathbb{K}}\left(T_\alpha\right) = k.$$

If $l = 1$ then by the rank-nullity theorem and the corollary above

$$\dim_{\mathbb{K}}\left(\ker T_\alpha \cap \ker T_\beta\right) = m - \mathrm{rk}_{\mathbb{K}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ A & 0 & 0 \\ 0 & I & 0 \\ D & \underline{D} & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } k = 0, \\ k-1 & \text{else.} \end{cases}$$

The lemma follows immediately. □

Finally, before we start the proofs for the main theorems of this section, let us see what kind of indicator functions $\chi_Y \in \mathbb{F}_4^X$ come from elements in $\mathbb{F}_2[A]$.

Example 5.2.25. Let $\mathbf{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ be the abelian group of three elements, and $A := \mathbf{Z}_3^2$ then $X \cong \mathbf{Z}_3^2$. Let $\mathbb{F}_4 = \mathbb{F}_2[1, z]$ with $z^2 = 1 + z$. We write

$$f_1 = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} \in \mathbb{F}_2[A],$$
$$f_2 = \binom{0}{1} + \binom{0}{2} \in \mathbb{F}_2[A],$$

then

$$\widehat{f_1}\left(\binom{\bar{0}}{\bar{0}}\right) = \bar{0}(0)\cdot\bar{0}(0) + \bar{0}(1)\cdot\bar{0}(0) + \bar{0}(2)\cdot\bar{0}(0) = 1 + 1 + 1 = 1,$$
$$\widehat{f_1}\left(\binom{\bar{1}}{\bar{0}}\right) = \bar{1}(0)\cdot\bar{0}(0) + \bar{1}(1)\cdot\bar{0}(0) + \bar{1}(2)\cdot\bar{0}(0) = 1 + z + (1 + z) = 0,$$
$$\widehat{f_1}\left(\binom{\bar{0}}{\bar{2}}\right) = \bar{0}(0)\cdot\bar{2}(0) + \bar{0}(1)\cdot\bar{2}(0) + \bar{0}(2)\cdot\bar{2}(0) = 1 + 1 + 1 = 1,$$
$$\widehat{f_2}\left(\binom{\bar{0}}{\bar{0}}\right) = \bar{0}(0)\cdot\bar{0}(1) + \bar{0}(0)\cdot\bar{0}(2) = 1 + 1 = 0,$$
$$\widehat{f_2}\left(\binom{\bar{1}}{\bar{0}}\right) = \bar{1}(0)\cdot\bar{0}(1) + \bar{1}(0)\cdot\bar{0}(2) = 1 + 1 = 0,$$
$$\widehat{f_2}\left(\binom{\bar{0}}{\bar{2}}\right) = \bar{0}(0)\cdot\bar{2}(1) + \bar{0}(0)\cdot\bar{2}(2) = (1 + z) + z = 1.$$

Now it becomes clear, that $\widehat{f_1}$ is the indicator function on $\left\{\binom{t_1}{t_2} \,\middle|\, t_1 = \bar{0}\right\}$, and $\widehat{f_1}$ is the indicator function on $\left\{\binom{t_1}{t_2} \,\middle|\, t_2 \in \{\bar{1}, \bar{2}\}\right\}$.

Thus $f_1 f_2 \in \mathbb{F}_2[A]$ and $\widehat{f_1 f_2} = \widehat{f_1}\widehat{f_2} = \chi_{\left\{\binom{\bar{0}}{\bar{1}}, \binom{\bar{0}}{\bar{2}}\right\}}$.

5.2.2. Proof of Theorem 5.2.1

We will proof the theorem by constructing an amenable group and an operator represented in the group ring with coefficients in $\mathbb{F}_2$. To calculate the Følner dimension of its kernel with Theorem 5.2.10, we will also construct an exhaustion of good basic graphs for this operator.

The following construction is inspired by section 5.1 from [**Gra14**] adopted to $\mathbf{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ with ideas from [**Gra10**]. Let $X := M^{\mathbb{Z}} \times S$ be a discrete abelian group with normalized Haar measure μ, $M := (\mathbf{Z}_3)^3$ as the set of symbols and $S := (\mathbf{Z}_3)^4$ as the set of states. For subsets $m_{-k}, \ldots, m_0, \ldots, m_l \subset M$ and $\sigma \subset S$

the set

$$\{(\dot{n}, \tau) \in M^{\mathbb{Z}} \times S : n_{-k} \in m_{-k}, \ldots, n_l \in m_l, \tau \in \sigma\}$$

is denoted by

$$[m_{-k} \ldots m_{-1}\underline{m_0}m_1 \ldots m_l][\sigma].$$

Furthermore, for $\sigma \subset S$, let

$$[\,][\sigma] := \bigcup_{m \subset M} [\underline{m}][\sigma],$$

and for $m \subset M$ let

$$[\underline{m}][\,] := \bigcup_{\sigma \subset S} [\underline{m}][\sigma].$$

With $\mathbf{0} := \{0\} \subset \mathbf{Z}_3$ and $\mathbf{1} := \{1, 2\} \subset \mathbf{Z}_3$ let $\bar{M} := \{\mathbf{0}, \mathbf{1}\}^3 \subset \mathcal{P}(M)$ be a partition of M and let $\bar{S} := \{\mathbf{0}, \mathbf{1}\}^4 \subset \mathcal{P}(S)$ be a partition of S. Then we can divide X into disjoint subsets

$$X = \bigsqcup_{m \in \bar{M}, \sigma \in \bar{S}} [\underline{m}][\sigma].$$

Let $A := \bigoplus_{\mathbb{Z}} M \times S$ then $X = \widehat{A} = \mathrm{Hom}(A, \mathbb{F}_4^*)$. Note, like in Example 5.2.25, that the indicator function $\chi_{[\underline{m}][\sigma]} \in \mathbb{F}_4^X$, for $m \in \bar{M}$ and $\sigma \in \bar{S}$, has a preimage in $\mathbb{F}_2[A] \subset \mathbb{F}_4[A]$.

Let

$$\beta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{Aut}(M)$$

and define $G := \left[\left(\langle\beta\rangle \times \ \mathbf{Z}_2^2\right) \wr \mathbb{Z}\right] \times \mathrm{Aut}(S)$, which is a finitely generated amenable group by Lemma 1.3.10.

Let $B_\kappa, \kappa \in \{1, 2\}$, be the generators of $\mathbf{Z}_2^2$, t be the generator of $\mathbb{Z}$ and e the neutral element. Fix two subsets of the natural numbers, $\Sigma = \{\Sigma_1, \Sigma_2\} \subset \mathcal{P}(\mathbb{N})$, and define the left-action $\rho_\Sigma : G \curvearrowright X$ component-wise. $\mathrm{Aut}(S)$ acts naturally on

S and the action $\left[\left(\langle\beta\rangle \times\ \mathbf{Z}_2^2\right) \wr \mathbb{Z}\right] \curvearrowright M^{\mathbb{Z}}$ is defined by

$$\left(\beta.(\dot{n})\right)_j = \begin{cases} n_j \text{ for } j \neq 0, \\ \beta(n_0) \text{ for } j = 0. \end{cases}$$

$$\left(B_\kappa.(\dot{n})\right)_j = \begin{cases} \beta(n_j) \text{ for } j \in \Sigma_\kappa, \\ n_j \text{ otherwise.} \end{cases}$$

$$\left(t.(\dot{n})\right)_j = n_{j+1}$$

For example

$$[\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}\ \cdot\ ][\sigma] \xrightarrow{\rho_\Sigma(t\cdot\beta)} [(\mathbf{0},\mathbf{0},\mathbf{1})\underline{\ \cdot\ }][\sigma].$$

This gives us $G \ltimes_\Sigma A$, which is finitely generated and amenable as the semidirect product, and thus extension, of two finitely generated amenable groups, see Lemma 1.3.10. Now we fix a subset of M of fixed points for β, $m_\beta := (\mathbf{0},\mathbf{0},\mathbf{0}) \cup (\mathbf{1},\mathbf{1},\mathbf{1})$. Note that $\mu\left([\underline{m_\beta}][]\right) = \frac{1}{3}$. Furthermore, we write m_β^k for a sequence of k entries with m_β. Similarly we fix subsets of S, $s_0 := (\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})$, $s_1 := (\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0})$, $s_2 := (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1})$, $s_3 := (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})$ as well as automorphism of S which are bijections from and to these sets.

$$(s_0 \to s_1) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(S), \quad (s_1 \to s_2) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{Aut}(S),$$

$$(s_2 \to s_3) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(S)$$

To define the operator we look at a partition of X into four sets, I, C, P and R with

$$\begin{aligned} I &:= [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_0], \\ C &:= [\underline{m_\beta}][s_1] \cup [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_1] \cup [\underline{m_\beta}][s_2] \cup [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_2] \cup [\underline{m_\beta}][s_3], \\ P &:= [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_3] \\ R &:= X \setminus (I \cup C \cup P). \end{aligned}$$

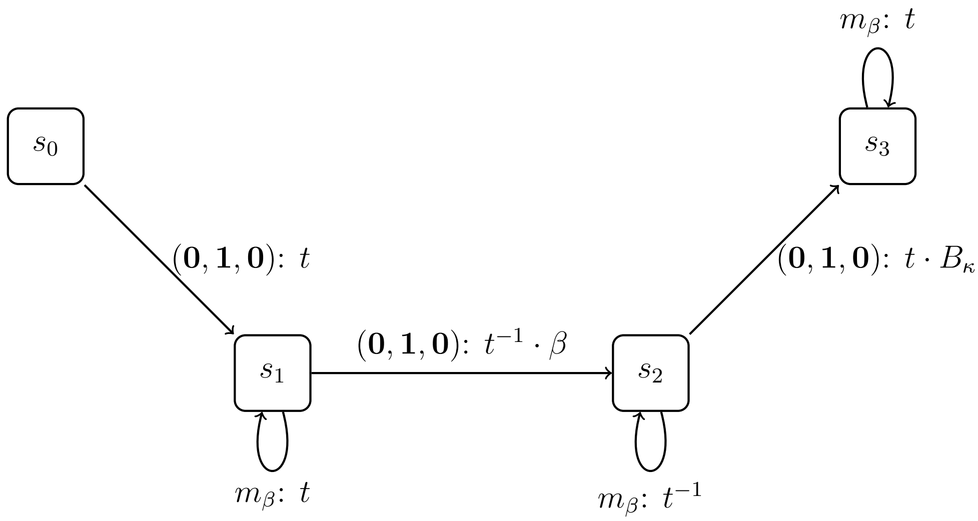

Figure 5.3. Graphical representation of T

Now we fix $\kappa \in \{1,2\}$ and define the operator $\breve{T} \in \mathbb{F}_2[G \ltimes_\Sigma A]$ by the formal sum $T :=$

$$
\begin{aligned}
& \big((s_0 \to s_1)\cdot t\big) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_0]} \\
+ \quad & (t) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_1]} \\
+ \quad & (t) \cdot\chi_{[\underline{(\mathbf{1},\mathbf{1},\mathbf{1})}][s_1]} \\
+ \quad & \big((s_1 \to s_2)\cdot t^{-1}\cdot\beta\big) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_1]} \\
+ \quad & \big(t^{-1}\big) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_2]} \\
+ \quad & \big(t^{-1}\big) \cdot\chi_{[\underline{(\mathbf{1},\mathbf{1},\mathbf{1})}][s_2]} \\
+ \quad & \big((s_2 \to s_3)\cdot t\cdot B_\kappa\big) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_2]} \\
+ \quad & (t) \cdot\chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_3]} \\
+ \quad & (t) \cdot\chi_{[\underline{(\mathbf{1},\mathbf{1},\mathbf{1})}][s_3]} \\
+ \quad & (e) \cdot\chi_{P\cup R}.
\end{aligned}
$$

It is also described by the Figure 5.3, where the trivial loops from the last summand are neglected. Because the supports of the summands give a disjoint partition of X, we may interpret T as a map from X to X.

