Fourier expansions of GL(3) Eisenstein series for congruence subgroups

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Deniz Balakci
aus Georgsmarienhütte

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Betreuungsausschuss

Prof. Dr. Valentin Blomer, Mathematisches Institut Göttingen

Prof. Dr. Jörg Brüdern, Mathematisches Institut Göttingen

Mitglieder der Prüfungskommission

Referent: Prof. Dr. Valentin Blomer, Mathematisches Institut Göttingen

Koreferent: Prof. Dr. Jörg Brüdern, Mathematisches Institut Göttingen

Weitere Mitglieder der Prüfungskommission

PD Dr. Ulf-Rainer Fiebig, Institut für Mathematische Stochastik

Prof. Dr. Ina Kersten, Mathematisches Institut Göttingen

Prof. Dr. Ingo Witt, Mathematisches Institut Göttingen

Prof. Dr. Thorsten Hohage, Institut für Numerische und Angewandte Mathematik

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Contents

Introduction

The theory of automorphic forms lies at the heart of number theory and has connections to many fields in theoretical mathematics like harmonic analysis, representation theory, spectral theory, partial differential equations, mathematical physics and algebraic geometry, thus there are several different “angles” one can look at automorphic forms. From the analytical point of view one is interested in the eigenfunctions of the Laplacian on a Riemannian manifold, in mathematical physics especially in the square integrable eigenstates. From the algebraic point of view one is interested in the modern adelic theory of automorphic representations of reductive groups and their connections to the Langlands program. On the geometrical side the fractional transformations induced by elements of the modular group are isometries of the hyperbolic upper half plane. The Eisenstein series span the orthogonal complement of the space of cusp forms. In contrast to the cusp forms, Eisenstein series can be written down explicitly through an infinite series and as a result of this the important properties like meromorphic continuation, functional equations and Fourier coefficients can be calculated exactly. There are several good reasons why one should study Eisenstein series in detail. We will quote a few here and explain how they fit in the general picture sketched above. In the theory of automorphic forms and the Langlands program Eisenstein series are used to study cusp forms. A way to do this is the Rankin-Selberg method (and the Langlands-Shahidi method for automorphic representations), which consists of integrating an Eisenstein series against a cusp form, unfolding the Eisenstein sum and then transfering the meromorphic continuation and functional equation from the Eisenstein series to the L-function of the cusp form. In analytic number theory the fact is used that the Fourier coefficients of Eisenstein series contain arithmetic information, for example the Siegel-Weil formula gives the ways a number can be represented by quadratic forms on average. In spectral theory direct integrals of Eisenstein series describe for a non cocompact lattice Γ the continous spectrum of the self adjoint extension of the Laplacian on the Hilbert space $\mathfrak{L}^2(\Gamma \setminus \mathfrak{H})$. In the language of representation theory this is equivalent to the decomposition of the right regular representation of $\mathfrak{L}^2(\Gamma \setminus \mathfrak{H})$ into irreducible subrepresentations.

First we give a short overview over the literature about Eisenstein series and their Fourier expansions (without guarantee of completeness). The theory of Eisenstein series was first treated in great generality for Lie groups in Langlands' famous work [**21**]. The treatment in [**21**] relies heavily on the use of representation theory and it is not trivial to translate this into the classical language. The standard reference for Eisenstein series and spectral theory for reductive groups in the modern adelic language is [**24**]. The theory of Eisenstein series for arbitrary Fuchsian groups of GL_2 was discussed in great detail in the classical language, see the standard references [**11**], [**12**]. For the general linear group GL_n explicit calculations for Eisenstein series in the classical language were mostly done for the lattice $SL_n(\mathbb{Z})$, we give now a short summary of the results. Langlands did some explicit calculations of the functional equations for $SL_n(\mathbb{Z})$ in the appendix of his book [**21**], but did not calculate explicit Fourier expansions. In [**13**] and [**26**] the Fourier expansion for the Eisenstein series twisted by a Maass cusp form is calculated for the lattice $SL_3(\mathbb{Z})$, though in these papers the space of positive definite matrices was used as a model of the generalized upper half plane and also generalized K-Bessel functions instead of Whittaker functions were used for the calculations. In [**23**] the same is done with Whittaker functions but in a more representation theoretic way. In [**6**] the Eisenstein series twisted by a constant Maass form for the lattice $SL_3(\mathbb{Z})$ is handled. The minimal Eisenstein series for the lattice $SL_3(\mathbb{Z})$ is discussed in great detail in [**4**] and [**27**]. In this thesis the Fourier expansions of all types of GL_3 Eisenstein series for the congruence subgroup $\Gamma_0(N)$ of $SL_3(\mathbb{Z})$ defined by

$$\Gamma_0(N) := \left\{ \gamma \in SL_3(\mathbb{Z}) \middle| \gamma \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \bmod N \right\} , \tag{0.1}$$

with N squarefree, are explicitly calculated. Further certain invariance properties of the Fourier coefficients are proved from which the functional equation can be deduced. This is explicitly carried out for the Eisenstein series twisted with a constant Maass form of prime level. To avoid to many notations we will use the notation of $\Gamma_0(N)$ also to denote the analogous congruence subgroup of $SL_2(\mathbb{Z})$. Note that, unlike the GL_2-version of $\Gamma_0(N)$, in the higher rank case $\Gamma_0(N)$ has not many symmetries. This means precisely that $\Gamma_0(N)$ is not invariant under the involution

$$z \mapsto z^t := \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} (z^{-1})^T \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} .$$

In order to facilitate the reading of this quite technical thesis we will give a brief introduction into the theory of automorphic forms on GL_3 in the classical language. The standard references for this topic are [**4**], [**7**] and [**9**, ch. 12], where also detailed proofs for the cited theorems can be found. For each introduced object, we give a short description how this object is used in the thesis. We begin with the introduction of the generalized upper half plane, a symmetric space on which our automorphic forms are defined. Note that the generalized upper half plane is a generalization of the well known upper plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$ in the complex plane. This becomes more transparent if one describes these spaces as quotients of $GL_3(\mathbb{R})$ by the orthogonal group O_3, which is a maximal compact subgroup, and the center of $GL_3(\mathbb{R})$.

Definition 0.1. The generalized upper half plane $\mathfrak{h}^3$ associated to $GL_3(\mathbb{R})$ is defined as the symmetric space

$$\begin{aligned}\mathfrak{h}^3 := &\left\{ z := x \cdot y := \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \middle| x_i \in \mathbb{R} \text{ and } y_j > 0 \text{ for } 1 \le i \le 3, 1 \le j \le 2 \right\} \\ \cong & GL_3(\mathbb{R})/O_3\mathbb{R}^\times .\end{aligned}$$

Related to the above definition is the Iwasawa decomposition of the group $GL_3(\mathbb{R})$, which states that every matrix $g \in GL_3(\mathbb{R})$ has a decomposition $g = hkr$ with elements $h \in \mathfrak{h}^3, k \in O_3, r \in \mathbb{R}^\times$. Note that the factor h is unique and the factors k and r are unique up to the multiplication with $\pm E_3$, where E_3 denotes the 3×3 identity matrix. In order to introduce Maass forms on $\mathfrak{h}^3$, note that the manifold $\mathfrak{h}^3$ has odd dimension five so there does not exist any complex structure, hence there does not exist an analogon of modular forms, so we have to introduce an analogon of the hyperbolic Laplace operator. We cite the introduction of GL_3-invariant differential operators in [**9**, def. 12.3.14], a more detailed treatment can be found in [**7**], [**8**]. For each $\alpha \in \mathfrak{g} := \mathfrak{gl}(3, \mathbb{R})$, where $\mathfrak{gl}(3, \mathbb{R})$ denotes the Lie algebra of $GL_3(\mathbb{R})$, we define a differential operator D_α acting on smooth functions $\phi : GL_3(\mathbb{R}) \to \mathbb{C}$ by the formula

$$D_\alpha \phi(g) := \lim_{t \to 0} \frac{1}{t} \left(\phi(g \cdot \exp(t\alpha)) - \phi(g) \right) \ . \tag{0.2}$$

Then the algebra of differential operators with real coefficients generated by the operators $D_\alpha, \alpha \in \mathfrak{g}$ is a realization of the universal enveloping algebra $U(\mathfrak{g})$. Its center, $\mathfrak{Z}(U(\mathfrak{g}))$ is isomorphic to a polynomial algebra in 3 generators. One choice of generators is given by the Casimir differential operators, see [**9**, def. 12.3.14]. In [**4**, ch. 2] the only relevant two generators (the third one acts trivially on functions defined on $\mathfrak{h}^3$, since it is differential operator induced by the identity matrix E_3) are calculated in detail. Note that one can generalize this construction to Lie groups. With these preparations we can state the notion of an automorphic form on the generalized upper half plane.

Definition 0.2. Let N be a positive integer. A Maass form of level N is a smooth function $\phi : \mathfrak{h}^3 \to \mathbb{C}$ satisfying the following conditions

(1) $$\phi(\gamma \cdot z) = \phi(z) \qquad \forall \gamma \in \Gamma_0(N) \ ,$$
(2) the function ϕ is an eigenfunction of every element of $\mathfrak{Z}(U(\mathfrak{g}))$ (the action is well defined),
(3) the function ϕ is of moderate growth, meaning for each fixed $\sigma \in GL_3(\mathbb{Q})$, there exist constants c, C and B such that
$$|f(\sigma \cdot z)| \leq C \cdot (y_1 y_2)^B$$
for all $z = x \cdot y \in \mathfrak{h}^3$ such that $\min(y_1, y_2) \geq c$.

Next we introduce the two maximal conjugated parabolic subgroups and the minimal parabolic subgroup of $SL_3(\mathbb{Z})$ by

$$P_{2,1} := \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} \in SL_3(\mathbb{Z}) \right\} \ ,$$
$$P_{1,2} := \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & a_{3,2} & a_{3,3} \end{pmatrix} \in SL_3(\mathbb{Z}) \right\} \ ,$$
$$P_{min} := \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ & 1 & a_{2,3} \\ & & 1 \end{pmatrix} \in SL_3(\mathbb{Z}) \right\} \ .$$

The parabolic subgroups are stabilizers of flags and play a tremendous role in the general theory of the constant terms of automorphic forms. Since we are working with the Fourier expansions in this thesis we go on without introducing the precise notion of a GL_3 Maass cusp form for $\Gamma_0(N)$ and the notion of the constant term. The general theory of the constant term is developed in detail for example in [**21**], [**28**] and [**29**]. The "translation" into the classical language can be found in [**9**, ch. 12]. To issue a warning in order to add clarity we will use the notation P_{min} also to denote the analogous subgroup of $SL_2(\mathbb{Z})$. Analogously to the GL_2 case where through separation of variables the obvious eigenfunction y^s of the hyperbolic Laplacian was constructed, see [**15**, ch. 1.7], one can construct a special eigenfunction of $\mathfrak{Z}(U(\mathfrak{g}))$, see [**7**, ch. 2.4, 10.4, 10.5] for details. For complex parameters s_1, s_2 the function

$$I_{(s_1,s_2)}(z) := y_1^{s_1+2s_2} y_2^{2s_1+s_2} \qquad \forall z \in \mathfrak{h}^3$$

is an eigenfunction of $\mathfrak{Z}(U(\mathfrak{g}))$ and invariant against left multiplication with real upper triangular matrices, whose diagonal entries are ± 1. Further for a complex parameter s the functions

$$I_{(s,-2s)}(z, P_{2,1}) := \left(y_1^2 y_2\right)^s \qquad \forall z \in \mathfrak{h}^3$$

and

$$I_{(2s,-s)}(z, P_{1,2}) := \left(y_2^2 y_1\right)^s \qquad \forall z \in \mathfrak{h}^3$$

are eigenfunctions of $\mathfrak{Z}(U(\mathfrak{g}))$ and invariant against left multiplication with real matrices in $P_{2,1}$, $P_{1,2}$, respectively. Now we have the tools to define the different types of GL_3 Eisenstein series. There are actually three different kinds of Eisenstein series, the minimal one associated to the minimal parabolic subgroup, the one twisted by a Maass cusp form associated to the two maximal parabolic subgroups and the one twisted by a constant Maass form associated to the two maximal parabolic subgroups. Note that since the congruence subgroups for GL_2 have no residual spectrum besides the point $s_0 = 1$, see [**15**, ch. 11.2] for details, there are only constant residual Maass forms. So there are only the twists with constant or cuspidal Maass forms, which occur in the spectral decomposition of $\mathfrak{L}^2(\Gamma_0(N) \setminus \mathfrak{h}^3)$. Analogously to the GL_2 situation we will define the Eisenstein series associated to the three different parabolic subgroups through averaging the various I_s functions defined above over the congruence subgroup $\Gamma_0(N)$, which implies trivially the $\Gamma_0(N)$ invariance. Associated to "cusps" $\alpha \in \Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{2,1}, \beta \in \Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{1,2}$ and a constant or cuspidal GL_2 Maass form ϕ of level M, with M a divisor of N, are the twisted Eisenstein series

$$\sum_{\gamma \in \alpha\Gamma(M,P_{2,1})\alpha^{-1} \cap \Gamma_0(N) \setminus \Gamma_0(N)} \phi\left(\mathfrak{m}_{P_{2,1}}\left(\alpha^{-1}\gamma z\right)\right) I_{(s,-2s)}\left(\alpha^{-1}\gamma z, P_{2,1}\right) \ ,$$

$$\sum_{\gamma \in \beta\Gamma(M,P_{1,2})\beta^{-1} \cap \Gamma_0(N) \setminus \Gamma_0(N)} \phi\left(\mathfrak{m}_{P_{1,2}}\left(\beta^{-1}\gamma z\right)\right) I_{(2s,-s)}\left(\beta^{-1}\gamma z, P_{1,2}\right) \ ,$$

where $\mathfrak{m}_{P_{2,1}}$, $\mathfrak{m}_{P_{1,2}}$, respectively, denotes the upper left, lower right, respectively, 2×2 block of a matrix in $P_{2,1}$, $P_{1,2}$, respectively. And $\Gamma(M, P_{2,1})$, $\Gamma(M, P_{1,2})$, respectively, consists of the matrices in $P_{2,1}$, $P_{1,2}$, respectively, whose upper left, lower right, respectively, 2×2 block matrix lies in $\Gamma_0(M)$. Note that the above Eisenstein series are well defined, this follows from the automorphy of ϕ and the invariance of the function $I_{(s,-2s)}(*, P_{2,1})$, $I_{(2s,-s)}(*, P_{1,2})$, respectively, against left multiplication with $P_{2,1}$, $P_{1,2}$, respectively. For a "cusp" $\delta \in \Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{min}$ define the associated minimal Eisenstein series by

$$\sum_{\gamma\in\delta P_{min}\delta^{-1}\cap\Gamma_0(N)\setminus\Gamma_0(N)} I_{(s_1,s_2)}\left(\delta^{-1}\gamma z\right) \ ,$$

again this is well defined. The minimal Eisenstein series is absolutely convergent for $\mathfrak{Re}(s_1), \mathfrak{Re}(s_2) > \frac{2}{3}$ in the case of $SL_3(\mathbb{Z})$, see [**4**, ch. 7] and [**7**, Prop. 10.4.3] for a proof following [**3**]. Since the $\delta P_{min}\delta^{-1} \cap \Gamma_0(N) \setminus \Gamma_0(N)$ cosets can be viewed as a subset of the cosets corresponding to $SL_3(\mathbb{Z})$, the minimal Eisenstein series for $\Gamma_0(N)$ is absolutely convergent in the same right half plane. The I_s-functions in the definition of the twisted Eisenstein series are in fact shifted versions of the function $I_{(s_1,s_2)}$, so since a constant or cuspidal Maass form for GL_2 has moderate growth, the absolute convergence is inherited from the minimal Eisenstein series. Note that the construction of the GL_3 Eisenstein series are carried out in the same way as in the GL_2 case through the method of images, hence summing up shifted versions of a special eigenfunction. In the calculation of the Fourier expansions one has to overcome several barriers. The first one is to calculate an explicit set of double coset representatives for the "cusps" for each parabolic subgroup. The second one is to have a set of left coset representatives for the sets in the summation condition and then to calculate the corresponding Iwasawa decompositions, so one can calculate the associated values of I_s-functions and Maass forms explicitly. This is solved by explicitly defining a set of elements in $SL_3(\mathbb{Z})$ in Bruhat decomposition for each parabolic subgroup such that each element in $SL_3(\mathbb{Z})$ is represented uniquely as the product of one element from this set and one element from the associated parabolic subgroup. Then we will show how to choose a suitable set of coset representatives from these matrices. These coset representatives are described by gcd-conditions on the primes dividing the level N. In contrast to the theory for $SL_3(\mathbb{Z})$ the calculations here are much more elaborate. Further the construction here will be in such a generality, that one can handle each parabolic subgroup, in the literature for $SL_3(\mathbb{Z})$ one used to construct coset representatives for the twisted and minimal Eisenstein series separately, see [**13**] for the twisted and [**4**], [**27**] for the minimal Eisenstein series. Once the above construction is achieved one has to calculate the $\mathfrak{h}^3$-part in the Iwasawa decomposition of those coset representatives and the corresponding values of the functions I_s and ϕ. Since one needs the terms ordered in a special way, for further calculations one cannot leave the whole calculations to programs like Mathematica. Note that these explicit constructions can also be applied to Poincaré series for $\Gamma_0(N)$ or generalized to other congruence subgroups of $SL_3(\mathbb{Z})$. After these preparations one can start with the calculation of the Fourier coefficients.

We now give a short review of the Fourier expansions of GL_3 automorphic forms. In the adelic language the Fourier expansion was first proved in [**18**], [**19**]. For the Fourier expansion of $SL_3(\mathbb{Z})$ invariant automorphic forms in the classical language see [**4**, Thm. page

65,(4.10)] for details. Note that in the Fourier expansion of $\Gamma_0(N)$ invariant automorphic forms there are no additional sums, since the congruence subgroup $\Gamma_0(N)$ contains the group $P_{2,1}$. Note that in contrast to the GL_2 theory, where the invariance against the element $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ implies the periodicity and the Fourier expansion of a Maass form, in the GL_3 case one has to make an effort to get the Fourier expansion and further the structure of the expansion is more complicated, since an additional sum over the left cosets of the lower rank group $SL_2(\mathbb{Z})$ occurs. This is due to the noncommutativity of the Heisenberg group.

THEOREM 0.3. *Let ϕ be an automorphic form on GL_3 for the congruence subgroup $\Gamma_0(N)$. Then ϕ possesses there a Fourier expansion*

$$\phi(z) = \sum_{m_2=0}^{\infty} \phi_{0,m_2}(z) + \sum_{\gamma \in P_{min} \backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} \phi_{m_1,m_2}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z \right)$$

with

$$\phi_{m_1,m_2}(z) = \int_0^1 \int_0^1 \int_0^1 \phi\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z \right) e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \,.$$

The existence of Whittaker models for GL_3 implies that $\phi_{m_1,m_2}(z)$ is a linear combination of six Whittaker functions. This is proven in [**4**, ch. 2] using the two differential equations ϕ satisfies and the invariance against the upper triangular matrices. The general theory of Whittaker functions was developed by Jacquet in [**17**] on arbitrary Chevalley groups and explicitly worked out for the group GL_3 in [**4**, ch. 3]. For $\mathfrak{Re}(\nu_1), \mathfrak{Re}(\nu_2) > \frac{1}{3}$, integers n_1, n_2 and with the notation $\xi_4 := \xi_1\xi_2 - \xi_3$, the explicit integral formulas in [**4**, (3.10)-(3.15)] for the Whittaker functions read as follows

$$W_{n_1,n_2}^{(\nu_1,\nu_2)}(z, w_0) = \begin{cases} \pi^{-3\nu_1-3\nu_2+\frac{1}{2}} \Gamma\left(\frac{3\nu_1}{2}\right) \Gamma\left(\frac{3\nu_2}{2}\right) \Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) I_{(\nu_1,\nu_2)}(z) \,, & \text{if } n_1 = n_2 = 0 \,; \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

$$W_{n_1,n_2}^{(\nu_1,\nu_2)}(z, w_1) = \pi^{-3\nu_1-3\nu_2+\frac{1}{2}} \Gamma\left(\frac{3\nu_1}{2}\right) \Gamma\left(\frac{3\nu_2}{2}\right) \Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) I_{(\nu_1,\nu_2)}(z) e(n_1 x_1 + n_2 x_2)$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\xi_3^2 + \xi_2^2 y_1^2 + y_1^2 y_2^2\right]^{-\frac{3\nu_1}{2}} \left[(\xi_3 - \xi_1\xi_2)^2 + \xi_1^2 y_2^2 + y_1^2 y_2^2\right]^{-\frac{3\nu_2}{2}} e(-n_1\xi_1 - n_2\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \,.$$

$$W_{n_1,n_2}^{(\nu_1,\nu_2)}(z, w_2) = \begin{cases} \pi^{-3\nu_1-3\nu_2+\frac{1}{2}} \Gamma\left(\frac{3\nu_1}{2}\right) \Gamma\left(\frac{3\nu_2}{2}\right) \Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) I_{(\nu_1,\nu_2)}(z) e(n_2 x_2) & \\ \int_{-\infty}^{\infty} \left[\xi_2^2 + y_2^2\right]^{-\frac{3\nu_2}{2}} e(-n_2\xi_2) \mathrm{d}\xi_2 \,, & \text{if } n_1 = 0 \,; \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

$$W^{(\nu_1,\nu_2)}_{n_1,n_2}(z,w_3)=\begin{cases}\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)I_{(\nu_1,\nu_2)}(z)e(n_1x_1)\\ \int_{-\infty}^{\infty}\left[\xi_1^2+y_1^2\right]^{-\frac{3\nu_1}{2}}e(-n_1\xi_1)\mathrm{d}\xi_1\ , & \text{if } n_2=0\ ;\\ 0\ , & \text{otherwise}\ .\end{cases}$$

$$W^{(\nu_1,\nu_2)}_{n_1,n_2}(z,w_4)=\begin{cases}\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)I_{(\nu_1,\nu_2)}(z)e(n_2x_2)\\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2\right]^{-\frac{3\nu_1}{2}}\left[\xi_2^2+y_2^2\right]^{-\frac{3\nu_2}{2}}e(-n_2\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_3\ , & \text{if } n_1=0\ ;\\ 0\ , & \text{otherwise}\ .\end{cases}$$

$$W^{(\nu_1,\nu_2)}_{n_1,n_2}(z,w_5)=\begin{cases}\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)I_{(\nu_1,\nu_2)}(z)e(n_1x_1)\\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\xi_1^2+y_1^2\right]^{-\frac{3\nu_1}{2}}\left[\xi_4^2+\xi_1^2y_2^2+y_1^2y_2^2\right]^{-\frac{3\nu_1}{2}}e(-n_1\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_4\ , & \text{if } n_2=0\ ;\\ 0\ , & \text{otherwise}\ .\end{cases}$$

The above Whittaker functions are the "building blocks" of the Fourier expansion of automorphic forms and only depend on the manifold GL_3, which means that $\phi_{m_1,m_2}(z)$ is a linear combination of Whittaker functions $W^{(\nu_1,\nu_2)}_{m_1,m_2}(z,*)$. The spectral parameters ν_1,ν_2 come from the eigenvalues of the automorphic form, see [**4**, thm. page 65] for a closer description of the dependencies. Further the Whittaker functions have meromorphic continuation and satisfy certain functional equations, see again [**4**, ch. 3] for the precise statements and proofs, which are used to obtain the functional equation of Eisenstein series. One main application of the Fourier expansion of the Eisenstein series is the derivation of the associated functional equation, by showing that each and every summand in the Fourier expansion has a certain invariance against the transformation $s\mapsto 1-s$, $s\mapsto w\cdot(s_1,s_2)$ (action of the Weyl group, see [**4**, (2.5)]), respectively. It turns out that through permuting the summands in the Fourier expansion, one gets the functional equation. In the Fourier expansion of our Eisenstein series the Fourier coefficients in the linear combination of these Whittaker functions depend on the complex parameter(s) s, s_1,s_2, respectively, and contain all the number theoretic and combinatorial information about the lattice $\Gamma_0(N)$. These Fourier coefficients are in fact Dirichlet series, which split into a number theoretic part, the L-function of the Maass cusp form in the twisted case, divisor sums in the other cases, and into a combinatorial part. The functional equations of the L-function and the divisor sums, see Theorem 18.3, Lemma 20.3, give the necessary invariance property of the number theoretic part. The combinatorial part is described through the introduction of certain families of power series in two variables associated to each prime number. The most important family of these power series occurs in the Fourier coefficients at the unramified primes (primes coprime to the level) after substituting certain number theoretic functions, Hecke eigenvalues, divisor sums, respectively, for the two variables. The other families of power series for the ramified primes (primes dividing the level) can be expressed as a sum of the power series with "good" transformation behaviour and a remaining term, which can be calculated explicitly. So for non trivial level one has to take linear combinations of the Eisenstein series for the different double coset representatives to compensate the absence of a nice transformation behaviour of the power series at the ramified primes. This is analogous to the treatment of GL_2 Eisenstein series, where one has to take the Eisenstein vector parametrized over the cusps and transform it with a certain scattering matrix, see [**15**] and [**20**] for a detailed treatment. We will apply this method to the Eisenstein series twisted by a constant Maass form for prime level.

The following summary lists the main results in this thesis. For the Eisenstein series twisted by a constant Maass form explicit Fourier expansions are proved in Theorem 11.3 and 12.3, further the functional equation is proved in Theorem 13.4. For the Eisenstein series twisted by a Maass cusp form Fourier expansions are stated in Theorem 14.3 and 15.2, the finer structure of the Fourier coefficients are examined in Lemma 14.5 and 15.3. Eventually the result, where the longest calculations are involved, is the Fourier expansion of the minimal Eisenstein series, stated in Theorem 17.2. The chosen normalization of the cusp forms and Whittaker functions in the above theorems is not canonical, the natural choice will be dictated by the functional equation.

Part I

Eisenstein series

CHAPTER 1

Double coset representatives

The first step of the calculation of the Fourier expansion for all three different kinds of Eisenstein series is to explicitly calculate a set of representatives for the double cosets $\Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{m,3-m}$ and $\Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{min}$. The double cosets above can be considered as the algebraic analogon of the cusps of the corresponding symmetric space $\Gamma_0(N) \setminus \mathfrak{h}^3$. In the classical theory for GL_2 it is well known that there is a one-to-one correspondence between the elements of the double coset $\Gamma_0(N) \setminus SL_2(\mathbb{Z})/P_{min}$ and the cusps for $\Gamma_0(N)$ (see [**22**] for a proof).

First we state two trivial lemmata which are often needed later in technical steps of calculations.

LEMMA 1.1. *Let d_1, d_2, N be integers with $N \geq 1$ and $(d_1, d_2) = 1$. Then there exists a positive integer s with the property $(d_1 + sd_2, N) = 1$.*

PROOF. If $N = 1$ we can choose s arbitrarily, so assume $N > 1$. Since $(d_1, d_2) = 1$, the number N has the following prime factor decomposition

$$N = \prod_{p|d_1} p^{\alpha_p} \prod_{p|d_2} p^{\alpha_p} \prod_{p\nmid d_1 d_2} p^{\alpha_p}.$$

This implies $\left(N, d_1 + d_2 \cdot \prod_{\substack{p|N \\ p\nmid d_1 d_2}} p \right) = 1$. Hence the choice $s = \prod_{\substack{p|N \\ p\nmid d_1 d_2}} p$ will do. □

The proof of the following is trivial and omitted.

LEMMA 1.2. *Let $\mathfrak{G}$ be a group and $\mathfrak{H}, \mathfrak{K}$ subgroups of the group $\mathfrak{G}$. Further let $a, b \in \mathfrak{G}$. The two double cosets $\mathfrak{H}a\mathfrak{K}$ and $\mathfrak{H}b\mathfrak{K}$ are equal if and only if there exists an element $k \in \mathfrak{K}$ such that $akb^{-1} \in \mathfrak{H}$.*

Now we return to our primary goal and begin with the calculation of a suitable set of double coset representatives for $\Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{m,3-m}$.

LEMMA 1.3. *Let m, N be positive integers such that $m < 3$. Further let f_1, f_2 be positive divisors of N. Then the two double cosets*

$$\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\f_1&&1\end{pmatrix}P_{m,3-m}\quad \textit{and}\quad \Gamma_0(N)\begin{pmatrix}1&&\\&1&\\f_2&&1\end{pmatrix}P_{m,3-m}$$

are equal if and only if $f_1 = f_2$.

Proof. Obviously we get equality if $f_1 = f_2$ holds, so there is only the "only if" part of the statement to prove. Using symmetry it is sufficient to prove $f_1 \mid f_2$. We use Lemma 1.2 and calculate directly

$$\begin{aligned}
&\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\f_1&&1\end{pmatrix}P_{m,3-m} = \Gamma_0(N)\begin{pmatrix}1&&\\&1&\\f_2&&1\end{pmatrix}P_{m,3-m}\\
\Leftrightarrow\quad &\exists\begin{pmatrix}A&X\\0&D\end{pmatrix}\in P_{m,3-m}:\quad \begin{pmatrix}1&&\\&1&\\f_1&&1\end{pmatrix}\begin{pmatrix}A&X\\0&D\end{pmatrix}\begin{pmatrix}1&&\\&1&\\f_2&&1\end{pmatrix}^{-1}\in\Gamma_0(N)\\
\Leftrightarrow\quad &\exists\begin{pmatrix}A&X\\0&D\end{pmatrix}\in P_{m,3-m}:\quad e_3^T\begin{pmatrix}1&&\\&1&\\f_1&&1\end{pmatrix}\begin{pmatrix}A&X\\0&D\end{pmatrix}\begin{pmatrix}1&&\\&1&\\-f_2&&1\end{pmatrix}\equiv(0,0,*)\bmod N\,.
\end{aligned}$$

Since f_1 divides N, reducing the last equation modulo f_1 implies:

$$\begin{aligned}
(-f_2,0,1)&\equiv(0,0,*)\bmod f_1\,,\quad \text{if } m=2\,,\\
(-f_2d_{2,2},d_{2,1},d_{2,2})&\equiv(0,0,*)\bmod f_1\,,\quad \text{if } m=1\,.
\end{aligned}$$

In the case $m = 2$ the congruence for the first entry immediately implies $f_1 \mid f_2$. For $m = 1$ notice that since D is an invertible matrix, the entries of the last row $(d_{2,1}, d_{2,2})$ are coprime. Now the above equation implies $f_1 \mid d_{2,1}$ so f_1 is coprime to $d_{2,2}$. But then the equation $-f_2d_{2,2} \equiv 0 \bmod f_1$ implies $f_1 \mid f_2$. □

Lemma 1.4. *Let* t, m, N *be positive integers such that* $m < 3$. *Then the two double cosets*

$$\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\t&&1\end{pmatrix}P_{m,3-m}\quad \textit{and}\quad \Gamma_0(N)\begin{pmatrix}1&&\\&1&\\(t,N)&&1\end{pmatrix}P_{m,3-m}$$

are equal.

Proof. We distinguish between two cases.

Case 1: $m = 2$

Choose two integers x, y such that $x\frac{t}{(t,N)} = 1 + y\frac{N}{(t,N)}$. Since $\left(x, \frac{N}{(t,N)}\right) = 1$ and $m = 2$ the row $\left(x, \frac{N}{(t,N)}\right)$ can be completed to a matrix $\begin{pmatrix} x & \frac{N}{(t,N)} \\ * & * \end{pmatrix}$ in $SL_2(\mathbb{Z})$. A short direct calculation gives the result:

$$
\begin{aligned}
&\Gamma_0(N)\cdot \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} \cdot P_{2,3-2} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} \begin{pmatrix} x & \frac{N}{(t,N)} & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} x & \frac{N}{(t,N)} & 0 \\ * & * & 0 \\ tx & N\frac{t}{(t,N)} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -(t,N) & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ (t,N) & & 1 \end{pmatrix} P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} x & \frac{N}{(t,N)} & 0 \\ * & * & 0 \\ tx-(t,N) & N\frac{t}{(t,N)} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ (t,N) & & 1 \end{pmatrix} P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} x & \frac{N}{(t,N)} & 0 \\ * & * & 0 \\ yN & N\frac{t}{(t,N)} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ (t,N) & & 1 \end{pmatrix} P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ (t,N) & & 1 \end{pmatrix} P_{2,1}\ .
\end{aligned}
$$

Case 2: $\quad m = 1$

Apply Lemma 1.1 and choose an integer s such that $\left(\frac{t}{(t,N)} - s\frac{N}{(t,N)}, N\right) = 1$. So the row $\left(N, \frac{t}{(t,N)} - s\frac{N}{(t,N)}\right)$ can be completed to a matrix $\begin{pmatrix} * & * \\ N & \frac{t}{(t,N)} - s\frac{N}{(t,N)} \end{pmatrix}$ in $SL_2(\mathbb{Z})$. Again a short direct calculation gives the result:

$$
\begin{aligned}
&\Gamma_0(N)\cdot \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} \cdot P_{1,3-1} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & N & \frac{t}{(t,N)} - s\frac{N}{(t,N)} \end{pmatrix} P_{1,2}
\end{aligned}
$$

$$
\begin{aligned}
=&\Gamma_0(N)\begin{pmatrix}1 & 0 & 0\\ 0 & * & *\\ t & N & \frac{t}{(t,N)}-s\frac{N}{(t,N)}\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ -(t,N) & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ (t,N) & & 1\end{pmatrix}P_{1,2}\\
=&\Gamma_0(N)\begin{pmatrix}1 & 0 & 0\\ * & * & *\\ sN & N & \frac{t}{(t,N)}-s\frac{N}{(t,N)}\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ (t,N) & & 1\end{pmatrix}P_{1,2}\\
=&\Gamma_0(N)\begin{pmatrix}1 & & \\ & 1 & \\ (t,N) & & 1\end{pmatrix}P_{1,2}\ .
\end{aligned}
$$

□

In the next lemma the main double coset decomposition is stated.

LEMMA 1.5. *Let m, N be positive integers with $m < 3$. The disjoint double coset decomposition*

$$
SL_3(\mathbb{Z}) = \dot{\bigcup_{f|N}}\Gamma_0(N)\cdot\begin{pmatrix}1 & & \\ & 1 & \\ f & & 1\end{pmatrix}\cdot P_{m,3-m}
$$

holds.

PROOF. Using Lemma 1.3 and Lemma 1.4 it is sufficient to show the following (in general not disjoint) double coset decomposition:

$$
SL_3(\mathbb{Z}) = \bigcup_{0\le t<N}\Gamma_0(N)\cdot\begin{pmatrix}1 & & \\ & 1 & \\ t & & 1\end{pmatrix}\cdot P_{m,3-m}\ .
$$

So let $A = (a_{i,j})_{i,j=1,2,3} \in SL_3(\mathbb{Z})$. Define the rows $a_1 := (a_{3,1}, ..., a_{3,m})$ and $a_2 := (a_{3,m+1}, ..., a_{3,3})$. Further define the corresponding gcd's of these rows, distinguishing between the cases that the row vanishes or not:

$$
d_1 = \begin{cases}(a_{3,1}, ..., a_{3,m}) & \text{if } a_1 \neq 0\\ 0 & \text{if } a_1 = 0\end{cases}\ ,
$$

$$
d_2 = \begin{cases}(a_{3,m+1}, ..., a_{3,3}) & \text{if } a_2 \neq 0\\ 0 & \text{if } a_2 = 0\ .\end{cases}
$$

Next we complete (if possible) the coprime rows $\frac{1}{d_1}a_1$ and $\frac{1}{d_2}a_2$ to matrices in $SL_m(\mathbb{Z})$ and $SL_{3-m}(\mathbb{Z})$ treating the case of vanishing separately:

$$B = \begin{cases} \begin{pmatrix} \frac{1}{d_1} a_1 \\ * \end{pmatrix} & \text{if } a_1 \neq 0 \\ \\ E_m & \text{if } a_1 = 0 \end{cases} \quad \text{and} \quad C = \begin{cases} \begin{pmatrix} * \\ \frac{1}{d_2} a_2 \end{pmatrix} & \text{if } a_2 \neq 0 \\ \\ E_{3-m} & \text{if } a_2 = 0 \,. \end{cases}$$

Since the matrix A is in $SL_3(\mathbb{Z})$, the entries of the last row are coprime and this gives that $(d_1, d_2) = 1$. Note that $d_1 = 0$ implies $d_2 = 1$ and vice versa. Lemma 1.1 guarantees the existence of an integer s such that $(d_1 + sd_2, N) = 1$, and there exists an integer $0 \leq t < N$ such that $t(d_1 + sd_2) \equiv d_1 \bmod N$. With these preparations the proof is easily completed, so that

$$\begin{aligned}
&\Gamma_0(N) \cdot A \cdot P_{m,3-m} \\
=&\Gamma_0(N) A \begin{pmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} P_{m,3-m} \\
=&\Gamma_0(N) \begin{pmatrix} * & * \\ a_1 B^{-1} & a_2 C^{-1} \end{pmatrix} P_{m,3-m} \\
=&\Gamma_0(N) \begin{pmatrix} * & * & * \\ * & * & * \\ d_1 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{pmatrix} P_{m,3-m} \\
=&\Gamma_0(N) \begin{pmatrix} * & * & * \\ * & * & * \\ d_1 & 0 & d_2 + sd_1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -t & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} P_{m,3-m} \\
=&\Gamma_0(N) \begin{pmatrix} * & * & * \\ * & * & * \\ d_1 - t(d_2 + sd_1) & 0 & d_2 + sd_1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} P_{m,3-m} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{pmatrix} P_{m,3-m} \,.
\end{aligned}$$

□

For the explicit calculations in the following chapters it is useful to have another similar set of double coset representatives as in Lemma 1.5 for $m = 2$.