Thus note that $T(x) = x, \forall x \in P\cup R$, so if $\exists N \in \mathbb{N}$ such that $T^{k+1}(x) = T^k(x)$, $\forall k \geq N$, we write $T^\infty(x) := T^N(x)$.

Lemma 5.2.26. *Let $\bar{X} := \{x \in X : T^\infty(x) \in P \cup R\}$, then*

$$\mu(\bar{X}) = 1.$$

Proof. Let $(\dot{n}, \tau) \in X \setminus \bar{X}$ then we see by looking at the diagram in Figure 5.3 that either $\forall k > 0 : n_k \in m_\beta$ or $\forall k < 0 : n_k \in m_\beta$, in both cases $(\dot{n}, \tau)$ lies in a set of measure zero. $\square$

We see that no element is mapped into I, $\{x \in X : T(x) \in I\} = \emptyset$.

Lemma 5.2.27. *Let $I_P := \{x \in I : T^\infty(x) \in P\}$ be the elements that start in I and arrive in P. Then*

$$I_P = \bigsqcup_{k \in \Sigma_\kappa} F_k$$

with $F_k := [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}m_\beta^{k-1}(\mathbf{0},\mathbf{1},\mathbf{0})][s_0]$. Furthermore $\mu(F_k) = \frac{2}{3^3} \cdot \frac{1}{3^{k-1}} \cdot \frac{2}{3^3} \cdot \frac{4}{3^4}$, and T is measure-preserving.

Proof. $\supseteq$: We chase the sets F_k for $k \in \Sigma_\kappa$ through the diagram in Figure 5.3

$$\begin{aligned}
T^0(F_k) &= [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}m_\beta^{k-1}(\mathbf{0},\mathbf{1},\mathbf{0})][s_0],\\
T^1(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})\underline{m_\beta}m_\beta^{k-2}(\mathbf{0},\mathbf{1},\mathbf{0})][s_1],\\
T^2(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})m_\beta\underline{m_\beta}m_\beta^{k-3}(\mathbf{0},\mathbf{1},\mathbf{0})][s_1],\\
T^k(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})m_\beta^{k-1}\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_1],\\
T^{k+1}(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})m_\beta^{k-2}\underline{m_\beta}(\mathbf{0},\mathbf{0},\mathbf{1})][s_2],\\
T^{2k}(F_k) &= [\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}m_\beta^{k-1}(\mathbf{0},\mathbf{0},\mathbf{1})][s_2],\\
T^{2k+1}(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})\underline{m_\beta}m_\beta^{k-2}(\mathbf{0},\mathbf{1},\mathbf{0})][s_3],\\
T^{3k}(F_k) &= [(\mathbf{0},\mathbf{1},\mathbf{0})m_\beta^{k-1}\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][s_3] \subset P.
\end{aligned}$$

$\subseteq$: Let $x \in I_P \subset [][s_0]$ be an element that arrives in $P \subset [][s_3]$. By definition this takes finitely many steps, and from the diagram in Figure 5.3 we can see that there exists $n \in \mathbb{N}$ such that $T^n(x) \in [][s_1]$ and $T^{n+1}(x) \in [][s_2]$. Thus the chase above shows that $x \in F_n$.

Now assume $n \notin \Sigma_\kappa$ then $T^{3n}(x) \in [(\mathbf{0},\mathbf{1},\mathbf{0})m_\beta^{k-1}\underline{(\mathbf{0},\mathbf{0},\mathbf{1})}][s_3] \subset R$ which is a contradiction to $x \in I_P$. To calculate the measure of F_k we have to multiply the measures of the subsets each position is restricted to. So

$$\mu(F_k) = \mu\left([\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][]\right)\mu\left([\underline{m_\beta}][]\right)^{k-1}\mu\left([\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][]\right)\mu\left([][s_0]\right) = \frac{2}{3^3}\cdot\frac{1}{3^{k-1}}\cdot\frac{2}{3^3}\cdot\frac{4}{3^4}.$$

T is measure-preserving since $\mu\left([][s_i]\right) = \mu\left([][s_j]\right)$ for $i, j \in \{0,1,2,3\}$ and $\mu\left([\underline{(\mathbf{0},\mathbf{1},\mathbf{0})}][]\right) = \mu\left([\underline{(\mathbf{0},\mathbf{0},\mathbf{1})}][]\right)$. $\square$

Let $S_1 := T + (e \cdot \chi_C)$ and $S_2 := e \cdot \chi_R$, we define good basic graphs for $\breve{S}_1$ and $\breve{S}_2$, $\Gamma_k = \{\breve{\chi}_{T^i(F_k)} : 0 \le i \le 3k\}$, $k \in \Sigma\kappa$, where we can easily check the conditions for a good basic graph are fulfilled according to Remark 5.2.6 and the proof above.

Thus, recalling the Figure 5.3, we see that $\mathcal{G} := \left(\mathcal{G}\left(\Gamma_k, \breve{S}_1\right), \mathcal{G}\left(\Gamma_k, \breve{S}_2\right)\right)$ is a nice graph of type $(1,0)$. Furthermore $\breve{S}_{1,\Gamma_k} = T_\alpha$ and $\breve{S}_{2,\Gamma_k} = T_\beta$, the adjancency operators for $\mathcal{G}$, since $\operatorname{supp}\Gamma_k \cap R = \emptyset$, and thus by Lemma 5.2.24

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,\Gamma_k} \cap \ker \breve{S}_{2,\Gamma_k}\right) = 1.$$

Note that $\mu\left(\operatorname{supp}(\Gamma_k)\right) = |\Gamma_k| \cdot \frac{2}{3^3} \cdot \frac{1}{3^{k-1}} \cdot \frac{2}{3^3} \cdot \frac{4}{3^4}$.

Let $Y = \left\{y \in \bar{X} \setminus I_P \;\middle|\; \nexists x \in X \colon T(x) = y\right\}$ then for every $x \in \bar{X} \setminus \left(\bigcup_{k\in\Sigma} \operatorname{supp}\Gamma_k\right)$ there exists $n \in \mathbb{N}$ and $y \in Y$ such that $T^n(y) = x$, that is all of $\bar{X}$ is reached from $Y \cup I_P$. We use this to find good basic graphs to exhaust $\bar{X}$.

For every $y \in Y$ the sequence $\left(T^n(y)\right)_{n\in\mathbb{N}}$ produces a sequence $\delta(y) = (\chi_n)_{n\in\mathbb{N}}$ of indicator functions corresponding to which summand of T applies at that step. Then the subset $Y_\delta := \left\{y \in Y \;\middle|\; \delta(y) = \delta\right\}$ generates a good basic graph H_δ for every such sequence δ. Clearly, $\delta \neq \delta'$ implies H_δ is disjoint from H'_δ and thus we produce an exhaustion of $\bar{X}$.

Now let H be such a good basic graph for $\breve{S}_1$ disjoint from Γ_k for $k \in \Sigma$. There are two cases to consider. In the first case $\operatorname{supp}(H) \cap I \neq \emptyset$, then by Lemma 5.2.27 $\operatorname{supp}(H) \cap R \neq \emptyset$ and so $\left(\mathcal{G}\left(H, \breve{S}_1\right), \mathcal{G}\left(H, \breve{S}_2\right)\right)$ is a nice graph of type $(1,1)$. Thus

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,H} \cap \ker \breve{S}_{2,H}\right) = 0.$$

In the second case $\operatorname{supp}(H) \cap I = \emptyset$, then $\left(\mathcal{G}\left(H, \breve{S}_1\right), \mathcal{G}\left(H, \breve{S}_2\right)\right)$ is a nice graph of type $(0,l)$. Thus

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,H} \cap \ker \breve{S}_{2,H}\right) = 0.$$

Since $\mu\left(\bar{X}\right) = 1$, we can amend $(\Gamma_k)_{k\in\Sigma}$ by such H_δ to get an exhausting sequence of good basic graphs for $\breve{S}_1$ and $\breve{S}_2$. So with Corollary 2.1.2 and Theorem 5.2.10 we get that

$$\dim_{\mathbb{F}_2[G\ltimes_\Sigma A]}\left(\ker\begin{pmatrix}\breve{S}_1\\ \breve{S}_2\end{pmatrix}\right) = \dim_{\mathbb{F}_4[G\ltimes_\Sigma A]}\left(\ker \breve{S}_1 \cap \ker \breve{S}_2\right) = \frac{8}{3^9} \cdot \sum_{k\in\Sigma_\kappa} \frac{1}{3^k}.$$

Now let $r \in \mathbb{R}_+$ and $r_0 \in [0,1] \subset \mathbb{R}$ such that $r \cdot \frac{3^9}{8}\frac{1}{r_0} = R \in \mathbb{N}$ then using the triadic expansion we fix $\Sigma = \{\Sigma_1, \Sigma_2\} \subset \mathcal{P}(\mathbb{N})$ such that

$$r_0 = \sum_{k \in \Sigma_1} \frac{1}{3^k} + \sum_{k \in \Sigma_2} \frac{1}{3^k}.$$

In the calculations above we fixed $\kappa \in \{1,2\}$. Now we write $\breve{S}_1^1, \breve{S}_2^1$ for $\kappa = 1$ and $\breve{S}_1^2, \breve{S}_2^2$ for $\kappa = 2$, then we define

$$\tilde{S} := \bigoplus_{i=1}^{R} \begin{pmatrix} \breve{S}_1^1 & 0 \\ \breve{S}_2^1 & 0 \\ 0 & \breve{S}_1^2 \\ 0 & \breve{S}_2^2 \end{pmatrix} \in \mathbb{F}_2[G \ltimes_\Sigma A]^{4R \times 2R}.$$

Then again with Corollary 2.1.2 and Theorem 5.2.10 we get that

$$\begin{aligned} \dim_{\mathbb{F}_2[G \ltimes_\Sigma A]} \left(\ker \tilde{S}\right) &= R \cdot \left(\dim_{\mathbb{F}_4[G \ltimes_\Sigma A]} \left(\ker \begin{pmatrix} \breve{S}_1^1 \\ \breve{S}_2^1 \end{pmatrix} \oplus \ker \begin{pmatrix} \breve{S}_1^2 \\ \breve{S}_2^2 \end{pmatrix} \right) \right) \\ &= R \cdot \left(\frac{8}{3^9} \cdot \sum_{k \in \Sigma_1} \frac{1}{3^k} + \frac{8}{3^9} \cdot \sum_{k \in \Sigma_2} \frac{1}{3^k} \right) \\ &= R \cdot \frac{8}{3^9} \cdot r_0 = r. \end{aligned}$$

5.2.3. Proof of Theorem 5.2.2

As with the previous proof this construction is inspired by section 5.1 from [**Gra14**] adopted to $\mathbf{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ with ideas from [**Gra10**]. Let $X := \left(M^{\mathbb{Z}}\right)^3 \times S$ be a discrete abelian group with normalized Haar measure μ, $M := \mathbf{Z}_3$ as the set of symbols and $S := (\mathbf{Z}_3)^4$ as the set of states.

We use notation similar to the previous section. For subsets $a_{-k_a}, \ldots, a_0, \ldots, a_{l_a}$, $b_{-k_b}, \ldots, b_0, \ldots, b_{l_b}$, $c_{-k_c}, \ldots, c_0, \ldots, c_{l_c} \subset M$ and $\sigma \subset S$ the set

$$\left\{ (\dot{x}, \dot{y}, \dot{z}, \tau) \in \left(M^{\mathbb{Z}}\right)^3 \times S \;\middle|\; \begin{array}{l} x_{-k} \in a_{-k_a}, \ldots, x_0 \in a_0, \ldots, x_l \in a_{l_a}, \\ y_{-k} \in b_{-k_b}, \ldots, y_0 \in b_0, \ldots, y_l \in b_{l_b}, \quad \tau \in \sigma \\ z_{-k} \in c_{-k_c}, \ldots, z_0 \in c_0, \ldots, z_l \in c_{l_c}, \end{array} \right\}$$

is denoted by

$$\begin{bmatrix} a_{-k_a} \dots \underline{a_0} \dots a_{l_a} \\ b_{-k_b} \dots \underline{b_0} \dots b_{l_b} \\ c_{-k_c} \dots \underline{c_0} \dots c_{l_c} \end{bmatrix} [\sigma].$$

Given $m = (m_1, m_2, m_3) \in \left(\mathcal{P}\left(M\right)\right)^3$ we may also write $[\underline{m}][\sigma]$ or $[\underline{(m_1, m_2, m_3)}][\sigma]$ for

$$\begin{bmatrix} \underline{m_1} \\ \underline{m_2} \\ \underline{m_3} \end{bmatrix} [\sigma].$$

With $\mathbf{0} := \{0\} \subset \mathbf{Z}_3$ and $\mathbf{1} := \{1, 2\} \subset \mathbf{Z}_3$ let $\bar{M} := \{\mathbf{0}, \mathbf{1}\} \subset \mathcal{P}(M)$ be a partition of M and let $\bar{S} := \{\mathbf{0}, \mathbf{1}\}^4 \subset \mathcal{P}(S)$ be a partition of S. Then we can divide X into disjoint subsets

$$X = \bigsqcup_{m \in \bar{M}^3, \sigma \in \bar{S}} [\underline{m}][\sigma].$$

Let $A := \left(\bigoplus_{\mathbb{Z}} M\right)^3 \times S$ then $X = \widehat{A} = \mathrm{Hom}(A, \mathbb{F}_4^*)$. Note that the indicator function $\chi_{[\underline{m}][\sigma]} \in \mathbb{F}_4^X$, for $m \in \bar{M}^3$ and $\sigma \in \bar{S}$, has a preimage in $\mathbb{F}_2[A] \subset \mathbb{F}_4[A]$.

Let $G := \mathbb{Z}^3 \times \mathrm{Aut}(S)$ with generators t_1, t_2, t_3 for $\mathbb{Z}^3$. Clearly, G is amenable by Lemma 1.3.10. Then there is a natural left action $\rho : G \curvearrowright X$ via componentwise shift on the tapes $\left(M^{\mathbb{Z}}\right)^3$ and natural action on S. For example

$$\begin{bmatrix} \cdot\, \underline{\mathbf{1}} \,\cdot \\ \cdot\, \underline{\mathbf{0}} \,\cdot \\ \cdot\, \underline{\mathbf{1}} \,\cdot \end{bmatrix} [\sigma] \xrightarrow{\rho(t_1 \cdot t_2^{-1})} \begin{bmatrix} \cdot\, \mathbf{1} \,\underline{\cdot} \\ \underline{\cdot}\, \mathbf{0} \,\cdot \\ \cdot\, \underline{\mathbf{1}} \,\cdot \end{bmatrix} [\sigma].$$

Let $G \curvearrowright A$ be the dual action, then $G \ltimes A \cong$ is amenable as the semidirect product, and thus extension, of two amenable groups, see Lemma 1.3.10.

We fix $s_0, s_1, s_2, s_3 \in S$ as before and to define the operator we look at a partition of X into four sets, I, C, P and R with

$$\begin{aligned} I :=& [\underline{(\mathbf{1}, \mathbf{1}, \mathbf{1})}][s_0], \\ C :=& \Big([\underline{(\mathbf{0}, \mathbf{0}, \mathbf{1})}][s_1] \cup [\underline{(\mathbf{1}, \mathbf{1}, \mathbf{1})}][s_1] \cup [\underline{(\mathbf{0}, \mathbf{0}, \mathbf{0})}][s_2] \\ & \cup [\underline{(\mathbf{1}, \mathbf{0}, \mathbf{0})}][s_2] \cup [\underline{(\mathbf{0}, \mathbf{0}, \mathbf{0})}][s_3] \cup [\underline{(\mathbf{1}, \mathbf{0}, \mathbf{0})}][s_3] \Big), \\ P :=& [\underline{(\mathbf{0}, \mathbf{1}, \mathbf{1})}][s_2] \\ R :=& X \setminus (I \cup C \cup P). \end{aligned}$$

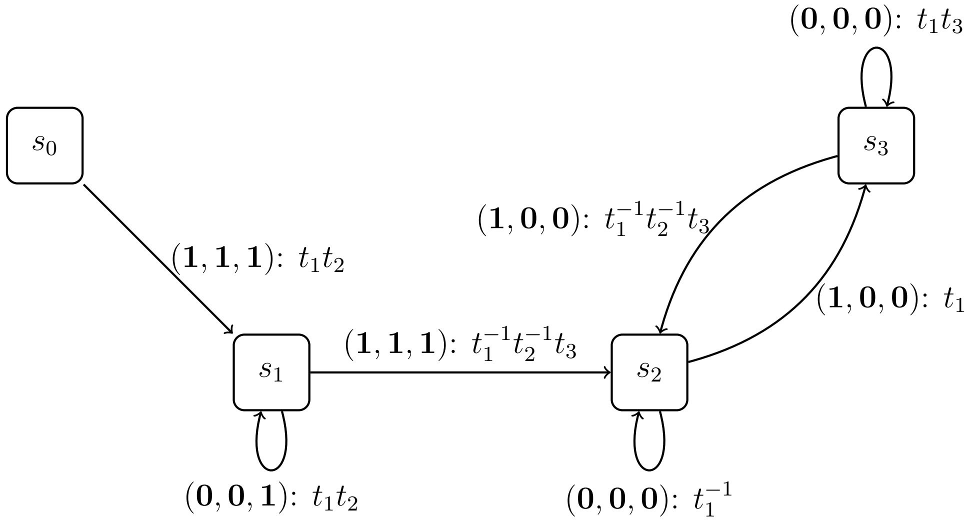

Figure 5.4. Graphical representation T

Now we define the operator $\breve{T} \in \mathbb{F}_2[G \ltimes A]$ by the formal sum $T :=$

$$
\begin{aligned}
& \big((s_0 \to s_1) \cdot t_1 \cdot t_2\big) \cdot \chi_{[\underline{(\mathbf{1},\mathbf{1},\mathbf{1})}][s_0]} \\
+ \quad & (t_1 \cdot t_2) \cdot \chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{1})}][s_1]} \\
+ \quad & \big((s_1 \to s_2) \cdot t_1^{-1} \cdot t_2^{-1} \cdot t_3\big) \cdot \chi_{[\underline{(\mathbf{1},\mathbf{1},\mathbf{1})}][s_1]} \\
+ \quad & \big(t_1^{-1}\big) \cdot \chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_2]} \\
+ \quad & \big((s_2 \to s_3) \cdot t_1\big) \cdot \chi_{[\underline{(\mathbf{1},\mathbf{0},\mathbf{0})}][s_2]} \\
+ \quad & (t_1 \cdot t_3) \cdot \chi_{[\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_3]} \\
+ \quad & \Big((s_2 \to s_3)^{-1} \cdot t_1^{-1} \cdot t_2^{-1} \cdot t_3\Big) \cdot \chi_{[\underline{(\mathbf{1},\mathbf{0},\mathbf{0})}][s_3]} \\
+ \quad & (e) \cdot \chi_{P \cup R}.
\end{aligned}
$$

It is also described by the Figure 5.4, where the trivial loops from the last summand are neglected. Since the supports of the summands do not intersect, we may interpret T as a map from X to X.

Thus note that $T(x) = x, \forall x \in P \cup R$ and if $\exists N \in \mathbb{N}$ such that $T^{k+1}(x) = T^k(x)$, $\forall k \geq N$, we write $T^\infty(x) := T^N(x)$.

Lemma 5.2.28. *Let* $\bar{X} := \{x \in X : T^\infty(x) \in P \cup R\}$, *then*

$$\mu(\bar{X}) = 1.$$

Proof. Let $w = (\dot{x}, \dot{y}, \dot{z}, \tau) \in X \setminus \bar{X}$ be an element that is caught in a cycle in Figure 5.4 then we see three cases for this.

For the first case, let $T^k(w) \in [][s_1]$ for $k > n_1 \in \mathbb{N}$, but this is only the case if $T^k(w) \in [\underline{(\mathbf{0},\mathbf{0},\mathbf{1})}][s_1]$ which implies that $x_i, y_i \in \mathbf{0}$ for $i > l_1 \in \mathbb{Z}$.

In the second case, let $T^k(w) \in [][s_2]$ for $k > n_2 \in \mathbb{N}$, but this is only the case if $T^k(w) \in [\underline{(\mathbf{0},\mathbf{0},\mathbf{0})}][s_2]$ which implies that $x_i \in \mathbf{0}$ for $i < l_2 \in \mathbb{Z}$.

The remaining case is that $T^k(w) \in [][s_3]$ for infinitely many k, but then $T^k(w) \in [\underline{(\cdot\,,\cdot\,,\mathbf{0})}][s_3]$ for infinitely many k. As t_3 but not t_3^{-1} appears in T, the third tape ($\dot{z}$) is shifted forward only. Thus follows that $z_i \in \mathbf{0}$ for infinitely many i.

In all three cases w lies in a set of measure zero. So $\mu(\bar{X}) = 1$. $\square$

By construction there are no elements which are mapped to I,

$$\{x \in X : T(x) \in I\} = \emptyset.$$

Lemma 5.2.29. *Let $I_P := \{x \in I : T^\infty(x) \in P\}$ be the elements that start in I and arrive in P. Then*

$$I_P = \bigsqcup_{k=1}^{\infty} F_k$$

with

$$F_k := \begin{bmatrix} \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^{k^2+k}\,\mathbf{1} \end{bmatrix} [s_0]. \tag{5.2.14}$$

Furthermore

$$\mu\left(F_k\right) = \left(\frac{2}{3}\cdot\frac{1}{3^k}\cdot\frac{2}{3}\right)^2\cdot\frac{2}{3}\cdot\frac{1}{3^{k^2+k}}\cdot\frac{2}{3}\cdot\frac{4}{3^4},$$

and T is measure-preserving.