Corollary 1.6. *Let N be a positive integer. The double coset decomposition*

$$SL_3(\mathbb{Z}) = \dot{\bigcup_{f|N}} \Gamma_0(N) \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \cdot P_{2,1}$$

holds.

PROOF. It is sufficient to show that the double cosets in Lemma 1.5 are equal to the double cosets corresponding to the representatives in the claimed decomposition above. This can be seen easily through a short calculation,

$$\begin{aligned}
&\Gamma_0(N)\cdot \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix} \cdot P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix} \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix} P_{2,1} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} P_{2,1} \, .
\end{aligned}$$

□

In conclusion, one sees that these results can be easily generalized to maximal parabolic subgroups for GL_n. Next we concentrate on the double cosets $\Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{min}$.

LEMMA 1.7. *Let N be a positive integer and $A \in SL_3(\mathbb{Z})$. Then there exist a positive divisor f of N and a matrix $B \in SL_2(\mathbb{Z})$ such that the double coset corresponding to A can be described through f and B as*

$$\Gamma_0(N) \cdot A \cdot P_{min} = \Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} B & \\ & 1 \end{pmatrix} P_{min} \, .$$

PROOF. Corollary 1.6 implies the existence of a positive divisor f of N and elements $\gamma \in \Gamma_0(N)$ and $p \in P_{2,1}$ such that

$$A = \gamma \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} p \, .$$

Decompose $p = \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{pmatrix}$ and note that $B := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The rest of the proof is a straightforward calculation absorbing γ and the factor of p in the unipotent radical into the corresponding subgroups

$$\Gamma_0(N) \cdot A \cdot P_{min}$$

$$
\begin{aligned}
=&\Gamma_0(N)\gamma \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} B & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{pmatrix} P_{min} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} B & \\ & 1 \end{pmatrix} P_{min} \ .
\end{aligned}
$$

□

To abbreviate notation we make the following definition.

Definition 1.8. Let N be a positive integer and f a positive divisor of N. The GL_2 congruence subgroup $\Gamma_0\left(\frac{N}{f}\right) \cap \Gamma_1\left(\left(\frac{N}{f}, f\right)\right)$ is denoted by $\Gamma(N, f)$.

Lemma 1.9. *Let N be a positive integer, f a positive divisor of N and $A \in SL_2(\mathbb{Z})$. Then the double coset*

$$
\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} P_{min}
$$

only depends on the double coset representative of A in $\Gamma(N, f) \setminus SL_2(\mathbb{Z})/P_{min}$.

Proof. Let $\gamma = (\gamma_{i,j})_{i,j=1,2} \in \Gamma(N, f)$ and $w \in \mathbb{Z}$. We need to show that the following two double cosets

$$
\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} P_{min} \quad \text{and} \quad \Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} \gamma A \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} & \\ & 1 \end{pmatrix} P_{min}
$$

are equal. First make a calculation simplifying the double coset on the right-hand side. We have

$$
\begin{aligned}
&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} \gamma A \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} & \\ & 1 \end{pmatrix} P_{min} \\
=&\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & w & \\ & 1 & \\ & & 1 \end{pmatrix} P_{min} \\
=&\Gamma_0(N) \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ \gamma_{2,1} f & \gamma_{2,2} f & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} P_{min}
\end{aligned}
$$

$$\begin{aligned}
&=\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\ \gamma_{2,1}f&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}P_{min}\\
&=\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}P_{min}\,.
\end{aligned}$$

Note that the last equation follows since $\gamma \in \Gamma(N,f)$ and this gives $\gamma_{2,1} \equiv 0 \bmod \frac{N}{f}$. To complete the proof let $z \in \mathbb{Z}$ be arbitrary. Since A is invertible, we can choose integers x, y depending on z such that $A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\z\end{pmatrix}$. Finally we apply Lemma 1.2 to prove the claimed form. We start with the chain of statements

$$\begin{aligned}
&\Gamma_0(N)\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}P_{min} = \Gamma_0(N)\begin{pmatrix}1&&\\&1&\\&f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}P_{min}\\
\Longleftrightarrow&\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}\begin{pmatrix}1&&x\\&1&y\\&&1\end{pmatrix}\left(\begin{pmatrix}1&&\\&1&\\&f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}\right)^{-1}\in\Gamma_0(N)\\
\Longleftrightarrow& e_3^T\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}A&\\&1\end{pmatrix}\begin{pmatrix}1&&x\\&1&y\\&&1\end{pmatrix}\begin{pmatrix}A^{-1}&\\&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&-f&1\end{pmatrix}\equiv(0,0,*) \bmod N\\
\Longleftrightarrow& e_3^T\begin{pmatrix}1&&\\&1&\\&\gamma_{2,2}f&1\end{pmatrix}\begin{pmatrix}1&&\\&1&z\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&-f&1\end{pmatrix}\equiv(0,0,*) \bmod N\\
\Longleftrightarrow&(0,(1-fz)\gamma_{2,2}f-f,*)\equiv(0,0,*) \bmod N\\
\Longleftrightarrow& f\gamma_{2,2}z\equiv\gamma_{2,2}-1 \bmod \frac{N}{f}\,.
\end{aligned}$$

Since $\gamma \in \Gamma(N,f)$ reducing the last equation modulo $\left(f,\frac{N}{f}\right)$ gives zero on both sides. Hence the previous statement is equivalent to

$$\frac{f}{\left(f,\frac{N}{f}\right)}\gamma_{2,2}z \equiv \frac{\gamma_{2,2}-1}{\left(f,\frac{N}{f}\right)} \bmod \frac{N}{f\left(f,\frac{N}{f}\right)}\,.$$

Since $\gamma \in \Gamma(N,f)$ implies that $\gamma_{2,2}\frac{f}{\left(f,\frac{N}{f}\right)}$ is a unit modulo $\frac{N}{f\left(f,\frac{N}{f}\right)}$, the last congruence is solvable in the variable z. □

At the end a suitable set of double coset representatives for $\Gamma_0(N) \setminus SL_3(\mathbb{Z})/P_{min}$ can be stated.

Lemma 1.10. *Let N be a positive integer. The following disjoint double coset decomposition*

$$SL_3(\mathbb{Z}) = \dot{\bigcup_{\substack{f|N \\ A\in\Gamma(N,f)\backslash SL_2(\mathbb{Z})/P_{min}}}} \Gamma_0(N)\cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} \cdot P_{min}$$

holds. Hence there is a parametrization of GL_3 cusps by the divisors of the level and the cusps of the GL_2 congruence subgroups $\Gamma(N,f)$.

Proof. Lemma 1.7 and Lemma 1.9 prove the decomposition. So it remains to prove that this decomposition is disjoint. Let f_1, f_2 be positive divisors of N and $\gamma, \delta \in SL_2(\mathbb{Z})$ such that the corresponding double cosets to these parameters are equal:

$$\Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f_1 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} P_{min} = \Gamma_0(N) \begin{pmatrix} 1 & & \\ & 1 & \\ & f_2 & 1 \end{pmatrix} \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} P_{min} .$$

With the same argument as in the proof of Lemma 1.3 reducing the double coset equation above modulo f_1 immediately gives us $f_1 \mid f_2$ and symmetry gives the reverse relation hence $f_1 = f_2$. For easier notation define $f := f_1 = f_2$. Using Lemma 1.2 we find integers $x_1, x_2, x_3 \in \mathbb{Z}$ such that:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} \right)^{-1} \in \Gamma_0(N) .$$

With the definitions $\begin{pmatrix} x \\ y \end{pmatrix} := \gamma \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \gamma \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \delta^{-1}$ the above statement is equivalent to the chain of statements

$$\begin{aligned} & \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -f & 1 \end{pmatrix} \in \Gamma_0(N) \\ \Longleftrightarrow & e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -f & 1 \end{pmatrix} \equiv (0,0,*) \bmod N \\ \Longleftrightarrow & \begin{array}{rcl} cf & \equiv & 0 \mod N \\ f(d-fy)-f & \equiv & 0 \mod N \end{array} \\ \Longleftrightarrow & \begin{array}{rcl} c & \equiv & 0 \mod \frac{N}{f} \\ d-1-fy & \equiv & 0 \mod \frac{N}{f} \end{array} \end{aligned}$$

$$\iff \begin{array}{rcl} c & \equiv & 0 \mod \frac{N}{f} \\ d-1 & \equiv & 0 \mod \left(f, \frac{N}{f}\right) \end{array} .$$

Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N, f)$, which implies that the corresponding double cosets to γ and δ in $\Gamma(N, f) \setminus SL_2(\mathbb{Z})/P_{min}$ are equal. □

At the end of this chapter we introduce special names for the calculated representatives in order to simplify later notations.

Definition 1.11. Let N be a positive integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. Define three kinds of matrices belonging to these parameters by the formulas

$$\alpha_f := \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix}, \qquad \beta_f := \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix}, \qquad \lambda_{f,h} := \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ h & 1 & \\ & & 1 \end{pmatrix} .$$

Note that in the case of a squarefree level N (which is the only case treated in this thesis) the identity $\Gamma(N, f) = \Gamma_0\left(\frac{N}{f}\right)$ holds and Lemma 20.4 in Appendix C tells us that the cusps for this congruence subgroup are indeed parameterized through the divisors of $\frac{N}{f}$.

CHAPTER 2

Definition of Eisenstein series for $\Gamma_0(N)$

In this chapter the various Eisenstein series (corresponding to the double coset representatives from the previous chapter) which occur in the spectral decomposition of $\mathfrak{L}^2(\Gamma_0(N) \setminus \mathfrak{h}^3)$ are defined. In fact there are three different types of Eisenstein series, which contribute to the spectral decomposition, namely Eisenstein series twisted by Maass cusp forms, Eisenstein series twisted by a constant Maass form and minimal Eisenstein series. The well-definedness and the region of absolute convergence of these Eisenstein series was discussed in the introduction. We begin with the definition of Eisenstein series twisted by Maass cusp forms.

Definition 2.1. Let N be a positive squarefree integer and M a positive divisor of N. Define the following subgroups of $P_{2,1}$ and $P_{1,2}$ by

$$\Gamma(M, P_{2,1}) := \left\{ \begin{pmatrix} A & & * \\ & & * \\ & & 1 \end{pmatrix} \in P_{2,1} \;\middle|\; A \in \Gamma_0(M) \right\} ,$$

$$\Gamma(M, P_{1,2}) := \left\{ \begin{pmatrix} 1 & * & * \\ & A & \end{pmatrix} \in P_{1,2} \;\middle|\; A \in \Gamma_0(M) \right\} .$$

Further we shortly recall here the definition of the principal character and Dirichlet L-series, for a detailed treatment see [**16**].

Definition 2.2. Let N be a positive integer. Denote by χ_N the principal character modulo N defined by

$$\chi_N : \mathbb{Z} \longrightarrow \mathbb{C}, \quad n \mapsto \begin{cases} 1, & \text{if } (n, N) = 1 , \\ 0, & \text{if } (n, N) > 1 , \end{cases}$$

and with $L_{\chi_N}(s) := \sum_{n \geq 1} \chi_N(n) n^{-s}$ the associated Dirichlet series, which is absolutely convergent for $\mathfrak{Re}(s) > 1$.

Definition 2.3. Let N be a positive squarefree integer and f, M positive divisors of N. Let ϕ be a Maass cusp form with eigenvalue $\nu(\nu - 1)$ for the congruence subgroup $\Gamma_0(M) \subseteq SL_2(\mathbb{Z})$. The Eisenstein series twisted by ϕ for the maximal parabolic subgroup $P_{2,1}$, $P_{1,2}$, respectively, is defined by

$$E(z,s,f,\phi,P_{2,1}) := \sum_{\gamma\in\Lambda(f,M,P_{2,1})\backslash\Gamma_0(N)} \phi\left(\mathfrak{m}_{P_{2,1}}\left(\alpha_f^{-1}\gamma z\right)\right) I_{(s,-2s)}\left(\alpha_f^{-1}\gamma z, P_{2,1}\right) ,$$

$$E(z,s,f,\phi,P_{1,2}) := \sum_{\gamma\in\Lambda(f,M,P_{1,2})\backslash\Gamma_0(N)} \phi\left(\mathfrak{m}_{P_{1,2}}\left(\beta_f^{-1}\gamma z\right)\right) I_{(2s,-s)}\left(\beta_f^{-1}\gamma z, P_{1,2}\right) ,$$

where the stabilizers are defined by

$$\Lambda(f,M,P_{2,1}) := \alpha_f\Gamma(M,P_{2,1})\alpha_f^{-1} \cap \Gamma_0(N) ,$$

$$\Lambda(f,M,P_{1,2}) := \beta_f\Gamma(M,P_{1,2})\beta_f^{-1} \cap \Gamma_0(N) .$$

The associated completed Eisenstein series are defined by

$$G(z,s,f,\phi,P_{2,1}) := \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)E(z,s,f,\phi,P_{2,1}) ,$$

$$G(z,s,f,\phi,P_{1,2}) := \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)E(z,s,f,\phi,P_{1,2}) .$$

Next the analogous definition of Eisenstein series twisted by constant Maass forms is given.

Definition 2.4. Let N be a positive squarefree integer and f a positive divisor of N. The Eisenstein series twisted by a constant Maass form for the maximal parabolic subgroup $P_{2,1}$, $P_{1,2}$, respectively, is defined by

$$E(z,s,f,P_{2,1}) := \sum_{\gamma\in\Lambda(f,1,P_{2,1})\backslash\Gamma_0(N)} I_{(s,-2s)}\left(\alpha_f^{-1}\gamma z, P_{2,1}\right) ,$$

$$E(z,s,f,P_{1,2}) := \sum_{\gamma\in\Lambda(f,1,P_{1,2})\backslash\Gamma_0(N)} I_{(2s,-s)}\left(\beta_f^{-1}\gamma z, P_{1,2}\right) .$$

The associated completed Eisenstein series are defined by

$$G(z,s,f,P_{2,1}) := \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)E(z,s,f,P_{2,1}) ,$$

$$G(z,s,f,P_{1,2}) := \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)E(z,s,f,P_{1,2}) .$$

Note that the functions $I_{(s,-2s)}(*,P_{2,1})$, $I_{(2s,-s)}(*,P_{1,2})$, respectively, are invariant against left multiplication with elements in $P_{2,1}$, $P_{1,2}$, respectively. Eventually the definition of the minimal Eisenstein series is given.

DEFINITION 2.5. Let N be a squarefree positive integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. The minimal Eisenstein series associated to the "cusp" $\lambda_{f,h}$ is defined by the formula

$$E(z,s_1,s_2,f,h,P_{min}) := \sum_{\gamma\in\Lambda(f,h,P_{min})\backslash\Gamma_0(N)} I_{(s_1,s_2)}\left(\lambda_{f,h}^{-1}\gamma z\right) .$$

With the stabilizer $\Lambda(f,h,P_{min}) := \lambda_{f,h}P_{min}\lambda_{f,h}^{-1}\cap\Gamma_0(N)$ and the associated completed Eisenstein series given by the formula

$$G(z,s_1,s_2,f,h,P_{min}) := \frac{1}{4}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$

$$\zeta(3s_1)\,\zeta(3s_2)\,\zeta(3s_1+3s_2-1)\,E(z,s_1,s_2,f,h,P_{min}) .$$

At the end of this chapter a lemma is stated, which gives a slightly easier description of the stabilizers occuring in the twisted Eisenstein series.

LEMMA 2.6. *For the stabilizers of the twisted Eisenstein series the following equations*

$$\Lambda(f,M,P_{2,1}) = \alpha_f\Gamma\left(\frac{M}{\left(\frac{N}{f},M\right)},P_{2,1}\right)\alpha_f^{-1}\cap\Gamma_0(N) ,$$

and

$$\Lambda(f,M,P_{1,2}) = \beta_f\Gamma\left(\frac{M}{(f,M)},P_{1,2}\right)\beta_f^{-1}\cap\Gamma_0(N) .$$

hold.

PROOF. The inclusion "$\subset$" is obvious. Hence it remains to prove the other inclusion.

(1) For $p = \begin{pmatrix} a & b & x\\ c & d & y\\ & & 1\end{pmatrix} \in \Gamma\left(\frac{M}{\left(\frac{N}{f},M\right)},P_{2,1}\right)$ the equivalence

$$\alpha_f p\alpha_f^{-1}\in\Gamma_0(N)$$

$$\Longleftrightarrow e_3^T\begin{pmatrix}1 & & \\ & 1 & \\ & f & 1\end{pmatrix}\begin{pmatrix} a & b & x\\ c & d & y\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -f & 1\end{pmatrix} \equiv (0,0,*) \bmod N$$

$$\begin{aligned}
&\iff \begin{array}{rcll} cf & \equiv & 0 & \bmod N \\ f(d-fy)-f & \equiv & 0 & \bmod N \end{array} \\
&\iff \begin{array}{rcll} c & \equiv & 0 & \bmod \frac{N}{f} \\ fy & \equiv & d-1 & \bmod \frac{N}{f} \end{array}
\end{aligned}$$

is valid. From the first congruence one sees immediately that $\frac{M\frac{N}{f}}{\left(\frac{N}{f},M\right)} \mid c$ so $p \in \Lambda(f, M, P_{2,1})$.

(2) For $p = \begin{pmatrix} 1 & x & y \\ & a & b \\ & c & d \end{pmatrix} \in \Gamma\left(\frac{M}{(f,M)}, P_{1,2}\right)$ an analogous calculation is done. We start with

$$\begin{aligned}
&\beta_f p \beta_f^{-1} \in \Gamma_0(N) \\
\iff & e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -f & & 1 \end{pmatrix} \equiv (0,0,*) \bmod N \\
\iff & \begin{array}{rcll} f(1-fy)-df & \equiv & 0 & \bmod N \\ fx+c & \equiv & 0 & \bmod N\,. \end{array}
\end{aligned}$$

Reducing the last congruence modulo f gives $f \mid c$, which implies $\frac{Mf}{(f,M)} \mid c$ so again $p \in \Lambda(f, M, P_{1,2})$.

□

Part II

Bruhat decompositions

CHAPTER 3

Bruhat decompositions for $SL_3(\mathbb{Z})$

The goal of this chapter is to construct a set of representatives in Bruhat decomposition for the cosets which occur in the summation of the Eisenstein series. First we explicitly construct for a given coprime row, column, respectively, a matrix in $SL_3(\mathbb{Z})$ such that this matrix has the given row, column, respectively, as its last row, first column, respectively. This will give us an explicit set of coset representatives in Bruhat decomposition for $P_{2,1}/SL_3(\mathbb{Z})$, $P_{1,2}/SL_3(\mathbb{Z})$, respectively.

LEMMA 3.1. *Let (a, b, c) be a row vector of integers with coprime entries.*

(1) If $a \neq 0$ then every pair of integers r, s which satisfies

$$r\frac{b}{(a,b)} \equiv -1 \bmod \frac{a}{(a,b)} \quad \textit{and} \quad sc \equiv -1 \bmod (a,b) \ ,$$

defines a matrix in $SL_3(\mathbb{Z})$ with the above row as the last row by

$$\gamma_{(a,b,c)} := \begin{pmatrix} 1 & & \frac{r}{a} \\ & 1 & \frac{s}{(a,b)} \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \begin{pmatrix} a & b & c \\ & \frac{(a,b)}{a} & \frac{-rc}{a} \\ & & \frac{1}{(a,b)} \end{pmatrix} \in SL_3(\mathbb{Z}) \ .$$

(2) If $a = 0$, $b \neq 0$, then every integer t which satisfies

$$tc \equiv -1 \bmod b \ ,$$

defines a matrix in $SL_3(\mathbb{Z})$ with the row $(0, b, c)$ as the last row by

$$\gamma_{(0,b,c)} := \begin{pmatrix} 1 & & \\ & 1 & \frac{t}{b} \\ & & 1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & -b & -c \\ & & -\frac{1}{b} \end{pmatrix} \in SL_3(\mathbb{Z}) \ .$$

(3) If $a = 0$, $b = 0$ then a matrix in $SL_3(\mathbb{Z})$ with the row $(0, 0, c) = (0, 0, \pm 1)$ as the last row is defined by

$$\gamma_{(0,0,\pm1)} := \begin{pmatrix} \pm1 & & \\ & 1 & \\ & & \pm1 \end{pmatrix} .$$

PROOF. First note that the two congruences in r, s in the first part can always be solved even if $c = 0$ or $b = 0$, because in this case the gcd-condition on the row implies $(a, b) = 1$, $\frac{a}{(a,b)} = 1$, respectively. An analogous argument is valid for the second part. The determinant multiplication formula implies that the matrix $\gamma_{(a,b,c)}$ has determinant 1 in all three cases. Hence it remains to check that all entries are integers and the last row is (a, b, c). This is done through a short calculation, which gives us the formulas

$$\gamma_{(a,b,c)} = \begin{pmatrix} r & \frac{br+(a,b)}{a} & 0 \\ \frac{as}{(a,b)} & \frac{bs}{(a,b)} & \frac{cs+1}{(a,b)} \\ a & b & c \end{pmatrix} \quad \text{and} \quad \gamma_{(0,b,c)} = \begin{pmatrix} -1 & & \\ & t & \frac{ct+1}{b} \\ & b & c \end{pmatrix} .$$

□

For further calculations we need that the integers r, s in the above lemma can be chosen in a way that they satisfy certain additional gcd-conditions.

LEMMA 3.2. *Let N be squarefree, f a positive divisor of N and (fa, fb, c) a row vector of integers with coprime entries.*

(1) If $a \neq 0$ then the integers r, s in Lemma 3.1 part (1) can be chosen in a way such that the following gcd-conditions hold.

(a) The integer r is coprime to $N(a, b)$.

(b) The integer s is coprime to $\frac{N}{f}$.

(2) If $a = 0$ and $b \neq 0$, then the integer t in Lemma 3.1 part (2) can be chosen in a way such that t is coprime to $\frac{N}{f}$.

PROOF. The proof of this lemma is done by the use of elementary modular arithmetics.

(1) (a) Choose integers r_1, t_1 such that $\frac{fb}{(fa,fb)} r_1 + 1 = t_1 \frac{fa}{(fa,fb)}$. Applying Lemma 1.1 with $d_1 = r_1$, $d_2 = \frac{a}{(a,b)}$ and $N = N(a, b)$ gives the existence of an integer w such that $\left(r_1 + w\frac{a}{(a,b)}, N(a, b)\right) = 1$. Hence choose $r := r_1 + w\frac{a}{(a,b)}$ and note that the equation

$$\frac{fb}{(fa, fb)} r + 1 = \left(t_1 + w\frac{fb}{(fa, fb)}\right) \frac{fa}{(fa, fb)}$$

holds.

(b) Choose integers s_1, t_1 such that $cs_1 + 1 = t_1(fa, fb)$. Applying Lemma 1.1 with $d_1 = s_1$, $d_2 = (fa, fb)$ and $N = \frac{N}{f}$ gives the existence of an integer w such that $\left(s_1 + w(fa, fb), \frac{N}{f}\right) = 1$. Hence choose $s := s_1 + w(fa, fb)$ and note that the equation

$$cs + 1 = (t_1 + cw)(fa, fb)$$

holds.

(2) Choose integers t_1, p_1 such that $ct_1 + 1 = p_1 fb$. Applying Lemma 1.1 with $d_1 = t_1$, $d_2 = fb$ and $N = \frac{N}{f}$ gives the existence of an integer w such that $\left(t_1 + wfb, \frac{N}{f}\right) = 1$. Hence choose $t := t_1 + wfb$ and note that the equation

$$ct + 1 = (p_1 + cw)fb$$

holds.

□

The next analogous calculations are performed for the case of a column vector of integers with coprime entries. First an analogous result to Lemma 3.1 is stated, but note that we need the inverse of the constructed matrix here.

Lemma 3.3. *Let $(a, b, c)^T$ be a column vector of integers with coprime entries.*

(1) If $a \neq 0$ then every pair of integers r, s which satisfies

$$r\frac{b}{(a,b)} \equiv -1 \bmod \frac{a}{(a,b)} \quad \textit{and} \quad sc \equiv -1 \bmod (a,b) \ ,$$

defines a matrix in $SL_3(\mathbb{Z})$ with the column $(c, b, a)^T$ as the first column by

$$\delta_{(a,b,c)} := \begin{pmatrix} \frac{1}{(a,b)} & -\frac{cr}{a} & c \\ & \frac{(a,b)}{a} & b \\ & & a \end{pmatrix} \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{(a,b)} & \frac{r}{a} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in SL_3(\mathbb{Z}) \ .$$

The inverse is given explicitly by

$$\begin{aligned} \delta^{-1}_{(a,b,c)} &= \begin{pmatrix} 1 & -\frac{s}{(a,b)} & -\frac{r}{a} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} (a,b) & cr & -c\frac{br+(a,b)}{a} \\ & \frac{a}{(a,b)} & -\frac{b}{(a,b)} \\ & & \frac{1}{a} \end{pmatrix} \\ &= \begin{pmatrix} -s & -r\frac{cs+1}{(a,b)} & \frac{bcrs+cs(a,b)+br+(a,b)}{a(a,b)} \\ (a,b) & cr & -c\frac{br+(a,b)}{a} \\ 0 & \frac{a}{(a,b)} & -\frac{b}{(a,b)} \end{pmatrix} . \end{aligned}$$

(2) If $a = 0$, $b \neq 0$, then every integer t which satisfies $tc \equiv 1 \bmod b$ defines a matrix in $SL_3(\mathbb{Z})$ with the column $(c, b, 0)^T$ as the first column by

$$\delta_{(0,b,c)} := \begin{pmatrix} \frac{1}{b} & c & \\ & b & \\ & & 1 \end{pmatrix} \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{b} & \\ & 1 & \\ & & 1 \end{pmatrix} \in SL_3(\mathbb{Z}) \, .$$

The inverse is given explicitly by

$$\delta^{-1}_{(0,b,c)} = \begin{pmatrix} 1 & -\frac{t}{b} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} b & -c & \\ & \frac{1}{b} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} t & \frac{1-ct}{b} & \\ -b & c & \\ & & 1 \end{pmatrix} .$$

(3) If $a = 0$, $b = 0$ then a matrix in $SL_3(\mathbb{Z})$ with the column $(c, 0, 0)^T = (\pm 1, 0, 0)^T$ as the first column is defined by

$$\delta_{(0,0,\pm 1)} := \begin{pmatrix} \pm 1 & & \\ & 1 & \\ & & \pm 1 \end{pmatrix} .$$

Proof. The proof is essentially the same as in Lemma 3.1. Again all congruences are solvable and the determinant multiplication formula implies that the matrix $\delta_{(a,b,c)}$ has determinant 1 in all three cases. Also again we check that all entries are integers and the first column is $(c, b, a)^T$:

$$\delta_{(a,b,c)} = \begin{pmatrix} c & \frac{cs+1}{(a,b)} & 0 \\ b & \frac{b}{(a,b)}s & \frac{rb+(a,b)}{a} \\ a & \frac{a}{(a,b)}s & r \end{pmatrix} \quad \text{and} \quad \delta_{(0,b,c)} = \begin{pmatrix} c & \frac{ct-1}{b} & \\ b & t & \\ & & 1 \end{pmatrix} .$$

The explicit formulas for the inverse matrices are checked through a trivial calculation. □

In calculations later we need that the integer r in the definition of $\delta_{(a,b,c)}$ satisfies a gcd-condition similarly to that in Lemma 3.2.

Lemma 3.4. *Let N be squarefree, f a positive divisor of N and (fa, b, c) a row vector of integers with coprime entries and with $a \neq 0$. The integer r in Lemma 3.3 part (1) can be chosen in a way such that $(r, (fa, b)) = 1$ holds.*

Proof. Choose an integer r such that

$$r\frac{b}{(fa,b)} \equiv -1 \bmod \frac{fa}{(fa,b)} .$$

Next applying Lemma 1.1 with $d_1 = r$, $d_2 = \frac{fa}{(fa,b)}$ and $N = (fa, b)$ gives the existence of an integer s such that $\left(r + s\frac{fa}{(fa,b)}, (fa,b)\right) = 1$. Passing from r to $r + s\frac{fa}{(fa,b)}$ does not change the above congruence.

□

In order to derive an explicit set of coset representatives in Bruhat decomposition for $P_{min}/SL_3(\mathbb{Z})$ we first state a Bruhat decomposition for $P_{min}/SL_2(\mathbb{Z})$.

Lemma 3.5. *Let (c, d) be a row vector of integers with coprime entries. For $c \neq 0$ every integer t which satisfies $td \equiv 1 \bmod c$ defines a matrix in $SL_2(\mathbb{Z})$ with the above row as the last row by*

$$\tau_{(c,d)} := \begin{pmatrix} 1 & \frac{t}{c} \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} c & d \\ & \frac{1}{c} \end{pmatrix} .$$

For $c = 0$ a matrix in $SL_2(\mathbb{Z})$ with the row $(0, d) = (0, \pm 1)$ as the last row is defined by

$$\tau_{(0,\pm 1)} := \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix} .$$

Further the matrices $\tau_{(c,d)}$ build a set of right coset representatives for $P_{min}/SL_2(\mathbb{Z})$.

Proof. A straightforward calculation shows that $\tau_{(c,d)} = \begin{pmatrix} t & \frac{dt-1}{c} \\ c & d \end{pmatrix}$ has integer entries. In conjunction with the determinant multiplication formula this gives the first claim. Hence it remains to prove that these matrices build a set of right coset representatives for $P_{min}/SL_2(\mathbb{Z})$. Since we have the equivalence of the conditions $P_{min}\tau_{(c_1,d_1)} = P_{min}\tau_{(c_2,d_2)}$ and $(c_1, d_1) = e_2^T \tau_{(c_1,d_1)} = e_2^T \tau_{(c_2,d_2)} = (c_2, d_2)$ we get the disjointedness of the corresponding left cosets. Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If $c = 0$ the determinant condition gives us $a = d = \pm 1$ hence γ is equivalent to $\tau_{(0,\pm 1)}$. If $c \neq 0$ a short calculation shows that γ is equivalent to $\tau_{(c,d)}$:

$$\gamma\tau_{(c,d)}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -\frac{dt-1}{c} \\ -c & t \end{pmatrix} = \begin{pmatrix} ad - bc & * \\ 0 & -c\frac{dt-1}{c} + dt \end{pmatrix} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \in P_{min} .$$

□

An easy corollary is the unique decomposition of elements in $SL_3(\mathbb{Z})$ as a product of an element of $P_{2,1}$, $P_{1,2}$, respectively, and a matrix of the type in Lemma 3.1, Lemma 3.3, respectively, and also a slightly more complicated decomposition for P_{min}.

COROLLARY 3.6. *Let* $\gamma = (\gamma_{i,j})_{i,j=1,2,3} \in SL_3(\mathbb{Z})$.

(1) There exists a unique row vector $(a,b,c) \in \mathbb{Z}^3$ *with coprime entries and a unique element* $p \in P_{2,1}$ *such that the decomposition* $\gamma = p\gamma_{(a,b,c)}$ *holds. Also there exists a unique column vector* $(a,b,c)^T \in \mathbb{Z}^3$ *with coprime entries and a unique element* $p \in P_{1,2}$ *such that the decomposition* $\gamma = \delta_{(a,b,c)}p$ *holds.*

(2) There exists a unique row vector $(a,b,c) \in \mathbb{Z}^3$ *with coprime entries and a unique element* $p \in P_{1,2}$ *such that the decomposition* $\gamma = p\delta^{-1}_{(a,b,c)}$ *holds.*

(3) There exist unique row vectors $(a,b,c) \in \mathbb{Z}^3$, $(e,f) \in \mathbb{Z}^2$ *with coprime entries and a unique element* $p \in P_{min}$ *such that the decomposition* $\gamma = p\begin{pmatrix} \tau_{(e,f)} & \\ & 1 \end{pmatrix}\gamma_{(a,b,c)}$ *holds.*

PROOF. (1) We only give the proof for $P_{2,1}$ since the proof for $P_{1,2}$ is essentially the same. Since γ and $\gamma_{(\gamma_{3,1},\gamma_{3,2},\gamma_{3,3})}$ have the same last row, $\gamma\gamma^{-1}_{(\gamma_{3,1},\gamma_{3,2},\gamma_{3,3})} \in P_{2,1}$ holds, which gives the existence. For the uniqueness assume $p_1\gamma_{(a_1,b_1,c_1)} = p_2\gamma_{(a_2,b_2,c_2)}$ with certain elements $p_1, p_2 \in P_{2,1}$. Using the fact $e_3^T p_i = e_3$ gives immediately that the two rows (a_i,b_i,c_i) are equal. Since $\gamma_{(a_i,b_i,c_i)}$ is invertible $p_1 = p_2$ follows and the proof is complete.

(2) Use part (1) for γ^{-1}, hence $\gamma^{-1} = \delta_{(a,b,c)}p$ with a suitable element $p \in P_{1,2}$. Taking inverses gives $\gamma = p^{-1}\delta_{(a,b,c)}$ with $p^{-1} \in P_{1,2}$. Since inversion is an involution part (1) also gives us uniqueness.

(3) Part (1) implies the decomposition $\gamma = p\gamma_{(a,b,c)}$ with $p = \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}\begin{pmatrix} A & \\ & 1 \end{pmatrix} \in P_{2,1}$. Decomposing $A = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\tau_{(e,f)}$ using Lemma 3.5 gives the existence. For the uniqueness assume $p_1\begin{pmatrix} \tau_{(e_1,f_1)} & \\ & 1 \end{pmatrix}\gamma_{(a_1,b_1,c_1)} = p_2\begin{pmatrix} \tau_{(e_2,f_2)} & \\ & 1 \end{pmatrix}\gamma_{(a_2,b_2,c_2)}$ with certain elements $p_1, p_2 \in P_{min}$. Part (1) gives immediately that the two rows (a_i,b_i,c_i) are equal. Since $p_1, p_2 \in P_{min}$ the uniqueness in Lemma 3.5 applied to the upper left 2×2 matrices in $p_1\begin{pmatrix} \tau_{(e_1,f_1)} & \\ & 1 \end{pmatrix} = p_2\begin{pmatrix} \tau_{(e_2,f_2)} & \\ & 1 \end{pmatrix}$ gives immediately that the two rows (e_i,f_i) are equal. Finally $p_1 = p_2$ follows and the proof is complete.

□

The reason why we constructed these matrices is that we want to give an explicit description of the cosets which occur in the summation of the Eisenstein series. The upcoming lemma shows that the parametrization over the last row, first column, respectively, is a suitable choice for this task.