Proof. $\supseteq$: We chase the sets F_k for $k \in \mathbb{N}$ through the diagram in Figure 5.4

$$T^0(F_k) = \begin{bmatrix} \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^{k^2+k}\,\mathbf{1} \end{bmatrix}[s_0], \qquad T^1(F_k) = \begin{bmatrix} \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k-1}\,\mathbf{1} \\ \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k-1}\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^{k^2+k}\,\mathbf{1} \end{bmatrix}[s_1],$$

$$T^{k+1}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^k\,\underline{\mathbf{1}} \\ \mathbf{1}\,\mathbf{0}^k\,\underline{\mathbf{1}} \\ \underline{\mathbf{1}}\,\mathbf{0}^{k^2+k}\,\mathbf{1} \end{bmatrix}[s_1], \qquad T^{k+2}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2+k-1}\,\mathbf{1} \end{bmatrix}[s_2],$$

$$T^{2k+2}(F_k) = \begin{bmatrix} \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2+k-1}\,\mathbf{1} \end{bmatrix}[s_2], \qquad T^{2k+3}(F_k) = \begin{bmatrix} \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k-1}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2+k-1}\,\mathbf{1} \end{bmatrix}[s_3],$$

$$T^{3k+3}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^k\,\underline{\mathbf{1}} \\ \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^k\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2-1}\,\mathbf{1} \end{bmatrix}[s_3], \qquad T^{3k+4}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k-2}\,\underline{\mathbf{0}}\,\mathbf{0}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k+1}\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2-2}\,\mathbf{1} \end{bmatrix}[s_2],$$

$$T^{(3k+4)+(2k+2)}(F_k) = T^{5k+6}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k-3}\,\underline{\mathbf{0}}\,\mathbf{0}\,\mathbf{0}\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{2k+2}\,\underline{\mathbf{0}}\,\mathbf{0}^{k^2-k-3}\,\mathbf{1} \end{bmatrix}[s_2],$$

$$T^{k(2k+2)+k+2} = T^{2k^2+3k+2}(F_k) = \begin{bmatrix} \mathbf{1}\,\mathbf{0}^{k-1}\,\underline{\mathbf{0}}\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^k\,\mathbf{1} \\ \mathbf{1}\,\mathbf{0}^{k^2+k}\,\underline{\mathbf{1}} \end{bmatrix}[s_2] \subset P.$$

$\subseteq$: Let $w \in I_P \subset [\,][s_0]$ be an element that arrives in $P = [\underline{(\mathbf{0}, \mathbf{1}, \mathbf{1})}][s_2]$. By definition this takes finitely many steps, and from the diagram in Figure 5.4 we can see that there exists $n \in \mathbb{N}$ such that $T^n(w) \in [\,][s_1]$ and $T^{n+1}(w) \in [\,][s_2]$.

Thus the chase above shows that

$$w \in \begin{bmatrix} \underline{\mathbf{1}}\,\mathbf{0}^{n-1}\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^{n-1}\,\mathbf{1} \\ \underline{\mathbf{1}} \end{bmatrix}[s_0].$$

If $n = 1$ then $T^2(w) \in [\underline{(\mathbf{1}, \mathbf{1}, \ \cdot)}][s_2] \subset R$. Now the second half of the diagram in Figure 5.4 shows that

$$w \in \begin{bmatrix} \underline{\mathbf{1}}\,\mathbf{0}^{n-1}\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^{n-1}\,\mathbf{1} \\ \underline{\mathbf{1}}\,\mathbf{0}^m\,\mathbf{1} \end{bmatrix}[s_0],$$

and for $m \neq (n-1)^2 + n - 1$ computation shows that $T^{2(n-1)^2+3(n-1)+2}(w) \in R$. So $w \in F_{n-1}$.

To calculate the measure of F_k we have to multiply the measures of the subsets each position is restricted to. T is measure-preserving since $\mu\left([][s_i]\right) = \mu\left([][s_j]\right)$ for $i, j \in \{0,1,2,3\}$. $\square$

As in the previous section, let $S_1 := T + (e \cdot \chi_C)$ and $S_2 := e \cdot \chi_R$. We define good basic graphs for $\breve{S}_1$ and $\breve{S}_2$, $\Gamma_k = \{\breve{\chi}_{T^i(F_k)} : 0 \leq i \leq 2k^2 + 3k + 2\}$, $k \in \mathbb{N}$. Then $\mathcal{G} := \left(\mathcal{G}\left(\Gamma_k, \breve{S}_1\right), \mathcal{G}\left(\Gamma_k, \breve{S}_2\right)\right)$ is a nice graph of type $(1,0)$. Furthermore $\breve{S}_{1,\Gamma_k} = T_\alpha$ and $\breve{S}_{2,\Gamma_k} = T_\beta$, the adjancency operators for $\mathcal{G}$, and thus by Lemma 5.2.24

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,\Gamma_k} \cap \ker \breve{S}_{2,\Gamma_k}\right) = 1.$$

Note that

$$\mu\left(\operatorname{supp}(\Gamma_k)\right) = |\Gamma_k| \cdot \left(\frac{2}{3} \cdot \frac{1}{3^k} \cdot \frac{2}{3}\right)^2 \cdot \frac{2}{3} \cdot \frac{1}{3^{k^2+k}} \cdot \frac{2}{3} \cdot \frac{4}{3^4} = |\Gamma_k| \cdot \frac{2^8}{3^{10}} \cdot \frac{1}{3^{k^2+3k}}.$$

We find an exhaustion of $\bar{X}$ by good basic graph as in the previous section. Let H be a good basic graph for $\breve{S}_1$ disjoint from Γ_k for $k \in \mathbb{N}$. There are two cases.

In the first case $\operatorname{supp}(H) \cap I \neq \emptyset$, then by Lemma 5.2.29 $\operatorname{supp}(H) \cap R \neq \emptyset$ and so $\left(\mathcal{G}\left(H, \breve{S}_1\right), \mathcal{G}\left(H, \breve{S}_2\right)\right)$ is a nice graph of type $(1,1)$. Thus

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,H} \cap \ker \breve{S}_{2,H}\right) = 0.$$

In the second case $\operatorname{supp}(H) \cap I = \emptyset$, then $\left(\mathcal{G}\left(H, \breve{S}_1\right), \mathcal{G}\left(H, \breve{S}_2\right)\right)$ is a nice graph of type $(0,l)$. Thus

$$\dim_{\mathbb{F}_4}\left(\ker \breve{S}_{1,H} \cap \ker \breve{S}_{2,H}\right) = 0.$$

Since $\mu\left(\bar{X}\right) = 1$, we thus amend $\{\Gamma_k\}_{k\in\mathbb{N}}$ to get an exhausting sequence of good basic graphs for $\breve{S}_1$ and $\breve{S}_2$. So with Theorem 5.2.10 we get the following result:

$$\dim_{\mathbb{F}_2[G\ltimes A]}\left(\ker\begin{pmatrix}\breve{S}_1\\ \breve{S}_2\end{pmatrix}\right) = \dim_{\mathbb{F}_4[G\ltimes A]}\left(\ker \breve{S}_1 \cap \ker \breve{S}_2\right) = \frac{2^8}{3^{10}} \cdot \sum_{k\in\mathbb{N}} \frac{1}{3^{k^2+3k}}.$$

As the series is a non-periodic triadic expansion the number above is irrational, but not covered by well known transcendence results.

Recall $G \ltimes A \cong \left(\mathbb{Z}^3 \ltimes \left(\bigoplus_{\mathbb{Z}} \mathbf{Z}_3\right)^3\right) \times \left(\mathrm{Aut}(S) \ltimes S\right)$, clearly $\left(\mathbb{Z}^3 \ltimes \left(\bigoplus_{\mathbb{Z}} \mathbf{Z}_3\right)^3\right)$ is finitely generated metabelian, by [**BBDZ13**, Theorem 7.1] there exists a finitely presented overgroup $\tilde{G}$ which is metabelian and thus amenable. Since $\left(\mathrm{Aut}(S) \ltimes S\right)$ is finite, $\tilde{G} \times \left(\mathrm{Aut}(S) \ltimes S\right)$ is also finitely presented.

Remark 5.2.30. Well known examples of transcendental numbers in form of a convergent series are of the type

$$\sum_k \beta^{r_k},$$

where $\beta \in (0,1)$ is algebraic and r_k is a sequence of exponential growth in k.

In Theorem 5.2.2 $\beta = \frac{1}{3}$ and r_k is a sequence of polynomial growth. Now if r_k is a polynomial in k, then the set of such numbers is countable, since the algebraic numbers are countable and the set of polynomials is countable. Therefore cardinality does not obstruct it from being a subset of the algebraic numbers and therefore the example being algebraic.

Remark 5.2.31. The example above corresponds to polynomial growth of order 2. From the construction used it seems natural, that the order can be increased with additional tapes, that is copies of $M^{\mathbb{Z}}$. However, there is little hope to make the jump to exponential growth while keeping $G \ltimes A$ metabelian-by-finite. This seems to restrict to 'read-only systems' as in the construction above in contrast to the underlying system for arbitrary real numbers seen in the proof of Theorem 5.2.1.

5.3. Spectral computation

In this section we adapt the role model for the computational tool of the previous section, sometimes called spectral computation, directly using the properties of the Følner dimension established in Chapter 4.

Thus with some algebraic effort we will see that the spectral computation from [**PSZ15**] can be translated to fields of finite characteristics. For this section we will only consider fields $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$. Note the similarity to the computational tool in the previous section.

5.3.1. General Setup

The following setup is meant to be a slight generalization of the setup in [**PSZ15**], also borrowing notation from [**Aus13**].

Let G be a discrete, finitely generated amenable Group and let U be a discrete, countable, abelian group, where each element has order 2, then all homomorphisms $U \to \mathbb{T}$ factor through $\{1,-1\}$. Let $X := \widehat{U} = \operatorname{Hom}(U,\mathbb{T})$ be its compact Pontryagin dual and μ the normalized Haar measure on X.

Let $f \in l^1(U)$, we define the Fourier transform

$$\mathcal{T}' f(x) := \sum_{u\in U} f(u)\, x(u), \text{ for } x \in X. \tag{5.3.1}$$

Theorem 5.3.1. [**Fol15**, Theorem 4.26] *Let U, X, μ be as above. Then the Fourier transform $\mathcal{T}'$ on $l^1(U) \cap l^2(U)$ extends uniquely to an unitary isomorphism*

$$\mathcal{T}\colon l^2(U) \to L^2(X,\mu). \tag{5.3.2}$$

This agrees with the following inverse Fourier transform,

$$\begin{aligned} \mathcal{T}^* F(u) &= \langle \mathcal{T}^* F, \delta_u \rangle_{l^2(U)} = \langle F, \mathcal{T}\delta_u \rangle_{L^2(X)} \\ &= \int_{x\in X} F(x) \overline{\sum_{v\in U} \delta_u(v)\, x(v)} d\mu(x) \\ &= \int_{x\in X} F(x)\, x(u)\, d\mu(x), \end{aligned} \tag{5.3.3}$$

for $F \in L^2(X)$ and $u \in U$.

Consider $\mathbb{C}[U] \subset \mathcal{B}\left(l^2(U)\right)$ by the left regular action of U and $L^\infty(X) \subset \mathcal{B}\left(L^2(X)\right)$ by pointwise multiplication.

Lemma 5.3.2. *Let U, X, μ be as above. Then*

(i) there is an algebra homomorphism $\mathbb{C}[U] \xrightarrow{\widehat{}} L^\infty(X)$ given by

$$\sum_{u\in U} c_u u \mapsto \left(x \mapsto \sum_{u\in U} c_u x(u) \right). \tag{5.3.4}$$

We write $\widehat{p}$ for the image of $p \in \mathbb{C}[U]$ in $L^\infty(X)$.

(ii) Let $E \subset U$ be a finite subset and $\Psi\colon E \to \{1,-1\}$, for these we define cylinder sets in X by $C_\Psi := \left\{ x \in X \;\middle|\; \forall u \in E\colon x(u) = \Psi(u) \right\}$. Then

$$p_\Psi := \prod_{u\in E} \left(\frac{e + \Psi(u)\, u}{2} \right) \in \mathbb{Z}[\tfrac{1}{2}][U] \tag{5.3.5}$$

maps to the indicator function of C_Ψ, $\widehat{p_\Psi} = 1_{C_\Psi}$.

(iii) For $p \in \mathbb{C}[U]$ we see that

$$p \circ \mathcal{T}^* = \mathcal{T}^* \circ \widehat{p}. \tag{5.3.6}$$

Proof. (i) Let $p=\sum_{u\in U}c_u u, q=\sum_{v\in U}d_v v\in\mathbb{C}[U]$ then

$$\begin{aligned}\widehat{pq}\left(x\right) &= \sum_{u,v\in U}\left(c_u d_v\right)x(uv)\\ &= \left(\sum_{u\in U}c_u x(u)\right)\left(\sum_{v\in U}d_v x(v)\right)\\ &= \widehat{p}\left(x\right)\widehat{q}\left(x\right)=\left(\widehat{p}\cdot\widehat{q}\right)\left(x\right).\end{aligned}$$

(ii) Let $E\subset U$ be a finite subset and $\Psi\colon E\to\{1,-1\}$ and let $x\in C_\Psi$ then

$$\begin{aligned}\widehat{p_\Psi}\left(x\right) &= \prod_{u\in E}\left(\frac{x\left(e\right)+\Psi\left(u\right)x\left(u\right)}{2}\right)\\ &= \prod_{u\in E}\left(\frac{1+x\left(u\right)^2}{2}\right)=\prod_{u\in E}\left(\frac{2}{2}\right)=1.\end{aligned}$$

On the other hand, let $x\notin C_\Psi$, then there exists $u\in E$ such that $x\left(u\right)\neq\Psi\left(u\right)$. Thus we see that

$$x\left(e\right)+\Psi\left(u\right)x\left(u\right)=1+(-1)=0,$$

and therefore $\widehat{p_\Psi}\left(x\right)=0$.

(iii) Let $p=\sum_{u\in U}c_u u\in\mathbb{C}[U]$, $F\in L^2\left(X\right)$ and $v\in U$, then

$$\begin{aligned}\mathcal{T}^*\left(\widehat{p}F\right)\left(v\right) &= \int_{x\in X}\widehat{p}\left(x\right)F\left(x\right)x\left(v\right)d\mu\left(x\right)\\ &= \int_{x\in X}\sum_{u\in U}c_u x(u)F\left(x\right)x\left(v\right)d\mu\left(x\right)\\ &= \sum_{u\in U}c_u\int_{x\in X}F\left(x\right)x\left(uv\right)d\mu\left(x\right)\\ &= \sum_{u\in U}c_u\mathcal{T}^*F\left(uv\right)\\ &= p\mathcal{T}^*F\left(v\right).\end{aligned}$$

□

We see that the image of $\mathbb{Z}[\frac{1}{2}][U]$ contains all $\chi\in L^\infty\left(X\right)$, which are $\mathbb{Z}[\frac{1}{2}]$-linear combinations of such indicator functions. Note that $\mathrm{im}\left(p_\Psi\right)\subseteq l^2\left(U\right)$ and

$$\mathrm{im}\left(p_\Psi\right)\cong\mathrm{im}\left(1_{C_\Psi}\right)\subseteq L^2\left(X,\mu\right).$$

Corollary 5.3.3. [**Tak02**, VII, Theorem 3.14] *We have a spatial isomorphism of von Neumann algebras by*

$$\mathcal{T}\mathcal{N}\left(U\right)\mathcal{T}^{*}=L^{\infty}\left(X\right). \tag{5.3.7}$$

Let $G\curvearrowright X$ be an action by automorphisms, then G acts on $L^{\infty}\left(X,\mu\right)$ by precomposition, $g.\chi\left(x\right)=\chi\left(g^{-1}.x\right)$, and $G\curvearrowright U$ such that the algebra homomorphism above is equivariant.

We follow [**Aus13**] and define the semi-direct product $U\rtimes G$ by the following multiplication on $U\times G$,

$$\left(u,g\right)\cdot\left(v,h\right):=\Big(\left(h^{-1}.u\right)v,gh\Big). \tag{5.3.8}$$

By the Fourier transform above, we get the unitary isomorphism

$$\bar{\mathcal{F}}\colon l^{2}\left(U\rtimes G\right)\to L^{2}\left(X\rtimes G,\mu\otimes\#_{G}\right)$$

via $l^{2}\left(U\rtimes G\right)\cong l^{2}\left(U\right)\otimes l^{2}\left(G\right)$,

$$l^{2}\left(U\right)\otimes l^{2}\left(G\right)\xrightarrow{\mathcal{F}\otimes\mathrm{Id}_{l^{2}\left(G\right)}}L^{2}\left(X\right)\otimes l^{2}\left(G\right)$$

and $L^{2}\left(X\right)\otimes l^{2}\left(G\right)\cong L^{2}\left(X\rtimes G,\mu\otimes\#_{G}\right)$.

For $f\in l^{1}\left(U\rtimes G\right)\cap l^{2}\left(U\rtimes G\right)$ we have the explicit formula

$$\bar{\mathcal{T}}f\left(x,g\right):=\sum_{u\in U}f\left(u,g\right)x\left(u\right), \tag{5.3.9}$$

and thus for $F\in L^{2}\left(X\rtimes G,\mu\otimes\#_{G}\right)$ we get the inverse

$$\begin{aligned}\bar{\mathcal{T}}^{*}F\left(u,g\right)&=\left\langle \bar{\mathcal{T}}^{*}F,\delta_{u,g}\right\rangle _{l^{2}\left(U\rtimes G\right)}\\&=\left\langle F,\bar{\mathcal{T}}\delta_{u,g}\right\rangle _{L^{2}\left(X\rtimes G\right)}\\&=\int_{\left(x,g'\right)\in X\times G}F\left(x,g'\right)\overline{\bar{\mathcal{T}}\delta_{u,g}\left(x,g'\right)}d\left(\mu\otimes\#_{G}\right)\left(x,g'\right)\\&=\int_{x\in X}\sum_{g'\in G}F\left(x,g'\right)\sum_{v\in U}\delta_{u,g}\left(v,g'\right)x\left(v\right)d\mu\left(x\right)\\&=\int_{x\in X}F\left(x,g\right)x\left(u\right)d\mu\left(x\right).\end{aligned} \tag{5.3.10}$$

Recall the group measure space construction [**BV95**, p.169]. The group measure space von Neumann algebra $L^{\infty}\left(X,\mu\right)\rtimes G$ is generated by

$$\left\{ M_{\chi}\mid\chi\in L^{\infty}\left(X,\mu\right)\right\} \cup\left\{ T_{h}\mid h\in G\right\},$$

where M_χ for $\chi \in L^\infty(X,\mu)$ is the twisted pointwise multiplication, that is for $F \in L^2(X \rtimes G, \mu \otimes \#_G)$

$$M_\chi F\big((x,g)\big) := g^{-1}.\chi(x)\, F\big((x,g)\big), \text{ for } (x,g) \in X \times G \tag{5.3.11}$$

and T_h for $h \in G$ is a translation operator,

$$T_h F\big((x,g)\big) := F\Big(\big(x, h^{-1}g\big)\Big). \tag{5.3.12}$$

Where the covariant relation $T_h M_\chi T_{h^{-1}} = M_{h.\chi}$ holds, indeed

$$\begin{aligned}
\big(T_h M_\chi T_h^{-1}\big) F(x,g) &= \big(M_\chi T_{h^{-1}} F\big)\big(x, h^{-1}g\big) \\
&= g^{-1}h.\chi(x)\,(T_{h^{-1}}F)\big(x,h^{-1}g\big) \\
&= g^{-1}.\,(h.\chi)(x)\,F(x,g) \\
&= M_{h.\chi}F(x,g).
\end{aligned}$$

Lemma 5.3.4. *We have a spatial isomorphism of von Neumann algebras by*

$$\bar{\mathcal{T}}\mathcal{N}(U \rtimes G)\,\bar{\mathcal{T}}^* = L^\infty(X,\mu) \rtimes G. \tag{5.3.13}$$

Proof. Let $F \in L^2(X \rtimes G, \mu \otimes \#_G)$, the left regular action $U \rtimes G \curvearrowright l^2(U \rtimes G)$ gives

$$\begin{aligned}
(v,e)\,\bar{\mathcal{T}}^* F\big((u,g)\big) = \bar{\mathcal{T}}^* F\big((v,e)\cdot(u,g)\big) &= \bar{\mathcal{T}}^* F\Big(\big(g^{-1}.v\big)u, g\Big) \\
&= \int_{x\in X} F(x,g)\, x\Big(\big(g^{-1}.v\big)u\Big)\, d\mu(x) \\
&= \int_{x\in X} F(x,g)\, x\big(g^{-1}.v\big)\, x(u)\, d\mu(x) \\
&= \int_{x\in X} g^{-1}.\widehat{v}(x)\, F(x,g)\, x(u)\, d\mu(x) \\
&= \int_{x\in X} M_{\widehat{v}} F(x,g)\, x(u)\, d\mu(x) \\
&= \bar{\mathcal{T}}^* M_{\widehat{v}} F\big((u,g)\big),
\end{aligned}$$

and

$$\begin{aligned}
(e,h)\,\bar{\mathcal{T}}^* F\big((x,g)\big) &= \bar{\mathcal{T}}^* F\big((x,hg)\big) \\
&= \int_{x\in X} F(x,hg)\, x(u)\, d\mu(x) \\
&= \int_{x\in X} T_h F(x,g)\, x(u)\, d\mu(x) \\
&= \bar{\mathcal{T}}^* T_h F\big((x,g)\big).
\end{aligned}$$

Let $A=\sum_{(v,h)\in U\rtimes G}a_{(v,h)}\left(v,h\right)\in\mathbb{C}[U\rtimes G]$, then

$$\begin{aligned}A\bar{\mathcal{T}}^{*}F\left(\left(u,g\right)\right)&=\left(\sum_{h\in G}\left(e,h\right)\right)\left(\sum_{v\in U}a_{(v,h)}\left(v,e\right)\right)\bar{\mathcal{T}}^{*}F\left(\left(u,g\right)\right)\\&=\sum_{h\in H}\left(e,h\right)\bar{\mathcal{T}}^{*}M_{\widehat{p_h}}F\left(\left(u,g\right)\right)\\&=\bar{\mathcal{T}}^{*}\sum_{h\in H}T_hM_{\widehat{p_h}}F\left(\left(u,g\right)\right),\end{aligned}$$

where $p_h:=\sum_{v\in U}a_{(v,h)}\left(v,e\right)\in\mathbb{C}[U]$. This together with

$$\mathbb{C}[U]\subset\mathcal{N}\left(U\right)\cong L^{\infty}\left(X\right)$$

shows the statement for the generators of both von Neumann algebras. □

Corollary 5.3.5. *Let $E\subset U$ be a finite subset and $\Psi\colon E\to\{1,-1\}$, then by $G\curvearrowright U$ we have $g.\Psi\colon g.E\to\{1,-1\}$ for any $g\in G$ such that by definition of cylinder sets $g.C_\Psi=C_{g.\Psi}$ and thus by the covariant relation*

$$\begin{aligned}\bar{\mathcal{T}}\left(e,g\right)\cdot p_\Psi\cdot\left(e,g^{-1}\right)\bar{\mathcal{T}}^{*}&=T_gM_{1_{C_\Psi}}T_{g^{-1}}=M_{g.1_{C_\Psi}}=M_{1_{C_{g.\Psi}}}\\&=\bar{\mathcal{T}}p_{g.\Psi}\bar{\mathcal{T}}^{*}.\end{aligned}\tag{5.3.14}$$