LEMMA 3.7. *Let N be a positive squarefree integer, f, M positive divisors of N and h a positive divisor of $\frac{N}{f}$.*

(1) *Let $p_i\gamma_{(a_i,b_i,c_i)} \in \alpha_f^{-1}\Gamma_0(N)$ for $i = 1, 2$ with elements $p_i \in P_{2,1}$ and row vectors (a_i, b_i, c_i) with coprime entries. Then the cosets $\Lambda(f, M, P_{2,1})\alpha_f p_i \gamma_{(a_i,b_i,c_i)}$ are equal if and only if the rows (a_i, b_i, c_i) are equal and the cosets $\Gamma_0\left(\frac{M}{\left(\frac{N}{f},M\right)}\right)\mathfrak{m}_{P_{2,1}}(p_i)$ are equal .*

(2) *Let $p_i\delta^{-1}_{(a_i,b_i,c_i)} \in \beta_f^{-1}\Gamma_0(N)$ for $i = 1, 2$ with elements $p_i \in P_{1,2}$ and column vectors $(a_i, b_i, c_i)^T$ with coprime entries. Then the cosets $\Lambda(f, M, P_{1,2})\beta_f p_i \delta^{-1}_{(a_i,b_i,c_i)}$ are equal if and only if the columns $(a_i, b_i, c_i)^T$ are equal and the cosets $\Gamma_0\left(\frac{M}{(f,M)}\right)\mathfrak{m}_{P_{1,2}}(p_i)$ are equal.*

(3) *Let $p_i\begin{pmatrix}\tau_{(x_i,y_i)} & \\ & 1\end{pmatrix}\gamma_{(a_i,b_i,c_i)} \in \lambda^{-1}_{f,h}\Gamma_0(N)$ for $i = 1, 2$ with elements $p_i \in P_{min}$ and row vectors (a_i, b_i, c_i), (x_i, y_i) with coprime entries. Then the cosets $\Lambda(f, h, P_{min})\lambda_{f,h}p_i\begin{pmatrix}\tau_{(x_i,y_i)} & \\ & 1\end{pmatrix}\gamma_{(a_i,b_i,c_i)}$ are equal if and only if the rows (a_i, b_i, c_i) as well as the rows (x_i, y_i) are equal.*

PROOF. (1) For the "if" direction assume $(a_1, b_1, c_1) = (a_2, b_2, c_2)$ and $\Gamma_0\left(\frac{M}{\left(\frac{N}{f},M\right)}\right)\mathfrak{m}_{P_{2,1}}(p_1) = \Gamma_0\left(\frac{M}{\left(\frac{N}{f},M\right)}\right)\mathfrak{m}_{P_{2,1}}(p_2)$. A short calculation

$$\underbrace{(\alpha_f p_1\gamma_{(a_1,b_1,c_1)})}_{\in\Gamma_0(N)}\underbrace{(\alpha_f p_2\gamma_{(a_2,b_2,c_2)})^{-1}}_{\in\Gamma_0(N)} = \alpha_f p_1 p_2^{-1}\alpha_f^{-1} \in \alpha_f\Gamma\left(\frac{M}{\left(\frac{N}{f},M\right)}, P_{2,1}\right)\alpha_f^{-1}\cap\Gamma_0(N) = \Lambda(f, M, P_{2,1}) ,$$

shows the equality of the cosets $\Lambda(f, M, P_{2,1})\alpha_f p_i\gamma_{(a_i,b_i,c_i)}$. In the last step Lemma 2.6 part (1) is used. For the "only if" direction assume the two cosets $\Lambda(f, M, P_{2,1})\alpha_f p_i\gamma_{(a_i,b_i,c_i)}$ are equal. Again a short calculation gives the equality of the rows (a_i, b_i, c_i):

$$\begin{aligned}\gamma_{(a_1,b_1,c_1)}\gamma^{-1}_{(a_2,b_2,c_2)} &= (\alpha_f p_1)^{-1}(\alpha_f p_1\gamma_{(a_1,b_1,c_1)})(\alpha_f p_2\gamma_{(a_2,b_2,c_2)})^{-1}(\alpha_f p_2)\\ &\in p_1^{-1}(\alpha_f^{-1}\Lambda(f, M, P_{2,1})\alpha_f)p_2 \subset p_1^{-1}\Gamma\left(\frac{M}{\left(\frac{N}{f},M\right)}, P_{2,1}\right)p_2 \subset P_{2,1} .\end{aligned}$$

In the last step Lemma 2.6 part (1) is used again. So there exists an element $p \in P_{2,1}$ such that $p\gamma_{(a_1,b_1,c_1)} = \gamma_{(a_2,b_2,c_2)}$ which implies $(a_1, b_1, c_1) = (a_2, b_2, c_2)$. With this result in mind the claimed equality of the cosets can be proved by

$$\begin{aligned}\mathfrak{m}_{P_{2,1}}(p_1)\,\mathfrak{m}_{P_{2,1}}(p_2)^{-1} &= \mathfrak{m}_{P_{2,1}}\left(p_1p_2^{-1}\right) = \mathfrak{m}_{P_{2,1}}\left(\alpha_f^{-1}\left(\alpha_f p_1\gamma_{(a_1,b_1,c_1)}\right)\left(\alpha_f p_2\gamma_{(a_2,b_2,c_2)}\right)^{-1}\alpha_f\right)\\ &\in \mathfrak{m}_{P_{2,1}}\left(\alpha_f^{-1}\Lambda(f,M,P_{2,1})\alpha_f\right) \subset \Gamma_0\left(\frac{M}{\left(\frac{N}{f},M\right)}\right) .\end{aligned}$$

(2) The proof of the second part is analogous to the proof of the first part using Lemma 2.6 part (2).

(3) For the "if" direction assume $(a_1, b_1, c_1) = (a_2, b_2, c_2)$ and $(x_1, y_1) = (x_2, y_2)$. A short calculation

$$\begin{aligned}&\underbrace{\lambda_{f,h}p_1\begin{pmatrix}\tau_{(x_1,y_1)} & \\ & 1\end{pmatrix}\gamma_{(a_1,b_1,c_1)}}_{\in\Gamma_0(N)}\underbrace{\left(\lambda_{f,h}p_2\begin{pmatrix}\tau_{(x_2,y_2)} & \\ & 1\end{pmatrix}\gamma_{(a_2,b_2,c_2)}\right)^{-1}}_{\in\Gamma_0(N)}\\ &= \lambda_{f,h}p_1p_2^{-1}\lambda_{f,h}^{-1} \in \lambda_{f,h}P_{min}\lambda_{f,h}^{-1} \cap \Gamma_0(N) = \Lambda(f,h,P_{min}) .\end{aligned}$$

shows the equality of the cosets $\Lambda(f,h,P_{min})\lambda_{f,h}p_i\begin{pmatrix}\tau_{(x_i,y_i)} & \\ & 1\end{pmatrix}\gamma_{(a_i,b_i,c_i)}$. For the "only if" direction assume the two cosets $\Lambda(f,h,P_{min})\lambda_{f,h}p_i\begin{pmatrix}\tau_{(x_i,y_i)} & \\ & 1\end{pmatrix}\gamma_{(a_i,b_i,c_i)}$ are equal. Again a short calculation gives the pairwise equality of the rows (a_i, b_i, c_i) and (x_i, y_i):

$$\begin{aligned}&\begin{pmatrix}\tau_{(x_1,y_1)} & \\ & 1\end{pmatrix}\gamma_{(a_1,b_1,c_1)}\left(\begin{pmatrix}\tau_{(x_2,y_2)} & \\ & 1\end{pmatrix}\gamma_{(a_2,b_2,c_2)}\right)^{-1}\\ =&(\lambda_{f,h}p_1)^{-1}\left(\lambda_{f,h}p_1\begin{pmatrix}\tau_{(x_1,y_1)} & \\ & 1\end{pmatrix}\gamma_{(a_1,b_1,c_1)}\right)\left(\lambda_{f,h}p_2\begin{pmatrix}\tau_{(x_2,y_2)} & \\ & 1\end{pmatrix}\gamma_{(a_2,b_2,c_2)}\right)^{-1}(\lambda_{f,h}p_2)\\ &\in p_1^{-1}(\lambda_{f,h}^{-1}\Lambda(f,h,P_{min})\lambda_{f,h})p_2 \subset p_1^{-1}P_{min}p_2 = P_{min} .\end{aligned}$$

With the same argument as in part (1) the equality of the rows (a_i, b_i, c_i) is obtained, after that using Lemma 3.5 gives the equality of the rows (x_i, y_i).

□

At the end of this chapter we collect the elements of the cosets in the summation of the Eisenstein series into sets concerning the Weyl element in their Bruhat decomposition.

DEFINITION 3.8. Let N be a positive squarefree integer, f, M positive divisors of N and h a positive divisor of $\frac{N}{f}$. Decompose each of the sets in the summation of the Eisenstein series in the sets Γ_i defined below, according to their Bruhat decomposition. Of course theses decompositions are not unique.

(1) Let

$$\dot{\bigcup_{1\le i\le 3}} \alpha_f \Gamma_i(f, M, P_{2,1})$$

be a system of left coset representatives for $\mathit{\Lambda}(f, M, P_{2,1}) \setminus \Gamma_0(N)$ with

$$\Gamma_1(f, M, P_{2,1}) \subseteq \left\{ p\gamma_{(a,b,c)} \middle| {}^{p\in P_{2,1}}_{a\neq 0} \right\} ,$$
$$\Gamma_2(f, M, P_{2,1}) \subseteq \left\{ p\gamma_{(0,b,c)} \middle| {}^{p\in P_{2,1}}_{b\neq 0} \right\} ,$$
$$\Gamma_3(f, M, P_{2,1}) \subseteq \left\{ p\gamma_{(0,0,\pm 1)} \middle| p \in P_{2,1} \right\} .$$

(2) Let

$$\dot{\bigcup_{1\le i\le 3}} \beta_f \Gamma_i(f, M, P_{1,2})$$

be a system of left coset representatives for $\mathit{\Lambda}(f, M, P_{1,2}) \setminus \Gamma_0(N)$ with

$$\Gamma_1(f, M, P_{1,2}) \subseteq \left\{ p\delta^{-1}_{(a,b,c)} \middle| {}^{p\in P_{1,2}}_{a\neq 0} \right\} ,$$
$$\Gamma_2(f, M, P_{1,2}) \subseteq \left\{ p\delta^{-1}_{(0,b,c)} \middle| {}^{p\in P_{1,2}}_{b\neq 0} \right\} ,$$
$$\Gamma_3(f, M, P_{1,2}) \subseteq \left\{ p\delta^{-1}_{(0,0,\pm 1)} \middle| p \in P_{1,2} \right\} .$$

(3) Let

$$\dot{\bigcup_{1\le i\le 6}} \lambda_{f,h} \Gamma_i(f, h, P_{min})$$

be a system of left coset representatives for $\mathit{\Lambda}(f, h, P_{min}) \setminus \Gamma_0(N)$ with

$$\Gamma_1(f, h, P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(a,b,c)} \middle| {}^{p\in P_{min}}_{\substack{a\neq 0 \\ d\neq 0}} \right\} ,$$
$$\Gamma_2(f, h, P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(0,\pm 1)} & \\ & 1 \end{pmatrix} \gamma_{(a,b,c)} \middle| {}^{p\in P_{min}}_{a\neq 0} \right\} ,$$
$$\Gamma_3(f, h, P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,b,c)} \middle| {}^{p\in P_{min}}_{\substack{b\neq 0 \\ d\neq 0}} \right\} ,$$

$$\Gamma_4(f,h,P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(0,\pm 1)} & \\ & 1 \end{pmatrix} \gamma_{(0,b,c)} \middle| {}^{p\in P_{min}}_{b\neq 0} \right\} ,$$
$$\Gamma_5(f,h,P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,0,\pm 1)} \middle| {}^{p\in P_{min}}_{d\neq 0} \right\} ,$$
$$\Gamma_6(f,h,P_{min}) \subseteq \left\{ p \begin{pmatrix} \tau_{(0,\pm 1)} & \\ & 1 \end{pmatrix} \gamma_{(0,0,\pm 1)} \middle| p \in P_{min} \right\} .$$

CHAPTER 4

Calculation of $\Gamma_i(f, M, P_{2,1})$

In this chapter we calculate a system of coset representatives for $\Lambda(f, M, P_{2,1})\backslash\Gamma_0(N)$ and also determine the corresponding sets $\Gamma_i(f, M, P_{2,1})$ in Definition 3.8. First we state an easy technical lemma which is needed in further calculations.

LEMMA 4.1. *Let N, M be positive squarefree integers and a an integer. There exist integers x, y, z with $(x, y) = 1$ such that the congruence*

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv z \begin{pmatrix} 0 \\ a \end{pmatrix} \bmod N \tag{4.1}$$

holds if and only if $(a, N) = 1$.

In the case of the existence of a solution, for any row vector (c, d) with coprime entries such that $d \mid \frac{M}{(M,N)}$ and $0 < c \leq \frac{\frac{M}{(M,N)}}{d}$, one can choose a solution in a way such that a suitable completed matrix $\begin{pmatrix} * & * \\ x & y \end{pmatrix} \in SL_2(\mathbb{Z})$ has the following three properties.

(1) The matrices $\begin{pmatrix} * & * \\ x & y \end{pmatrix}$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ lie in the same left coset of $\Gamma_0\left(\frac{M}{(M,N)}\right) \backslash SL_2(\mathbb{Z})$.

(2) The matrices $\begin{pmatrix} * & * \\ x & y \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ (M,N)\left(c, \frac{M}{(N,M)d}\right) & 1 \end{pmatrix}$ lie in the same double coset of $\Gamma_0(M)\diagdown SL_2(\mathbb{Z})\diagup P_{min}$. Further there exist a decomposition

$$\begin{pmatrix} * & * \\ x & y \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 \\ (M,N)\left(c, \frac{M}{(N,M)d}\right) & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

with $\gamma \in \Gamma_0(M)$ and the integer α depends only on the integers c, d, M, N modulo $\frac{M}{(M,N)\left(c, \frac{M}{(N,M)d}\right)}$.

PROOF. Since the integers x, y must be coprime the *gcd*-condition $(a, N) = 1$ is obviously necessary, so it remains to prove that it is sufficient. We do this by constructing a solution which also fulfills the two properties described above. The squarefreeness of M guarantees that the integers dN and $\frac{M}{(M,N)d}$ are coprime. So choose an integer $\tilde{N}$ such that $\tilde{N}N \equiv 1 \bmod \frac{M}{(M,N)d}$ holds, then choose an integer k such that the integer

$$\tilde{N}c + k \cdot \frac{M}{(M,N)\,d} \tag{4.2}$$

is coprime to d. Now define the integers x, y and z by

$$x := N\left(\tilde{N}c + k \cdot \frac{M}{(M,N)\,d}\right) , \tag{4.3}$$

$$y := d \tag{4.4}$$

and

$$z := d\tilde{a} \tag{4.5}$$

with $a\tilde{a} \equiv 1 \bmod N$.

We first show that the integers x and y are coprime. In order to fulfill this task we use the coprimeness of the integer in (4.2) and d. We have

$$(x, y) = \left(N\left(\tilde{N}c + k \cdot \frac{M}{(M,N)\,d}\right), d\right) = (N, d) = 1 \ .$$

Note that the squarefreeness of M and the divisibility condition $d \mid \frac{M}{(M,N)}$ imply $(N, d) = 1$. Next we show that the integers x, y and z fulfill the congruence (4.1). We have

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} N\left(\tilde{N}c + k \cdot \frac{M}{(M,N)d}\right) \\ d \end{pmatrix} \equiv \begin{pmatrix} 0 \\ d(\tilde{a}a) \end{pmatrix} \equiv z \begin{pmatrix} 0 \\ a \end{pmatrix} \bmod N \ .$$

Now we use Lemma 1.2 to show that the integers x, y also satisfy the claimed properties:

$$\begin{aligned}
& \Gamma_0\left(\frac{M}{(M,N)}\right)\begin{pmatrix} * & * \\ x & y \end{pmatrix} = \Gamma_0\left(\frac{M}{(M,N)}\right)\begin{pmatrix} * & * \\ c & d \end{pmatrix} \\
\Longleftrightarrow \quad & \begin{pmatrix} * & * \\ x & y \end{pmatrix}\begin{pmatrix} * & * \\ c & d \end{pmatrix}^{-1} \in \Gamma_0\left(\frac{M}{(M,N)}\right) \\
\Longleftrightarrow \quad & e_2^T \begin{pmatrix} * & * \\ x & y \end{pmatrix}\begin{pmatrix} d & * \\ -c & * \end{pmatrix} e_1 \equiv 0 \bmod \frac{M}{(M,N)} \\
\Longleftrightarrow \quad & dx - cy \equiv 0 \bmod \frac{M}{(M,N)} \\
\Longleftrightarrow \quad & dN\left(\tilde{N}c + k \cdot \frac{M}{(M,N)\,d}\right) - cd \equiv 0 \bmod \frac{M}{(M,N)} \\
\Longleftrightarrow \quad & \left(N\tilde{N} - 1\right) c \equiv 0 \bmod \frac{M}{(M,N)\,d} \ .
\end{aligned}$$

Obvious the last congruence is true. In order to check the second property we first calculate the gcd

$$\begin{aligned}(x, M) &= (x, (M, N)) \left(x, \frac{M}{(M,N)\,d}\right)(x, d)\\ &= (M, N) \left(N\left(\tilde{N}c + k \cdot \frac{M}{(M,N)\,d}\right), \frac{M}{(M,N)\,d}\right)\\ &= (M, N)\left(c, \frac{M}{(M,N)\,d}\right) \; . \end{aligned} \tag{4.6}$$

In the above splitting we used the squarefreeness of M. Finally we can prove the second property. The following equivalence transformations

$$\begin{aligned} & \Gamma_0\,(M) \begin{pmatrix} * & * \\ x & y \end{pmatrix} P_{min} = \Gamma_0\,(M) \begin{pmatrix} 1 & 0 \\ (x, M) & 1 \end{pmatrix} P_{min}\\ \Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad & \begin{pmatrix} * & * \\ x & y \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (x, M) & 1 \end{pmatrix}^{-1} \in \Gamma_0(M)\\ \Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad & x\,(1 - (x, M)\,\alpha) - (x, M)\,y \equiv 0 \bmod M\\ \Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad & \frac{x}{(x, M)} - x\alpha - d \equiv 0 \bmod \frac{M}{(x, M)} \end{aligned}$$

hold. The squarefreeness of M gives that x is a unit modulo $\frac{M}{(x,M)}$, hence the last congruence is solvable. The explicit formula in (4.6) finishes the proof of the second property. $\square$

Now we are well prepared to start our calculations.

Lemma 4.2. *Let N be a squarefree integer and f, M positive divisors of N. A possible choice of the sets $\Gamma_i(f, M, P_{2,1})$ is given by*

(1)

$$\Gamma_1(f, M, P_{2,1}) = \left\{ \begin{pmatrix} p_{(a,b,c,d,e)} & * \\ & * \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \;\middle|\; \begin{matrix} (fa,fb,c)=1 \\ a\neq 0 \\ \left(a,b,\frac{N}{f}\right)=1 \\ e|(f,M) \\ 0<d\leq \frac{(f,M)}{e} \\ (d,e)=1 \end{matrix} \right\} .$$

(2) The set $\Gamma_2(f, M, P_{2,1})$ is given by

$$\Gamma_2(f, M, P_{2,1}) = \left\{ \begin{pmatrix} p_{(0,b,c,d,e)} & * \\ & * \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} \;\middle|\; \begin{matrix} (0,fb,c)=1 \\ b\neq 0 \\ \left(b,\frac{N}{f}\right)=1 \\ e|(f,M) \\ 0<d\leq \frac{(f,M)}{e} \\ (d,e)=1 \end{matrix} \right\} ,$$

(3) The set $\Gamma_3(f, M, P_{2,1})$ is given by

$$\Gamma_3(f,M,P_{2,1}) = \begin{cases} \varnothing & \text{if } f \neq N\,, \\ \left\{ \begin{pmatrix} p_{(0,0,\pm1,d,e)} & & * \\ & & * \\ & & 1 \end{pmatrix} \gamma_{(0,0,\pm 1)} \middle| \begin{smallmatrix} e \mid M \\ 0<d\leq \frac{M}{e} \\ (d,e)=1 \end{smallmatrix} \right\} & \text{if } f = N\,, \end{cases}$$

with $p_{(a,b,c,d,e)} \in SL_2(\mathbb{Z})$ satisfying the following three properties.

(1) The matrices $p_{(a,b,c,d,e)}$ and $\begin{pmatrix} * & * \\ d & e \end{pmatrix}$ lie in the same right coset of $\Gamma_0((f,M)) \setminus SL_2(\mathbb{Z})$.

(2) The matrices $p_{(a,b,c,d,e)}$ and $\begin{pmatrix} 1 & 0 \\ \left(M, \frac{N}{f}\right)\left(d, \frac{(f,M)}{e}\right) & 1 \end{pmatrix}$ lie in the same double coset of $\Gamma_0(M) \diagdown SL_2(\mathbb{Z}) \diagup P_{min}$. Further there exist a decomposition

$$p_{(a,b,c,d,e)} = \gamma \begin{pmatrix} 1 & 0 \\ \left(M, \frac{N}{f}\right)\left(d, \frac{(f,M)}{e}\right) & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

with $\gamma \in \Gamma_0(M)$ and the integer α depends only on the integers $d, e, M, \frac{N}{f}$ modulo $\frac{(M,f)}{\left(d, \frac{(f,M)}{e}\right)}$.

Proof. (1) For our calculations we use part (1) in Corollary 3.6 and Lemma 3.7 and determine which last rows occur in $\Gamma_1(f, M, P_{2,1})$. Reducing modulo f immediately gives that the last rows must have the form (fa, fb, c), so it remains to investigate which additional conditions must be fulfilled. This is done by the equivalence transformations

$$\exists p \in P_{2,1}: \quad p\gamma_{(fa,fb,c)} \in \alpha_f^{-1}\Gamma_0(N)$$

$$\begin{aligned}
\iff\ & \exists p \in P_{2,1}: \quad e_3^T \alpha_f p \gamma_{(fa,fb,c)} \equiv (0,0,*) \bmod N \\
\iff\ & \exists x,y,z \in \mathbb{Z}, (x,y)=1: \\
& e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ x & y & z \\ & & 1 \end{pmatrix} \begin{pmatrix} r & \frac{br+(a,b)}{a} & 0 \\ \frac{as}{(a,b)} & \frac{bs}{(a,b)} & \frac{cs+1}{f(a,b)} \\ fa & fb & c \end{pmatrix} \equiv (0,0,*) \bmod N \\
\iff\ & \exists x,y,z \in \mathbb{Z}, (x,y)=1:
\end{aligned}$$

$$\begin{aligned} fxr + fy\tfrac{as}{(a,b)} + (1+fz)fa &\equiv 0 \bmod N \\ fx\tfrac{br+(a,b)}{a} + fy\tfrac{bs}{(a,b)} + (1+fz)fb &\equiv 0 \bmod N\,. \end{aligned}$$

Divide by f in each equation and write the system in matrix form, so the equivalent system

$$\exists x,y,z \in \mathbb{Z}, (x,y)=1: \quad \begin{pmatrix} r & \frac{as}{(a,b)} \\ \frac{br+(a,b)}{a} & \frac{bs}{(a,b)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv -(fz+1) \begin{pmatrix} a \\ b \end{pmatrix} \bmod \frac{N}{f} \tag{4.7}$$

is obtained. For further calculations we need the determinant of the above 2×2 matrix:

$$\det \begin{pmatrix} r & \frac{as}{(a,b)} \\ \frac{br+(a,b)}{a} & \frac{bs}{(a,b)} \end{pmatrix} = r\frac{bs}{(a,b)} - \frac{as}{(a,b)}\frac{br+(a,b)}{a} = -s\,.$$

Finally we solve the matrix equation in (4.7) by multiplying with the complementary matrix, which is an equivalence transformation since Lemma 3.2 states that s is coprime to $\frac{N}{f}$. So we get the equivalent system

$$\exists x,y,z \in \mathbb{Z}, (x,y)=1: \quad s\begin{pmatrix} x \\ y \end{pmatrix} \equiv (fz+1) \begin{pmatrix} \frac{bs}{(a,b)} & -\frac{as}{(a,b)} \\ -\frac{br+(a,b)}{a} & r \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \bmod \frac{N}{f}\,. \tag{4.8}$$

The squarefreeness of N gives that $fz+1$ is again any integer modulo $\frac{N}{f}$ after a suitable choice of z. After simplifying the system (4.8) we get the equivalent system

$$\exists x,y,z \in \mathbb{Z}, (x,y)=1: \quad \begin{pmatrix} x \\ y \end{pmatrix} \equiv z \begin{pmatrix} 0 \\ -(a,b) \end{pmatrix} \bmod \frac{N}{f}\,. \tag{4.9}$$

Lemma 4.1 with $N = \frac{N}{f}$, $M = M$ and $a = -(a,b)$ implies that the above system is solvable if and only if $\left(a, b, \frac{N}{f}\right) = 1$. In the case of solvability, Lemma 4.1 guarantees that one can choose rows (x,y) in such a way that the completed matrix $p_{(a,b,c,d,e)} = \begin{pmatrix} * & * \\ x & y \end{pmatrix}$ has the properties claimed in the theorem, note that the squarefreeness of N and $M \mid N$ implies $\frac{M}{\left(\frac{N}{f}, M\right)} = (f,M)$. The explicit description of the left cosets of $\Gamma_0(N)$ in Lemma 20.4 in conjunction with Lemma 3.7 guarantees that the cosets in $\Lambda(f,M,P_{2,1}) \setminus \Gamma_0(N)$, associated to the above constructed representatives, are disjoint and the so constructed representatives exhaust $\Gamma_1(f,M,P_{2,1})$.

(2) We proceed analogously to part (1) and use Corollary 3.6 and Lemma 3.7 and determine which last rows occur in $\Gamma_2(f,M,P_{2,1})$. Again reducing modulo f immediately gives that the last rows must have the form $(0, fb, c)$, so it remains to investigate which additional conditions must be fulfilled. This is done by the equivalence transformations

$$\begin{aligned}
& \exists p \in P_{2,1}: \quad p\gamma_{(0,fb,c)} \in \alpha_f^{-1}\Gamma_0(N) \\
\iff & \exists p \in P_{2,1}: \quad e_3^T \alpha_f p \gamma_{(0,fb,c)} \equiv (0,0,*) \bmod N \\
\iff & \exists x,y,z \in \mathbb{Z}, (x,y)=1: \\
& e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ & f & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ x & y & z \\ & & 1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & t & \frac{ct+1}{fb} \\ & fb & c \end{pmatrix} \equiv (0,0,*) \bmod N \\
\iff & \exists x,y,z \in \mathbb{Z}, (x,y)=1:
\end{aligned}$$

$$\begin{aligned}
-fx &\equiv 0 \bmod N \\
fyt + (1+fz)fb &\equiv 0 \bmod N \,.
\end{aligned}$$

Divide by f in both equations. Lemma 3.2 part (2) gives that t is a unit modulo $\frac{N}{f}$ and the squarefreeness of N gives that $fz+1$ is again any integer modulo $\frac{N}{f}$ after a suitable choice of z. After simplifying the above system we get the equivalent system

$$\exists x,y,z \in \mathbb{Z}, (x,y)=1: \quad \begin{pmatrix} x \\ y \end{pmatrix} \equiv z \begin{pmatrix} 0 \\ b \end{pmatrix} \bmod \frac{N}{f} \,. \tag{4.10}$$

Lemma 4.1 with $N = \frac{N}{f}$, $M = M$ and $a = b$ implies that the system (4.10) is solvable if and only if $\left(b, \frac{N}{f}\right) = 1$. In the case of solvability argue as in part (1).

(3) Same procedure as in the last two cases. We start with

$$\exists p \in P_{2,1}: \quad p\gamma_{(0,0,\pm 1)} \in \alpha_f^{-1}\Gamma_0(N) \,.$$

Reducing modulo N and comparing the last rows of both sides immediately gives $f = N$. So it is necessary that $f = N$ holds and in this case the result is easily verified through absorbing the $\alpha_f, \gamma_{(0,0,\pm 1)}$ into $\Gamma_0(N)$. Again Lemma 20.4 in conjunction with Lemma 4.1 gives the claimed result.

□

CHAPTER 5

Calculation of $\Gamma_i(f, M, P_{1,2})$

In this chapter we calculate a system of coset representatives for $\Lambda(f, M, P_{1,2}) \backslash \Gamma_0(N)$ and also determine the corresponding sets $\Gamma_i(f, M, P_{1,2})$ in Definition 3.8. Again we state an easy technical lemma, which is needed in further calculations.

LEMMA 5.1. *Let M, f_1, f_2 be positive squarefree integers with $(f_1, f_2) = 1$, further let a, b be integers with $(a, f_1) = 1$. For any row vector (c, d) with coprime entries such that $d \mid \frac{M}{(M, f_1 f_2)}$ and $0 < c \leq \frac{\frac{M}{(M, f_1 f_2)}}{d}$, one can choose a solution x, y of the congruence*

$$ax + by \equiv 0 \bmod f_1$$

with $(f_2 x, y) = 1$ in a way such that a suitable completed matrix $\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix} \in SL_2(\mathbb{Z})$ has the following three properties.

(1) The matrices $\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix}$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ lie in the same left coset of $\Gamma_0\left(\frac{M}{(M, f_1 f_2)}\right) \backslash SL_2(\mathbb{Z})$.

(2) The matrices $\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ (f_2, M)(b, M, f_1)\left(c, \frac{M}{(M, f_1 f_2) d}\right) & 1 \end{pmatrix}$ lie in the same double coset of $\Gamma_0(M) \diagdown SL_2(\mathbb{Z}) \diagup P_{min}$. Further there exist a decomposition

$$\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 \\ (f_2, M)(b, M, f_1)\left(c, \frac{M}{(M, f_1 f_2) d}\right) & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

with $\gamma \in \Gamma_0(M)$ and the integer α depends on a, b, c, d, f_1, f_2, M only modulo $\frac{M}{(f_2, M)(b, M, f_1)\left(c, \frac{M}{(M, f_1 f_2) d}\right)}$.

PROOF. The squarefreeness of M guarantees that the integers f_1 and $\frac{M}{(M, f_1 f_2) d}$ are coprime. So there exist integers l, k such that

$$\tilde{f}_2 c + l \cdot \frac{M}{(M, f_1 f_2) d} = -\tilde{a} b d + k \cdot f_1 \tag{5.1}$$

where $\tilde{a} a \equiv 1 \bmod f_1$ and $\tilde{f}_2 \cdot f_2 \equiv 1 \bmod \frac{M}{(M, f_1 f_2) d}$ holds. Since d and $\frac{M}{(M, f_1 f_2) d}$ are coprime choose a suitable integer m such that $\left(k + m \frac{M}{(M, f_1 f_2) d}, d\right) = 1$. Through passing in (5.1) from k, l, respectively, to $k + m\frac{M}{(M,f_1f_2)d}$, $l + mf_1$, respectively, we can assume w.l.o.g. that $(k, d) = 1$. Now define the integers x and y by

$$x := \tilde{f}_2 c + l \cdot \frac{M}{(M, f_1 f_2)\, d} = -\tilde{a} bd + k \cdot f_1 \tag{5.2}$$

and

$$y := d\ . \tag{5.3}$$

We first show using the gcd-condition $(k, d) = 1$ that the integers $f_2 x$ and y are coprime. We have

$$(f_2 x, y) = (f_2\left(-\tilde{a}bd + kf_1\right), d) = (kf_1 f_2, d) = (f_1 f_2, d) = 1\ .$$

Note that the squarefreeness of M and the divisibility condition $d \mid \frac{M}{(M,f_1f_2)}$ imply $(f_1 f_2, d) = 1$. Next we show that the integers $f_2 x$ and y fulfill the claimed congruence

$$ax + by \equiv a(-\tilde{a}bd + kf_1) + bd = -(\tilde{a}a)bd + bd \equiv 0 \bmod f_1\ .$$

Now we use Lemma 1.2 to show that the integers $f_2 x, y$ also satisfy the claimed properties:

$$\begin{aligned}
& \Gamma_0\left(\frac{M}{(M, f_1 f_2)}\right)\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix} = \Gamma_0\left(\frac{M}{(M, f_1 f_2)}\right)\begin{pmatrix} * & * \\ c & d \end{pmatrix} \\
\iff\quad & \begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix}\begin{pmatrix} * & * \\ c & d \end{pmatrix}^{-1} \in \Gamma_0\left(\frac{M}{(M, f_1 f_2)}\right) \\
\iff\quad & e_2^T \begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix}\begin{pmatrix} d & * \\ -c & * \end{pmatrix} e_1 \equiv 0 \bmod \frac{M}{(M, f_1 f_2)} \\
\iff\quad & df_2 x - cy \equiv 0 \bmod \frac{M}{(M, f_1 f_2)} \\
\iff\quad & df_2\left(\tilde{f}_2 c + l\frac{M}{(M, f_1 f_2)\, d}\right) - cd \equiv 0 \bmod \frac{M}{(M, f_1 f_2)} \\
\iff\quad & \left(f_2 \tilde{f}_2 - 1\right) c \equiv 0 \bmod \frac{M}{(M, f_1 f_2)\, d}\ .
\end{aligned}$$

Obvious the last congruence is true. In order to check the second property we first calculate the gcd

$$\begin{aligned}
(f_2 x, M) &= (f_2, M)\left(x, \frac{M}{(M, f_1 f_2)\, d}\right)(x, d)\,(x, (M, f_1)) \\
&= (f_2, M)\left(\tilde{f}_2 c + l\frac{M}{(M, f_1 f_2)\, d}, \frac{M}{(M, f_1 f_2)\, d}\right)(-\tilde{a}bd + kf_1, M, f_1)
\end{aligned}$$

$$= (f_2, M)\left(c, \frac{M}{(M, f_1 f_2)\, d}\right)(b, M, f_1)\ . \tag{5.4}$$

In the above splitting we used the squarefreeness of M and the fact that f_1 and f_2 are coprime. Finally we can prove the second property. The following equivalence transformations

$$\begin{aligned}
&\Gamma_0\left(M\right)\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix} P_{min} = \Gamma_0\left(M\right)\begin{pmatrix} 1 & 0 \\ (f_2 x, M) & 1 \end{pmatrix} P_{min} \\
\Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad &\begin{pmatrix} * & * \\ f_2 x & y \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (f_2 x, M) & 1 \end{pmatrix}^{-1} \in \Gamma_0(M) \\
\Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad &f_2 x \left(1 - (f_2 x, M)\, \alpha\right) - (f_2 x, M)\, y \equiv 0 \bmod M \\
\Longleftrightarrow \exists \alpha \in \mathbb{Z}: \quad &\frac{f_2 x}{(f_2 x, M)} - f_2 x \alpha - d \equiv 0 \bmod \frac{M}{(f_2 x, M)}\ .
\end{aligned}$$

hold. The squarefreeness of M gives that $f_2 x$ is a unit modulo $\frac{M}{(f_2 x, M)}$, hence the last congruence is solvable. The explicit formula in (5.4) finishes the proof of the second property.

□

Now we can state the main result of this chapter.

LEMMA 5.2. *Let N be a squarefree integer and f, M positive divisors of N. A possible choice of the sets $\Gamma_i(f, M, P_{1,2})$ is given by*

(1)

$$\Gamma_1(f, M, P_{1,2}) = \left\{ \begin{pmatrix} 1 & * \quad * \\ 0 & \\ 0 & q_{(a,b,c,d,e)} \end{pmatrix} \delta_{(fa,b,c)}^{-1} \,\middle|\, \begin{matrix} (fa,b,c)=1 \\ a \neq 0 \\ \left(a, \frac{N}{f}\right) = 1 \\ e \mid \left(\frac{N}{f}, M\right) \\ 0 < d \leq \frac{\left(\frac{N}{f}, M\right)}{e} \\ (d,e)=1 \end{matrix} \right\},$$

with $q_{(a,b,c,d,e)} \in \Gamma_0\left(\frac{f}{(f,b)}\right)$ satisfying the following four properties.

(a) The matrices $q_{(a,b,c,d,e)}$ and $\begin{pmatrix} * & * \\ d & e \end{pmatrix}$ lie in the same right coset of $\Gamma_0\left(\left(\frac{N}{f}, M\right)\right)\backslash SL_2\left(\mathbb{Z}\right)$.

(b) The matrices $q_{(a,b,c,d,e)}$ *and* $\begin{pmatrix} 1 & 0 \\ \left(\frac{f}{(f,b)}, M\right)\left(\frac{(f,b)a}{(fa,b)}, M, f, b\right)\left(d, \frac{\left(\frac{N}{f}, M\right)}{e}\right) & 1 \end{pmatrix}$ *lie in the same double coset of* $\Gamma_0(M)\backslash SL_2(\mathbb{Z})/P_{min}$*. There exist a decomposition*

$$q_{(a,b,c,d,e)} = \gamma \begin{pmatrix} 1 & 0 \\ \left(\frac{f}{(f,b)}, M\right)\left(\frac{(f,b)a}{(fa,b)}, M, f, b\right)\left(d, \frac{\left(\frac{N}{f}, M\right)}{e}\right) & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

with $\gamma \in \Gamma_0(M)$ *and the integer* α *depends on* $\frac{(f,b)a}{(fa,b)}, (f,b), c, d, e, f, M$ *only modulo* $\frac{M}{\left(\frac{f}{(f,b)}, M\right)\left(\frac{(f,b)a}{(fa,b)}, M, f, b\right)\left(d, \frac{\left(\frac{N}{f}, M\right)}{e}\right)}$.

(2) The set $\Gamma_2(f, M, P_{1,2})$ *is given by*

$$\Gamma_2(f, M, P_{1,2}) = \begin{cases} \varnothing & \text{if } f \neq N\,, \\ \left\{ \delta^{-1}_{(0,b,c)} \,\middle|\, \substack{(0,b,c)=1 \\ b \neq 0} \right\} & \text{if } f = N\,. \end{cases}$$

(3) The set $\Gamma_3(f, M, P_{1,2})$ *is given by*

$$\Gamma_3(f, M, P_{1,2}) = \begin{cases} \varnothing & \text{if } f \neq N\,, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\} & \text{if } f = N\,. \end{cases}$$

PROOF. The proof of this lemma is analogous to the proof of Lemma 4.2.