Furthermore

$$\begin{aligned}\bar{\mathcal{T}}p_{g.\Psi}\cdot l^2\left(U\rtimes G\right)&=\bar{\mathcal{T}}p_{g.\Psi}\bar{\mathcal{T}}^{*}\bar{\mathcal{T}}\left(l^2\left(U\rtimes G\right)\right)\\&=M_{g.1_{C_\Psi}}\left(L^2\left(X\rtimes G\right)\right)\\&=T_gM_{1_{C_\Psi}}\left(L^2\left(X\rtimes G\right)\right)\\&=\bar{\mathcal{T}}\left(e,g\right)\cdot p_\Psi\cdot l^2\left(U\rtimes G\right).\end{aligned}\tag{5.3.15}$$

Finally note that for $\chi\in L^{\infty}\left(X,\mu\right)$ the canonical von Neumann trace for $\mathcal{N}\left(U\rtimes G\right)$ gives

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}(U\rtimes G)}\left(\bar{\mathcal{T}}^{*}M_\chi\bar{\mathcal{T}}\right)&=\left\langle\bar{\mathcal{T}}^{*}M_\chi\bar{\mathcal{T}}\delta_{(e,e)},\delta_{(e,e)}\right\rangle_{l^2(U\rtimes G)}\\&=\left\langle M_\chi\bar{\mathcal{T}}\delta_{(e,e)},\bar{\mathcal{T}}\delta_{(e,e)}\right\rangle_{L^2(X\rtimes G,\mu\otimes\#_G)}\\&=\int_{x\in X}\sum_{g\in G}g^{-1}.\chi\left(x\right)\sum_{u\in U}\delta_{(e,e)}\left(u,g\right)x\left(u\right)\overline{\sum_{v\in U}\delta_{(e,e)}\left(v,g\right)x\left(v\right)}d\mu\left(x\right)\\&=\int_{x\in X}\chi\left(x\right)d\mu\left(x\right),\end{aligned}$$

and for $h \neq e \in G$

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}(U\rtimes G)}\left(\bar{\mathcal{T}}^{*}T_{h}\bar{\mathcal{T}}\right) &= \left\langle \bar{\mathcal{T}}^{*}T_{h}\bar{\mathcal{T}}\delta_{(e,e)},\delta_{(e,e)}\right\rangle_{l^{2}(U\rtimes G)}\\ &= \left\langle T_{h}\bar{\mathcal{T}}\delta_{(e,e)},\bar{\mathcal{T}}\delta_{(e,e)}\right\rangle_{L^{2}(X\rtimes G,\mu\otimes\#_{G})}\\ &= \int_{x\in X}\sum_{g\in G}\sum_{u\in U}\delta_{(e,e)}\left(u,h^{-1}g\right)x\left(u\right)\overline{\sum_{v\in U}\delta_{(e,e)}\left(v,g\right)x\left(v\right)}d\mu\left(x\right)\\ &= \int_{x\in X}\sum_{u\in U}\delta_{(e,e)}\left(u,h^{-1}\right)d\mu\left(x\right)=0.\end{aligned}$$

This fixes our choice of von Neumann trace for $L^{\infty}\left(X,\mu\right)\rtimes G$. We will also write

$$\widehat{A}:=\bar{\mathcal{T}}A\bar{\mathcal{T}}^{*} \tag{5.3.16}$$

for $A\in\mathcal{N}\left(U\rtimes G\right)$.

5.3.2. Decomposition

Definition 5.3.6. We call an operator $A\in\mathbb{C}[U\rtimes G]$ regional if

(i) there exists a sequence of pairs $\mathcal{C}_{r}:=\left(\Psi_{r},G_{r}\right)$, where $\Psi_{r}\colon E\to\{1,-1\}$ with $E\subset U$ finite and $\{e\}\subset G_{r}\subset G$ finite. Which gives a disjoint Borel partition of X

$$X=C_{0}\cup\bigcup_{r=1}^{\infty}\bigcup_{g\in G_{r}}g.C_{\Psi_{r}}\cup C_{\infty}, \tag{5.3.17}$$

where $C_{\Psi_{r}}$ are cylinder sets as before and $\mu\left(C_{\infty}\right)=0$. Furthermore this partition must induce an $\widehat{A}$-invariant decomposition into right Hilbert $U\rtimes G$-modules

$$L^{2}\left(X\rtimes G,\mu\otimes\#_{G}\right)=\mathcal{H}_{0}\oplus\left(\bigoplus_{r=1}^{\infty}\mathcal{H}_{\mathcal{C}_{r}}\right)\oplus\mathcal{H}_{\infty}, \tag{5.3.18}$$

where $\mathcal{H}_{0}=\operatorname{im}\left(M_{1_{C_{0}}}\right)$, $\mathcal{H}_{\infty}=\operatorname{im}\left(M_{1_{C_{\infty}}}\right)$ and

$$\mathcal{H}_{\mathcal{C}_{r}}=\bigoplus_{g\in G_{r}}T_{g}\mathcal{H}_{\Psi_{r}},\text{ with }\mathcal{H}_{\Psi_{r}}=\operatorname{im}\left(M_{1_{C_{\Psi_{r}}}}\right).$$

Note that by Corollary 5.3.5

$$T_{g}\mathcal{H}_{\Psi_{r}}=\mathcal{H}_{g.\Psi_{r}}.$$

(ii) $\widehat{A}|_{\mathcal{H}_{0}}=0$.

(iii) Consider the natural basis $\left(b_{i}\right)$ of $\mathbb{C}^{|G_{r}|}$ corresponding to an enumeration of $g_{i}\in G_{r}$ for $i=1,\ldots,|G_{r}|$. There is an unitary isomorphism of right

Hilbert $U \rtimes G$-modules

$$\mathcal{H}_{\Psi_r} \otimes \mathbb{C}^{|G_r|} \cong \mathcal{H}_{\mathcal{C}_r}$$

given by $F \otimes b_i \mapsto T_{g_i} F$ for $F \in \mathcal{H}_{\Psi_r}$, such that this implies an unitary equivalence of Hilbert $U \rtimes G$-endomorphisms

$$\widehat{A}|_{\mathcal{H}_{\mathcal{C}_r}} \simeq \mathrm{Id}_{\mathcal{H}_{\Psi_r}} \otimes A^{\mathcal{C}_r},$$

where $A^{\mathcal{C}_r} \in \mathbb{C}^{|G_r| \times |G_r|}$ is a finite dimensional matrix. We also write $(A, \mathcal{R})$ for a regional operator where $\mathcal{R} := (\mathcal{C}_r)_{r=1}^{\infty}$ is the sequence of finite families.

Lemma 5.3.7. *Let* $\mathbb{K}$ *be a field with* $\mathrm{char}(\mathbb{K}) \neq 2$, *then*

$$\begin{aligned}
\dim_{\mathbb{K}[U \rtimes G]} \left(\overline{p_\Psi} \cdot \mathbb{K}[U \rtimes G]\right) &= \dim_{\mathbb{Q}[U \rtimes G]} \left(p_\Psi \cdot \mathbb{Q}[U \rtimes G]\right) \\
&= \dim_{\mathbb{C}[U \rtimes G]} \left(p_\Psi \cdot \mathbb{C}[U \rtimes G]\right) \\
&= \dim_{\mathcal{N}(U \rtimes G)} \left(p_\Psi \cdot l^2 (U \rtimes G)\right) \\
&= \dim_{L^\infty(X,\mu) \rtimes G} \left(\mathcal{H}_\Psi\right) \\
&= \mu\left(C_\Psi\right).
\end{aligned} \tag{5.3.19}$$

Proof. Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$. From Lemma 5.3.2 we deduce $p_\Psi = p_\Psi^2 \in \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and $p_\Psi \cdot l^2(U \rtimes G) \cong \mathcal{H}_\Psi$. Furthermore we denote by $\overline{p_\Psi}$ the image of the canonical homomorphism in $\mathbb{K}[U \rtimes G]$. Then we have by Corollary 4.4.2 and Theorem 4.4.5 that

$$\begin{aligned}
\dim_{\mathbb{K}[U \rtimes G]} \left(\overline{p_\Psi} \cdot \mathbb{K}[U \rtimes G]\right) &= \dim_{\mathbb{Q}[U \rtimes G]} \left(p_\Psi \cdot \mathbb{Q}[U \rtimes G]\right) \\
&= \dim_{\mathbb{C}[U \rtimes G]} \left(p_\Psi \cdot \mathbb{C}[U \rtimes G]\right) \\
&= \dim_{\mathcal{N}(U \rtimes G)} \left(p_\Psi \cdot l^2 (U \rtimes G)\right).
\end{aligned}$$

On the other hand by definition of the von Neumann dimension

$$\begin{aligned}
\dim_{\mathcal{N}(U \rtimes G)} \left(p_\Psi \cdot l^2 (U \rtimes G)\right) &= \dim_{L^\infty(X,\mu) \rtimes G} \left(\mathcal{H}_\Psi\right) \\
&= \mathrm{tr}_{L^\infty(X,\mu) \rtimes G} \left(M_{1_{C_\Psi}}\right) \\
&= \mathrm{tr}_{\mathcal{N}(U \rtimes G)} \left(\bar{\mathcal{T}}^* M_{1_{C_\Psi}} \bar{\mathcal{T}}\right) \\
&= \int_{x \in X} 1_{C_\Psi}(x) \, d\mu(x) \\
&= \mu\left(C_\Psi\right).
\end{aligned}$$

□

Lemma 5.3.8. *Let $\mathbb{K}$ be a field with* $\operatorname{char}(\mathbb{K}) \neq 2$. *Let $(A, \mathcal{R})$ be a regional operator with $A \in \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and its image under the canonical homomorphism $\overline{A} \in \mathbb{K}[U \rtimes G]$. Then we get the following module isomorphisms,*

$$A \cdot \sum_{g \in G_r} p_{g.\Psi_r} \cdot \mathbb{C}[U \rtimes G] \cong p_{\Psi_r} \cdot \mathbb{C}[U \rtimes G] \otimes_{\mathbb{C}} \operatorname{im} A^{\mathcal{C}_r}, \tag{5.3.20}$$

$$\overline{A} \cdot \sum_{g \in G_r} \overline{p_{g.\Psi_r}} \cdot \mathbb{K}[U \rtimes G] \cong \overline{p_{\Psi_r}} \cdot \mathbb{K}[U \rtimes G] \otimes_{\mathbb{K}} \operatorname{im} \overline{A^{\mathcal{C}_r}}, \tag{5.3.21}$$

where $\overline{A^{\mathcal{C}_r}} \in \mathbb{K}^{|G_r| \times |G_r|}$ is the image of $A^{\mathcal{C}_r} \in \mathbb{Z}[\frac{1}{2}]^{|G_r| \times |G_r|}$ under the canonical homomorphism.

Note that the left hand side is a submodule of $\mathbb{K}[U \rtimes G]$, but the right hand side is a submodule of $\mathbb{K}[U \rtimes G]^{|G_r|}$.