(1) For our calculations we use part (2) in Corollary 3.6, Lemma 3.7 and determine which first columns occur in $\Gamma_1(f, M, P_{1,2})$. We start with the statement

$$\exists p \in P_{1,2}: \quad p\delta^{-1}_{(a,b,c)} \in \beta_f^{-1}\Gamma_0(N)\,. \tag{5.5}$$

Multiplying from the right with $\delta_{(a,b,c)}$ in (5.5), then reducing modulo f gives immediately that the entry with indices $(3,1)$ vanishes modulo f, hence $f \mid a$. So (5.5) is equivalent to the chain of statements

$$\begin{aligned}
& \exists p \in P_{1,2}: \quad e_3^T \beta_f p \delta_{(fa,b,c)}^{-1} \equiv (0,0,*) \bmod N \\
\iff \quad & \exists x,y,u,w \in \mathbb{Z}, (u,w)=1: \\
& e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & * & * \\ & u & w \end{pmatrix} \begin{pmatrix} -s & -r\frac{cs+1}{(fa,b)} & \frac{bcrs+cs(fa,b)+br+(fa,b)}{fa(fa,b)} \\ (fa,b) & cr & -c\frac{br+(fa,b)}{fa} \\ 0 & \frac{fa}{(fa,b)} & -\frac{b}{(fa,b)} \end{pmatrix} \equiv (0,0,*) \bmod N \\
\iff \quad & \exists x,y,u,w \in \mathbb{Z}, (u,w)=1:
\end{aligned}$$

$$\begin{aligned}
-sf + (fa,b)(fx+u) &\equiv 0 \bmod N \\
-rf\frac{cs+1}{(fa,b)} + cr(fx+u) + \frac{fa}{(fa,b)}(fy+w) &\equiv 0 \bmod N\ .
\end{aligned}$$

Writing the above equations in matrix form gives us the equivalent system

$$\begin{gathered}
\exists x,y,u,w \in \mathbb{Z}, (u,w)=1: \\
\begin{pmatrix} (fa,b) & 0 \\ cr & \frac{fa}{(fa,b)} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} sf-(fa,b)fx \\ rf\frac{cs+1}{(fa,b)} - crxf - \frac{f^2a}{(fa,b)}y \end{pmatrix} \bmod N\ . \qquad (5.6)
\end{gathered}$$

It remains to prove that (5.6) is solvable if and only if a is coprime to $\frac{N}{f}$. We begin with proving the "only if" part through contradiction. Since our precondition is that the matrix equations in (5.6) have a solution we multiply with the complementary matrix, then calculate the matrix product and dividing out f. Note that the multiplication with the complementary matrix is in general not a equivalence transformation, since its determinant fa need not be coprime to N. This gives us the system

$$\begin{aligned}
& \exists x,y,u,w \in \mathbb{Z}, (u,w)=1: \\
& fa\begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} \frac{fa}{(fa,b)} & 0 \\ -cr & (fa,b) \end{pmatrix} \begin{pmatrix} sf-(fa,b)fx \\ rf\frac{cs+1}{(fa,b)} - crxf - \frac{f^2a}{(fa,b)}y \end{pmatrix} \bmod N \\
\iff \quad & \exists x,y,u,w \in \mathbb{Z}, (u,w)=1:
\end{aligned}$$

$$a\begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} \frac{fa}{(fa,b)}s - fax \\ r - fay \end{pmatrix} \bmod \frac{N}{f}\ . \qquad (5.7)$$

Assume now $\left(a, \frac{N}{f}\right) > 1$ and let p be a prime which divides $\left(a, \frac{N}{f}\right)$. If $p \mid \frac{fa}{(fa,b)}$, reducing (5.7) modulo p implies $r \equiv 0 \bmod p$, which is a contradiction to the congruence obstruction $r\frac{b}{(fa,b)} \equiv -1 \bmod \frac{fa}{(fa,b)}$ in Lemma 3.3, so $\left(\frac{fa}{(fa,b)}, \frac{N}{f}\right) = 1$. Otherwise $p \mid (fa,b)$ holds, again reducing (5.7) modulo p implies $\frac{fa}{(fa,b)}s \equiv$

0 mod p, hence $s \equiv 0 \bmod p$ which is a contradiction to the congruence obstruction $sc \equiv -1 \bmod (fa, b)$ in Lemma 3.3.

For the proof of the "if" part assume $\left(a, \frac{N}{f}\right) = 1$. The squarefreeness of N allows us to split the system (5.6) into one system modulo f and one modulo $\frac{N}{f}$. Hence (5.6) is equivalent to

$$\exists x, y, u, w \in \mathbb{Z}, (u, w) = 1 :$$
$$\begin{pmatrix} (fa, b) & 0 \\ cr & \frac{fa}{(fa,b)} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \bmod f \tag{5.8}$$

$$\wedge \quad \begin{pmatrix} (fa, b) & 0 \\ cr & \frac{fa}{(fa,b)} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} sf - (fa, b)fx \\ rf\frac{cs+1}{(fa,b)} - crxf - \frac{f^2 a}{(fa,b)} y \end{pmatrix} \bmod \frac{N}{f} . \tag{5.9}$$

The squarefreeness of N and the precondition $\left(a, \frac{N}{f}\right) = 1$ implies that fa is a unit modulo $\frac{N}{f}$, so the system (5.9) can be transformed with the same methods as in the "only if" part. This gives us the equivalent system

$$\exists x, y, u, w \in \mathbb{Z}, (u, w) = 1 :$$
$$\begin{pmatrix} (fa, b) & 0 \\ cr & \frac{fa}{(fa,b)} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \bmod f \tag{5.10}$$

$$\wedge \quad a \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} \frac{fa}{(fa,b)} s - fax \\ r - fay \end{pmatrix} \bmod \frac{N}{f} . \tag{5.11}$$

Again since fa is a unit modulo $\frac{N}{f}$, in the system (5.11) for any row (u, w) consisting of coprime integers one can choose integers x, y such that the system is fulfilled. So the two systems (5.10) and (5.11) simplify to one system

$$\exists u, w \in \mathbb{Z}, (u, w) = 1 : \quad \begin{pmatrix} (fa, b) & 0 \\ cr & \frac{fa}{(fa,b)} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \bmod f . \tag{5.12}$$

In order to construct a solution of (5.12) split $f = (f, b) \cdot \frac{f}{(f,b)}$. Considering only u's, which are divisible by $\frac{f}{(f,b)}$, the first equation in the above system (5.12) is true and the whole system collapses to a single equation

$$\begin{aligned} &\exists u, w \in \mathbb{Z}, \left(\frac{f}{(f, b)} u, w\right) = 1 : && cr\frac{f}{(f, b)} u + w\frac{fa}{(fa, b)} \equiv 0 \bmod f \\ \Longleftrightarrow &\exists u, w \in \mathbb{Z}, \left(\frac{f}{(f, b)} u, w\right) = 1 : && cru + w\frac{(f, b)a}{(fa, b)} \equiv 0 \bmod (f, b) . \end{aligned}$$

The column $(fa, b, c)^T$ consists of coprime entries and Lemma 3.4 implies that r is coprime to (fa, b), so cr is a unit modulo (f, b). So Lemma 5.1 guarantees that one can choose rows $\left(\frac{f}{(f,b)}u, w\right)$ in such a way that the completed matrix $q_{(a,b,c,d,e)} = \begin{pmatrix} * & * \\ \frac{f}{(f,b)}u & w \end{pmatrix}$ has the properties claimed in the theorem, note that the squarefreeness of N and $M \mid N$ implies $\frac{M}{(f,M)} = \left(\frac{N}{f}, M\right)$. The explicit description of the left cosets of $\Gamma_0(N)$ in Lemma 20.4 in conjunction with Lemma 3.7 guarantees that the cosets in $\Lambda(f, M, P_{1,2}) \setminus \Gamma_0(N)$, associated to the above constructed representatives, are disjoint and the so constructed representatives exhaust $\Gamma_1(f, M, P_{1,2})$.

(2) For our calculations we use part (2) in Corollary 3.6, Lemma 3.7 and determine which first columns occur in $\Gamma_2(f, M, P_{1,2})$. We start with the statement

$$\exists p \in P_{1,2} : \quad p\delta^{-1}_{(0,b,c)} \in \beta_f^{-1}\Gamma_0(N)\ . \tag{5.13}$$

Since $\delta^{-1}_{(0,b,c)} \in \Gamma_0(N)$ this element can be absorbed into $\Gamma_0(N)$. So the statement (5.13) above is equivalent to the chain of statements

$$\begin{aligned} & \exists p \in P_{1,2} : \quad e_3^T \beta_f p \equiv (0, 0, *) \bmod N \\ \Longleftrightarrow\quad & \exists x, y, z \in \mathbb{Z}, (x, y) = 1 : \quad e_3^T \begin{pmatrix} 1 & & \\ & 1 & \\ f & & 1 \end{pmatrix} \begin{pmatrix} 1 & z & * \\ & * & * \\ & x & y \end{pmatrix} \equiv (0, 0, *) \bmod N \\ \Longleftrightarrow\quad & \exists x, y, z \in \mathbb{Z}, (x, y) = 1 : \end{aligned}$$

$$\begin{aligned} f &\equiv 0 \bmod N \\ fz + x &\equiv 0 \bmod N\ . \end{aligned}$$

Obviously the last system is solvable if and only if $f = N$ and $N \mid x$ and one can choose p as the identity matrix. Since in this case $\Gamma_0\left(\frac{M}{(M,f)}\right) = SL_2(\mathbb{Z})$ holds, Lemma 3.7 guarantees that the so constructed coset representatives in $\Lambda(f, M, P_{1,2}) \setminus \Gamma_0(N)$ are disjoint and that they exhaust $\Gamma_2(f, M, P_{1,2})$.

(3) Since $\delta^{-1}_{(0,0,\pm 1)} \in \Gamma_0(N)$ we proceed analogously as in the second case.

□

CHAPTER 6

Calculation of $\Gamma_i(f, h, P_{min})$

It remains to calculate a system of coset representatives for $\Lambda(f, h, P_{min}) \setminus \Gamma_0(N)$ and the corresponding sets $\Gamma_i(f, h, P_{min})$ in Definition 3.8.

LEMMA 6.1. *Let N be a squarefree integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. With the notation $\tau_{(c,d)}$ from Lemma 3.5 a possible choice of the sets $\Gamma_i(f, h, P_{min})$ is given by*

(1)

$$\Gamma_1(f,h,P_{min}) = \left\{ r_{(a,b,c,d,e)} \begin{pmatrix} \tau_{(dh,e)} & \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \middle| \substack{(fa,fb,c)=1 \\ a\neq 0 \\ \left(a,b,\frac{N}{f}\right)=1 \\ (dh,e)=1 \\ d\neq 0 \\ \left(d,\frac{N}{fh}\right)=1} \right\},$$

with $r_{(a,b,c,d,e)} \in P_{min}$.

(2) The set $\Gamma_2(f, h, P_{min})$ is given by

$$\Gamma_2(f,h,P_{min}) = \begin{cases} \varnothing & \text{if } h \neq \frac{N}{f}, \\ \left\{ r_{(a,b,c,0,\pm1)} \begin{pmatrix} \tau_{(0,\pm1)} & \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \middle| \substack{(fa,fb,c)=1 \\ a\neq 0 \\ \left(a,b,\frac{N}{f}\right)=1} \right\} & \text{if } h = \frac{N}{f}, \end{cases}$$

with $r_{(a,b,c,0,\pm1)} \in P_{min}$.

(3) The set $\Gamma_3(f, h, P_{min})$ is given by

$$\Gamma_3(f,h,P_{min}) = \left\{ r_{(0,b,c,d,e)} \begin{pmatrix} \tau_{(dh,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} \middle| \substack{(fb,c)=1 \\ b\neq 0 \\ \left(b,\frac{N}{f}\right)=1 \\ (dh,e)=1 \\ d\neq 0 \\ \left(d,\frac{N}{fh}\right)=1} \right\},$$

with $r_{(0,b,c,d,e)} \in P_{min}$.

(4) *The set* $\Gamma_4(f,h,P_{min})$ *is given by*

$$\Gamma_4(f,h,P_{min}) = \begin{cases} \varnothing & \text{if } h \neq \frac{N}{f} \ , \\ \left\{ r_{(0,b,c,0,\pm1)} \begin{pmatrix} \tau_{(0,\pm1)} & \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} \middle| \substack{(0,fb,c)=1 \\ b\neq 0 \\ \left(b,\frac{N}{f}\right)=1} \right\} & \text{if } h = \frac{N}{f} \ , \end{cases}$$

with $r_{(0,b,c,0,\pm1)} \in P_{min}$.

(5) *The set* $\Gamma_5(f,h,P_{min})$ *is given by*

$$\Gamma_5(f,h,P_{min}) = \begin{cases} \varnothing & \text{if } f \neq N \ , \\ \left\{ \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,0,\pm1)} \middle| \substack{(d,e)=1 \\ d\neq 0} \right\} & \text{if } f = N \ . \end{cases}$$

(6) *The set* $\Gamma_6(f,h,P_{min})$ *is given by*

$$\Gamma_6(f,h,P_{min}) = \begin{cases} \varnothing & \text{if } f \neq N \ , \\ \left\{ \begin{pmatrix} \tau_{(0,\pm1)} & \\ & 1 \end{pmatrix} \gamma_{(0,0,\pm1)} \right\} & \text{if } f = N \ . \end{cases}$$

Proof. Note that in all six calculations Lemma 3.7 ensures that the calculated coset representatives are disjoint and exhaust the respective $\Gamma_i(f,h,P_{min})$.

(1) For our calculations we use part (3) in Corollary 3.6 and determine which rows occur in $\Gamma_1(f,h,P_{min})$. Reducing modulo f immediately gives that the last row in $\gamma_{(a,b,c)}$ must have the form (fa, fb, c), so it remains to investigate which additional conditions the two rows (fa, fb, c) and (d,e) must fulfill. We start with the statement

$$\exists p \in P_{min}: \quad p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \in \lambda_{f,h}^{-1}\Gamma_0(N) \ . \tag{6.1}$$

With the defintion of the matrix $\tau_{(d,e)}$ in Lemma 3.5 we pass from (6.1) to the equivalent condition

$$\begin{aligned}
&\exists x,y,z\in\mathbb{Z}: && \begin{pmatrix}1&&\\h&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x&y\\&1&z\\&&1\end{pmatrix}\begin{pmatrix}t&\frac{et-1}{d}&\\d&e&\\&&1\end{pmatrix}\gamma_{(fa,fb,c)}\in\alpha_f^{-1}\Gamma_0(N)\\
\iff\ &\exists x,y,z\in\mathbb{Z}: && \begin{pmatrix}t+dx&\frac{et-1}{d}+ex&y\\th+hdx+d&\frac{et-1}{d}h+hex+e&hy+z\\&&1\end{pmatrix}\gamma_{(fa,fb,c)}\in\alpha_f^{-1}\Gamma_0(N)\ .
\end{aligned}$$

Now we can resort to the calculations in the proof of part (1) in Lemma 4.2. All the calculations there can be transfered to this situation. So using (4.9) implies that the above condition is equivalent to the following condition

$$\exists x,y,z\in\mathbb{Z}:\quad \begin{pmatrix}th+hdx+d\\\frac{et-1}{d}h+hex+e\end{pmatrix}\equiv-(f(hy+z)+1)\begin{pmatrix}0\\(a,b)\end{pmatrix}\mod\frac{N}{f}\ . \tag{6.2}$$

Since N is squarefree we can achieve that $-(f(hy+z)+1)$ is any integer modulo $\frac{N}{f}$ by choosing a suitable integer z. So (6.2) is equivalent to the condition

$$\exists x,u\in\mathbb{Z}:\quad \begin{pmatrix}th+hdx+d\\\frac{et-1}{d}h+hex+e\end{pmatrix}\equiv u\begin{pmatrix}0\\(a,b)\end{pmatrix}\mod\frac{N}{f}\ . \tag{6.3}$$

Note that the row $\left(th+hdx+d,\frac{et-1}{d}h+hex+e\right)$ has coprime entries since it is a row of an invertible 2×2 matrix, which occurs above as an upper left block in a 3×3 matrix in $P_{2,1}$. So Lemma 4.1 implies that it is necessary for the solvability of the system (6.3) that the gcd condition

$$\left(a,b,\frac{N}{f}\right)=1 \tag{6.4}$$

holds. In this case a suitable integer u can be chosen so that the second equation of (6.3) is satisfied. Hence the system (6.3) collapses to a single congruence

$$\exists x\in\mathbb{Z}:\quad th+hdx+d\equiv0\mod\frac{N}{f}\ . \tag{6.5}$$

Since h is a divisor of $\frac{N}{f}$ reducing the congruence in (6.5) modulo h immediately gives the divisibility relation

$$h\mid d\ . \tag{6.6}$$

Write $d=hd_1$ and substitute this into (6.5). So we get the equivalent system

$$\begin{aligned}
&\exists x\in\mathbb{Z}: && t+d_1(hx+1)\equiv0\mod\frac{N}{fh}\\
\iff\ &\exists x\in\mathbb{Z}: && t+d_1x\equiv0\mod\frac{N}{fh}\ .
\end{aligned}$$

The last equivalence holds since N is squarefree. Reducing the above congruence modulo $\left(d_1,\frac{N}{fh}\right)$ gives $\left(d_1,\frac{N}{fh}\right)\mid t$. But since (t,hd_1) is the first column of the invertible matrix $\tau_{(d,e)}$, which implies that the integers t,hd_1 are coprime, we get

$\left(d_1, \frac{N}{fh}\right) = 1$. Obviously this last equation is solvable for the variable x if the condition

$$\left(d_1, \frac{N}{fh}\right) = 1 \tag{6.7}$$

holds. Collecting the conditions (6.4), (6.6) and (6.7) gives the proof of the claim.

(2) We use the same argumentation as in part (1) and will omit steps and arguments which are similar. We have the following chain of equivalences

$$\begin{aligned}
& \exists p \in P_{min}: \quad p \begin{pmatrix} \tau_{(0,\pm1)} & \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \in \lambda_{f,h}^{-1}\Gamma_0(N) \\
\iff \quad & \exists x,y,z \in \mathbb{Z}: \quad \begin{pmatrix} 1 & & \\ h & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} \pm1 & & \\ & \pm1 & \\ & & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \in \alpha_f^{-1}\Gamma_0(N) \\
\iff \quad & \exists x,y,z \in \mathbb{Z}: \quad \begin{pmatrix} \pm1 & \pm x & y \\ \pm h & \pm(hx+1) & hy+z \\ & & 1 \end{pmatrix} \gamma_{(fa,fb,c)} \in \alpha_f^{-1}\Gamma_0(N) \\
\iff \quad & \exists x,u \in \mathbb{Z}: \quad \begin{pmatrix} \pm h \\ \pm(hx+1) \end{pmatrix} \equiv u \begin{pmatrix} 0 \\ (a,b) \end{pmatrix} \bmod \frac{N}{f} \,.
\end{aligned}$$

The first congruence in the above system is solvable if and only if $h = \frac{N}{f}$ and in this case the second congruence is equivalent to the congruence

$$\pm 1 \equiv u(a,b) \bmod \frac{N}{f} \,.$$

This congruence is solvable if and only if $\left(a, b, \frac{N}{f}\right) = 1$.

(3) Reducing modulo f immediately gives that the last row in $\gamma_{(0,b,c)}$ must have the form $(0, fb, c)$. To obtain the requested result we will use a similar calculation as in the previous cases. We have the following chain of equivalences

$$\begin{aligned}
& \exists p \in P_{min}: \quad p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} \in \lambda_{f,h}^{-1}\Gamma_0(N) \\
\iff \quad & \exists x,y,z \in \mathbb{Z}: \quad \begin{pmatrix} t_1 + dx & \frac{et_1-1}{d} + ex & y \\ t_1 h + hdx + d & \frac{et_1-1}{d}h + hex + e & hy+z \\ & & 1 \end{pmatrix} \gamma_{(0,fb,c)} \in \alpha_f^{-1}\Gamma_0(N) \,.
\end{aligned}$$

Now we can resort to the calculations in the proof of part (2) in Lemma 4.2. All the calculations can be transfered to this situation. So we obtain that the above condition is equivalent to the system

$$\exists x, y, z \in \mathbb{Z} : \quad t \begin{pmatrix} t_1 h + hdx + d \\ \frac{et_1 - 1}{d} h + hex + e \end{pmatrix} \equiv (f(hy + z) + 1) \begin{pmatrix} 0 \\ b \end{pmatrix} \bmod \frac{N}{f} . \tag{6.8}$$

We now argue analogously as in part (1). First we use the squarefreeness of N to choose a suitable integer z such that $t^{-1}(f(hy+z)+1)$ is any integer modulo $\frac{N}{f}$. So the system in (6.8) is equivalent to the system

$$\exists x, u \in \mathbb{Z} : \quad \begin{pmatrix} t_1 h + hdx + d \\ \frac{et_1 - 1}{d} h + hex + e \end{pmatrix} \equiv u \begin{pmatrix} 0 \\ b \end{pmatrix} \bmod \frac{N}{f} . \tag{6.9}$$

Next with the same argument as in part (1) we obtain that the row $(t_1 h + hdx + d, \frac{et_1-1}{d} h + hex + e)$ has coprime entries. Again Lemma 4.1 implies that it is necessary for the solvability of the system (6.9) that the gcd condition

$$\left(b, \frac{N}{f} \right) = 1 \tag{6.10}$$

holds. In this case a suitable integer u solves the last congruence of (6.9). Hence the system (6.9) collapses to a single congruence

$$\exists x \in \mathbb{Z} : \quad t_1 h + hdx + d \equiv 0 \bmod \frac{N}{f} . \tag{6.11}$$

Since h is a divisor of $\frac{N}{f}$, reducing (6.11) modulo h immediately gives the gcd condition

$$h \mid d . \tag{6.12}$$

Write $d = hd_1$, so we obtain with the same steps as in part (1) the congruence

$$\exists x \in \mathbb{Z} : \quad t_1 + d_1 x \equiv 0 \bmod \frac{N}{fh} , \tag{6.13}$$

which is equivalent to (6.11). Again this last congruence is solvable in the variable x if and only the gcd condition

$$\left(d_1, \frac{N}{fh} \right) = 1 \tag{6.14}$$

holds. Collecting the gcd conditions (6.10),(6.12) and (6.14) gives the proof.

(4) We adopt most of the steps in part (3) and will omit arguments, which are of the same nature as the ones in part (3). We have the following chain of equivalences

$$\begin{aligned} & \exists p \in P_{min} : \quad p \begin{pmatrix} \tau_{(0,\pm1)} & \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} \in \lambda_{f,h}^{-1} \Gamma_0(N) \\ \iff \quad & \exists x, y, z \in \mathbb{Z} : \quad \begin{pmatrix} \pm 1 & \pm x & y \\ \pm h & \pm(hx+1) & hy+z \\ & & 1 \end{pmatrix} \gamma_{(0,fb,c)} \in \alpha_f^{-1} \Gamma_0(N) \\ \iff \quad & \exists x, u \in \mathbb{Z} : \quad \begin{pmatrix} \pm h \\ \pm(hx+1) \end{pmatrix} \equiv u \begin{pmatrix} 0 \\ b \end{pmatrix} \bmod \frac{N}{f} . \end{aligned}$$

The first congruence in the above system is solvable if and only if $h = \frac{N}{f}$ and in this case the second congruence is equivalent to the congruence

$$\exists u \in \mathbb{Z} : \quad \pm 1 \equiv ub \bmod \frac{N}{f} \ . \tag{6.15}$$

The congruence (6.15) is solvable if and only if $\left(b, \frac{N}{f}\right) = 1$.

(5) In the last two cases we must handle the rows $(0, 0, \pm 1)$. We start with the statement

$$\exists p \in P_{min} : \quad p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \gamma_{(0,0,\pm 1)} \in \lambda_{f,h}^{-1} \Gamma_0(N) \ . \tag{6.16}$$

Since $\gamma_{(0,0,\pm 1)} \in \Gamma_0(N)$ reducing modulo N in (6.16) gives immediately $f = N$, hence $h = 1$. In this case absorb $\lambda_{f,h}^{-1}$ and $\gamma_{(0,0,\pm 1)}$ into $\Gamma_0(N)$ which gives the condition

$$\exists p \in P_{min} : \quad p \begin{pmatrix} \tau_{(d,e)} & \\ & 1 \end{pmatrix} \in \Gamma_0(N) \ , \tag{6.17}$$

which is equivalent to condition (6.16). Selecting p as the identity matrix satisfies the condition (6.17).

□

Part III

Iwasawa decompositions

CHAPTER 7

Iwasawa decompositions for $\Gamma_i(f, M, P_{2,1})$ and $\Gamma_i(f, M, P_{1,2})$

In this chapter we will recall a well known algorithm based on the Schmidt procedure in linear algebra and compute the $\mathfrak{h}^3$-part of the Iwasawa decomposition of the elements in the sets $\Gamma_i(f, M, P_{2,1})$ and $\Gamma_i(f, M, P_{1,2})$. These results are needed for the explicit calculation of the Fourier coefficients of the Eisenstein series. Note that most of the calculations in this and the following chapter can also be executed by a scientific computer program, e.g. mathematica, but in order to compute the Fourier expansions we need the terms sorted in a special way, which is not automatically done by those programs. So I decided to do all the calculations by hand. The reader only interested in the results can go straight to the Lemmata 7.4, 8.3, 8.6 and skip all the calculations.

LEMMA 7.1. *Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$ and $\| \cdot \|$ the induced norm. Let $A \in GL_n(\mathbb{R})$ and let a_i denotes its i-th row. Then the Iwasawa decomposition of the matrix A is given by the formula*

$$A = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \cdots & \lambda_{1,n} \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\|b_1\|} b_1 \\ \vdots \\ \frac{1}{\|b_n\|} b_n \end{pmatrix} \| b_n \| ,$$

where the coefficients $\lambda_{i,j}$ and the vectors b_k are defined by

(1)
$$b_{n-i} := a_{n-i} - \sum_{k=n-i+1}^{n} \frac{\langle a_{n-i}, b_k \rangle}{\langle b_k, b_k \rangle} b_k \qquad , \quad 0 \leq i < n .$$

(2)
$$\lambda_{i,j} := \begin{cases} \frac{\langle a_i, b_j \rangle}{\|b_j\| \|b_n\|} & ,if \quad 1 \leq i < j \leq n , \\ \frac{\|b_i\|}{\|b_n\|} & ,if \quad i = j . \end{cases}$$

PROOF. Since A is invertible we can apply the Schmidt procedure for the inner product $\langle \cdot, \cdot \rangle$ to the basis $a_n, ..., a_1$ of $\mathbb{R}^n$. So we obtain an orthogonal basis $b_n, ..., b_1$ of $\mathbb{R}^n$, where each vector is explicitly given by the formula

$$b_{n-i} := a_{n-i} - \sum_{k=n-i+1}^{n} \frac{\langle a_{n-i}, b_k \rangle}{\langle b_k, b_k \rangle} b_k .$$

Writing the above formulas in matrix form gives us

$$\begin{aligned}
A &= \begin{pmatrix} 1 & \frac{\langle a_1,b_2\rangle}{\langle b_2,b_2\rangle} & \cdots & \cdots & \frac{\langle a_1,b_n\rangle}{\langle b_n,b_n\rangle} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \frac{\langle a_{n-1},b_n\rangle}{\langle b_n,b_n\rangle} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{\langle a_1,b_2\rangle}{\langle b_2,b_2\rangle} & \cdots & \cdots & \frac{\langle a_1,b_n\rangle}{\langle b_n,b_n\rangle} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \frac{\langle a_{n-1},b_n\rangle}{\langle b_n,b_n\rangle} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{\|b_1\|}{\|b_n\|} & & & \\ & \ddots & & \\ & & \frac{\|b_{n-1}\|}{\|b_n\|} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\|b_1\|} b_1 \\ \vdots \\ \frac{1}{\|b_n\|} b_n \end{pmatrix} \|b_n\| \\
&= \begin{pmatrix} \frac{\|b_1\|}{\|b_n\|} & \frac{\langle a_1,b_2\rangle}{\|b_2\|\|b_n\|} & \cdots & \cdots & \frac{\langle a_1,b_n\rangle}{\|b_n\|^2} \\ & \frac{\|b_2\|}{\|b_n\|} & & & \vdots \\ & & \ddots & & \vdots \\ & & & \frac{\|b_{n-1}\|}{\|b_n\|} & \frac{\langle a_{n-1},b_n\rangle}{\|b_n\|^2} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\|b_1\|} b_1 \\ \vdots \\ \frac{1}{\|b_n\|} b_n \end{pmatrix} \|b_n\| \,,
\end{aligned}$$

which is the claimed Iwasawa decomposition of A.

□

With this algorithm at our hand we are able to calculate the $\mathfrak{h}^3$-part in the Iwasawa decomposition of any invertible matrix. Since all relevant matrices for our calculations are already in Bruhat decomposition, we only have to calculate the $\mathfrak{h}^3$-part for matrices in $\mathfrak{h}^3$ transformed with elements from the Weyl group. The next lemma gives explicit formulas of the $\mathfrak{h}^3$-part for these transformed matrices.

LEMMA 7.2. *Let* $z = \begin{pmatrix} y_2 & x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$ *and* ω *be an element of the Weyl group of* $GL_3(\mathbb{R})$. *Let* $a_{i,j}$ *denote the coefficients of the* $\mathfrak{h}^3$*-part in the Iwasawa decomposition of* ωz, *i.e.*

$$\omega z \equiv \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} \mod O_3\mathbb{R}^\times \,.$$

Then the following explicit formulas for the coefficients $a_{i,j}$ are valid.

(1) *For* $\omega = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$:

$$
\begin{aligned}
a_{1,1} =& \frac{y_1 y_2 \sqrt{y_2^2 + x_2^2}}{\sqrt{y_2^2 + x_2^2 + x_3^2}\sqrt{(y_1 x_2 x_3 - x_1(x_2^2 + y_2^2))^2 + y_1^2 y_2^2 (y_2^2 + x_2^2 + x_3^2)}} \ , \\
a_{1,2} =& \frac{\sqrt{y_2^2 + x_2^2}(y_1 x_2 x_3 - x_1(x_2^2 + y_2^2))}{(y_2^2 + x_2^2 + x_3^2)\sqrt{(y_1 x_2 x_3 - x_1(x_2^2 + y_2^2))^2 + y_1^2 y_2^2 (y_2^2 + x_2^2 + x_3^2)}} \ , \\
a_{1,3} =& \frac{-x_3}{y_2^2 + x_2^2 + x_3^2} \ , \\
a_{2,2} =& \frac{\sqrt{(y_1 x_2 x_3 - x_1(x_2^2 + y_2^2))^2 + y_1^2 y_2^2 (y_2^2 + x_2^2 + x_3^2)}}{\sqrt{y_2^2 + x_2^2}(y_2^2 + x_2^2 + x_3^2)} \ , \\
a_{2,3} =& \frac{y_1 x_2 + x_1 x_3}{y_2^2 + x_2^2 + x_3^2} \ .
\end{aligned}
$$

(2) *For* $\omega = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$:

$$
\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{y_1 y_2}{\sqrt{y_2^2+x_2^2+x_3^2}\sqrt{y_2^2+x_2^2}} & \frac{-y_1 x_2 x_3 + x_1(y_2^2+x_2^2)}{(y_2^2+x_2^2+x_3^2)\sqrt{y_2^2+x_2^2}} & \frac{y_1 x_2 + x_1 x_3}{y_2^2+x_2^2+x_3^2} \\ & \frac{\sqrt{y_2^2+x_2^2}}{y_2^2+x_2^2+x_3^2} & \frac{x_3}{y_2^2+x_2^2+x_3^2} \\ & & 1 \end{pmatrix} \ .
$$

(3) *For* $\omega = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$:

$$
\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{y_1 y_2}{\sqrt{y_1^2+x_1^2}\sqrt{(y_1^2+x_1^2)y_2^2+(x_1 x_2 - y_1 x_3)^2}} & -\frac{y_1(x_1 x_2 - y_1 x_3)}{(y_1^2+x_1^2)\sqrt{(y_1^2+x_1^2)y_2^2+(x_1 x_2 - y_1 x_3)^2}} & \frac{x_1}{y_1^2+x_1^2} \\ & \frac{\sqrt{(y_1^2+x_1^2)y_2^2+(x_1 x_2 - y_1 x_3)^2}}{x_1^2+y_1^2} & \frac{y_1 x_2 + x_1 x_3}{y_1^2+x_1^2} \\ & & 1 \end{pmatrix} \ .
$$

(4) *For* $\omega = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}$:

$$
\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{y_2}{\sqrt{y_1^2+x_1^2}} & \frac{-x_1 x_2 + x_3 y_1}{y_1^2+x_1^2} & \frac{x_2 y_1 + x_1 x_3}{x_1^2+y_1^2} \\ & \frac{y_1}{y_1^2+x_1^2} & \frac{x_1}{x_1^2+y_1^2} \\ & & 1 \end{pmatrix} \ .
$$

(5) For $\omega = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{y_1 y_2}{\sqrt{y_2^2+x_2^2}} & -\frac{y_1 x_2}{\sqrt{y_2^2+x_2^2}} & x_1 \\ & \sqrt{y_2^2+x_2^2} & -x_3 \\ & & 1 \end{pmatrix}.$$

Proof. A (long) straightforward computation using Lemma 7.1 which we omit. ∎

A corollary of the above results are the explicit Iwasawa decompositions for the blocks in the Langlands Levi decomposition of the above matrix products. These formulas are needed for the calculations of the Fourier expansions of the Eisenstein series twisted by a Maass cusp form.

Corollary 7.3. *Let* $z = \begin{pmatrix} y_2 & x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$*. The formulas below for the Iwasawa decompositions of the Levi components of the products* ωz *for certain Weyl elements* ω *are valid.*

For $\omega = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$*:* $\quad \mathfrak{m}_{P_{2,1}}(\omega z) \equiv \begin{pmatrix} \frac{y_1 y_2 \sqrt{y_2^2+x_2^2+x_3^2}}{y_2^2+x_2^2} & x_1 - \frac{y_1 x_2 x_3}{y_2^2+x_2^2} \\ & 1 \end{pmatrix} \bmod O_2\mathbb{R}^\times .$

For $\omega = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}$*:* $\quad \mathfrak{m}_{P_{2,1}}(\omega z) \equiv \begin{pmatrix} \frac{y_2}{y_1}\sqrt{y_1^2+x_1^2} & x_3 - \frac{x_1 x_2}{y_1} \\ & 1 \end{pmatrix} \bmod O_2\mathbb{R}^\times .$

For $\omega = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$*:* $\quad \mathfrak{m}_{P_{1,2}}(\omega z) \equiv \begin{pmatrix} \frac{\sqrt{(y_1^2+x_1^2)y_2^2+(x_1x_2-y_1x_3)^2}}{x_1^2+y_1^2} & \frac{y_1x_2+x_1x_3}{y_1^2+x_1^2} \\ & 1 \end{pmatrix} \bmod O_2\mathbb{R}^\times .$

For $\omega = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}$*:* $\quad \mathfrak{m}_{P_{1,2}}(\omega z) \equiv \begin{pmatrix} \sqrt{y_2^2+x_2^2} & -x_3 \\ & 1 \end{pmatrix} \bmod O_2\mathbb{R}^\times .$

Proof. Use Lemma 7.2 and multiply in the first and second formula with a suitable element in $\mathbb{R}^\times$ to obtain the GL_2-Iwasawa form. ∎

Finally we give explicit formulas for the Levi components of the elements in the sets $\Gamma_i(f, M, P_{2,1})$ and $\Gamma_i(f, M, P_{1,2})$.