Proof. Let $\mathbb{K}$ be a field with $\operatorname{char}(\mathbb{K}) \neq 2$. Let $(A, \mathcal{R})$ be a regional operator with $A \in \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and its image under the canonical homomorphism $\overline{A} \in \mathbb{K}[U \rtimes G]$. Recall the definition of regional operator and Corollary 5.3.5. Let $F \in \mathcal{H}_{\mathcal{C}_r}$ then $F = \sum_{i=1}^{|G_r|} T_{g_i} F_{g_i}$ with $F_{g_i} \in \mathcal{H}_{\Psi_r}$, where $g_i \in G_r$ is the enumeration corresponding to the natural basis (b_i) of $\mathbb{C}^{|G_r|}$. Now we get

$$\begin{aligned} F &\mapsto \sum_{i=1}^{|G_r|} F_{g_i} \otimes b_i \in \mathcal{H}_{\Psi_r} \otimes \mathbb{C}^{|G_r|}, \\ \widehat{A}(F) &\mapsto \sum_{i=1}^{|G_r|} F_{g_i} \otimes A^{\mathcal{C}_r}(b_i) \\ &= \sum_{i=1}^{|G_r|} \sum_{j=1}^{|G_r|} F_{g_i} \otimes \left\langle A^{\mathcal{C}_r}(b_i), b_j \right\rangle_{\mathbb{C}} b_j, \end{aligned}$$

and thus

$$\widehat{A}(F) = \sum_{i=1}^{|G_r|} \sum_{j=1}^{|G_r|} \left\langle A^{\mathcal{C}_r}(b_i), b_j \right\rangle_{\mathbb{C}} T_{g_j} F_{g_i}.$$

Furthermore assume $f := \bar{\mathcal{T}}^* F \in \sum_{g\in G_r} p_{g.\Psi_r} \cdot \mathbb{C}[U \rtimes G]$, then $f = \sum_{i=1}^{|G_r|} (e, g_i)\, f_{g_i}$ for some $f_{g_i} \in p_{\Psi_r} \cdot \mathbb{C}[U \rtimes G]$, and $f_{g_i} = \bar{\mathcal{T}}^* F_{g_i}$. Thus

$$\begin{aligned} A \cdot f &= A \cdot \bar{\mathcal{T}}^* F = \bar{\mathcal{T}}^* \widehat{A}\,(F) \\ &= \bar{\mathcal{T}}^* \left(\sum_{i=1}^{|G_r|} \sum_{j=1}^{|G_r|} \left\langle A^{\mathcal{C}_r}\,(b_i)\,, b_j \right\rangle_{\mathbb{C}} T_{g_j} F_{g_i} \right) \\ &= \sum_{i=1}^{|G_r|} \sum_{j=1}^{|G_r|} \left\langle A^{\mathcal{C}_r}\,(b_i)\,, b_j \right\rangle_{\mathbb{C}} (e, g_j)\, f_{g_i}. \end{aligned}$$

So we get an isomorphism of $\mathbb{C}[U \rtimes G]$-modules

$$A \cdot \sum_{g\in G_r} p_{g.\Psi_r} \cdot \mathbb{C}[U \rtimes G] \cong p_{\Psi_r} \cdot \mathbb{C}[U \rtimes G] \otimes_{\mathbb{C}} \operatorname{im} A^{\mathcal{C}_r},$$

under the isomorphism

$$p_{\Psi_r} \cdot \mathbb{C}[U \rtimes G] \otimes \mathbb{C}^{|G_r|} \cong \sum_{g\in G_r} p_{g.\Psi_r} \cdot \mathbb{C}[U \rtimes G]$$

defined by

$$f \otimes b_i \mapsto (e, g_i) \cdot f$$

for $f \in p_{\Psi_r} \cdot \mathbb{C}[U \rtimes G]$.

Now consider the canonical homomorphism $\mathbb{C}[U \rtimes G] \supset \mathbb{Z}[\frac{1}{2}][U \rtimes G] \to \mathbb{K}[U \rtimes G]$. Similarly the natural basis of $\mathbb{C}^{|G_r|}$ maps bijectively to the natural basis of $\mathbb{K}^{|G_r|}$. Let $f \in p_{g_i.\Psi_r} \cdot \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ then $f = (e, g_i) \cdot f_i$ for some $f_i \in p_{\Psi_r} \cdot \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and thus

$$\begin{aligned} A \cdot (e, g_i) \cdot f_i &= \sum_{j=1}^{|G_r|} \left\langle A^{\mathcal{C}_r}\,(b_i)\,, b_j \right\rangle_{\mathbb{C}} (e, g_j) \cdot f_i, \\ \overline{A} \cdot (e, g_i) \cdot \overline{f_i} &= \sum_{j=1}^{|G_r|} \left\langle \overline{A^{\mathcal{C}_r}} \left(\overline{b_i}\right), \overline{b_j} \right\rangle_{\mathbb{K}} (e, g_j) \cdot \overline{f_i}. \end{aligned}$$

The statement of the lemma follows. □

Remark 5.3.9. Let $\mathbb{K}$ be a field with $\operatorname{char}(\mathbb{K}) = q \notin \{0, 2\}$ and $Y < q \in \mathbb{N}$. Furthermore let $A' \in \mathbb{Z}[\frac{1}{2}][\sqrt{Y}][U \rtimes G] \subset \mathbb{C}[U \rtimes G]$ such that $(A', \mathcal{R})$ is a regional operator with the same sequence of finite families as $(A, \mathcal{R})$ in the lemma above. Then (5.3.21) holds for $\mathbb{K}[\sqrt{Y}]$ where $\overline{A} \in \mathbb{K}[\sqrt{Y}][U \rtimes G]$ is the image under the canonical homomorphism.

Theorem 5.3.10. *Let G and U be as in the previous section. Let $\mathbb{K}$ be a field with* $\operatorname{char}(\mathbb{K}) = q \notin \{0,2\}$ *and $b < q \in \mathbb{N}$. Let $(A, \mathcal{R})$ be a regional operator with $A \in \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and its image under the canonical homomorphism $\overline{A} \in \mathbb{K}[U \rtimes G]$.*

Recall the subsets $C_{\Psi_r} \subset \operatorname{Hom}(U, \mathbb{T})$ and the matrices $A^{\mathcal{C}_r}$ from the definition of regional operator.

(i) Then

$$\dim_{\mathbb{C}[U \rtimes G]} \ker (A + b) = \sum_{r=1}^{\infty} \mu (C_{\Psi_r}) \dim_{\mathbb{C}} \ker \left(A^{\mathcal{C}_r} + b \cdot \mathbb{1}\right), \tag{5.3.22}$$

$$\dim_{\mathbb{K}[U \rtimes G]} \ker \left(\overline{A + b}\right) = \sum_{r=1}^{\infty} \mu (C_{\Psi_r}) \dim_{\mathbb{K}} \ker \left(\overline{A^{\mathcal{C}_r}} + b \cdot \mathbb{1}\right). \tag{5.3.23}$$

(ii) Let $Y < q \in \mathbb{N}$ and let $A' \in \mathbb{Z}[\frac{1}{2}][\sqrt{Y}][U \rtimes G] \subset \mathbb{C}[U \rtimes G]$ such that $(A', \mathcal{R})$ is a regional operator with the same sequence of finite families. Then

$$\dim_{\mathbb{C}[U \rtimes G]} \ker \left(A' + b\right) = \sum_{r=1}^{\infty} \mu (C_{\Psi_r}) \dim_{\mathbb{C}} \ker \left(A'^{\mathcal{C}_r} + b \cdot \mathbb{1}\right), \tag{5.3.24}$$

$$\dim_{\mathbb{K}[\sqrt{Y}][U \rtimes G]} \ker \left(\overline{A' + b}\right) = \sum_{r=1}^{\infty} \mu (C_{\Psi_r}) \dim_{\mathbb{K}[\sqrt{Y}]} \ker \left(\overline{A'^{\mathcal{C}_r}} + b \cdot \mathbb{1}\right). \tag{5.3.25}$$

Proof. Let $\mathbb{K}$ be a field with $\operatorname{char}(\mathbb{K}) = q \notin \{0,2\}$ and $b < q \in \mathbb{N}$. Let $(A, \mathcal{R})$ be a regional operator with $A \in \mathbb{Z}[\frac{1}{2}][U \rtimes G]$ and its image under the canonical homomorphism $\overline{A} \in \mathbb{K}[U \rtimes G]$. We only treat $\dim_{\mathbb{K}[U \rtimes G]} \ker \left(\overline{A + b}\right)$, the other cases behave the same way.

By definition we have $\widehat{A}|_{\mathcal{H}_0} = 0$, thus we may add the identity operator b times and see that $\mathcal{H}_0$ has empty intersection with the kernel of $\widehat{A + b}$. Together with $\mu (C_\infty) = 0$, we have that

$$\lim_{R \to \infty} \dim_{\mathbb{K}[U \rtimes G]} \left(\ker \left(\overline{A + b}\right) \cap \bigcap_{r=1}^{R} \left(\left(1 - \sum_{\Psi \in \mathcal{C}_r} \overline{p_\Psi} \right) \cdot \mathbb{K}[U \rtimes G] \right) \right) = 0.$$

It remains that with Lemma 5.3.8 we get these isomorphisms of modules

$$\begin{aligned} \ker \left(\overline{A + b}\right) \cap \left(\sum_{\Psi \in \mathcal{C}_r} \overline{p_\Psi} \cdot \mathbb{K}[U \rtimes G] \right) &\cong \overline{p_{\Psi_r}} \cdot \mathbb{K}[U \rtimes G] \otimes_{\mathbb{K}} \ker \left(\overline{A^{\mathcal{C}_r}} + b \cdot \mathbb{1}\right) \\ &\cong \bigoplus_{i=1}^{\dim_{\mathbb{K}} \ker \left(\overline{A^{\mathcal{C}}} + b \cdot \mathbb{1}\right)} \overline{p_{\Psi_r}} \cdot \mathbb{K}[U \rtimes G]. \end{aligned}$$

Now continuity from below (4.2.5) implies

$$\dim_{\mathbb{K}[U\rtimes G]}\ker\left(\overline{A+b}\right)=\lim_{R\to\infty}\dim_{\mathbb{K}[U\rtimes G]}\left(\bigoplus_{r=1}^{R}\ker\left(\overline{A+b}\right)\cap\left(\sum_{\Psi\in\mathcal{C}_r}\overline{p_\Psi}\cdot\mathbb{K}[U\rtimes G]\right)\right).$$

Thus the theorem follows by Lemma 5.3.7 and additivity (4.2.7). □

5.3.3. Calculations for different fields

Finally, we look at the actual setup from [**PSZ15**] where

$$G:=\mathbb{Z}\wr\mathbb{Z}=\left\langle s_1,s_2\,\middle|\,\forall k\in\mathbb{Z}\colon\ \left[s_2^k s_1 s_2^{-k},s_1\right]=1\right\rangle$$

is the wreath product of $\mathbb{Z}$ by $\mathbb{Z}$, and $U:=\mathbf{Z}_2^{\oplus G}/V_I$, for some G-action invariant subgroup $V_I\lhd\mathbf{Z}_2^{\oplus G}$ depending on a subset $I\subset\mathbb{N}$. The operator in question from [**PSZ15**] is given by

$$\widehat{A}=\sum_{s\in\left\{s_1^{\pm1},s_2^{\pm1}\right\}}\left(T_{s^{-1}}M_{F_s}+M_{F_{s^{-1}}}T_{s^{-1}}\right),\tag{5.3.26}$$

where $\left(F_s\colon X\to\mathbb{Z}[\tfrac12]\right)\in L^\infty(X)$ and $F_s(x)$ depends only on the evaluation of x at a fixed finite subset of U. Therefore we can write F_s as a $\mathbb{Z}[\tfrac12]$-linear combination of indicator functions for cylinder sets as described in Lemma 5.3.2.

This in turn implies that $A\in\mathbb{Z}[\tfrac12][U\rtimes G]$ and with $\mathcal{R}$ given by Section 4 in [**PSZ15**] Theorem 5.3.10 applies. Indeed it only remains to check [**PSZ15**, Lemma 6.1].

By [**PSZ15**, Proposition 5.3] the matrices $A^{\mathcal{C}}+2\cdot\mathbb{1}\in\mathbb{Q}^{l+1\times l+1}$ are parametrized by triples (l,i,j) and take the form

$$\begin{pmatrix}
2 & \alpha & 0 & \cdots & & & & \\
\alpha & 2 & 2 & 0 & \cdots & & & \\
0 & 2 & \ddots & \ddots & \ddots & & & \\
\vdots & 0 & \ddots & & & \ddots & \vdots & \\
 & \vdots & \ddots & & & \ddots & 0 & \vdots \\
 & & & \ddots & \ddots & \ddots & 2 & 0 \\
 & & & \cdots & 0 & 2 & 2 & \beta \\
 & & & & \cdots & 0 & \beta & 2
\end{pmatrix},$$

where $l\geq 2\in\mathbb{N}$ and $1\leq i\leq j\leq 2$. Depending on the definition of F_s we have that $\alpha=2$ for $i=1$ and $\alpha=2^{-1}$ for $i=2$, the same holds for j and β. Any vector x in the kernel must be mapped to zero by each row, thus if there exists any such vector we may fix the first coordinate and use the first l rows to get

$$\begin{aligned}
x_0 &= 2\alpha,\\
x_1 &= \alpha^{-1}\left(-2x_0\right) = -2^2,\\
x_2 &= 2^{-1}\left(-\alpha x_0 - 2x_1\right) = -\alpha^2 + 2^2,\\
x_3 &= 2^{-1}\left(-2x_1 - 2x_2\right) = 2^2 + \alpha^2 - 2^2 = \alpha^2,\\
x_4 &= 2^{-1}\left(-2x_2 - 2x_3\right) = \alpha^2 - 2^2 - \alpha^2 = -2^2,\\
x_5 &= 2^{-1}\left(-2x_3 - 2x_4\right) = -\alpha^2 + 2^2,\\
&\vdots\\
x_l &= \beta^{-1}\left(-2x_{l-2} - 2x_{l-1}\right)\\
&= \begin{cases} \beta^{-1}\left(2^3 + 2\alpha^2 - 2^3\right) = 2\alpha^2\beta^{-1} & \text{if } l \equiv 0 \pmod 3,\\ \beta^{-1}\left(2\alpha^2 - 2^3 - 2\alpha^2\right) = -2^3\beta^{-1} & \text{if } l \equiv 1 \pmod 3,\\ \beta^{-1}\left(-2\alpha^2 + 2^3\right) & \text{if } l \equiv 2 \pmod 3. \end{cases}
\end{aligned}$$

It remains to be checked that the last row maps this vector to zero to show that the kernel is non-zero. That is

$$\begin{aligned}
0 &= \beta x_{l-1} + 2x_l\\
&= \begin{cases} \beta\left(-\alpha^2 + 2^2\right) + 2^2\alpha^2\beta^{-1} = \beta^{-1}\left(2^2\beta^2 - \alpha^2\beta^2 + 2^2\alpha^2\right) & \text{for } l \equiv 0 \pmod 3,\\ \beta\alpha^2 - 2^4\beta^{-1} = \beta^{-1}\left(\alpha^2\beta^2 - 2^4\right) & \text{for } l \equiv 1 \pmod 3,\\ -2^2\beta + 2\beta^{-1}\left(-2\alpha^2 + 2^3\right) = 2^2\beta^{-1}\left(2^2 - \alpha^2 - \beta^2\right) & \text{for } l \equiv 2 \pmod 3. \end{cases}
\end{aligned}$$

Now we treat the three cases for i, j explicitly.

(i) For $i = j = 1$ and therefore $\alpha = \beta = 2$ we get

$$\beta x_{l-1} + 2x_l = \begin{cases} 2^3 \neq 0 & \text{for } l \equiv 0 \pmod 3,\\ 2^3\left(1 - 1\right) = 0 & \text{for } l \equiv 1 \pmod 3,\\ -2^3 \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(ii) For $i = 1, j = 2$ and therefore $\alpha = 2, \beta = 2^{-1}$ we get

$$\beta x_{l-1} + 2x_l = \begin{cases} 2^5 \neq 0 & \text{for } l \equiv 0 \pmod 3,\\ 2\left(1 - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3,\\ -2 \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(iii) For $i = j = 2$ and therefore $\alpha = \beta = 2^{-1}$ we get

$$\beta x_{l-1} + 2x_l = \begin{cases} 2^{-3}\left(2^5 - 1\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ 2^{-3}\left(1 - 2^8\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\left(2^3 - 1\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

This confirms the result of [**PSZ15**, Lemma 6.1] that only in the case of $i = j = 1$ and $l \equiv 1 \pmod 3$ the kernel is non-trivial and in particular has dimension 1.

Now for a field $\mathbb{K}$ with $\operatorname{char}(\mathbb{K}) = q \neq 2$ the same calculations $\pmod q$ hold for $\overline{A} + 2 \cdot \mathbb{1} \in \mathbb{K}^{l+1 \times l+1}$. Then we can read from the equations above, that for

$$q \nmid \left(2^3 - 1\right), \left(2^4 - 1\right), \left(2^5 - 1\right), \left(2^8 - 1\right)$$

the same results hold without adjusting the maps F_s.

In turn this means that we have to find examples for $q \in \{3, 5, 7, 17, 31\}$ separately and adjust F_s. For that we see in [**PSZ15**, Proposition 3.5] that the value 2^{-1} of F_s corresponding to $i, j = 2$ has no influence on anything but the value of α and β in the calculations of [**PSZ15**, Lemma 6.1] above. Therefore we may freely change this value without changing the sequence of finite families $\mathcal{R}$.

First we treat $q \in \{17, 31\}$ and change 2^{-1} to 3. Subsequently the computations above change in two of the three cases.

(i) For $i = 1, j = 2$ and therefore $\alpha = 2, \beta = 3$ we get

$$\beta x_{l-1} + 2x_l = \begin{cases} 2^4 3^{-1} \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ 3^{-1} 2^2\left(3^2 - 2^2\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ -2^2 3 \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(ii) For $i = j = 2$ and therefore $\alpha = \beta = 3$ we get

$$\beta x_{l-1} + 2x_l = \begin{cases} 3\left(-3^2 + 2^3\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ 3^{-1}\left(3^4 - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^3 3^{-1}\left(-3^2 + 2\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

This takes care of $q \in \{17, 31\}$, since $81 \equiv 13 \not\equiv 16 \pmod{17}$ and $81 \equiv 19 \not\equiv 16 \pmod{31}$.

For the cases of $q \in \{3, 5, 7\}$ we have to allow $F_s : X \to \mathbb{Z}[\frac{1}{2}][\sqrt{Y}]$, $Y < q \in \mathbb{N}$ and thus arrive in the second case of Theorem 5.3.10.

In particular, let $\mathbb{K} = \mathbb{F}_7$ be the field with 7 elements and $Y = 6$. We change 2^{-1} to $1 + \sqrt{6}$.

(i) For $i = 1, j = 2$ and therefore $\alpha = 2, \beta^2 = 1 + 2\sqrt{6} + 6 = 2\sqrt{6} \in \mathbb{F}_7[\sqrt{6}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} 2^4\beta^{-1} \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ 2^2\beta^{-1}\left(2\sqrt{6} - 2^2\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\beta^{-1}\left(-2\sqrt{6}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(ii) For $i = j = 2$ and therefore $\alpha^2 = \beta^2 = 2\sqrt{6} \in \mathbb{F}_7[\sqrt{6}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} \beta^{-1}\left(2^4\sqrt{6} - 2^2 \cdot 6\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ 2^2\beta^{-1}\left(6 - 2^2\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\beta^{-1}\left(2^2 - 2\sqrt{6} - 2\sqrt{6}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

Next let $\mathbb{K} = \mathbb{F}_5$ be the field with 5 elements and $Y = 2$. We change 2^{-1} to $1 + 2\sqrt{2}$.

(i) For $i = 1, j = 2$ and therefore $\alpha = 2, \beta^2 = 4 + 4\sqrt{2} \in \mathbb{F}_5[\sqrt{2}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} \beta^{-1}\left(2^4\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ \beta^{-1}\left(3 + 2\sqrt{2} - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ -2^2\beta^{-1}\left(4 + 4\sqrt{2}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(ii) For $i = j = 2$ and therefore $\alpha^2 = \beta^2 = 4 + 4\sqrt{2} \in \mathbb{F}_5[\sqrt{2}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} \beta^{-1}\left(2 - 3\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ \beta^{-1}\left(3 + 2\sqrt{2} - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\beta^{-1}\left(2^2 - 3 - 3\sqrt{2}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

Finally let $\mathbb{K} = \mathbb{F}_3$ be the field with 3 elements and $Y = 2$. We change 2^{-1} to $1 + \sqrt{2}$.

(i) For $i = 1, j = 2$ and therefore $\alpha = 2, \beta^2 = 1 + 2\sqrt{2} + 2 = 2\sqrt{2} \in \mathbb{F}_3[\sqrt{2}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} \beta^{-1}\left(2^4\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ \beta^{-1}\left(2^3\sqrt{2} - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\beta^{-1}\left(-2\sqrt{2}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

(ii) For $i = j = 2$ and therefore $\alpha^2 = \beta^2 = 2\sqrt{6} \in \mathbb{F}_3[\sqrt{2}]$ we have

$$\beta x_{l-1} + 2x_l = \begin{cases} \beta^{-1}\left(2^4\sqrt{2} - 2^3\right) \neq 0 & \text{for } l \equiv 0 \pmod 3, \\ \beta^{-1}\left(2^3 - 2^4\right) \neq 0 & \text{for } l \equiv 1 \pmod 3, \\ 2^2\beta^{-1}\left(2^2 - 2^2\sqrt{2}\right) \neq 0 & \text{for } l \equiv 2 \pmod 3. \end{cases}$$

Thus we have rebuild the example from [**PSZ15**] for all finite characteristics except 2.

In particular we also get the result of [**PSZ15**, Theorem 10.1] for

$$I = \{k! \mid k \in \mathbb{N}\},$$

that is

$$\dim_{\mathbb{K}[U \rtimes G]} \ker\left(\overline{A+2}\right) = \beta_1 + \beta_2 \sum_{k=1}^{\infty} 2^{-6(k!)+k},$$

where β_1, β_2 are rational numbers. It is a known result that this number is transcendental as $\sum_{k=1}^{\infty} 2^{-6(k!)+k}$ is a Liouville number [**Lio51**].

Remark 5.3.11. Note that it is also possible to rebuild the example for characteristics 2 but requires a change in setup similar to the previous section.

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