Lemma 7.4. *Let N be a squarefree integer and f a positive divisor of N. Further let $z = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. With the notation in Lemma 4.2, Lemma 5.2, respectively, the explicit formulas for the Levi components of the elements in $\Gamma_i(f, M, P_{2,1})$ and $\Gamma_i(f, M, P_{1,2})$ below are valid.*

(1) We have

$$\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(a,b,c,d,e)} & \begin{matrix} * \\ * \end{matrix} \\ & 1 \end{pmatrix} \gamma_{(fa,fb,c)} z\right) = p_{(a,b,c,d,e)} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} ,$$

with the coefficients:

$$\begin{aligned} y =& (a,b)^2 y_2 \frac{\sqrt{f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3 + fbx_1 + c)^2}}{a^2y_2^2 + (ax_2+b)^2} , \\ x =& \frac{(a,b)fx_1 - rc}{\frac{a}{(a,b)}} - \frac{\frac{(a,b)^2}{a}(ax_2+b)(fax_3+fbx_1+c)}{a^2y_2^2 + (ax_2+b)^2} . \end{aligned}$$

(2) We have

$$\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(0,b,c,d,e)} & \begin{matrix} * \\ * \end{matrix} \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} z\right) = p_{(0,b,c,d,e)} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} ,$$

with the coefficients:

$$\begin{aligned} y =& y_2\sqrt{y_1^2b^2f^2 + (fbx_1+c)^2} , \\ x =& -bfx_3 + (fbx_1+c)x_2 . \end{aligned}$$

(3) We have

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix} 1 & * & * \\ 0 & & \\ 0 & \multicolumn{2}{c}{q_{(a,b,c,d,e)}} \end{pmatrix} \delta^{-1}_{(fa,b,c)} z\right) = q_{(a,b,c,d,e)} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} ,$$

with the coefficients:

$$\begin{aligned} y =& y_1(af,b)^2 \frac{\sqrt{f^2a^2y_1^2y_2^2 + y_2^2(afx_1-b)^2 + ((fax_1-b)x_2 - fax_3 + c)^2}}{(fax_1-b)^2 + f^2a^2y_1^2} , \\ x =& \frac{(af,b)cr}{fa} + \frac{(af,b)^2}{fa}\frac{f^2a^2y_1^2x_2 + (fax_3-c)(fax_1-b)}{f^2a^2y_1^2 + (fax_1-b)^2} . \end{aligned}$$

(4) We have

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & * & * \\ 0 & \multicolumn{2}{c}{q_{(0,b,c,d,e)}} \\ 0 & & \end{pmatrix}\delta_{(0,b,c)}^{-1}z\right) = q_{(0,b,c,d,e)}\begin{pmatrix}y & x \\ & 1\end{pmatrix},$$

with the coefficients:

$$\begin{aligned} y =& y_1\sqrt{b^2y_2^2+(bx_2-c)^2}\,, \\ x =& cx_1 - bx_3\,. \end{aligned}$$

PROOF. (1) First we use the Bruhat decomposition in Definition 3.1 and do a brute force calculation

$$\begin{aligned} &\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(a,b,c,d,e)} & \begin{matrix}* \\ *\end{matrix} \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right) \\ =& p_{(a,b,c,d,e)}\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix}1 & & \frac{r}{fa} \\ & 1 & \frac{s}{f(a,b)} \\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} fa & fb & c \\ & \frac{(a,b)}{a} & \frac{-rc}{fa} \\ & & \frac{1}{f(a,b)}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1\end{pmatrix}\right) \\ =& p_{(a,b,c,d,e)}\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix}f^2a(a,b)y_1y_2 & f^2(a,b)y_1(ax_2+b) & f(a,b)(fax_3+fbx_1+c) \\ & \frac{(a,b)^2fy_1}{a} & \frac{(a,b)^2fx_1}{a}-\frac{rc(a,b)}{a} \\ & & 1\end{pmatrix}\right). \end{aligned}$$

Now we apply Corollary 7.3 to recast the previous equation as

$$\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(a,b,c,d,e)} & \begin{matrix}* \\ *\end{matrix} \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right) = p_{(a,b,c,d,e)}\begin{pmatrix}y & x \\ & 1\end{pmatrix},$$

with the coefficients y, x given by the two formulas (before and after cancellation)

$$\begin{aligned} y &= f^3(a,b)^3y_1^2y_2\frac{\sqrt{f^4(a,b)^2a^2y_1^2y_2^2+f^4(a,b)^2y_1^2(ax_2+b)^2+f^2(a,b)^2(fax_3+fbx_1+c)^2}}{f^4(a,b)^2a^2y_1^2y_2^2+f^4(a,b)^2y_1^2(ax_2+b)^2} \\ &= (a,b)^2y_2\frac{\sqrt{f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2}}{a^2y_2^2+(ax_2+b)^2}\,, \end{aligned}$$

$$\begin{aligned} x &= \frac{(a,b)^2fx_1-rc(a,b)}{a} - \frac{\frac{(a,b)^4f^4y_1^2}{a}(ax_2+b)(fax_3+fbx_1+c)}{f^4(a,b)^2a^2y_1^2y_2^2+f^4(a,b)^2y_1^2(ax_2+b)^2} \\ &= \frac{(a,b)fx_1-rc}{\frac{a}{(a,b)}} - \frac{\frac{(a,b)^2}{a}(ax_2+b)(fax_3+fbx_1+c)}{a^2y_2^2+(ax_2+b)^2}\,. \end{aligned}$$

(2) Again we use the Bruhat decomposition in Definition 3.1 and calculate directly

$$\begin{aligned}
&\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(0,b,c,d,e)} & \begin{matrix} * \\ * \end{matrix} \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} z\right) \\
=&p_{(0,b,c,d,e)}\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} 1 & & \\ & 1 & \frac{t}{fb} \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ & -fb & -c \\ & & -\frac{1}{fb} \end{pmatrix}\begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}\right) \\
=&p_{(0,b,c,d,e)}\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}\begin{pmatrix} -bfy_1y_2 & -bfx_2y_1 & -bfx_3 \\ & f^2b^2y_1 & f^2b^2x_1+cfb \\ & & 1 \end{pmatrix}\right) .
\end{aligned}$$

Corollary 7.3 implies the formula

$$\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix} p_{(0,b,c,d,e)} & \begin{matrix} * \\ * \end{matrix} \\ & 1 \end{pmatrix} \gamma_{(0,fb,c)} z\right) = p_{(0,b,c,d,e)} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} ,$$

with the coefficients y, x given by the two formulas

$$\begin{aligned}
y =& -\frac{y_1y_2bf}{f^2b^2y_1}\sqrt{(y_1b^2f^2)^2+(f^2b^2x_1+cfb)^2} , \\
x =& -bfx_3 - \frac{(f^2b^2x_1+cfb)(-x_2y_1bf)}{f^2b^2y_1} .
\end{aligned}$$

Using the right invariance against the orthogonal group O_3 and multiplying with $\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ from the right eliminates the minus sign in entry $(1,1)$. Cancellation gives us finally the claimed formulas

$$\begin{aligned}
y =& y_2\sqrt{y_1^2b^2f^2+(fbx_1+c)^2} , \\
x =& -bfx_3 + (fbx_1+c)x_2 .
\end{aligned}$$

(3) Use the Bruhat decomposition in Lemma 3.3 and proceed similarly as in part (1). For easier readabilitiy use the abbreviation $\omega = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$ and calculate directly

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix} 1 & * & * \\ 0 & & \\ 0 & q_{(a,b,c,d,e)} & \end{pmatrix} \delta^{-1}_{(fa,b,c)} z\right)$$

$$=q_{(a,b,c,d,e)}\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & -\frac{s}{(fa,b)} & -\frac{r}{fa}\\ & 1 & 0\\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix}(fa,b) & cr & -c\frac{br+(fa,b)}{fa}\\ & \frac{fa}{(fa,b)} & -\frac{b}{(fa,b)}\\ & & \frac{1}{fa}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\right)$$

$$=q_{(a,b,c,d,e)}\mathfrak{m}_{P_{1,2}}\left(\omega\begin{pmatrix}fa(fa,b)y_1y_2 & fa(fa,b)x_2y_1+facry_1 & fa(fa,b)x_3+facrx_1-cbr-c(fa,b)\\ & \frac{f^2a^2y_1}{(fa,b)} & \frac{f^2a^2x_1-fab}{(fa,b)}\\ & & 1\end{pmatrix}\right).$$

Now we apply Corollary 7.3 to recast the previous equation as

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & * & *\\ 0 & & \\ 0 & q_{(a,b,c,d,e)} & \end{pmatrix}\delta^{-1}_{(fa,b,c)}z\right)=q_{(a,b,c,d,e)}\begin{pmatrix}y & x\\ & 1\end{pmatrix},$$

with the coefficients y, x given by the two formulas (before and after cancellation)

$$y=\left[\left(\left(\frac{f^2a^2y_1}{(fa,b)}\right)^2+\left(\frac{f^2a^2x_1-fab}{(fa,b)}\right)^2\right)(fa(fa,b)y_1y_2)^2+\left(\left(\frac{f^2a^2x_1-fab}{(fa,b)}\right)(fa(fa,b)x_2y_1+facry_1)\right.\right.$$
$$\left.\left.-\left(\frac{f^2a^2y_1}{(fa,b)}\right)(fa(fa,b)x_3+facrx_1-cbr-c(fa,b))\right)^2\right]^{\frac{1}{2}}\left(\left(\frac{f^2a^2x_1-fab}{(fa,b)}\right)^2+\left(\frac{f^2a^2y_1}{(fa,b)}\right)^2\right)^{-1}$$

$$=\quad y_1(af,b)^2\frac{\sqrt{f^2a^2y_1^2y_2^2+y_2^2(afx_1-b)^2+((fax_1-b)x_2-fax_3+c)^2}}{(fax_1-b)^2+f^2a^2y_1^2},$$

$$x=\frac{\left(\frac{f^2a^2y_1}{(fa,b)}\right)(fa(fa,b)x_2y_1+facry_1)+\left(\frac{f^2a^2x_1-fab}{(fa,b)}\right)(fa(fa,b)x_3+facrx_1-cbr-c(fa,b))}{\left(\frac{f^2a^2y_1}{(fa,b)}\right)^2+\left(\frac{f^2a^2x_1-fab}{(fa,b)}\right)^2}$$
$$=\frac{(af,b)cr}{fa}+\frac{(af,b)^2}{fa}\frac{f^2a^2y_1^2x_2+(fax_3-c)(fax_1-b)}{f^2a^2y_1^2+(fax_1-b)^2}.$$

(4) The last case is analogous to the previous one. Again use the Bruhat decomposition in Lemma 3.3 getting

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & * & *\\ 0 & & \\ 0 & q_{(0,b,c,d,e)} & \end{pmatrix}\delta^{-1}_{(0,b,c)}z\right)$$
$$=q_{(0,b,c,d,e)}\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & -\frac{t}{b} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}b & -c & \\ & \frac{1}{b} & \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\right)$$
$$=q_{(0,b,c,d,e)}\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}by_1y_2 & y_1(bx_2-c) & bx_3-cx_1\\ & \frac{y_1}{b} & \frac{x_1}{b}\\ & & 1\end{pmatrix}\right).$$

Again apply Corollary 7.3 to recast the previous equation as

$$\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1 & * & * \\ 0 & & \\ 0 & q_{(0,b,c,d,e)} & \end{pmatrix}\delta_{(0,b,c)}^{-1}z\right) = q_{(0,b,c,d,e)}\begin{pmatrix}y & x \\ & 1\end{pmatrix}\ ,$$

with the coefficients y, x given by the two formulas (before and after cancellation)

$$\begin{aligned} y =& \sqrt{(by_1y_2)^2 + (y_1(bx_2 - c))^2} = y_1\sqrt{b^2y_2^2 + (bx_2 - c)^2}\ , \\ x =& cx_1 - bx_3\ . \end{aligned}$$

□

CHAPTER 8

Values of various I_s-functions

In this chapter we calculate the values of the three different I_s-functions for the elements of the associated sets Γ_i. We begin with a trivial lemma which allows us to calculate the determinant and the scaling factor of certain Iwasawa decompositions directly.

LEMMA 8.1. *Let $\gamma \in GL_3(\mathbb{R})^+$ and $\| \cdot \|$ be the euclidean norm on $\mathbb{R}^3$. There exists a unique Iwasawa decomposition such that $\gamma = zkr$ with $z = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3, k \in SO_3, r > 0$ and the determinant of the matrix z and the scaling factor r are given explicitly through the formulas*

$$\begin{aligned} r &= \| e_3^T \gamma \| \\ \det(z) &= \det(\gamma) \| e_3^T \gamma \|^{-3} \quad . \end{aligned}$$

PROOF. First note that since the determinant of γ is positive we can achieve $k \in SO_3$ and $r > 0$ in the Iwasawa decomposition (maybe we have to change the sign of r). For the formula of the scaling factor we calculate directly using the fact that the rows of the orthogonal matrix k are normalized:

$$\| e_3^T \gamma \| = \| e_3^T zkr \| = r \| e_3^T k \| = r \ .$$

The formula for the determinant follows immediately applying the determinant multiplication rule

$$\det(z) = \det(\gamma) \det(k)^{-1} r^{-3} = \det(\gamma) \| e_3^T \gamma \|^{-3} \quad .$$

□

With this preparation the explicit formulas for the I_s-functions occuring in the twisted Eisenstein series can be proved. Note that the following formulas are also proved in [**7**].

LEMMA 8.2. *Let $\gamma \in GL_3(\mathbb{R})^+$ and $\| \cdot \|$ be the euclidean norm on $\mathbb{R}^3$, then the explicit values for the functions $I_{(s,-2s)}(*, P_{2,1})$ and $I_{(2s,-s)}(*, P_{1,2})$ at γ are given through the formulas*

$$I_{(s,-2s)}(\gamma, P_{2,1}) = \det(\gamma)^s \parallel e_3^T \gamma \parallel^{-3s} ,$$

$$I_{(2s,-s)}(\gamma, P_{1,2}) = \det(\gamma)^{-s} \parallel \gamma^{-1} e_1 \parallel^{-3s} .$$

PROOF. First write down the Iwasawa decomposition $\gamma = zkr$ with $z = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$, $k \in SO_3$, $r > 0$. We calculate explicitly using Lemma 8.1:

$$I_{(s,-2s)}(\gamma, P_{2,1}) = I_{(s,-2s)}(z, P_{2,1}) = (y_1^2 y_2)^s = \det(z)^s = \det(\gamma)^s \parallel e_3^T \gamma \parallel^{-3s} . \tag{8.1}$$

For the proof of the second part we have to pass to the element $\omega(\gamma^{-1})^T\omega$ where $\omega = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$ is the long Weyl element. Note that the Iwasawa decomposition transforms

$$\omega \left(\gamma^{-1}\right)^T \omega = \left(\omega(z^{-1})^T \omega y_1 y_2\right) \cdot \left(\omega(k^{-1})^T \omega\right) \cdot \left(r^{-1} y_1^{-1} y_2^{-1}\right) , \tag{8.2}$$

with $\omega \left(z^{-1}\right)^T \omega y_1 y_2 = \begin{pmatrix} y_2 y_1 & -y_2 x_1 & x_1 x_2 - x_3 \\ & y_2 & -x_2 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$ and $\omega(k^{-1})^T\omega \in SO_3$.

So combining (8.1) and (8.2) and afterwards using the formulas in Lemma 8.1 gives the result

$$\begin{aligned} & I_{(2s,-s)}(\gamma, P_{1,2}) \\ =& I_{(2s,-s)}(z, P_{1,2}) = (y_2^2 y_1)^s \\ =& \det\left(\omega(z^{-1})^T \omega y_1 y_2\right)^s \\ =& \det\left(\omega \left(\gamma^{-1}\right)^T \omega\right)^s \parallel e_3^T \omega(\gamma^{-1})^T \omega \parallel^{-3s} \\ =& \det(\gamma)^{-s} \parallel \gamma^{-1} e_1 \parallel^{-3s} . \end{aligned}$$

Note that the ω and the transposition in the last term can be dropped, since neither a permutation of the entries of a vector nor the transposition changes its euclidean norm.

□

Now the values of the I_s-functions for the sets $\Gamma_i(f, M, P_{2,1})$ and $\Gamma_i(f, M, P_{1,2})$ can be given in an explicit form.

LEMMA 8.3. *Let N be a squarefree integer and f a positive divisor of N. Further let* $z = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. *With the notation in Lemma 4.2, Lemma 5.2, respectively, the functions $I_{(s,-2s)}(*, P_{2,1})$ and $I_{(2s,-s)}(*, P_{1,2})$ take the following explicit values*

(1)

$$I_{(s,-2s)}\left(\begin{pmatrix} p_{(a,b,c,d,e)} & * \\ & * \\ & 1 \end{pmatrix}\gamma_{(fa,fb,c)}z, P_{2,1}\right)$$
$$=(y_1^2y_2)^s\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3 + fbx_1 + c)^2\right]^{-\frac{3s}{2}},$$

(2)

$$I_{(s,-2s)}\left(\begin{pmatrix} p_{(0,b,c,d,e)} & * \\ & * \\ & 1 \end{pmatrix}\gamma_{(0,fb,c)}z, P_{2,1}\right) = (y_1^2y_2)^s\left[f^2y_1^2b^2 + (fbx_1 + c)^2\right]^{-\frac{3s}{2}},$$

(3)

$$I_{(2s,-s)}\left(\begin{pmatrix} 1 & * & * \\ 0 & & \\ 0 & q_{(a,b,c,d,e)} & \end{pmatrix}\delta^{-1}_{(fa,b,c)}z, P_{1,2}\right)$$
$$=(y_1y_2^2)^s\left[f^2a^2y_1^2y_2^2 + y_2^2(b - fax_1)^2 + (c - bx_2 + fa(x_1x_2 - x_3))^2\right]^{-\frac{3s}{2}},$$

(4)

$$I_{(2s,-s)}\left(\begin{pmatrix} 1 & * & * \\ 0 & & \\ 0 & q_{(0,b,c,d,e)} & \end{pmatrix}\delta^{-1}_{(0,b,c)}z, P_{1,2}\right) = (y_1y_2^2)^s\left[y_2^2b^2 + (c - bx_2)^2\right]^{-\frac{3s}{2}}.$$

PROOF. We treat the parts (1),(2) and (3),(4) together.

(1) First use the invariance of $I_{(s,-2s)}(*, P_{2,1})$ against the group $P_{2,1}$, then Lemma 8.2 getting

$$I_{(s,-2s)}\left(\begin{pmatrix} p_{(a,b,c,d,e)} & * \\ & * \\ & 1 \end{pmatrix}\gamma_{(fa,fb,c)}z, P_{2,1}\right) = I_{(s,-2s)}\left(\gamma_{(fa,fb,c)}z, P_{2,1}\right)$$
$$= \det(\gamma_{(fa,fb,c)}z)^s \parallel e_3^T\gamma_{(fa,fb,c)}z \parallel^{-3s} = (y_1^2y_2)^s \parallel (fa, fb, c)z \parallel^{-3s}$$

$$=(y_1^2y_2)^s\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2\right]^{-\frac{3s}{2}}\ .$$

Considering the cases $a=0$ and $a\neq 0$ separately gives the proof.

(2) Use the invariance of $I_{(2s,-s)}(*,P_{1,2})$ against the group $P_{1,2}$, then Lemma 8.2 getting

$$I_{(2s,-s)}\left(\begin{pmatrix}1 & * & * \\ 0 & \multicolumn{2}{c}{q_{(a,b,c,d,e)}} \\ 0 & & \end{pmatrix}\delta^{-1}_{(fa,b,c)}z,P_{1,2}\right)=I_{(2s,-s)}\left(\delta^{-1}_{(fa,b,c)}z,P_{1,2}\right)$$
$$=\det\left(\delta^{-1}_{(fa,b,c)}z\right)^{-s}\parallel(\delta^{-1}_{(fa,b,c)}z)^{-1}e_1\parallel^{-3s}=(y_1^2y_2)^{-s}\parallel z^{-1}(c,b,fa)^T\parallel^{-3s}$$
$$=(y_1^2y_2)^{-s}\left[f^2a^2+y_1^{-2}(b-fax_1)^2+y_1^{-2}y_2^{-2}(c-bx_2+fa(x_1x_2-x_3))^2\right]^{-\frac{3s}{2}}$$
$$=(y_1y_2^2)^s\left[f^2a^2y_1^2y_2^2+y_2^2(b-fax_1)^2+(c-bx_2+fa(x_1x_2-x_3))^2\right]^{-\frac{3s}{2}}\ .$$

Again considering the cases $a=0$ and $a\neq 0$ separately gives the proof.

□

Next we handle the $I_{(s_1,s_2)}$-function occuring in the definition of the minimal Eisenstein series. Again we first state a trivial fact concerning the calculation of the $I_{(s_1,s_2)}$-function.

Lemma 8.4. *Let* $z=\begin{pmatrix}y_2 & x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\in\mathfrak{h}^3$. *Then the formula*

$$I_{(s_1,s_2)}(z)=y_1^{s_2-s_1}y_2^{2s_1+s_2}$$

for the calculation of the $I_{(s_1,s_2)}$*-function holds.*

Proof. Write $z=\begin{pmatrix}1 & \frac{x_2}{y_1} & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}\begin{pmatrix}y_1\left(\frac{y_2}{y_1}\right) & & \\ & y_1 & \\ & & 1\end{pmatrix}$ and use the definition of the $I_{(s_1,s_2)}$-function.

$$I_{(s_1,s_2)}(z)=y_1^{s_1+2s_2}\left(\frac{y_2}{y_1}\right)^{2s_1+s_2}=y_1^{s_2-s_1}y_2^{2s_1+s_2}\ .$$

□

For completeness we restate the explicit formulas for the values of the $I_{(s_1,s_2)}$-function on Weyl elements in [**4**, (3.4)-(3.9)]. Note that our definition of the $I_{(s_1,s_2)}$-function differs from the definition in [**4**, page 19] in this respect that the complex variables s_1,s_2 are swapped.

Lemma 8.5. *Let* $z = \begin{pmatrix} y_2 & x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$ *and denote with* ω *certain Weyl elements. Then we have the following explicit calculations of the values of the* $I_{(s_1,s_2)}$*-function at* ωz.

(1) *For* $\omega = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$ *we have*

$$I_{(s_1,s_2)}(\omega z) = (y_2^2 + x_2^2 + x_3^2)^{-\frac{3s_2}{2}} \left[y_2^2 + x_1^2 \left(\frac{y_2}{y_1} \right)^2 + \left(x_3 - \frac{x_1 x_2}{y_1} \right)^2 \right]^{-\frac{3s_1}{2}} y_1^{s_2 - s_1} y_2^{2s_1 + s_2} \ .$$

(2) *For* $\omega = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$ *we have*

$$I_{(s_1,s_2)}(\omega z) = (y_2^2 + x_2^2 + x_3^2)^{-\frac{3s_2}{2}} (y_2^2 + x_2^2)^{-\frac{3s_1}{2}} y_1^{2s_1 + s_2} y_2^{2s_1 + s_2} \ .$$

(3) *For* $\omega = \begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}$ *we have*

$$I_{(s_1,s_2)}(\omega z) = (y_1^2 + x_1^2)^{-\frac{3s_2}{2}} \left[y_2^2 + x_1^2 \left(\frac{y_2}{y_1} \right)^2 + \left(x_3 - \frac{x_1 x_2}{y_1} \right)^2 \right]^{-\frac{3s_1}{2}} y_1^{s_2 - s_1} y_2^{2s_1 + s_2} \ .$$

(4) *For* $\omega = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}$ *we have*

$$I_{(s_1,s_2)}(\omega z) = (y_1^2 + x_1^2)^{-\frac{3s_2}{2}} y_1^{s_2 - s_1} y_2^{2s_1 + s_2} \ .$$

(5) *For* $\omega = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}$ *we have*

$$I_{(s_1,s_2)}(\omega z) = (y_2^2 + x_2^2)^{-\frac{3s_1}{2}} y_1^{2s_1 + s_2} y_2^{2s_1 + s_2} \ .$$

Proof. We use Lemma 8.4 and the explicit Iwasawa decompositions in Lemma 7.2 and calculate directly, note that trivial simplifications of the terms are omitted.

(1) For $\omega = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$ we have

$$\begin{aligned}
I_{(s_1,s_2)}(\omega z) = &\left[\frac{y_1 y_2\sqrt{y_2^2+x_2^2}}{\sqrt{y_2^2+x_2^2+x_3^2}\sqrt{(y_1x_2x_3-x_1(x_2^2+y_2^2))^2+y_1^2y_2^2(y_2^2+x_2^2+x_3^2)}}\right]^{2s_1+s_2} \\
&\left[\frac{\sqrt{(y_1x_2x_3-x_1(x_2^2+y_2^2))^2+y_1^2y_2^2(y_2^2+x_2^2+x_3^2)}}{\sqrt{y_2^2+x_2^2}(y_2^2+x_2^2+x_3^2)}\right]^{s_2-s_1} \\
=&(y_1y_2)^{2s_1+s_2}(y_2^2+x_2^2+x_3^2)^{-\frac{3s_2}{2}} \\
&\left[\frac{y_1^2x_2^2x_3^2-2x_1x_2x_3y_1(x_2^2+y_2^2)+x_1^2(x_2^2+y_2^2)^2+y_1^2y_2^2(y_2^2+x_2^2+x_3^2)}{y_2^2+x_2^2}\right]^{-\frac{3s_1}{2}} \\
=&(y_2^2+x_2^2+x_3^2)^{-\frac{3s_2}{2}}\left[y_2^2+x_1^2\left(\frac{y_2}{y_1}\right)^2+\left(x_3-\frac{x_1x_2}{y_1}\right)^2\right]^{-\frac{3s_1}{2}} y_1^{s_2-s_1}y_2^{2s_1+s_2}\ .
\end{aligned}$$

(2) For $\omega = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$ we have

$$\begin{aligned}
I_{(s_1,s_2)}(\omega z) &= \left[\frac{y_1y_2}{\sqrt{y_2^2+x_2^2+x_3^2}\sqrt{y_2^2+x_2^2}}\right]^{2s_1+s_2}\left[\frac{\sqrt{y_2^2+x_2^2}}{y_2^2+x_2^2+x_3^2}\right]^{s_2-s_1} \\
&= (y_2^2+x_2^2+x_3^2)^{-\frac{3s_2}{2}}(y_2^2+x_2^2)^{-\frac{3s_1}{2}}y_1^{2s_1+s_2}y_2^{2s_1+s_2}\ .
\end{aligned}$$

(3) For $\omega = \begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}$ we have

$$\begin{aligned}
&I_{(s_1,s_2)}(\omega z) \\
=&\left[\frac{y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{(y_1^2+x_1^2)y_2^2+(x_1x_2-y_1x_3)^2}}\right]^{2s_1+s_2}\left[\frac{\sqrt{(y_1^2+x_1^2)y_2^2+(x_1x_2-y_1x_3)^2}}{x_1^2+y_1^2}\right]^{s_2-s_1} \\
=&(y_1^2y_2^2+x_1^2y_2^2+x_1^2x_2^2-2y_1x_1x_2x_3+y_1^2x_3^2)^{-\frac{3s_1}{2}}(x_1^2+y_1^2)^{-\frac{3s_2}{2}}(y_1y_2)^{2s_1+s_2} \\
=&(y_1^2+x_1^2)^{-\frac{3s_2}{2}}\left[y_2^2+x_1^2\left(\frac{y_2}{y_1}\right)^2+\left(x_3-\frac{x_1x_2}{y_1}\right)^2\right]^{-\frac{3s_1}{2}} y_1^{s_2-s_1}y_2^{2s_1+s_2}\ .
\end{aligned}$$

(4) For $\omega = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}$ we have

$$I_{(s_1,s_2)}(\omega z) = \left[\frac{y_2}{\sqrt{y_1^2+x_1^2}}\right]^{2s_1+s_2} \left[\frac{y_1}{y_1^2+x_1^2}\right]^{s_2-s_1} = (y_1^2+x_1^2)^{-\frac{3s_2}{2}} y_1^{s_2-s_1} y_2^{2s_1+s_2} .$$

(5) For $\omega = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}$ we have

$$I_{(s_1,s_2)}(\omega z) = \left[\frac{y_1 y_2}{\sqrt{y_2^2+x_2^2}}\right]^{2s_1+s_2} \left[\sqrt{y_2^2+x_2^2}\right]^{s_2-s_1} = (y_2^2+x_2^2)^{-\frac{3s_1}{2}} y_1^{2s_1+s_2} y_2^{2s_1+s_2} .$$

□

At the end of this chapter we give an analagous result to Lemma 8.3 by doing the calculation of the values of the $I_{(s_1,s_2)}$-function for the sets $\Gamma_i(f,h,P_{min})$.

LEMMA 8.6. *Let N be a squarefree integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. Further let $z = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. With the notation in Lemma 6.1 the function $I_{(s_1,s_2)}(*)$ takes the following explicit values on the sets $\Gamma_i(f,h,P_{min})$.*

(1) For the elements in $\Gamma_1(f,h,P_{min})$ the $I_{(s_1,s_2)}$-function takes the value

$$\begin{aligned} & I_{(s_1,s_2)}\left(r_{(a,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right) \\ = & \left(\frac{|a|}{h\,|d|\,(a,b)}\right)^{3s_1} y_1^{2s_2+s_1} y_2^{2s_1+s_2}\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}} \\ & \left[f^2a^2y_1^2y_2^2 + a^2y_2^2\left(fx_1 - \frac{rc}{(a,b)} + \frac{ea}{hd(a,b)^2}\right)^2\right. \\ & \left. + \left((fax_3+fbx_1+c) - \left(fx_1 - \frac{rc}{(a,b)} + \frac{ea}{hd(a,b)^2}\right)(ax_2+b)\right)^2\right]^{-\frac{3s_1}{2}} . \end{aligned}$$

(2) For the elements in $\Gamma_2\left(f,\frac{N}{f},P_{min}\right)$ the $I_{(s_1,s_2)}$-function takes the value

$$\begin{aligned} & I_{(s_1,s_2)}\left(r_{(a,b,c,0,\pm1)}\begin{pmatrix}\tau_{(0,\pm1)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right) \\ = & (a,b)^{3s_1} y_1^{s_1+2s_2} y_2^{s_2+2s_1}\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}} \\ & \left[a^2y_2^2 + (ax_2+b)^2\right]^{-\frac{3s_1}{2}} . \end{aligned}$$

(3) For the elements in $\Gamma_3(f,h,P_{min})$ the $I_{(s_1,s_2)}$-function takes the value

$$\begin{aligned}
&I_{(s_1,s_2)}\left(r_{(0,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)\\
=&y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\\
&\left[h^2f^2d^2b^2y_1^2y_2^2+h^2d^2y_2^2(fbx_1+c)^2+(e-hfdbx_3+hd(fbx_1+c)x_2)^2\right]^{-\frac{3s_1}{2}} .
\end{aligned}$$

(4) For the elements in $\Gamma_4\left(f,\frac{N}{f},P_{min}\right)$ the $I_{(s_1,s_2)}$-function takes the value

$$I_{(s_1,s_2)}\left(r_{(0,b,c,0,\pm1)}\begin{pmatrix}\tau_{(0,\pm1)} & \\ & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)=y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}} .$$

(5) For the elements in $\Gamma_5(N,1,P_{min})$ the $I_{(s_1,s_2)}$-function takes the value

$$I_{(s_1,s_2)}\left(\begin{pmatrix}\tau_{(d,e)} & \\ & 1\end{pmatrix}\gamma_{(0,0,\pm1)}z\right)=y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[d^2y_2^2+(x_2d\pm e)^2\right]^{-\frac{3s_1}{2}} .$$

Proof. Our main tool for this proof is the explicit formula in Lemma 8.5 for the $I_{(s_1,s_2)}$-function in conjunction with the left-invariance of the $I_{(s_1,s_2)}$-function against P_{min} and diagonal matrices with entries ±1 on the diagonal. Further we use the Bruhat decomposition for $\tau_{(d,e)}$ and $\gamma_{(a,b,c)}$ in Lemma 3.5 and 3.1.

(1) The proof is performed by a straightforward calculation

$$\begin{aligned}
&I_{(s_1,s_2)}\left(r_{(a,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix}1 & \frac{t}{dh} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right)\\
=&\quad I_{(s_1,s_2)}\left(\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \frac{r}{fa}\\ & 1 & \frac{s}{f(a,b)}\\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}\begin{pmatrix}fa & fb & c\\ & \frac{(a,b)}{a} & \frac{-rc}{fa}\\ & & \frac{1}{f(a,b)}\end{pmatrix}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & *\\ & 1 & *\\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}\begin{pmatrix}fa & fb & c\\ & \frac{(a,b)}{a} & \frac{-rc}{fa}\\ & & \frac{1}{f(a,b)}\end{pmatrix}z\right)
\end{aligned}$$

$$=I_{(s_1,s_2)}\left(\begin{pmatrix}1 & & * \\ & 1 & * \\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix} & dh & e \\ & & \frac{1}{dh} \\ 1 & & \end{pmatrix}\begin{pmatrix}fa & fb & c \\ & \frac{(a,b)}{a} & \frac{-rc}{fa} \\ & & \frac{1}{f(a,b)}\end{pmatrix}z\right)$$

$$=I_{(s_1,s_2)}\left(\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix} & dh & e \\ & & \frac{1}{dh} \\ 1 & & \end{pmatrix}\begin{pmatrix}fa & fb & c \\ & \frac{(a,b)}{a} & \frac{-rc}{fa} \\ & & \frac{1}{f(a,b)}\end{pmatrix}z\right)$$

$$=I_{(s_1,s_2)}\left(\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix}1 & & \\ & dh & e \\ & & \frac{1}{dh}\end{pmatrix}\begin{pmatrix}fa & fb & c \\ & \frac{(a,b)}{a} & \frac{-rc}{fa} \\ & & \frac{1}{f(a,b)}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1\end{pmatrix}\right)$$

$$=I_{(s_1,s_2)}\left(\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix}f^2hda(a,b)y_1y_2 & f^2hd(a,b)y_1(ax_2+b) & fhd(a,b)(fax_3+fbx_1+c) \\ & \frac{fh^2(a,b)^2d^2y_1}{a} & \frac{h^2d^2(a,b)^2}{a}\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right) \\ & & 1\end{pmatrix}\right).$$

Now apply the explicit formula in Lemma 8.5 and simplify the terms. This is done in the calculation

$$I_{(s_1,s_2)}\left(r_{(a,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right)$$

$$= \left[(f^2hda(a,b)y_1y_2)^2+(f^2hd(a,b)y_1(ax_2+b))^2+(fhd(a,b)(fax_3+fbx_1+c))^2\right]^{-\frac{3s_2}{2}}$$
$$\Bigg[(f^2hda(a,b)y_1y_2)^2+\left(\frac{h^2d^2(a,b)^2}{a}\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)\right)^2\left(\frac{f^2hda(a,b)y_1y_2}{\frac{fh^2(a,b)^2d^2y_1}{a}}\right)^2$$
$$+\left((fhd(a,b)(fax_3+fbx_1+c))-\frac{\left(\frac{h^2d^2(a,b)^2}{a}\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)\right)(f^2hd(a,b)y_1(ax_2+b))}{\left(\frac{fh^2(a,b)^2d^2y_1}{a}\right)}\right)^2\Bigg]^{-\frac{3s_1}{2}}$$
$$\left(\frac{fh^2(a,b)^2d^2y_1}{|a|}\right)^{s_2-s_1}\left(f^2h\,|d|\,|a|\,(a,b)y_1y_2\right)^{2s_1+s_2}$$

$$=(f^2h^2d^2(a,b)^2)^{-\frac{3s_2}{2}}\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}$$
$$\Bigg[(f^2hda(a,b)y_1y_2)^2+(fhd(a,b)ay_2)^2\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)^2$$
$$+\left((fhd(a,b))(fax_3+fbx_1+c)-(fhd(a,b))\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)(ax_2+b)\right)^2\Bigg]^{-\frac{3s_1}{2}}$$
$$(fh(a,b)\,|d|)^{3s_2}\,(f\,|a|)^{3s_1}\,y_1^{2s_2+s_1}y_2^{2s_1+s_2}$$

$$=\left(\frac{|a|}{h\,|d|\,(a,b)}\right)^{3s_1}y_1^{2s_2+s_1}y_2^{2s_1+s_2}\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}$$

$$\left[f^2a^2y_1^2y_2^2 + a^2y_2^2\left(fx_1 - \frac{rc}{(a,b)} + \frac{ea}{hd(a,b)^2}\right)^2\right.$$
$$\left.+\left((fax_3+fbx_1+c) - \left(fx_1 - \frac{rc}{(a,b)} + \frac{ea}{hd(a,b)^2}\right)(ax_2+b)\right)^2\right]^{-\frac{3s_1}{2}}.$$

(2) We do a straightforward calculation, which is similar to that in the first part, but shorter since $\tau_{(0,\pm1)}$ is an upper triangular matrix. We have

$$I_{(s_1,s_2)}\left(r_{(a,b,c,0,\pm1)}\begin{pmatrix}\tau_{(0,\pm1)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix}1 & & \frac{r}{fa} \\ & 1 & \frac{s}{f(a,b)} \\ & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix}fa & fb & c \\ & \frac{(a,b)}{a} & \frac{-rc}{fa} \\ & & \frac{1}{f(a,b)}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1\end{pmatrix}\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix}f^2a(a,b)y_1y_2 & f^2(a,b)y_1(ax_2+b) & f(a,b)(fax_3+fbx_1+c) \\ & \frac{f(a,b)^2}{a}y_1 & \frac{(a,b)}{a}(f(a,b)x_1-rc) \\ & & 1\end{pmatrix}\right).$$

Apply the explicit formula in Lemma 8.5 and simplify the terms. This is done in the calculation

$$I_{(s_1,s_2)}\left(r_{(a,b,c,0,\pm1)}\begin{pmatrix}\tau_{(0,\pm1)} & \\ & 1\end{pmatrix}\gamma_{(fa,fb,c)}z\right)$$
$$= \left[(f^2a(a,b)y_1y_2)^2 + (f^2(a,b)y_1(ax_2+b))^2 + (f(a,b)(fax_3+fbx_1+c))^2\right]^{-\frac{3s_2}{2}}$$
$$\left[(f^2a(a,b)y_1y_2)^2 + (f^2(a,b)y_1(ax_2+b))^2\right]^{-\frac{3s_1}{2}}\left(\left(\frac{f(a,b)^2}{a}y_1\right)(f^2a(a,b)y_1y_2)\right)^{2s_1+s_2}$$

$$=(f(a,b))^{-3s_2}\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}$$
$$(f^2(a,b)y_1)^{-3s_1}\left[a^2y_2^2 + (ax_2+b)^2\right]^{-\frac{3s_1}{2}}\left(f^3(a,b)^3y_1^2y_2\right)^{2s_1+s_2}$$

$$=(a,b)^{3s_1}y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}$$
$$\left[a^2y_2^2 + (ax_2+b)^2\right]^{-\frac{3s_1}{2}}.$$

(3) A short calculation gives

$$
\begin{aligned}
&I_{(s_1,s_2)}\left(r_{(0,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix}1 & \frac{t}{dh} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \frac{t}{fb} \\ & & 1\end{pmatrix}\begin{pmatrix}-1 & & \\ & & -1 \\ & -1 & \end{pmatrix}\begin{pmatrix}1 & & \\ & -fb & -c \\ & & -\frac{1}{fb}\end{pmatrix}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix}1 & & * \\ & 1 & * \\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}dh & e & \\ & \frac{1}{dh} & \\ & & 1\end{pmatrix}\begin{pmatrix}-1 & & \\ & & -1 \\ & -1 & \end{pmatrix}\begin{pmatrix}1 & & \\ & -fb & -c \\ & & -\frac{1}{fb}\end{pmatrix}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix}-1 & & \\ & & -1 \\ & -1 & \end{pmatrix}\begin{pmatrix}-dh & & -e \\ & & -\frac{1}{dh} \\ & -1 & \end{pmatrix}\begin{pmatrix}1 & & \\ & -fb & -c \\ & & -\frac{1}{fb}\end{pmatrix}z\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix}dh & & e \\ & 1 & \\ & & \frac{1}{dh}\end{pmatrix}\begin{pmatrix}1 & & \\ & -fb & -c \\ & & -\frac{1}{fb}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1\end{pmatrix}\right)\\
=&I_{(s_1,s_2)}\left(\begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix}h^2fd^2by_1y_2 & -h^2fd^2by_1x_2 & hd(e-hfdbx_3) \\ & f^2hb^2dy_1 & hfbd(fbx_1+c) \\ & & 1\end{pmatrix}\right).
\end{aligned}
$$

In the last step the right invariance against the maximal compact subgroup O_3 was used to kill the minus sign in the $(1,1)$ entry. So it remains to apply the explicit formula in Lemma 8.5 and simplify the terms. We have

$$
\begin{aligned}
&I_{(s_1,s_2)}\left(r_{(0,b,c,d,e)}\begin{pmatrix}\tau_{(dh,e)} & \\ & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)\\
=&(h^2fd^2\,|b|\,y_1y_2)^{2s_1+s_2}(f^2hb^2dy_1)^{s_2-s_1}\left[(f^2hb^2dy_1)^2+(hfbd(fbx_1+c))^2\right]^{-\frac{3s_2}{2}}\\
&\left[(h^2fd^2by_1y_2)^2+\left(\frac{h^2fd^2by_1y_2}{f^2hb^2dy_1}\right)^2(hfbd(fbx_1+c))^2\right.\\
&\left.+\left((hd(e-hfdbx_3))-\frac{(hfbd(fbx_1+c))(-h^2fd^2by_1x_2)}{f^2hb^2dy_1}\right)^2\right]^{-\frac{3s_1}{2}}\\
=&y_1^{s_1+2s_2}y_2^{2s_1+s_2}(h\,|d|)^{3s_1+3s_2}(f\,|b|)^{3s_2}(hf\,|b|\,|d|)^{-3s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\\
&\left[(h^2fd^2by_1y_2)^2+h^4d^4y_2^2(fbx_1+c)^2+\left(hd(e-hfdbx_3)+h^2d^2(fbx_1+c)x_2\right)^2\right]^{-\frac{3s_1}{2}}
\end{aligned}
$$

$$=y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}$$
$$\left[h^2f^2d^2b^2y_1^2y_2^2+h^2d^2y_2^2(fbx_1+c)^2+(e-hfdbx_3+hd(fbx_1+c)x_2)^2\right]^{-\frac{3s_1}{2}}\quad .$$

(4) This case is similarly to part (2). The proof is given through the calculation

$$I_{(s_1,s_2)}\left(r_{(0,b,c,0,\pm1)}\begin{pmatrix}\tau_{(0,\pm1)} & \\ & 1\end{pmatrix}\gamma_{(0,fb,c)}z\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix}1 & & \\ & 1 & \frac{t}{fb}\\ & & 1\end{pmatrix}\begin{pmatrix}-1 & & \\ & & -1\\ & -1 & \end{pmatrix}\begin{pmatrix}1 & & \\ & -fb & -c\\ & & -\frac{1}{fb}\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix}-1 & & \\ & & -1\\ & -1 & \end{pmatrix}\begin{pmatrix}fby_1y_2 & -fby_1x_2 & -fbx_3\\ & f^2b^2y_1 & fb(fbx_1+c)\\ & & 1\end{pmatrix}\right)$$
$$=\left(f^2b^2y_1\right)^{s_2-s_1}(fby_1y_2)^{2s_1+s_2}\left[(f^2b^2y_1)^2+(fb(fbx_1+c))^2\right]^{-\frac{3s_2}{2}}$$
$$=y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\quad .$$

(5) Finally we handle the last case. We have

$$I_{(s_1,s_2)}\left(\begin{pmatrix}\tau_{(d,e)} & \\ & 1\end{pmatrix}\gamma_{(0,0,\pm1)}z\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix}1 & \frac{t}{d} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}d & e & \\ & \frac{1}{d} & \\ & & 1\end{pmatrix}\begin{pmatrix}\pm1 & & \\ & 1 & \\ & & \pm1\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\right)$$
$$=I_{(s_1,s_2)}\left(\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2d & y_1(e\pm x_2d) & dx_3\pm ex_1\\ & \frac{y_1}{d} & \pm\frac{x_1}{d}\\ & & 1\end{pmatrix}\right)\quad .$$

Similarly to part (3) we used the right invariance against the maximal compact subgroup O_3 to ensure that all diagonal entries are positive. We have

$$I_{(s_1,s_2)}\left(\begin{pmatrix}\tau_{(d,e)} & \\ & 1\end{pmatrix}\gamma_{(0,0,\pm1)}z\right)$$
$$=\left((y_1y_2d)\left(\frac{y_1}{d}\right)\right)^{2s_1+s_2}\left[(y_1y_2d)^2+(y_1(e\pm x_2d))^2\right]^{-\frac{3s_1}{2}}$$
$$=y_1^{s_1+2s_2}y_2^{2s_1+s_2}\left[d^2y_2^2+(x_2d\pm e)^2\right]^{-\frac{3s_1}{2}}\quad .$$

□

Part IV

Power series

CHAPTER 9

Power series for unramified primes

In this section we will take a closer look at certain families of power series in two independent variables, which will occur in the unramified parts of the Fourier coefficients in the Eisenstein series. We will introduce these power series in a generality such that we can handle both the twisted and the minimal Eisenstein series together through substituting certain number theoretic functions into the two independent variables. It turns out that the power series associated to the degenerate Fourier coefficients can be easily calculated and the ones occuring in the non degenerate Fourier coefficients are even rational polynomials, which conform to a certain transformation rule. This transformation rule, which is of purely combinatorial nature, can be viewed as the combinatorial part of the functional equation of the Eisenstein series. The terminology and notation, which are used through the whole chapter is introduced in the following definition.

Definition 9.1. Let p be a prime number. Let $(A_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}[X,Y]$ be a sequence of polynomials in the two variables X, Y, which has the initial value $A_0 = 1$ and fulfills the recursion

$$A_{n+2} = A_1 A_{n+1} - Y A_n \quad \forall n \in \mathbb{N} \,.$$

Associated to the prime p and the sequence $(A_n)_{n\in\mathbb{N}}$ are the family of power series $(F_p(\alpha,\beta))_{(\alpha,\beta)\in\mathbb{N}_0\times(\{0\}\cup p^{\mathbb{N}_0})} \subset \mathbb{C}(X,Y)$ and the two degree three polynomials S_p, T_p defined by

(1)
$$F_p(\alpha,\beta) := \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k\leq \alpha+n}} c_{p^{n-k}}(\beta) c_{p^k}(p^\alpha) A_{n+\alpha-2k} Y^k \,,$$

(2)
$$S_p := 1 - A_1 X + Y X^2 \,,$$

(3)
$$T_p := 1 - pA_1 X + p^2 Y X^2 \,,$$

where $c_q(h)$ denotes the Ramanujan sum.

Our main results in this chapter are formulas for the power series $F_p(\alpha,\beta)$, distinguishing whether β vanishes or not, which are of a more explicit nature than the definition. In order to reach this goal we first state two technical lemmata.

LEMMA 9.2. *For any nonnegative integers α, β the identity*

$$A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^\beta\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^\beta\right) Y\right] = S_p \sum_{n=0}^{\beta} A_{\alpha+1+n} p^n X^n$$

holds.

PROOF. The proof is given through induction over β.

(1) Basis: $\beta = 0$.
We use the properties of the Ramanujan sums summarized in Lemma 18.3 in Appendix A. We have

$$\begin{aligned} & A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^0\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^0\right) Y\right] \\ =& A_{\alpha+1} + X\left[A_{1+\alpha+1}(-1) - A_{1+\alpha-1} Y\right] + X^2\left[0 - A_{2+\alpha-1}(-1)Y\right] \\ =& A_{\alpha+1} - X A_1 A_{\alpha+1} + A_{\alpha+1} Y X^2 \\ =& S_p \sum_{n=0}^{0} A_{\alpha+1+n} p^n X^n \ . \end{aligned}$$

In the second last equation the recursion formula for the sequence A_n was used.

(2) Induction step:

$$\begin{aligned} & A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^{\beta+1}\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^{\beta+1}\right) Y\right] \\ =& A_{\alpha+1} \quad + \sum_{n\geq 2} X^n A_{n+\alpha+1} c_{p^n}\left(p^{\beta+1}\right) - \sum_{n\geq 3} X^n A_{n+\alpha-1} c_{p^{n-1}}\left(p^{\beta+1}\right) Y \\ & + X^1\left[A_{1+\alpha+1} c_{p^1}\left(p^{\beta+1}\right) - A_{1+\alpha-1} c_{p^{1-1}}\left(p^{\beta+1}\right) Y\right] - X^2 A_{2+\alpha-1} c_{p^{2-1}}\left(p^{\beta+1}\right) Y \\ =& A_{\alpha+1} + \sum_{n\geq 2} X^n A_{n+\alpha+1} p c_{p^{n-1}}\left(p^\beta\right) - \sum_{n\geq 3} X^n A_{n+\alpha-1} p c_{p^{n-2}}\left(p^\beta\right) Y \\ & + X\left[A_{\alpha+2}(p-1) - A_\alpha Y\right] - (p-1) A_{\alpha+1} X^2 Y \ . \end{aligned}$$

The reduction property of the Ramanujan sum in Lemma 18.3 part (4) was applied in the last equation. Make the index shift $n \to n-1$ in each sum and add suitable terms to extract the power series in the induction hypothesis. We have

$$
\begin{aligned}
&A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^{\beta+1}\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^{\beta+1}\right) Y\right] \\
=&A_{\alpha+1} + \sum_{n\geq 1} X^{n+1} A_{n+\alpha+2} p c_{p^n}\left(p^{\beta}\right) - \sum_{n\geq 2} X^{n+1} A_{n+\alpha} p c_{p^{n-1}}\left(p^{\beta}\right) Y \\
&+ X\left[A_{\alpha+2}(p-1) - A_\alpha Y\right] - (p-1) A_{\alpha+1} X^2 Y \\
=&pX\left[A_{(\alpha+1)+1} + \sum_{n\geq 1} X^n \left[A_{n+(\alpha+1)+1} c_{p^n}\left(p^{\beta}\right) - A_{n+(\alpha+1)-1} c_{p^{n-1}}\left(p^{\beta}\right) Y\right]\right] \\
&+ pA_{\alpha+1} X^2 Y - pA_{\alpha+2} X + A_{\alpha+1} + X\left[A_{\alpha+2}(p-1) - A_\alpha Y\right] - (p-1) A_{\alpha+1} X^2 Y\ .
\end{aligned}
$$

Now apply the induction hypothesis for the sum and the recursion for the sequence A_n for the additional terms. Finally the proof is completed by the calculation

$$
\begin{aligned}
&A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^{\beta+1}\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^{\beta+1}\right) Y\right] \\
=&pXS_p \sum_{n=0}^{\beta} A_{(\alpha+1)+1+n} p^n X^n + A_{\alpha+1} S_p \\
=&S_p \sum_{n=0}^{\beta+1} A_{\alpha+1+n} p^n X^n\ .
\end{aligned}
$$

□

LEMMA 9.3. *For any nonnegative integer α the identity*

$$
A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1}\phi\left(p^n\right) - A_{n+\alpha-1}\phi\left(p^{n-1}\right) Y\right] = \frac{S_p}{T_p}\left(A_{\alpha+1} - pA_\alpha XY\right)
$$

holds, where ϕ denotes Euler's ϕ-function.

PROOF. Similarly as in the proof of the previous lemma we calculate directly. We have

$$
\begin{aligned}
&T_p\left(A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1}\phi\left(p^n\right) - A_{n+\alpha-1}\phi\left(p^{n-1}\right) Y\right]\right) \\
=&T_p A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1}\phi\left(p^n\right) - A_{n+\alpha-1}\phi\left(p^{n-1}\right) Y\right] \\
&- \sum_{n\geq 1} X^{n+1}\left[A_1 A_{n+\alpha+1} p\phi\left(p^n\right) - A_1 A_{n+\alpha-1} p\phi\left(p^{n-1}\right) Y\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n\geq 1} X^{n+2}\left[A_{n+\alpha+1}p^2\phi\left(p^n\right)Y - A_{n+\alpha-1}p^2\phi\left(p^{n-1}\right)Y^2\right]\\
=&(1-pA_1X+p^2YX^2)A_{\alpha+1} + X\left[A_{\alpha+2}\phi\left(p\right) - A_\alpha Y\right] + X^2\left[A_{\alpha+3}\phi\left(p^2\right) - A_{\alpha+1}\phi\left(p\right)Y\right]\\
&- X^2\left[A_1A_{\alpha+2}p\phi\left(p\right) - A_1A_\alpha pY\right]\\
&+\sum_{n\geq 3} X^n\Big[\left(A_{n+\alpha+1}\phi\left(p^n\right) - A_1A_{n+\alpha}p\phi\left(p^{n-1}\right) + A_{n+\alpha-1}p^2\phi\left(p^{n-2}\right)Y\right)\\
&- Y\left(A_{n+\alpha-1}\phi\left(p^{n-1}\right) - A_1A_{n+\alpha-2}p\phi\left(p^{n-2}\right) + A_{n+\alpha-3}p^2\phi\left(p^{n-3}\right)Y\right)\Big]\,.
\end{aligned}
$$

Using the recursion formula for the sequence A_n implies that the sum vanishes up to the last summand for $n = 3$. So finally we get the formula

$$
\begin{aligned}
&T_p\left(A_{\alpha+1} + \sum_{n\geq 1} X^n\left[A_{n+\alpha+1}\phi\left(p^n\right) - A_{n+\alpha-1}\phi\left(p^{n-1}\right)Y\right]\right)\\
=&A_{\alpha+1} - X\left(pYA_\alpha + A_1A_{\alpha+1}\right) + X^2Y\left(A_{\alpha+1} + pA_1A_\alpha\right) - pA_\alpha Y^2X^3\\
=&S_p\left(A_{\alpha+1} - pA_\alpha XY\right)\ .
\end{aligned}
$$

□

Now we can state an explicit formula for the polynomials F_p in the case β is non-zero. This formula does not include complicated obstructions like the inner sum in the original definition of F_p.

Lemma 9.4. *The explicit formula*

$$
F_p\left(\alpha, p^\beta\right) = S_p\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha} p^{k+l}A_{\alpha+k-l}X^{k+l}Y^l\ ,
$$

for the polynomials F_p holds.

Proof. The proof is given through double induction over α and β.

(1) Basis: $\alpha = 0$.
We start calculating the base case. We have

$$
\begin{aligned}
F_p\left(0, p^0\right) =&\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k\leq 0+n}} c_{p^{n-k}}(p^0)c_{p^k}(p^0)A_{n+0-2k}Y^k\\
=&A_0 + A_1c_p\left(p^0\right)c_{p^0}\left(p^0\right)X + A_0c_p\left(p^0\right)c_p\left(p^0\right)YX^2
\end{aligned}
$$

$$
\begin{aligned}
&= S_p \\
&= S_p \sum_{k=0}^{0} \sum_{l=0}^{0} p^{k+l} A_{0+k-l} X^{k+l} Y^l \ .
\end{aligned}
$$

Next we continue with the calculation for prime powers greater than 1, so let β be a nonnegative integer. The identity

$$
\begin{aligned}
F_p\left(0, p^{\beta+1}\right) &= \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n \\ 2k\leq 0+n}} c_{p^{n-k}}(p^{\beta+1}) c_{p^k}(p^0) A_{n+0-2k} Y^k \\
&= A_0 + A_1 c_p\left(p^{\beta+1}\right) X + \sum_{n\geq 2} X^n \left[c_{p^n}\left(p^{\beta+1}\right) A_n + c_{p^{n-1}}\left(p^{\beta+1}\right) c_p(p^0) A_{n-2} Y\right]
\end{aligned}
$$

holds. Use the reduction property of the Ramanujan sums to transform the above sum into

$$
\begin{aligned}
F_p\left(0, p^{\beta+1}\right) =& 1 + A_1 c_p\left(p^{\beta+1}\right) X - c_p\left(p^{\beta+1}\right) A_0 Y X^2 + \sum_{n\geq 2} X^n p c_{p^{n-1}}\left(p^\beta\right) A_n \\
&- \sum_{n\geq 3} X^n p c_{p^{n-2}}\left(p^\beta\right) A_{n-2} Y \ .
\end{aligned}
$$

Then make the index shift $n \to n+1$ in both sums to transform the above term into

$$
\begin{aligned}
F_p\left(0, p^{\beta+1}\right) =& 1 + A_1(p-1)X - (p-1)YX^2 + \sum_{n\geq 1} X^{n+1} p c_{p^n}\left(p^\beta\right) A_{n+1} \\
&- \sum_{n\geq 2} X^{n+1} p c_{p^{n-1}}\left(p^\beta\right) A_{n-1} Y \\
=& 1 - A_1 X - (p-1)YX^2 + c_{p^0}\left(p^\beta\right) A_0 p Y X^2 \\
&+ pX\left[A_1 + \sum_{n\geq 1} X^n \left[c_{p^n}\left(p^\beta\right) A_{n+1} - c_{p^{n-1}}\left(p^\beta\right) A_{n-1} Y\right]\right] \ .
\end{aligned}
$$

Finally Lemma 9.2 gives us the result

$$
\begin{aligned}
F_p\left(0, p^{\beta+1}\right) &= S_p + pXS_p \sum_{n=0}^{\beta} A_{0+1+n} p^n X^n \\
&= S_p \sum_{k=0}^{\beta+1} \sum_{l=0}^{0} p^{k+l} A_{0+k-l} X^{k+l} Y^l \ .
\end{aligned}
$$

(2) Induction step:

The proof is given through a straightforward calculation. First use the reduction property of the Ramanujan sums. We have the identity

$$\begin{aligned}
F_p\left(\alpha+1,p^\beta\right) =& \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k\leq(\alpha+1)+n}} c_{p^{n-k}}\left(p^\beta\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+(\alpha+1)-2k} Y^k \\
=& \sum_{n\geq 0} X^n \sum_{\substack{2-1\leq k-1\leq n-1\\ 2(k-1)\leq \alpha+(n-1)}} c_{p^{(n-1)-(k-1)}}\left(p^\beta\right) p c_{p^{k-1}}\left(p^\alpha\right) A_{(n-1)+\alpha-2(k-1)} Y^{k-1} Y \\
&+ \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq \min(1,n)\\ 2k\leq \alpha+n+1}} c_{p^{n-k}}\left(p^\beta\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+\alpha+1-2k} Y^k .
\end{aligned}$$

In the first sum make the index shifts $n \to n+1$ and $k \to k+1$ and add the missing term for $k=0$ to obtain $F_p\left(\alpha,p^\beta\right)$. This transforms the above sum into

$$\begin{aligned}
F_p\left(\alpha+1,p^\beta\right) =& pXY \sum_{n\geq 1} X^n \sum_{\substack{1\leq k\leq n\\ 2k\leq \alpha+n}} c_{p^{n-k}}\left(p^\beta\right) c_{p^k}\left(p^\alpha\right) A_{n+\alpha-2k} Y^k \\
&+ \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq \min(1,n)\\ 2k\leq \alpha+n+1}} c_{p^{n-k}}\left(p^\beta\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+\alpha+1-2k} Y^k \\
=& pXYF_p\left(\alpha,p^\beta\right) - pXY \sum_{n\geq 0} X^n c_{p^n}\left(p^\beta\right) A_{n+\alpha} \\
&+ \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq \min(1,n)\\ 2k\leq \alpha+n+1}} c_{p^{n-k}}\left(p^\beta\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+\alpha+1-2k} Y^k .
\end{aligned}$$

Next shift the index $n \to n+1$ in the first sum and evaluate the inner sum in the second sum. We have

$$\begin{aligned}
F_p\left(\alpha+1,p^\beta\right) =& pXYF_p\left(\alpha,p^\beta\right) - \sum_{n\geq 1} X^n p c_{p^{n-1}}\left(p^\beta\right) A_{n+\alpha-1} Y \\
&+ A_{\alpha+1} + \sum_{n\geq 1} X^n \left[c_{p^n}\left(p^\beta\right) A_{n+\alpha+1} + c_{p^{n-1}}\left(p^\beta\right)(p-1) A_{n+\alpha-1} Y\right] \\
=& pXYF_p\left(\alpha,p^\beta\right) + A_{\alpha+1} + \sum_{n\geq 1} X^n \left[A_{n+\alpha+1} c_{p^n}\left(p^\beta\right) - A_{n+\alpha-1} c_{p^{n-1}}\left(p^\beta\right) Y\right] .
\end{aligned}$$

Now apply the induction hypothesis and Lemma 9.2. This gives us

$$F_p\left(\alpha+1,p^\beta\right) = pXYS_p\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}A_{\alpha+k-l}X^{k+l}Y^l + S_p\sum_{n=0}^{\beta}A_{\alpha+1+n}p^nX^n$$
$$= S_p\sum_{k=0}^{\beta}\left[A_{(\alpha+1)+k}p^{k+0}X^{k+0}Y^0 + \sum_{l=0}^{\alpha}p^{k+(l+1)}A_{(\alpha+1)+k-(l+1)}X^{k+(l+1)}Y^{l+1}\right] .$$

Finally make the index shift $l \to l+1$ in the inner sum and get the requested result. We have the identity

$$F_p\left(\alpha+1,p^\beta\right) = S_p\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha+1}p^{k+l}A_{(\alpha+1)+k-l}X^{k+l}Y^l .$$

□

An important transformation law for the polynomials F_p can now easily derived.

LEMMA 9.5. *The transformation law*

$$\left(\frac{F_p\left(\alpha,p^\beta\right)}{S_p}\right)\left(p^{-2}X^{-1}Y^{-1},Y\right) = p^{-\alpha-\beta}X^{-\alpha-\beta}Y^{-\beta}\left(\frac{F_p\left(\beta,p^\alpha\right)}{S_p}\right)(X,Y)$$

for the polynomials F_p holds.

PROOF. Use the formula in Lemma 9.4 and calculate directly

$$\left(\frac{F_p\left(\alpha,p^\beta\right)}{S_p}\right)\left(p^{-2}X^{-1}Y^{-1},Y\right)$$
$$= \sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}A_{\alpha+k-l}\left(p^{-2}X^{-1}Y^{-1}\right)^{k+l}Y^l$$
$$= p^{-\alpha-\beta}X^{-\alpha-\beta}Y^{-\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{(\beta-k)+(\alpha-l)}A_{\beta+(\alpha-l)-(\beta-k)}X^{(\beta-k)+(\alpha-l)}Y^{\beta-k} .$$

The index shifts $\beta-k \to k$ and $\alpha-l \to \alpha$ gives the result

$$\left(\frac{F_p\left(\alpha,p^\beta\right)}{S_p}\right)\left(p^{-2}X^{-1}Y^{-1},Y\right)$$

$$= p^{-\alpha-\beta} X^{-\alpha-\beta} Y^{-\beta} \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} p^{k+l} A_{\beta+l-k} X^{k+l} Y^{k}$$
$$= p^{-\alpha-\beta} X^{-\alpha-\beta} Y^{-\beta} \left(\frac{F_p\left(\beta, p^{\alpha}\right)}{S_p} \right) (X, Y) \ .$$

□

At the end of this chapter an explicit formula for the polynomials F_p in the case that β vanishes is given.

LEMMA 9.6. *The explicit formula*

$$F_p\left(\alpha, 0\right) = \frac{S_p}{T_p} A_{\alpha}$$

for the polynomials F_p holds.

PROOF. Again the proof is given through induction over α.

(1) Basis: $\alpha = 0$.
We do a straightforward calculation

$$T_p F_p\left(0,0\right) = T_p \sum_{n \geq 0} X^n \sum_{\substack{0 \leq k \leq n \\ 2k \leq 0+n}} \phi\left(p^{n-k}\right) c_{p^k}\left(p^0\right) A_{n+0-2k} Y^k$$
$$= T_p \left(1 + (p-1) A_1 X + \sum_{n \geq 2} X^n \left[\phi\left(p^n\right) A_n - \phi\left(p^{n-1}\right) A_{n-2} Y \right] \right) \ .$$

Make an index shift $n \to n-1$ and use the reduction property of Euler's ϕ-function to transform the above terms into

$$T_p F_p\left(0,0\right) = T_p \left(1 + (p-1) A_1 X + \sum_{n \geq 1} X^{n+1} \left[\phi\left(p^{n+1}\right) A_{n+1} - \phi\left(p^n\right) A_{n-1} Y \right] \right)$$
$$= pX T_p \left(A_1 + \sum_{n \geq 1} X^n \left[\phi\left(p^n\right) A_{n+1} - \phi\left(p^{n-1}\right) A_{n-1} Y \right] \right)$$
$$+ T_p \left(1 - A_1 X + pX^2 Y - \phi\left(p\right) X^2 Y \right) \ .$$

To get the result apply Lemma 9.3 to the above sum. We have

$$T_p F_p\left(0,0\right) = pX S_p \left(A_1 - pXY\right) + T_p S_p = S_p A_0 \ .$$

(2) Induction step:

The proof is given through a straightforward calculation. First use the reduction property of the Ramanujan sums. We have the identity

$$\begin{aligned}T_pF_p(\alpha+1,0) =& T_p\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k\leq(\alpha+1)+n}} \phi\left(p^{n-k}\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+(\alpha+1)-2k}Y^k\\ =& T_p\sum_{n\geq 0} X^n \sum_{\substack{2-1\leq k-1\leq n-1\\ 2(k-1)\leq\alpha+(n-1)}} \phi\left(p^{(n-1)-(k-1)}\right) pc_{p^{k-1}}\left(p^{\alpha}\right) A_{(n-1)+\alpha-2(k-1)}Y^{k-1}Y\\ &+ T_p\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq \min(1,n)\\ 2k\leq\alpha+n+1}} \phi\left(p^{n-k}\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+\alpha+1-2k}Y^k\ .\end{aligned}$$

In the first sum make the index shifts $n \to n+1$ and $k \to k+1$ and add the missing term for $k=0$ to obtain $F_p(\alpha,0)$. We have

$$\begin{aligned}T_pF_p(\alpha+1,0) =& pXYT_p\sum_{n\geq 1} X^n \sum_{\substack{1\leq k\leq n\\ 2k\leq\alpha+n}} \phi\left(p^{n-k}\right) c_{p^k}\left(p^{\alpha}\right) A_{n+\alpha-2k}Y^k\\ &+ T_p\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq \min(1,n)\\ 2k\leq\alpha+n+1}} \phi\left(p^{n-k}\right) c_{p^k}\left(p^{\alpha+1}\right) A_{n+\alpha+1-2k}Y^k\\ =& pXYT_pF_p(\alpha,0) - pXYT_p\sum_{n\geq 0} X^n\phi\left(p^n\right) A_{n+\alpha}\\ &+ T_p\left(A_{\alpha+1} + \sum_{n\geq 1} X^n\left[\phi\left(p^n\right) A_{n+\alpha+1} + \phi\left(p^{n-1}\right)(p-1)A_{n+\alpha-1}Y\right]\right)\\ =& pXYT_pF_p(\alpha,0) + T_p\left(A_{\alpha+1} + \sum_{n\geq 1} X^n\left[A_{n+\alpha+1}\phi\left(p^n\right) - A_{n+\alpha-1}\phi\left(p^{n-1}\right)Y\right]\right)\ .\end{aligned}$$

The induction hypothesis in conjunction with Lemma 9.3 applied to the second series gives the result

$$T_pF_p(\alpha+1,0) = pXYS_pA_\alpha + S_p\left(A_{\alpha+1} - pA_\alpha XY\right) = S_pA_{\alpha+1}\ .$$

□

CHAPTER 10

Power series for ramified primes

In this section we analyze the power series, which will occur in the ramified parts of the Fourier coefficients in the Eisenstein series. It will turn out that these power series are closely related to the power series associated to the unramified primes. Analogously to the explicit calculation of $F_p(\alpha, \beta)$ we will calculate some of these power series in the "degenerate" case explicitly. We adopt the terminology and notation from the last chapter.

Definition 10.1. Associated to the prime p and the sequence $(A_n)_{n\in\mathbb{N}}$ are the families of power series $(G_p(\alpha,\beta))_{(\alpha,\beta)}, (H_p(\alpha,\beta))_{(\alpha,\beta)} \subset \mathbb{C}(X,Y)$ defined by

$$G_p(\alpha,\beta) := \sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n \\ 2k+1\leq \alpha+n}} c_{p^{n-k}}(\beta) c_{p^{k+1}}(p^\alpha) A_{n+\alpha-2k-1} Y^k ,$$
$$H_p(\alpha,\beta) := \sum_{n\geq 0} X^n A_{\alpha+n} c_{p^n}(\beta) ,$$

with $(\alpha,\beta) \in \mathbb{N}_0 \times \left(\{0\} \cup p^{\mathbb{N}_0}\right)$.

The calculation of these power series shows that they are expressible as a sum consisting of the power series F_p and a rest term of a quite elementary form. Since the power series G_p also occurs in the minimal Eisenstein series, we will present this technique by calculating G_p in the case of a vanishing β. The other power series occuring in the Eisenstein series twisted by a Maass cusp form can be handled similarly.

Lemma 10.2. *With the convention $A_{-1} := 0$ the explicit formulas for the power series G_p and H_p given by*

(1)
$$H_p(\alpha,0) = \frac{A_\alpha - (A_{\alpha+1} + pA_{\alpha-1}Y)X + pA_\alpha Y X^2}{T_p} ,$$

(2)
$$G_p(\alpha,0) = (p-1)\frac{A_{\alpha-1} - A_\alpha X}{T_p} ,$$

hold.

Proof. The proof is given by reduction of G_p to F_p and applying the explicit formula in Lemma 9.6.

(1) We begin with the calculation of H_p. Combining the reduction formula for Ramanujan sums and the recursion for the sequence A_n will give the proof. We have

$$\begin{aligned}
T_pH_p(\alpha,0) =& \left(1 - pA_1X + p^2YX^2\right)\sum_{n\geq 0} X^n\phi(p^n)A_{n+\alpha} \\
=&\sum_{n=0}^{2} X^n\phi(p^n)A_{n+\alpha} - pA_1X\sum_{n=0}^{1} X^n\phi(p^n)A_{n+\alpha} + p^2YX^2A_\alpha \\
&+\sum_{n\geq 3} X^n\left[\phi(p^n)A_{n+\alpha} - p\phi\left(p^{n-1}\right)A_1A_{n+\alpha-1} + p^2\phi\left(p^{n-2}\right)A_{n+\alpha-2}Y\right] \\
=&\left(A_\alpha + X\phi(p)A_{\alpha+1} + X^2\phi\left(p^2\right)A_{\alpha+2}\right) - \left(pA_1A_\alpha X + p\phi(p)A_1A_{\alpha+1}X^2\right) + p^2YX^2A_\alpha \\
=&A_\alpha - \left(A_{\alpha+1} + pA_{\alpha-1}Y\right)X + pA_\alpha YX^2\ .
\end{aligned}$$

(2) Next we continue with the calculation of G_p. For $\alpha = 0$ evaluate G_p directly using the recursion formula for the sequence A_n. The identiy

$$\begin{aligned}
T_pG_p(0,0) =&T_p\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k+1\leq 0+n}} c_{p^{n-k}}(0)c_{p^{k+1}}\left(p^0\right)A_{n+0-2k-1}Y^k \\
=&-\left(1 - pA_1X + p^2YX^2\right)\sum_{n\geq 1} X^n\phi(p^n)A_{n-1} \\
=&-\left(X\phi(p)A_0 + X^2\phi\left(p^2\right)A_1\right) + pA_1X\left(X\phi(p)A_0\right) \\
&-\sum_{n\geq 3} X^n\left[\phi(p^n)A_{n-1} - p\phi\left(p^{n-1}\right)A_1A_{n-2} + p^2\phi\left(p^{n-2}\right)A_{n-3}Y\right] \\
=&-(p-1)X \\
=&(p-1)\left(A_{-1} - XA_0\right)
\end{aligned}$$

holds. For $\alpha > 0$ use the reduction property of the Ramanujan sums to trace back the polynomials G_p to the polynomials F_p and H_p. This gives us

$$T_pG_p(\alpha+1,0) = T_p\sum_{n\geq 0} X^n \sum_{\substack{0\leq k\leq n\\ 2k+1\leq(\alpha+1)+n}} c_{p^{n-k}}(0)\,c_{p^{k+1}}\left(p^{\alpha+1}\right)A_{n+(\alpha+1)-2k-1}Y^k\ .$$

Split the inner sum into two sums according to whether k vanishes or not, then apply the reduction formula for Ramanujan sums and the recursion for the sequence A_n. After that use Lemma 9.6 to evaluate F_p and part (1) for the evaluation of H_p. We have

$$T_pG_p(\alpha+1,0) =pT_p\sum_{n\geq 0} X^n \sum_{\substack{0<k\leq n\\ 2k\leq\alpha+n}} c_{p^{n-k}}(0)c_{p^k}\left(p^\alpha\right)A_{n+\alpha-2k}Y^k + (p-1)T_p\sum_{n\geq 0} X^n c_{p^n}(0)\,A_{n+\alpha}$$

$$
\begin{aligned}
=&pT_pF_p\left(\alpha,0\right)-pT_p\sum_{n\geq 0}X^n\phi\left(p^n\right)A_{n+\alpha}+(p-1)T_p\sum_{n\geq 0}X^n\phi\left(p^n\right)A_{n+\alpha}\\
=&pS_pA_\alpha-\big(A_\alpha-\left(A_{\alpha+1}+pA_{\alpha-1}Y\right)X+pA_\alpha YX^2\big)\\
=&(p-1)\left(A_\alpha-A_{\alpha+1}X\right)\ .
\end{aligned}
$$

□

Part V

Fourier expansion and functional equation for the Eisenstein series twisted by a constant Maass form

CHAPTER 11

Fourier expansion of the Eisenstein series $E(z, s, f, P_{2,1})$

In this chapter the Fourier expansion of the Eisenstein series $E(z, s, f, P_{2,1})$ is calculated. As a preparation we evaluate the Dirichlet series which occur in the Fourier coefficients of the Eisenstein series $E(z, s, f, P_{2,1})$.

Definition 11.1. Let N be a squarefree integer, f a positive divisor of N and m an integer. Define the Dirichlet series $A_m(s, f, P_{2,1})$ associated to these parameters by

$$A_m(s, f, P_{2,1}) := \sum_{a\geq 1} a^{-s} \sum_{\substack{0\leq t<a \\ \left(\frac{N}{f},a,t\right)=1}} e\left(m\frac{t}{a}\right) \sum_{\substack{0\leq q<fa \\ (f(a,t),q)=1}} 1 \ .$$

The next step is to evaluate these Dirichlet series.

Lemma 11.2. *Let N be a squarefree integer, f a positive divisor of N and m a non-zero integer. The following explicit formulas for the Dirichlet series $A_*(s, f, P_{2,1})$ are valid.*

(1) For a non-zero index of the above Dirichlet series we have

$$A_m(s, f, P_{2,1}) = \prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}(1-p^{1-s})\sigma_{2-s}(|m|)L_{\chi_N}(s)^{-1} \ ,$$

(2) For a vanishing index of the above Dirichlet series we have

$$A_0(s, f, P_{2,1}) = \prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}(1-p^{1-s})L_{\chi_N}(s)^{-1}\zeta(s-2) \ .$$

Proof. We handle part (1) and (2) together and assume for the present that m is an arbitrary integer. The first step is to evaluate the inner sum. In order to do this split the summation over $q = f(a,t)k + l$ with $0 \leq k < \frac{a}{(a,t)}$ and $0 \leq l < f(a,t)$. We have

$$A_m(s, f, P_{2,1}) = \sum_{a\geq 1} a^{-s} \sum_{\substack{0\leq t<a \\ \left(\frac{N}{f},a,t\right)=1}} e\left(m\frac{t}{a}\right) \sum_{0\leq k<\frac{a}{(a,t)}} \sum_{\substack{0\leq l<f(a,t) \\ (f(a,t),l)=1}} 1$$

$$= \sum_{a\geq 1} a^{-s} \sum_{\substack{0\leq t<a \\ \left(\frac{N}{f},a,t\right)=1}} e\left(m\frac{t}{a}\right) \frac{a}{(a,t)}\phi(f(a,t)) \,.$$

In the second step split $t = t_1t_2$ with $t_1 \mid a$ and $(\frac{a}{t_1}, t_2) = 1$. Note that the gcd-conditions transforms as $(a,t) = t_1$ and $1 = \left(\frac{N}{f}, a, t_1t_2\right) = \left(\frac{N}{f}, t_1\right)$. This gives us

$$A_m(s,f,P_{2,1}) = \sum_{a\geq 1} a^{-s} \sum_{\substack{t_1\mid a \\ \left(\frac{N}{f},t_1\right)=1}} \sum_{\substack{0\leq t_2<\frac{a}{t_1} \\ \left(\frac{a}{t_1},t_2\right)=1}} e\left(m\frac{t_2}{\frac{a}{t_1}}\right)\frac{a}{t_1}\phi(ft_1) = \sum_{a\geq 1} a^{-s} \sum_{\substack{t_1\mid a \\ \left(\frac{N}{f},t_1\right)=1}} c_{\frac{a}{t_1}}(m)\frac{a}{t_1}\phi(ft_1) \,.$$

In the third step split the summation over $a = a_1a_2a_3$ with $a_1 \mid \left(\frac{N}{f}\right)^\infty$, $a_2 \mid f^\infty$, $(a_3, N) = 1$ and do the same for the summation over t_1. We have

$$A_m(s,f,P_{2,1}) = \sum_{\substack{a_1\mid\left(\frac{N}{f}\right)^\infty \\ a_2\mid f^\infty \\ (a_3,N)=1}} (a_1a_2a_3)^{-s} \sum_{\substack{t_1\mid a_1 \\ t_2\mid a_2 \\ t_3\mid a_3 \\ \left(\frac{N}{f},t_1t_2t_3\right)=1}} c_{\frac{a_1a_2a_3}{t_1t_2t_3}}(m)\frac{a_1a_2a_3}{t_1t_2t_3}\phi(ft_1t_2t_3) \,.$$

In the fourth step note that $1 = \left(\frac{N}{f}, t_1t_2t_3\right) = \left(\frac{N}{f}, t_1\right)$ and $t_1 \mid \left(\frac{N}{f}\right)^\infty$ implies $t_1 = 1$. Then use the multiplicativity of the Ramanujan sums to factorize the Dirichlet series. We have

$$A_m(s,f,P_{2,1}) = \left(\sum_{a_1\mid\left(\frac{N}{f}\right)^\infty} a_1^{1-s}c_{a_1}(m)\right)\left(\sum_{a_2\mid f^\infty} a_2^{-s}\sum_{t_2\mid a_2} c_{\frac{a_2}{t_2}}(m)\frac{a_2}{t_2}\phi(ft_2)\right)$$
$$\left(\sum_{(a_3,N)=1} a_3^{-s}\sum_{t_3\mid a_3} c_{\frac{a_3}{t_3}}(m)\frac{a_3}{t_3}\phi(t_3)\right) \,.$$

In the fifth step factorize the second and third Dirichlet series, which are both convolutions of two Dirichlet series. We have

$$A_m(s,f,P_{2,1}) = \left(\sum_{a\mid\left(\frac{N}{f}\right)^\infty} c_a(m)a^{1-s}\right)\left(\sum_{a\mid f^\infty} c_a(m)a^{1-s}\right)\left(\sum_{a\mid f^\infty} \phi(fa)a^{-s}\right)$$

$$\cdot\left(\sum_{(a,N)=1} c_a(m)a^{1-s}\right)\left(\sum_{(a,N)=1}\phi(a)a^{-s}\right)$$
$$=\left(\sum_{a\geq 1} c_a(m)a^{1-s}\right)\left(\sum_{a|f^\infty}\phi(fa)a^{-s}\right)\left(\sum_{(a,N)=1}\phi(a)a^{-s}\right)$$
$$=\left(\sum_{a\geq 1} c_a(m)a^{1-s}\right)\left(\prod_{p|f}\frac{p-1}{1-p^{1-s}}\right)\left(\frac{L_{\chi_N}(s-1)}{L_{\chi_N}(s)}\right)$$
$$=\prod_{p|f}(p-1)\frac{L_{\chi_{\frac{N}{f}}}(s-1)}{L_{\chi_N}(s)}\left(\sum_{a\geq 1} a^{1-s}c_a(m)\right) .$$

For the evaluation of the second Dirichlet sum Lemma 18.5 part (2) was used and for the third one Lemma 18.4 part (4). In the evaluation of the last Dirichlet series one has to distinguish between the cases m vanishes or not. In the case $m \neq 0$ use Lemma 18.4 part (2). So in this case we finally get the claimed formula

$$\begin{aligned} A_m(s,f,P_{2,1}) &= \prod_{p|f}(p-1)L_{\chi_{\frac{N}{f}}}(s-1)L_{\chi_N}(s)^{-1}\sigma_{1-(s-1)}(|m|)\zeta(s-1)^{-1} \\ &= \prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}(1-p^{1-s})\sigma_{2-s}(|m|)L_{\chi_N}(s)^{-1} . \end{aligned}$$

In the case $m = 0$ use Lemma 18.4 part (4). Again we get the claimed formula

$$\begin{aligned} A_0(s,f,P_{2,1}) &= \prod_{p|f}(p-1)L_{\chi_{\frac{N}{f}}}(s-1)L_{\chi_N}(s)^{-1}\frac{\zeta((s-1)-1)}{\zeta(s-1)} \\ &= \prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}(1-p^{1-s})L_{\chi_N}(s)^{-1}\zeta(s-2) . \end{aligned}$$

□

So now the main result of this chapter, the Fourier expansion of the Eisenstein series $E(z,s,f,P_{2,1})$, can be stated. In the case of the lattice $SL_3(\mathbb{Z})$ this result was first established in [**6**].

THEOREM 11.3. *Let N be a positive squarefree integer and f a positive divisor of N. The Eisenstein series $G(z,s,f,P_{2,1})$ satisfies the explicit Fourier expansion*

$$G(z,s,f,P_{2,1}) = \sum_{m_2=0}^{\infty} G_{0,m_2}(z,s,f,P_{2,1}) + \sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty}\sum_{m_2=0}^{\infty} G_{m_1,m_2}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z,s,f,P_{2,1}\right) .$$

Let m_1, m_2 be positive integers and let $m_1 = \prod_p p^{\alpha_p}$ be the prime decomposition of m_1. Then the Fourier coefficients satisfy the explicit formulas

(1)

$$G_{m_1,m_2}(z,s,f,P_{2,1}) = 0 ,$$

(2)

$$G_{m_1,0}(z,s,f,P_{2,1}) = 4y_1^{\frac{s+1}{2}} y_2^s f^{-3s} |m_1|^{\frac{3s-1}{2}} \prod_{p|f} \frac{1-p^{(1-3s)(\alpha_p+1)} - p + p^{1+(1-3s)\alpha_p}}{p^{(1-3s)(\alpha_p+1)}-1} \prod_{p|\frac{N}{f}} \frac{p^{1-3s}-1}{p^{(1-3s)(\alpha_p+1)}-1}$$
$$\sigma_{1-3s}(|m_1|) K_{\frac{3s-1}{2}}(2\pi |m_1| y_1) e(m_1 x_1) ,$$

(3)

$$G_{0,m_2}(z,s,f,P_{2,1}) = 4y_1^{1-s} y_2^{1-\frac{s}{2}} f^{-3s} \prod_{p|f}(p-1) \prod_{p|\frac{N}{f}} \left(1-p^{1-3s}\right) \sigma_{2-3s}(|m_2|) |m_2|^{\frac{3s}{2}-1}$$
$$K_{\frac{3s}{2}-1}(2\pi |m_2| y_2) e(m_2 x_2) ,$$

(4)

$$G_{0,0}(z,s,f,P_{2,1}) = 2\pi^{1-\frac{3s}{2}} y_1^{1-s} y_2^{2-2s} f^{-3s} \prod_{p|f} \frac{p-1}{1-p^{2-3s}} \prod_{p|\frac{N}{f}} \frac{1-p^{1-3s}}{1-p^{2-3s}} \Gamma\left(\frac{3s}{2}-1\right) L_{\chi_N}(3s-2)$$
$$+ 2\pi^{\frac{1-3s}{2}} y_1^{1-s} y_2^s f^{-3s} \prod_{p|f} \frac{p-1}{1-p^{1-3s}} \Gamma\left(\frac{3s-1}{2}\right) L_{\chi_N}(3s-1)$$
$$+ 2\delta_{f,N} \pi^{-\frac{3s}{2}} y_1^{2s} y_2^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) .$$

PROOF. Since the unipotent part of z can be absorbed into the integration, one can assume without loss of generality that $z = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix} \in \mathfrak{h}^3$. We start the calculation of the Fourier coefficients by dividing the summation in the Eisenstein series into several cases according to the Bruhat decomposition in Definition 3.8. Assume for the present that m_1, m_2 are arbitrary integers, then we have

$$G_{m_1,m_2}(z,s,f,P_{2,1}) = \int_0^1\int_0^1\int_0^1 G\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix} z,s,f,P_{2,1}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} \sum_{\gamma \in \Gamma_i(f,1,P_{2,1})} \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \\
&\int_0^1 \int_0^1 \int_0^1 I_{(s,-2s)}\left(\gamma \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, P_{2,1}\right) e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
&=: \sum_{i=1}^{3} G^{(i)}_{m_1,m_2}(z,s) \ .
\end{aligned}
$$

Using the explicit description of the sets $\Gamma_i(f,1,P_{2,1})$ in Lemma 4.2 and the explicit formula for the values of the $I_{(s,-2s)}(*,P_{2,1})$-function on these sets in Lemma 8.3 we start to calculate each of the three summands above.

(1) We start with the most difficult part the calculation of $G^{(1)}_{m_1,m_2}$. We have

$$
\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{a \neq 0} \sum_{\substack{b \in \mathbb{Z} \\ \left(\frac{N}{f},(a,b)\right)=1}} \sum_{\substack{c \in \mathbb{Z} \\ (fa,fb,c)=1}} \\
&\int_0^1 \int_0^1 \int_0^1 (y_1^2 y_2)^s \left[f^2 a^2 y_1^2 y_2^2 + f^2 y_1^2 (a x_2 + b)^2 + (fa x_3 + fb x_1 + c)^2\right]^{-\frac{3s}{2}} \\
&e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ .
\end{aligned}
$$

In the first step pass to the condition $a \geq 1$ extracting a factor 2. After that split the summation over c in this way $c = fak + r$ with $k \in \mathbb{Z}$, $0 \leq r < fa$. Note that the gcd-condition transforms as follows $1 = (fa, fb, c) = (fa, fb, fak + r) = (fa, fb, r)$. This gives us

$$
\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s) = 2\pi^{-\frac{3s}{2}} (y_1^2 y_2)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{a \geq 1} \sum_{\substack{b \in \mathbb{Z} \\ \left(\frac{N}{f},(a,b)\right)=1}} \sum_{k \in \mathbb{Z}} \sum_{\substack{0 \leq r < fa \\ (fa,fb,r)=1}} \\
&\int_0^1 \int_0^1 \int_0^1 \left[f^2 a^2 y_1^2 y_2^2 + f^2 y_1^2 (a x_2 + b)^2 + (fa(x_3 + k) + fb x_1 + r)^2\right]^{-\frac{3s}{2}} \\
&e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ .
\end{aligned}
$$

In the second step pass to an infinite integral in the variable x_3 by the shift $x_3 \to x_3 + k$. Then substitute $x_3 \to x_3 + \frac{fbx_1 + r}{fa}$ in the infinite integral in the variable x_3. We have

$$
G^{(1)}_{m_1,m_2}(z,s) = 2\pi^{-\frac{3s}{2}} \left(y_1^2 y_2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{a \geq 1} \sum_{\substack{b \in \mathbb{Z} \\ \left(\frac{N}{f},(a,b)\right)=1}} \sum_{\substack{0 \leq r < fa \\ (fa,fb,r)=1}}
$$

$$\int_{-\infty}^{\infty}\int_0^1\int_0^1 \left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+f^2a^2x_3^2\right]^{-\frac{3s}{2}} e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

After that split the summation over b in this way $b = ak+t$ with $k \in \mathbb{Z}$, $0 \le t < a$. Note that the two gcd-conditions transform as follows $1 = (fa, fb, r) = (fa, f(ka+t), r) = (fa, ft, r)$ and $1 = \left(\frac{N}{f}, (a,b)\right) = \left(\frac{N}{f}, (a, ka+t)\right) = \left(\frac{N}{f}, (a,t)\right)$. This gives us

$$G^{(1)}_{m_1,m_2}(z,s) = 2\pi^{-\frac{3s}{2}}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\sum_{a\ge1}\sum_{k\in\mathbb{Z}}\sum_{\substack{0\le t<a\\ \left(\frac{N}{f},(a,t)\right)=1}}\sum_{\substack{0\le r<fa\\ (fa,ft,r)=1}}$$

$$\int_{-\infty}^{\infty}\int_0^1\int_0^1 \left[f^2a^2y_1^2y_2^2+f^2y_1^2(a(x_2+k)+t)^2+f^2a^2x_3^2\right]^{-\frac{3s}{2}} e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

In the fourth step pass to an infinite integral in the variable x_2 through the shift $x_2 \to x_2+k$. Then substitute $x_2 \to x_2 + \frac{t}{a}$ in the infinite integral in the variable x_2 and pick up an exponential $e\left(m_2\frac{t}{a}\right)$. We have

$$\begin{aligned}G^{(1)}_{m_1,m_2}(z,s) =&2\pi^{-\frac{3s}{2}}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\sum_{a\ge1}\sum_{\substack{0\le t<a\\ \left(\frac{N}{f},(a,t)\right)=1}} e\left(m_2\frac{t}{a}\right)\sum_{\substack{0\le r<fa\\ (fa,ft,r)=1}}\int_0^1 e(-m_1x_1)\mathrm{d}x_1\\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f^2a^2y_1^2y_2^2+f^2a^2y_1^2x_2^2+f^2a^2x_3^2\right]^{-\frac{3s}{2}} e(-m_2x_2)\mathrm{d}x_2\mathrm{d}x_3\ .\end{aligned}$$

Now evaluate the exponential integral in the variable x_1 and pull out the factor f^2a^2 in the infinite double integral. This gives the identity

$$\begin{aligned}G^{(1)}_{m_1,m_2}(z,s) =&2\delta_{0,m_1}\pi^{-\frac{3s}{2}}f^{-3s}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)A_{m_2}(3s,f,P_{2,1})\\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{-\frac{3s}{2}} e(-m_2x_2)\mathrm{d}x_2\mathrm{d}x_3\ .\end{aligned}$$

Use Lemma 11.2 to evaluate the Dirichlet series and Lemma 19.3 for the evaluation of the double integral. For $m_2 \neq 0$ the formula

$$\begin{aligned}G^{(1)}_{m_1,m_2}(z,s) =&2\delta_{0,m_1}\pi^{-\frac{3s}{2}}f^{-3s}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\\ &\left(\prod_{p\mid f}(p-1)\prod_{p\mid\frac{N}{f}}\left(1-p^{1-3s}\right)\sigma_{2-3s}(|m_2|)L_{\chi_N}(3s)^{-1}\right.\end{aligned}$$

$$\left(2\pi^{\frac{3s}{2}} y_1^{1-2\left(\frac{3s}{2}\right)} y_2^{1-\frac{3s}{2}} \left|m_2\right|^{\frac{3s}{2}-1} \Gamma\left(\frac{3s}{2}\right)^{-1} K_{\frac{3s}{2}-1}\left(2\pi \left|m_2\right| y_2\right)\right)$$
$$= 4\delta_{0,m_1} y_1^{1-s} y_2^{1-\frac{s}{2}} f^{-3s} \prod_{p|f}(p-1) \prod_{p|\frac{N}{f}} \left(1-p^{1-3s}\right) \sigma_{2-3s}(|m_2|) \left|m_2\right|^{\frac{3s}{2}-1} K_{\frac{3s}{2}-1}\left(2\pi \left|m_2\right| y_2\right)$$

is valid. For $m_2 = 0$ the formula

$$\begin{aligned} G^{(1)}_{m_1,0}(z,s) =& 2\delta_{0,m_1} \pi^{-\frac{3s}{2}} f^{-3s} \left(y_1^2 y_2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \\ & \left(\prod_{p|f}(p-1) \prod_{p|\frac{N}{f}} \left(1-p^{1-3s}\right) L_{\chi_N}(3s)^{-1} \zeta(3s-2)\right) \\ & \left(\pi y_1^{1-2\left(\frac{3s}{2}\right)} y_2^{2-2\left(\frac{3s}{2}\right)} \frac{\Gamma\left(\frac{3s}{2}-1\right)}{\Gamma\left(\frac{3s}{2}\right)}\right) \\ =& 2\delta_{0,m_1} \pi^{1-\frac{3s}{2}} y_1^{1-s} y_2^{2-2s} f^{-3s} \prod_{p|f} \frac{p-1}{1-p^{2-3s}} \prod_{p|\frac{N}{f}} \frac{1-p^{1-3s}}{1-p^{2-3s}} \Gamma\left(\frac{3s}{2}-1\right) L_{\chi_N}(3s-2) \end{aligned}$$

is valid.

(2) We proceed with the calculation of $G^{(2)}_{m_1,m_2}$. Since many steps are similar to those in the calculation before we will take an abbreviation here. We have

$$\begin{aligned} G^{(2)}_{m_1,m_2}(z,s) =& \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{b\neq 0 \\ \left(b,\frac{N}{f}\right)=1}} \sum_{\substack{c\in\mathbb{Z} \\ (fb,c)=1}} \\ & \int_0^1 \int_0^1 \int_0^1 (y_1^2 y_2)^s \left[f^2 b^2 y_1^2 + (fbx_1 + c)^2\right]^{-\frac{3s}{2}} e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \,. \end{aligned}$$

Again pass to the condition $b \geq 1$ extracting a factor 2. After that split the summation over c in this way $c = fbk + r$ with $k \in \mathbb{Z}$, $0 \leq r < fb$ and note that the gcd-condition transforms as follows $1 = (fb, c) = (fb, fbk + r) = (fb, r)$. This gives us

$$\begin{aligned} G^{(2)}_{m_1,m_2}(z,s) =& 2(y_1^2 y_2)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{b\geq 1 \\ \left(b,\frac{N}{f}\right)=1}} \sum_{k\in\mathbb{Z}} \sum_{\substack{0\leq r<fb \\ (fb,r)=1}} \\ & \int_0^1 \int_0^1 \int_0^1 \left[f^2 b^2 y_1^2 + (fb(x_1+k) + r)^2\right]^{-\frac{3s}{2}} e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \,. \end{aligned}$$

Pass to an infinite integral in the variable x_1 through the shift $x_1 \to x_1 + k$. Then substitute $x_1 \to x_1 + \frac{r}{fb}$ in the infinite integral in the variable x_1 and pick up an exponential $e\left(m_1 \frac{r}{fb}\right)$. This gives the identity

$$\begin{aligned}G^{(2)}_{m_1,m_2}(z,s) =& 2(y_1^2 y_2)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{b\geq 1 \\ \left(b,\frac{N}{f}\right)=1}} \sum_{\substack{0\leq r<fb \\ (fb,r)=1}} e\left(m_1 \frac{r}{fb}\right) \\ & \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \left[f^2 b^2 y_1^2 + f^2 b^2 x_1^2\right]^{-\frac{3s}{2}} e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\ =& 2\left(y_1^2 y_2\right)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) f^{-3s} \sum_{\substack{b\geq 1 \\ \left(b,\frac{N}{f}\right)=1}} b^{-3s} c_{fb}(m_1) \int_0^1 \mathrm{d}x_3 \int_0^1 e(-m_2 x_2) \mathrm{d}x_2 \\ & \int_{-\infty}^{\infty} \left[y_1^2 + x_1^2\right]^{-\frac{3s}{2}} e(-m_1 x_1) \mathrm{d}x_1 \,.\end{aligned}$$

Evaluate the integrals in the variables x_2 and x_3 and use the multiplicativity of the Ramanujan sums to split the Dirichlet series. We have

$$\begin{aligned}G^{(2)}_{m_1,m_2}(z,s) =& 2\delta_{0,m_2} \left(y_1^2 y_2\right)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) f^{-3s} \sum_{\substack{b\geq 1 \\ (b,N)=1}} b^{-3s} c_b(m_1) \sum_{b|f^\infty} b^{-3s} c_{fb}(m_1) \\ & \int_{-\infty}^{\infty} \left[y_1^2 + x_1^2\right]^{-\frac{3s}{2}} e(-m_1 x_1) \mathrm{d}x_1 \,.\end{aligned}$$

First assume $m_1 \neq 0$ with prime factor decomposition $m_1 = \prod_p p^{\alpha_p}$. Use Lemma 18.4 part (2) to evaluate the first Dirichlet series, Lemma 18.5 part (1) for the second one and use Lemma 19.3 part (3) for the evaluation of the integral. The formula

$$\begin{aligned}& G^{(2)}_{m_1,m_2}(z,s) \\ =& 2\delta_{0,m_2} \left(y_1^2 y_2\right)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) f^{-3s} \left(\sigma_{1-3s}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_N}(3s)^{-1}\right. \\ & \left.\prod_{p|f} \left((p-1)\sigma_{1-3s}\left(p^{\alpha_p}\right) - p^{1+(1-3s)\alpha_p}\right)\right) \left(2\pi^{\frac{3s}{2}} \left|m_1\right|^{\frac{3s}{2}-\frac{1}{2}} y_1^{\frac{1}{2}-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right)^{-1} K_{\frac{3s}{2}-\frac{1}{2}}\left(2\pi \left|m_1\right| y_1\right)\right) \\ =& 4\delta_{0,m_2} y_1^{\frac{s+1}{2}} y_2^s f^{-3s} \left|m_1\right|^{\frac{3s-1}{2}} \prod_{p|f} \left((p-1)\sigma_{1-3s}\left(p^{\alpha_p}\right) - p^{1+(1-3s)\alpha_p}\right)\end{aligned}$$

$$\sigma_{1-3s}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) K_{\frac{3s-1}{2}}\left(2\pi\left|m_1\right| y_1\right)$$
$$=4\delta_{0,m_2} y_1^{\frac{s+1}{2}} y_2^s f^{-3s}\left|m_1\right|^{\frac{3s-1}{2}} \prod_{p|f} \frac{1-p^{(1-3s)(\alpha_p+1)}-p+p^{1+(1-3s)\alpha_p}}{p^{(1-3s)(\alpha_p+1)}-1} \prod_{p|\frac{N}{f}} \frac{p^{1-3s}-1}{p^{(1-3s)(\alpha_p+1)}-1}$$
$$\sigma_{1-3s}\left(\left|m_1\right|\right) K_{\frac{3s-1}{2}}\left(2\pi\left|m_1\right| y_1\right)$$

is valid. Next assume $m_1 = 0$. Use Lemma 18.4 part (4) to evaluate the first Dirichlet series, Lemma 18.5 part (2) for the second one and use Lemma 19.3 part (4) for the evaluation of the integral. The formula

$$G_{0,m_2}^{(2)}(z,s) = 2\delta_{0,m_2}\left(y_1^2 y_2\right)^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) f^{-3s}\left(\frac{L_{\chi_N}(3s-1)}{L_{\chi_N}(3s)}\right)$$
$$\left(\prod_{p|f} \frac{p-1}{1-p^{1-3s}}\right)\left(\sqrt{\pi} y_1^{1-2\left(\frac{3s}{2}\right)} \frac{\Gamma\left(\frac{3s}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{3s}{2}\right)}\right)$$
$$=2\delta_{0,m_2} \pi^{\frac{1-3s}{2}} y_1^{1-s} y_2^s f^{-3s} \prod_{p|f} \frac{p-1}{1-p^{1-3s}} \Gamma\left(\frac{3s-1}{2}\right) L_{\chi_N}(3s-1)$$

is valid.

(3) It remains to do the easiest part namely the calculation of $G_{m_1,m_2}^{(3)}(z,s)$. Again Lemma 4.2 gives the description of the set $\Gamma_3(f,1,P_{2,1})$, which consists of two diagonal matrices at most. Use Lemma 8.2 to calculate the corresponding values of the $I_{(s,-2s)}(*,P_{2,1})$-function directly. We have

$$G_{m_1,m_2}^{(3)}(z,s) = \delta_{f,N} \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \int_0^1 \int_0^1 \int_0^1 \left[I_{(s,-2s)}\left(\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix} z, P_{2,1}\right)\right.$$
$$\left. + I_{(s,-2s)}\left(\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix} z, P_{2,1}\right)\right] e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$
$$=\delta_{f,N} \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \int_0^1 \int_0^1 \int_0^1 \left[\det\left(\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}\right)^s \| e_3^T \begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix} \|^{-3s}\right.$$
$$\left. + \det\left(\begin{pmatrix}-y_1y_2&-y_1x_2&-x_3\\&y_1&x_1\\&&-1\end{pmatrix}\right)^s \| e_3^T \begin{pmatrix}-y_1y_2&-y_1x_2&-x_3\\&y_1&x_1\\&&-1\end{pmatrix} \|^{-3s}\right] e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$
$$=2\delta_{f,N}\delta_{0,m_1}\delta_{0,m_2} \pi^{-\frac{3s}{2}} \left(y_1^2y_2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s).$$

So it remains to collect the results in the three calculations to complete the proof. □

CHAPTER 12

Fourier expansion of the Eisenstein series $E(z,s,f,P_{1,2})$

In this chapter the same work as in the previous chapter is done for the Eisenstein series $E(z,s,f,P_{1,2})$. Again we start with the definition and calculation of a Dirichlet series which occurs in the Fourier coefficients of the Eisenstein series $E(z,s,f,P_{1,2})$.

DEFINITION 12.1. Let N be a squarefree integer, f a positive divisor of N and m an integer. Define the Dirichlet series $A_m(s,f,P_{1,2})$ associated to these parameters by

$$A_m(s,f,P_{1,2}) := \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s} \sum_{0\leq t<fa} e\left(m\frac{t}{fa}\right) \sum_{\substack{0\leq q<fa\\ ((fa,t),q)=1}} 1\ .$$

Again the next step is to evaluate this Dirichlet series.

LEMMA 12.2. *Let N be a squarefree integer, f a positive divisor of N and $m = \prod_p p^{\alpha_p}$ a non-zero integer with corresponding prime factor decomposition. The following explicit formulas for the Dirichlet series $A_*(s,f,P_{1,2})$ are valid.*

(1) For a non-zero index of the above defined Dirichlet series we have

$$A_m(s,f,P_{1,2}) = \prod_{p|f}\left((p^2-1)\sigma_{2-s}\left(p^{\alpha_p}\right) - p^{2+(2-s)\alpha_p}\right)\sigma_{2-s}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_N}(s)^{-1}\ .$$

(2) For a vanishing index of the above Dirichlet series we have

$$A_0(s,f,P_{1,2}) = \prod_{p|f}\left(p^2-1\right)\frac{L_{\chi_{\frac{N}{f}}}(s-2)}{L_{\chi_N}(s)}\ .$$

PROOF. We proceed similarly as in the proof of Lemma 11.2 and handle both parts together, so assume for the present that m is an arbitrary integer. Again the first step is to evaluate the inner sum, so split the summation over $q=(fa,t)k+l$ with $0\leq k<\frac{fa}{(fa,t)}$ and $0\leq l<(fa,t)$. This gives us

$$A_m(s,f,P_{1,2}) = \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s} \sum_{0\leq t<fa} e\left(m\frac{t}{fa}\right)\frac{fa}{(fa,t)}\phi((fa,t))\ .$$

Again the second step is to split the summation over $t=t_1t_2$ with $t_1 \mid fa$ and $\left(\frac{fa}{t_1},t_2\right)=1$. We have

$$A_m(s,f,P_{1,2}) = \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s} \sum_{t_1\mid fa} \sum_{\substack{0\leq t_2<\frac{fa}{t_1}\\ \left(t_2,\frac{fa}{t_1}\right)=1}} e\left(m\frac{t_2}{\frac{fa}{t_1}}\right)\frac{fa}{t_1}\phi(t_1) = \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s}\sum_{t\mid fa} c_{\frac{fa}{t}}(m)\frac{fa}{t}\phi(t)\ .$$

So the next step is to split the summation over $a=a_1a_2$ with $a_1 \mid f^\infty$, $(a_2,N)=1$ and do the same for the summation over t. Then factorize the Dirichlet series and also the convolutions of Dirichlet series occuring in the following calculation. This gives us the identity

$$\begin{aligned}
A_m(s,f,P_{1,2}) &= \sum_{\substack{a_1\mid f^\infty\\ (a_2,N)=1}} (a_1a_2)^{-s} \sum_{\substack{t_1\mid fa_1\\ t_2\mid a_2}} c_{\frac{fa_1}{t_1}\frac{a_2}{t_2}}(m)\frac{fa_1}{t_1}\frac{a_2}{t_2}\phi(t_1t_2)\\
&= \left(\sum_{a\mid f^\infty} a^{-s}\sum_{t\mid fa} c_{\frac{fa}{t}}(m)\frac{fa}{t}\phi(t)\right)\left(\sum_{(a,N)=1} a^{-s}\sum_{t\mid a} c_{\frac{a}{t}}(m)\frac{a}{t}\phi(t)\right)\\
&= \left(\sum_{a\mid f^\infty} a^{-s}\sum_{t\mid fa} c_{\frac{fa}{t}}(m)\frac{fa}{t}\phi(t)\right)\left(\sum_{(a,N)=1} a^{1-s}c_a(m)\right)\left(\sum_{(a,N)=1} a^{-s}\phi(a)\right)\\
&= \left(\sum_{a\mid f^\infty} a^{-s}\sum_{t\mid fa} c_{\frac{fa}{t}}(m)\frac{fa}{t}\phi(t)\right)\left(\sum_{(a,N)=1} a^{1-s}c_a(m)\right)\frac{L_{\chi_N}(s-1)}{L_{\chi_N}(s)}\ .
\end{aligned}$$

For the calculation of the third Dirichlet series Lemma 18.4 part (4) was used.
First assume $m\neq 0$ and use Lemma 18.7 part (1) for the calculation of the first Dirichlet series and Lemma 18.4 part (2) for the second one. The formula

$$\begin{aligned}
A_m(s,f,P_{1,2}) =& \prod_{p\mid f}\left((p^2-1)\sigma_{2-s}\left(p^{\alpha_p}\right)-p^{2+(2-s)\alpha_p}\right)\sigma_{1-(s-1)}\left(\prod_{(p,N)=1} p^{\alpha_p}\right)L_{\chi_N}(s-1)^{-1}\frac{L_{\chi_N}(s-1)}{L_{\chi_N}(s)}\\
=& \prod_{p\mid f}\left((p^2-1)\sigma_{2-s}\left(p^{\alpha_p}\right)-p^{2+(2-s)\alpha_p}\right)\sigma_{2-s}\left(\prod_{(p,N)=1} p^{\alpha_p}\right)L_{\chi_N}(s)^{-1}
\end{aligned}$$

is valid. So finally assume $m=0$ and use Lemma 18.7 part (2) for the calculation of the first Dirichlet series and Lemma 18.4 part (4) for the second one. The formula

$$A_0(s,f,P_{1,2}) = \left(\prod_{p|f} \frac{p^2-1}{1-p^{2-s}}\right) \frac{L_{\chi_N}((s-1)-1)}{L_{\chi_N}(s-1)} \frac{L_{\chi_N}(s-1)}{L_{\chi_N}(s)} = \left(\prod_{p|f} (p^2-1)\right) \frac{L_{\chi_{\frac{N}{f}}}(s-2)}{L_{\chi_N}(s)}$$

is valid.

□

With these preparations we can proof the explicit Fourier expansion for the Eisenstein series $E(z,s,f,P_{1,2})$ analogously to Theorem 11.3.

THEOREM 12.3. *Let N be a positive squarefree integer and f a positive divisor of N. The Eisenstein series $G(z,s,f,P_{1,2})$ satisfies the explicit Fourier expansion*

$$G(z,s,f,P_{1,2}) = \sum_{m_2=0}^{\infty} G_{0,m_2}(z,s,f,P_{1,2}) + \sum_{\gamma \in P_{min} \backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1,m_2}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, s, f, P_{1,2}\right) .$$

Let m_1, m_2 be a positive integer with corresponding prime decomposition $m_1 = \prod_p p^{\alpha_p}$. Then the Fourier coefficients satisfy the explicit formulas

(1)

$$G_{m_1,m_2}(z,s,f,P_{1,2}) = 0 ,$$

(2)

$$G_{m_1,0}(z,s,f,P_{1,2}) = 4 y_1^{1-\frac{s}{2}} y_2^{1-s} f^{-3s} \prod_{p|f} \frac{1-p^{(2-3s)(\alpha_p+1)} - p^2 + p^{2+(2-3s)\alpha_p}}{p^{(2-3s)(\alpha_p+1)}-1} \prod_{p|\frac{N}{f}} \frac{p^{2-3s}-1}{p^{(2-3s)(\alpha_p+1)}-1}$$
$$\sigma_{2-3s}(|m_1|)\, |m_1|^{\frac{3s}{2}-1} K_{\frac{3s}{2}-1}(2\pi |m_1| y_1) e(m_1 x_1) ,$$

(3)

$$G_{0,m_2}(z,s,f,P_{1,2}) = 4\delta_{f,N} y_1^s y_2^{\frac{1+s}{2}} \prod_{p|N} \left(1-p^{-3s}\right) |m_2|^{\frac{3s-1}{2}} \sigma_{1-3s}(|m_2|) K_{\frac{3s-1}{2}}(2\pi |m_2| y_2)\, e(m_2 x_2) ,$$

(4)

$$\begin{aligned} G_{0,0}(z,s,f,P_{1,2}) =& 2\pi^{1-\frac{3s}{2}} \prod_{p|f} \frac{p^2-1}{1-p^{2-3s}} f^{-3s} y_1^{2-2s} y_2^{1-s} \Gamma\left(\frac{3s}{2}-1\right) L_{\chi_N}(3s-2) \\ &+ 2\delta_{f,N} \pi^{\frac{1-3s}{2}} y_1^s y_2^{1-s} \prod_{p|N} \frac{1-p^{-3s}}{1-p^{1-3s}} \Gamma\left(\frac{3s-1}{2}\right) L_{\chi_N}(3s-1) \\ &+ 2\delta_{f,N} \pi^{-\frac{3s}{2}} y_1^s y_2^{2s} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) . \end{aligned}$$

PROOF. Since the unipotent part of z can be absorbed into the integration, one can assume without loss of generality that $z = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. We start the calculation of the Fourier coefficients dividing the summation in the Eisenstein series into several cases according to the Bruhat decomposition in Definition 3.8. Assume for the present, that m_1, m_2 are arbitrary integers, then we have

$$\begin{aligned} G_{m_1,m_2}(z,s,f,P_{1,2}) =& \int_0^1\int_0^1\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, s, f, P_{1,2}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ =& \sum_{i=1}^{3} \sum_{\gamma\in\Gamma_i(f,1,P_{1,2})} \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \\ & \int_0^1\int_0^1\int_0^1 I_{(2s,-s)}\left(\gamma\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, P_{1,2}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ =:& \sum_{i=1}^{3} G^{(i)}_{m_1,m_2}(z,s) \ . \end{aligned}$$

Using the explicit description of the sets $\Gamma_i(f,1,P_{1,2})$ in Lemma 5.2 and the explicit formula for the values of the $I_{(2s,-s)}(*,P_{1,2})$-function on these sets in Lemma 8.3 we proceed analogously as in the proof of Theorem 11.3 and start calculating each of the three summands above.

(1) Again we start with the most difficult part, the calculation of $G^{(1)}_{m_1,m_2}$. We have

$$\begin{aligned} G^{(1)}_{m_1,m_2}(z,s) =& \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{a\neq 0 \\ \left(a,\frac{N}{f}\right)=1}} \sum_{b\in\mathbb{Z}} \sum_{\substack{c\in\mathbb{Z} \\ (fa,b,c)=1}} \\ & \int_0^1\int_0^1\int_0^1 \left(y_1y_2^2\right)^s \left[f^2a^2y_1^2y_2^2 + y_2^2(b - fax_1)^2 + (c - bx_2 + fa(x_1x_2 - x_3))^2\right]^{-\frac{3s}{2}} \\ & e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \ . \end{aligned}$$

In the first step pass to the condition $a \geq 1$ extracting a factor 2. After that split the summation over c in this way $c = fak + r$ with $k \in \mathbb{Z}$, $0 \leq r < fa$. Note the transformation of the gcd-condition $1 = (fa,b,c) = (fa,fb,fak+r) = (fa,b,r)$. Then we have

$$G^{(1)}_{m_1,m_2}(z,s) = 2\pi^{-\frac{3s}{2}} \left(y_1 y_2^2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} \sum_{b\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{\substack{0\leq r<fa\\ (fa,b,r)=1}}$$
$$\int_0^1\int_0^1\int_0^1 \left[f^2a^2y_1^2y_2^2 + y_2^2(b-fax_1)^2 + (r-bx_2+fax_1x_2-fa(x_3-k))^2\right]^{-\frac{3s}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

In the second step pass to an infinite integral in the variable x_3 through the shift $x_3 \to x_3 + k$. Then substitute $x_3 \to x_3 + \frac{-r+bx_2-fax_1x_2}{fa}$ in the infinite integral in the variable x_3. After that split the summation over b in this way $b = fak + t$ with $k \in \mathbb{Z}$, $0 \leq t < fa$. Note again the transformation of the gcd-condition $1 = (fa, b, r) = (fa, fak + t, r) = (fa, t, r)$. This gives us

$$G^{(1)}_{m_1,m_2}(z,s) = 2\pi^{-\frac{3s}{2}} \left(y_1 y_2^2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} \sum_{0\leq t<fa} \sum_{k\in\mathbb{Z}} \sum_{\substack{0\leq r<fa\\ (fa,t,r)=1}} \int_0^1 e(-m_2x_2)\mathrm{d}x_2$$
$$\int_{-\infty}^{\infty}\int_0^1 \left[f^2a^2y_1^2y_2^2 + y_2^2(t-fa(x_1-k))^2 + f^2a^2x_3^2\right]^{-\frac{3s}{2}} e(-m_1x_1)\mathrm{d}x_1\mathrm{d}x_3\ .$$

In the next step, first evaluate the exponential integral in the variable x_2. Then pass to an infinite integral in the variable x_1 through the shift $x_1 \to x_1 - k$, after that substitute $x_1 \to x_1 - \frac{t}{fa}$ in the infinite integral in the variable x_1 and pick up an exponential $e\left(-m_1\frac{t}{fa}\right)$. Finally pull the factor fa out of the double integral. We have

$$G^{(1)}_{m_1,m_2}(z,s) = 2\delta_{0,m_2}\pi^{-\frac{3s}{2}} \left(y_1 y_2^2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) f^{-3s} \sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-3s} \sum_{0\leq t<fa} e\left(-m_1\frac{t}{fa}\right) \sum_{\substack{0\leq r<fa\\ (fa,t,r)=1}}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_2^2x_1^2 + x_3^2\right]^{-\frac{3s}{2}} e(-m_1x_1)\mathrm{d}x_1\mathrm{d}x_3\ .$$

To evaluate the Dirichlet series we use Lemma 12.2 and for the double integral use Lemma 19.3, note that y_1 and y_2 are exchanged. For $m_1 \neq 0$ the formula

$$G^{(1)}_{m_1,m_2}(z,s) = 2\delta_{0,m_2}\pi^{-\frac{3s}{2}} \left(y_1 y_2^2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s)$$
$$f^{-3s}\left(\prod_{p|f}\left((p^2-1)\sigma_{2-3s}(p^{\alpha_p}) - p^{2+(2-3s)\alpha_p}\right)\sigma_{2-3s}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_N}(3s)^{-1}\right)$$

$$\left(2\pi^{\frac{3s}{2}} y_2^{1-2\left(\frac{3s}{2}\right)} y_1^{1-\frac{3s}{2}} |m_1|^{\frac{3s}{2}-1} \Gamma\left(\frac{3s}{2}\right)^{-1} K_{\frac{3s}{2}-1}(2\pi |m_1| y_1)\right)$$
$$=4\delta_{0,m_2} y_1^{1-\frac{s}{2}} y_2^{1-s} f^{-3s} \prod_{p|f} \left((p^2-1)\sigma_{2-3s}(p^{\alpha_p}) - p^{2+(2-3s)\alpha_p}\right)$$
$$\sigma_{2-3s}\left(\prod_{(p,N)=1} p^{\alpha p}\right) |m_1|^{\frac{3s}{2}-1} K_{\frac{3s}{2}-1}(2\pi |m_1| y_1)$$
$$=4\delta_{0,m_2} y_1^{1-\frac{s}{2}} y_2^{1-s} f^{-3s} \prod_{p|f} \frac{1-p^{(2-3s)(\alpha_p+1)} - p^2 + p^{2+(2-3s)\alpha_p}}{p^{(2-3s)(\alpha_p+1)}-1} \prod_{p|\frac{N}{f}} \frac{p^{2-3s}-1}{p^{(2-3s)(\alpha_p+1)}-1}$$
$$\sigma_{2-3s}(|m_1|) |m_1|^{\frac{3s}{2}-1} K_{\frac{3s}{2}-1}(2\pi |m_1| y_1)$$

is valid. For $m_1 = 0$ the formula

$$G^{(1)}_{0,m_2}(z,s) = 2\delta_{0,m_2} \pi^{-\frac{3s}{2}} \left(y_1 y_2^2\right)^s \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s)$$
$$f^{-3s}\left(\prod_{p|f}(p^2-1)\frac{L_{\chi_{\frac{N}{f}}}(3s-2)}{L_{\chi_N}(3s)}\right)\left(\pi y_2^{1-2\left(\frac{3s}{2}\right)} y_1^{2-2\left(\frac{3s}{2}\right)} \frac{\Gamma\left(\frac{3s}{2}-1\right)}{\Gamma\left(\frac{3s}{2}\right)}\right)$$
$$=2\delta_{0,m_2}\pi^{1-\frac{3s}{2}} \prod_{p|f}\left(p^2-1\right) f^{-3s} y_1^{2-2s} y_2^{1-s} \Gamma\left(\frac{3s}{2}-1\right) L_{\chi_{\frac{N}{f}}}(3s-2)$$
$$=2\delta_{0,m_2}\pi^{1-\frac{3s}{2}} \prod_{p|f} \frac{p^2-1}{1-p^{2-3s}} f^{-3s} y_1^{2-2s} y_2^{1-s} \Gamma\left(\frac{3s}{2}-1\right) L_{\chi_N}(3s-2)$$

is valid.

(2) Next we do the calculation of $G^{(2)}_{m_1,m_2}$. We have

$$G^{(2)}_{m_1,m_2}(z,s) = \delta_{f,N} \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) L_{\chi_N}(3s) \sum_{b\neq 0} \sum_{\substack{c\in\mathbb{Z}\\(b,c)=1}}$$
$$\int_0^1\int_0^1\int_0^1 \left(y_1 y_2^2\right)^s \left[y_2^2 b^2 + (c-bx_2)^2\right]^{-\frac{3s}{2}} e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

In the first step pass to the condition $b \geq 1$ extracting a factor 2 and factorize the integral. After that split the summation over c in this way $c = bk + r$ with $k \in \mathbb{Z}$, $0 \leq r < b$ and note $1 = (b,c) = (b, kb+r) = (b,r)$. This gives us

$$G^{(2)}_{m_1,m_2}(z,s) = 2\delta_{f,N}\left(y_1y_2^2\right)^s \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\sum_{b\geq 1}\sum_{k\in\mathbb{Z}}\sum_{\substack{0\leq r<b\\(b,r)=1}}$$
$$\int_0^1\left[y_2^2b^2+(r-b(x_2-k))^2\right]^{-\frac{3s}{2}}e(-m_2x_2)\mathrm{d}x_2\int_0^1 e(-m_1x_1)\mathrm{d}x_1\int_0^1\mathrm{d}x_3\ .$$

In the second step pass to an infinite integral in the variable x_2 through the shift $x_2 \to x_2+k$, then substitute $x_2 \to x_2+\frac{r}{b}$ in the infinite integral in the variable x_2 and pick up an exponential $e\left(-m_2\frac{r}{b}\right)$. After that evaluate the other two integrals and pull out the factor b in the integral in the variable x_2. Then we have

$$G^{(2)}_{m_1,m_2}(z,s) = 2\delta_{f,N}\delta_{0,m_1}\left(y_1y_2^2\right)^s \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\sum_{b\geq 1}b^{-3s}\sum_{\substack{0\leq r<b\\(b,r)=1}}e\left(-m_2\frac{r}{b}\right)$$
$$\int_{-\infty}^{\infty}\left[y_2^2+x_2^2\right]^{-\frac{3s}{2}}e(-m_2x_2)\mathrm{d}x_2\ .$$

For the evaluation of the Dirichlet series use Lemma 18.4 and for the evaluation of the integral use Lemma 19.3. For $m_2\neq 0$ the formula

$$\begin{aligned}G^{(2)}_{m_1,m_2}(z,s) =& 2\delta_{f,N}\delta_{0,m_1}\left(y_1y_2^2\right)^s \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\left(\sigma_{1-3s}(m_2)\zeta(3s)^{-1}\right)\\&\left(2\pi^{\frac{3s}{2}}\left|m_2\right|^{\frac{3s-1}{2}}y_2^{\frac{1-3s}{2}}\Gamma\left(\frac{3s}{2}\right)^{-1}K_{\frac{3s-1}{2}}\left(2\pi\left|m_2\right|y_2\right)\right)\\=&4\delta_{f,N}\delta_{0,m_1}y_1^sy_2^{\frac{1+s}{2}}\prod_{p|N}\left(1-p^{-3s}\right)\left|m_2\right|^{\frac{3s-1}{2}}\sigma_{1-3s}(|m_2|)K_{\frac{3s-1}{2}}\left(2\pi\left|m_2\right|y_2\right)\end{aligned}$$

is valid. For $m_2=0$ the formula

$$\begin{aligned}G^{(2)}_{m_1,0}(z,s) =& 2\delta_{f,N}\delta_{0,m_1}\left(y_1y_2^2\right)^s \pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\frac{\zeta(3s-1)}{\zeta(3s)}\left(\sqrt{\pi}y_2^{1-2\left(\frac{3s}{2}\right)}\frac{\Gamma\left(\frac{3s-1}{2}\right)}{\Gamma\left(\frac{3s}{2}\right)}\right)\\=&2\delta_{f,N}\delta_{0,m_1}\pi^{\frac{1-3s}{2}}y_1^sy_2^{1-s}\prod_{p|N}\left(1-p^{-3s}\right)\Gamma\left(\frac{3s-1}{2}\right)\zeta(3s-1)\\=&2\delta_{f,N}\delta_{0,m_1}\pi^{\frac{1-3s}{2}}y_1^sy_2^{1-s}\prod_{p|N}\frac{1-p^{-3s}}{1-p^{1-3s}}\Gamma\left(\frac{3s-1}{2}\right)L_{\chi_N}(3s-1)\end{aligned}$$

is valid.

(3) It remains to calculate $G^{(3)}_{m_1,m_2}(z,s)$. Again Lemma 5.2 gives a description of the set $\Gamma_3(f,1,P_{1,2})$, which consists of two diagonal matrices at most. Use Lemma 8.2 to calculate the values of the $I_{(2s,-s)}(*,P_{1,2})$-function directly. We have

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N}\pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\int_0^1\int_0^1\int_0^1\left[I_{(2s,-s)}\left(\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z,P_{1,2}\right)\right.$$

$$\left. + I_{(2s,-s)}\left(\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z,P_{1,2}\right)\right]e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$= \delta_{f,N}\pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\int_0^1\int_0^1\int_0^1\left[\det\left(\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}\right)^{-s}\left\|\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}^{-1}e_1\right\|^{-3s}\right.$$

$$\left. + \det\left(\begin{pmatrix}-y_1y_2&-y_1x_2&-x_3\\&y_1&x_1\\&&-1\end{pmatrix}\right)^{-s}\left\|\begin{pmatrix}-y_1y_2&-y_1x_2&-x_3\\&y_1&x_1\\&&-1\end{pmatrix}^{-1}e_1\right\|^{-3s}\right]e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$= 2\delta_{f,N}\delta_{0,m_1}\delta_{0,m_2}\pi^{-\frac{3s}{2}}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\left(y_1^2y_2\right)^{-s}\left(y_1^{-1}y_2^{-1}\right)^{-3s}$$

$$= 2\delta_{f,N}\delta_{0,m_1}\delta_{0,m_2}\pi^{-\frac{3s}{2}}y_1^sy_2^{2s}\Gamma\left(\frac{3s}{2}\right)L_{\chi_N}(3s)\ .$$

□

CHAPTER 13

Functional equation for the Eisenstein series twisted by a constant Maass form

Now we will use the Fourier expansions in the Theorems 11.3 and 12.3 to obtain the meromorphic continuation and the functional equations for the Eisenstein series twisted by a constant Maass form. In order to simplify the calculation we will only handle the case where the level N is prime. To simplify the upcoming calculations we split the Fourier coefficients of the Eisenstein series into a product of two functions. For these functions we will prove some invariance properties which will imply the functional equations.

Definition 13.1. Let N be a squarefree integer and f a positive divisor of N, further let $m_1 = \prod_p p^{\alpha_p}$ and $m_2 = \prod_p p^{\beta_p}$ be positive integers in prime decomposition.

(1) Define a family of functions on the generalized upper half plane associated to these integers by

$$g_{m_1,0}(z,s) := 4y_1^{\frac{s+1}{2}} y_2^{s} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{\frac{3s-1}{2}} \sigma_{1-3s} \left(\prod_{(p,N)=1} p^{\alpha_p} \right) K_{\frac{3s-1}{2}} \left(2\pi \left|m_1\right| y_1\right) e\left(m_1 x_1\right) ,$$

$$g_{0,m_2}(z,s) := 4y_1^{1-s} y_2^{1-\frac{s}{2}} \left(\prod_{(p,N)=1} p^{\beta_p} \right)^{\frac{3s}{2}-1} \sigma_{2-3s} \left(\prod_{(p,N)=1} p^{\beta_p} \right) K_{\frac{3s}{2}-1} \left(2\pi \left|m_2\right| y_2\right) e\left(m_2 x_2\right) ,$$

$$g_{0,0}^{(1)}(z,s) := 2y_1^{1-s} y_2^{2-2s} \pi^{1-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}-1\right) \zeta(3s-2) ,$$

$$g_{0,0}^{(2)}(z,s) := 2y_1^{1-s} y_2^{s} \pi^{\frac{1-3s}{2}} \Gamma\left(\frac{3s-1}{2}\right) \zeta(3s-1) ,$$

$$g_{0,0}^{(3)}(z,s) := 2y_1^{2s} y_2^{s} \pi^{-\frac{3s}{2}} \Gamma\left(\frac{3s}{2}\right) \zeta(3s) .$$

(2) Define the following holomorphic functions:

$$a_{m_1,0}(s,f) := f^{-3s} \left(\prod_{p|N} p^{\alpha_p} \right)^{\frac{3s-1}{2}} \prod_{p|f} \left((p-1)\sigma_{1-3s}\left(p^{\alpha_p}\right) - p^{1+(1-3s)\alpha_p} \right) ,$$

$$a_{0,m_2}(s,f) := f^{-3s}\left(\prod_{p|N} p^{\beta_p}\right)^{\frac{3s}{2}-1} \sigma_{2-3s}\left(\prod_{p|N} p^{\beta_p}\right)\prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}\left(1-p^{1-3s}\right) ,$$

$$a_{0,0}^{(1)}(s,f) := f^{-3s}\prod_{p|f}(p-1)\prod_{p|\frac{N}{f}}\left(1-p^{1-3s}\right) ,$$

$$a_{0,0}^{(2)}(s,f) := a_{0,0}^{(1)}(s,f) ,$$

$$a_{0,0}^{(3)}(s,f) := \delta_{N,f}\prod_{p|N}\left(1-p^{-3s}\right) ,$$

$$b_{m_1,0}(s,f) := f^{-3s}\left(\prod_{p|N} p^{\alpha_p}\right)^{\frac{3s}{2}-1}\prod_{p|f}\left((p^2-1)\sigma_{2-3s}\left(p^{\alpha_p}\right)-p^{2+(2-3s)\alpha_p}\right) ,$$

$$b_{0,m_2}(s,f) := \delta_{N,f}\left(\prod_{p|N} p^{\beta_p}\right)^{\frac{3s-1}{2}} \sigma_{1-3s}\left(\prod_{p|N} p^{\beta_p}\right)\prod_{p|N}\left(1-p^{-3s}\right) ,$$

$$b_{0,0}^{(1)}(s,f) := f^{-3s}\prod_{p|f}(p^2-1)\prod_{p|\frac{N}{f}}(1-p^{2-3s}) ,$$

$$b_{0,0}^{(2)}(s,f) := \delta_{f,N}\prod_{p|N}\left(1-p^{-3s}\right) ,$$

$$b_{0,0}^{(3)}(s,f) := b_{0,0}^{(2)}(s,f) .$$

Using the functional equation of the divisor sum and the K-Bessel function, we will derive some functional equations for the above defined family of functions g_*.

LEMMA 13.2. *With the notation in Definition 13.1 for the family of functions g_* the functional equations*

(1)
$$g_{m_1,0}\left(z^t,1-s\right) = g_{0,m_1}(z,s) ,$$

(2)
$$g_{0,0}^{(1)}\left(z^t,1-s\right) = g_{0,0}^{(3)}(z,s) ,$$

(3)
$$g_{0,0}^{(2)}\left(z^t,1-s\right) = g_{0,0}^{(2)}(z,s) ,$$

(4)
$$g_{0,0}^{(3)}\left(z^t,1-s\right) = g_{0,0}^{(1)}(z,s) ,$$

hold.

PROOF. (1) The proof is given through a straightforward calculation. We have

$$
\begin{aligned}
g_{m_1,0}(z^t,1-s) =& 4y_2^{\frac{(1-s)+1}{2}} y_1^{1-s} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{\frac{3(1-s)-1}{2}} \\
& \sigma_{1-3(1-s)} \left(\prod_{(p,N)=1} p^{\alpha_p} \right) K_{\frac{3(1-s)-1}{2}} \left(2\pi \left|m_1\right| y_2\right) e\left(m_1 x_2\right) \\
=& 4y_1^{1-s} y_2^{1-\frac{s}{2}} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{1-\frac{3s}{2}} \sigma_{-(2-3s)} \left(\prod_{(p,N)=1} p^{\alpha_p} \right) K_{-\left(\frac{3s}{2}-1\right)} \left(2\pi \left|m_1\right| y_2\right) e\left(m_1 x_2\right) \ .
\end{aligned}
$$

Now use the functional equation for the divisor sum in Lemma 18.3, Appendix A and the functional equation for the K-Bessel function in Lemma 19.2, Appendix B. Then we have

$$
\begin{aligned}
g_{m_1,0}(z^t,1-s) =& 4y_1^{1-s} y_2^{1-\frac{s}{2}} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{1-\frac{3s}{2}} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{-(2-3s)} \sigma_{2-3s} \left(\prod_{(p,N)=1} p^{\alpha_p} \right) \\
& K_{\frac{3s}{2}-1} \left(2\pi \left|m_1\right| y_2\right) e\left(m_1 x_2\right) \\
=& 4y_1^{1-s} y_2^{1-\frac{s}{2}} \left(\prod_{(N,p)=1} p^{\alpha_p} \right)^{\frac{3s}{2}-1} \sigma_{2-3s} \left(\prod_{(p,N)=1} p^{\alpha_p} \right) K_{\frac{3s}{2}-1} \left(2\pi \left|m_1\right| y_2\right) e\left(m_1 x_2\right) \\
=& g_{0,m_1}(z,s) \ .
\end{aligned}
$$

(2) The parts 2) and 4) are dual to each other. The functional equation follows immediately from the functional equation of the completed Riemann zeta function $\tilde{\zeta}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \tilde{\zeta}(1-s)$. We have

$$
\begin{aligned}
g_{0,0}^{(1)}\left(z^t,1-s\right) =& 2y_2^{1-(1-s)} y_1^{2-2(1-s)} \tilde{\zeta}\left(3(1-s)-2\right) \\
=& 2y_1^{2s} y_2^{s} \tilde{\zeta}\left(1-3s\right) \\
=& 2y_1^{2s} y_2^{s} \tilde{\zeta}\left(3s\right) \\
=& g_{0,0}^{(3)}\left(z,s\right) \ .
\end{aligned}
$$

(3) A short straightforward calculation gives the result. We have

$$
\begin{aligned}
g_{0,0}^{(2)}\left(z^t,1-s\right) =& 2y_2^{1-(1-s)} y_1^{1-s} \tilde{\zeta}\left(3(1-s)-1\right) \\
=& 2y_1^{1-s} y_2^{s} \tilde{\zeta}\left(2-3s\right) \\
=& 2y_1^{1-s} y_2^{s} \tilde{\zeta}\left(1-(2-3s)\right) \\
=& 2y_1^{1-s} y_2^{s} \tilde{\zeta}\left(3s-1\right)
\end{aligned}
$$

$$=g^{(2)}_{0,0}\,(z,s)\ .$$

□

Next we show that there exist some scattering matrices such that certain vectors of the holomorphic functions a_* and b_* fulfill some functional equations.

LEMMA 13.3. *Let $N = p$ be prime. With the notation in Definition 13.1 for the family of functions a_* and b_* the functional equations*

(1)
$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b_{m_1,0}(s,1) \\ b_{m_1,0}(s,p) \end{pmatrix} = \begin{pmatrix} a_{m_1,0}(1-s,1) \\ a_{m_1,0}(1-s,p) \end{pmatrix} \ ,$$

(2)
$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b_{0,m_2}(s,1) \\ b_{0,m_2}(s,p) \end{pmatrix} = \begin{pmatrix} a_{0,m_2}(1-s,1) \\ a_{0,m_2}(1-s,p) \end{pmatrix} \ ,$$

(3)
$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b^{(1)}_{0,0}(s,1) \\ b^{(1)}_{0,0}(s,p) \end{pmatrix} = \begin{pmatrix} a^{(3)}_{0,0}(1-s,1) \\ a^{(3)}_{0,0}(1-s,p) \end{pmatrix} \ ,$$

(4)
$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b^{(2)}_{0,0}(s,1) \\ b^{(2)}_{0,0}(s,p) \end{pmatrix} = \begin{pmatrix} a^{(2)}_{0,0}(1-s,1) \\ a^{(2)}_{0,0}(1-s,p) \end{pmatrix} \ ,$$

(5)
$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b^{(3)}_{0,0}(s,1) \\ b^{(3)}_{0,0}(s,p) \end{pmatrix} = \begin{pmatrix} a^{(1)}_{0,0}(1-s,1) \\ a^{(1)}_{0,0}(1-s,p) \end{pmatrix} \ ,$$

hold with the matrix entries:

(1)
$$a(s) = \frac{p^2-1}{p^2-p^{2-3s}} \ ,$$

(2)
$$b(s) = \frac{p^2-p^{3s}}{p^2-p^{2-3s}} \ ,$$

(3)
$$c(s) = \frac{1-p^{3s-1}}{p^2-p^{2-3s}} \ ,$$

(4)
$$d(s) = \frac{(p-1)p^{3s-1}}{p^2-p^{2-3s}} \ .$$

PROOF. The proof is given through checking each row in a straightforward calulation.

(1) Checking the first row:

$$a(s)b_{m_1,0}(s,1)+b(s)b_{m_1,0}(s,p)$$

$$=\left(\frac{p^2-1}{p^2-p^{2-3s}}\right)\cdot p^{\alpha_p\left(\frac{3s}{2}-1\right)}+\left(\frac{p^2-p^{3s}}{p^2-p^{2-3s}}\right)\cdot p^{-3s}p^{\alpha_p\left(\frac{3s}{2}-1\right)}\left[(p^2-1)\sigma_{2-3s}\left(p^{\alpha_p}\right)-p^{2+(2-3s)\alpha_p}\right]$$

$$=\frac{p^{\alpha_p\left(\frac{3s}{2}-1\right)}}{p^2-p^{2-3s}}\left[p^2-1-\left(1-p^{2-3s}\right)\left((p^2-1)\frac{1-p^{(\alpha_p+1)(2-3s)}}{1-p^{2-3s}}-p^{2+(2-3s)\alpha_p}\right)\right]$$

$$=\frac{p^{\alpha_p\left(\frac{3s}{2}-1\right)}}{p^2-p^{2-3s}}\left[(p^2-1)p^{(\alpha_p+1)(2-3s)}+p^{2+(2-3s)\alpha_p}\left(1-p^{2-3s}\right)\right]$$

$$=\frac{p^{\alpha_p\left(\frac{3s}{2}-1\right)}}{p^2-p^{2-3s}}p^{\alpha_p(2-3s)}\left(p^2-p^{2-3s}\right)$$

$$=p^{\alpha_p\left(\frac{3(1-s)-1}{2}\right)}$$

$$=a_{m_1,0}(1-s,1)\ .$$

Checking the second row:

$$c(s)b_{m_1,0}(s,1)+d(s)b_{m_1,0}(s,p)-a_{m_1,0}(1-s,p)$$

$$=\left(\frac{1-p^{3s-1}}{p^2-p^{2-3s}}\right)\cdot p^{\alpha_p\left(\frac{3s}{2}-1\right)}+\left(\frac{(p-1)p^{3s-1}}{p^2-p^{2-3s}}\right)\cdot p^{-3s}p^{\alpha_p\left(\frac{3s}{2}-1\right)}\left[(p^2-1)\sigma_{2-3s}\left(p^{\alpha_p}\right)-p^{2+(2-3s)\alpha_p}\right]$$
$$-p^{-3(1-s)}p^{\alpha_p\left(\frac{3(1-s)-1}{2}\right)}\left[(p-1)\sigma_{1-3(1-s)}\left(p^{\alpha_p}\right)-p^{1+(1-3(1-s))\alpha_p}\right]\ .$$

Use the functional equation for the divisor sum in the second summand. We have

$$c(s)b_{m_1,0}(s,1)+d(s)b_{m_1,0}(s,p)-a_{m_1,0}(1-s,p)$$

$$=\frac{1-p^{3s-1}}{p^2-p^{2-3s}}p^{\alpha_p\left(\frac{3s}{2}-1\right)}+\frac{(p-1)p^{-1}}{p^2-p^{2-3s}}p^{\alpha_p\left(\frac{3s}{2}-1\right)}\left[(p^2-1)\sigma_{3s-2}\left(p^{\alpha_p}\right)p^{(2-3s)\alpha_p}-p^{2+(2-3s)\alpha_p}\right]$$
$$-p^{3s-3}p^{\alpha_p\left(\frac{2-3s}{2}\right)}\left[(p-1)\sigma_{3s-2}\left(p^{\alpha_p}\right)-p^{1+(3s-2)\alpha_p}\right]\ .$$

Finally evaluate the divisor sums. This gives us

$$c(s)b_{m_1,0}(s,1)+d(s)b_{m_1,0}(s,p)-a_{m_1,0}(1-s,p)$$

$$=\frac{p^{\alpha_p\left(\frac{2-3s}{2}\right)}}{p^2-p^{2-3s}}\left[\left(1-p^{3s-1}\right)p^{\alpha_p(3s-2)}+p^{-1}(p-1)\left[\left(p^2-1\right)\frac{1-p^{(\alpha_p+1)(3s-2)}}{1-p^{3s-2}}-p^2\right]\right.$$
$$\left.-p^{3s-3}\left(p^2-p^{2-3s}\right)\left[(p-1)\frac{1-p^{(\alpha_p+1)(3s-2)}}{1-p^{3s-2}}-p^{1+(3s-2)\alpha_p}\right]\right]$$

$$=\frac{p^{\alpha_p\left(\frac{2-3s}{2}\right)}}{\left(p^2-p^{2-3s}\right)\left(1-p^{3s-2}\right)}\left[\left(1-p^{3s-2}\right)\left(1-p^{3s-1}\right)p^{\alpha_p(3s-2)}\right.$$

$$
\begin{aligned}
&+p^{-1}(p-1)\left[\left(p^2-1\right)\left(1-p^{(\alpha_p+1)(3s-2)}\right)-p^2\left(1-p^{3s-2}\right)\right]\\
&-p^{3s-3}\left(p^2-p^{2-3s}\right)\left[(p-1)\left(1-p^{(\alpha_p+1)(3s-2)}\right)-\left(1-p^{3s-2}\right)p^{1+(3s-2)\alpha_p}\right]\Bigg]\\
=&\frac{p^{\alpha_p\left(\frac{2-3s}{2}\right)}}{\left(p^2-p^{2-3s}\right)\left(1-p^{3s-2}\right)}\Bigg[\left(1-p^{3s-2}\right)\left(1-p^{3s-1}\right)p^{\alpha_p(3s-2)}\\
&+p^{-1}(p-1)\left[p^{3s}-1-\left(p^2-1\right)p^{3s-2}p^{\alpha_p(3s-2)}\right]\\
&-p^{-1}\left(p^{3s}-1\right)\left[\left(p^{3s-2}-p\right)p^{\alpha_p(3s-2)}+p-1\right]\Bigg]\\
=&0\ .
\end{aligned}
$$

Finally we order the terms and see that the result vanishes.

(2) First note that $b_{0,m_2}(s,1)=0$. Here the calculations are shorter and we handle the whole matrix in one calculation. Again we use the functional equation for the divisor sum. Then we have

$$
\begin{aligned}
\begin{pmatrix} a_{0,m_2}(1-s,1)\\ a_{0,m_2}(1-s,p)\end{pmatrix}
&=\begin{pmatrix} p^{\beta_p\left(\frac{3(1-s)}{2}-1\right)}\sigma_{2-3(1-s)}\left(p^{\beta_p}\right)\left(1-p^{1-3(1-s)}\right)\\ p^{-3(1-s)}p^{\beta_p\left(\frac{3(1-s)}{2}-1\right)}\sigma_{2-3(1-s)}\left(p^{\beta_p}\right)(p-1)\end{pmatrix}\\
&=\begin{pmatrix} p^{\beta_p\left(\frac{1-3s}{2}\right)}\sigma_{1-3s}\left(p^{\beta_p}\right)p^{-\beta_p(1-3s)}\left(1-p^{3s-2}\right)\\ p^{3s-3}(p-1)p^{\beta_p\left(\frac{1-3s}{2}\right)}\sigma_{1-3s}\left(p^{\beta_p}\right)p^{-\beta_p(1-3s)}\end{pmatrix}\\
&=\begin{pmatrix} \left(\frac{p^2-p^{3s}}{p^2-p^{2-3s}}\right)\cdot\left(p^{\beta_p\left(\frac{3s-1}{2}\right)}\sigma_{1-3s}\left(p^{\beta_p}\right)\left(1-p^{-3s}\right)\right)\\ \left(\frac{(p-1)p^{3s-1}}{p^2-p^{2-3s}}\right)\cdot\left(p^{\beta_p\left(\frac{3s-1}{2}\right)}\sigma_{1-3s}\left(p^{\beta_p}\right)\left(1-p^{-3s}\right)\right)\end{pmatrix}\\
&=\begin{pmatrix} a(s)\cdot 0+b(s)b_{0,m_2}(s,p)\\ c(s)\cdot 0+d(s)b_{0,m_2}(s,p)\end{pmatrix}\\
&=\begin{pmatrix} a(s) & b(s)\\ c(s) & d(s)\end{pmatrix}\cdot\begin{pmatrix} b_{0,m_2}(s,1)\\ b_{0,m_2}(s,p)\end{pmatrix}\ .
\end{aligned}
$$

(3) Note that $a^{(3)}_{0,0}(s,1)=0$ and do a straightforward matrix calculation:

$$
\begin{aligned}
\begin{pmatrix} a(s) & b(s)\\ c(s) & d(s)\end{pmatrix}\cdot\begin{pmatrix} b^{(1)}_{0,0}(s,1)\\ b^{(1)}_{0,0}(s,p)\end{pmatrix}
&=\begin{pmatrix} a(s)b^{(1)}_{0,0}(s,1)+b(s)b^{(1)}_{0,0}(s,p)\\ c(s)b^{(1)}_{0,0}(s,1)+d(s)b^{(1)}_{0,0}(s,p)\end{pmatrix}\\
&=\begin{pmatrix} \left(\frac{p^2-1}{p^2-p^{2-3s}}\right)\cdot\left(1-p^{2-3s}\right)+\left(\frac{p^2-p^{3s}}{p^2-p^{2-3s}}\right)\cdot\left(p^{-3s}(p^2-1)\right)\\ \left(\frac{1-p^{3s-1}}{p^2-p^{2-3s}}\right)\cdot\left(1-p^{2-3s}\right)+\left(\frac{(p-1)p^{3s-1}}{p^2-p^{2-3s}}\right)\cdot\left(p^{-3s}(p^2-1)\right)\end{pmatrix}\\
&=\begin{pmatrix} 0\\ \frac{\left(1-p^{2-3s}-p^{3s-1}+p\right)+\left(p^2-p-1+p^{-1}\right)}{p^2-p^{2-3s}}\end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} 0 \\ 1 - p^{-3(1-s)} \end{pmatrix}$$
$$= \begin{pmatrix} a^{(3)}_{0,0}(1-s,1) \\ a^{(3)}_{0,0}(1-s,p) \end{pmatrix} .$$

(4) Note that $b^{(2)}_{0,0}(s,1) = 0$ and do again a straightforward calculation. We have

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b^{(2)}_{0,0}(s,1) \\ b^{(2)}_{0,0}(s,p) \end{pmatrix} = \begin{pmatrix} a(s)b^{(2)}_{0,0}(s,1) + b(s)b^{(2)}_{0,0}(s,p) \\ c(s)b^{(2)}_{0,0}(s,1) + d(s)b^{(2)}_{0,0}(s,p) \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{p^2 - p^{3s}}{p^2 - p^{2-3s}}\right) \cdot \left(1 - p^{-3s}\right) \\ \left(\frac{(p-1)p^{3s-1}}{p^2 - p^{2-3s}}\right) \cdot \left(1 - p^{-3s}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 1 - p^{1-3(1-s)} \\ p^{-3(1-s)}(p-1) \end{pmatrix}$$
$$= \begin{pmatrix} a^{(2)}_{0,0}(1-s,1) \\ a^{(2)}_{0,0}(1-s,p) \end{pmatrix} .$$

(5) Again note that $b^{(3)}_{0,0}(s,1) = 0$. We have

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b^{(3)}_{0,0}(s,1) \\ b^{(3)}_{0,0}(s,p) \end{pmatrix} = \begin{pmatrix} a(s)b^{(3)}_{0,0}(s,1) + b(s)b^{(3)}_{0,0}(s,p) \\ c(s)b^{(3)}_{0,0}(s,1) + d(s)b^{(3)}_{0,0}(s,p) \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{p^2 - p^{3s}}{p^2 - p^{2-3s}}\right) \cdot \left(1 - p^{-3s}\right) \\ \left(\frac{(p-1)p^{3s-1}}{p^2 - p^{2-3s}}\right) \cdot \left(1 - p^{-3s}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 1 - p^{1-3(1-s)} \\ p^{-3(1-s)}(p-1) \end{pmatrix}$$
$$= \begin{pmatrix} a^{(1)}_{0,0}(1-s,1) \\ a^{(1)}_{0,0}(1-s,p) \end{pmatrix} .$$

□

With these preparation we can easily prove the functional equation of the Eisenstein series $E(z,s,f,P_{2,1})$ and $E(z,s,f,P_{1,2})$.

THEOREM 13.4. *Let $N = p$ be a prime. Then the four Eisenstein series twisted by a constant Maass form corresponding to the level p have meromorphic continuation and satisfy the functional equation*

$$\begin{pmatrix} G(z,1-s,1,P_{2,1}) \\ G(z,1-s,p,P_{2,1}) \\ G(z,1-s,1,P_{1,2}) \\ G(z,1-s,p,P_{1,2}) \end{pmatrix} = \begin{pmatrix} & & a(s) & b(s) \\ & & c(s) & d(s) \\ e(s) & f(s) & & \\ g(s) & h(s) & & \end{pmatrix} \cdot \begin{pmatrix} G(z,s,1,P_{2,1}) \\ G(z,s,p,P_{2,1}) \\ G(z,s,1,P_{1,2}) \\ G(z,s,p,P_{1,2}) \end{pmatrix}$$

with the scattering matrix:

$$\begin{pmatrix} & & a(s) & b(s) \\ & & c(s) & d(s) \\ e(s) & f(s) & & \\ g(s) & h(s) & & \end{pmatrix} = \begin{pmatrix} & & \frac{p^2-1}{p^2-p^{2-3s}} & \frac{p^2-p^{3s}}{p^2-p^{2-3s}} \\ & & \frac{1-p^{3s-1}}{p^2-p^{2-3s}} & \frac{(p-1)p^{3s-1}}{p^2-p^{2-3s}} \\ \frac{p^2-p}{p^2-p^{2-3s}} & \frac{p^2-p^{3s+1}}{p^2-p^{2-3s}} & & \\ \frac{p-p^{3s-1}}{p^2-p^{2-3s}} & \frac{(p^2-1)p^{3s-1}}{p^2-p^{2-3s}} & & \end{pmatrix}.$$

The variable transformation $s \longmapsto 1-s$ *maps the lower left most* 2×2*-block in the scattering matrix to the inverse of the upper right most* 2×2*-block.*

Proof. The meromorphic continuation of the Eisenstein series follows immediately from the explicit Fourier expansion in Theorem 11.3 and 12.3. Since the above scattering matrix has block form we start to handle the upper right most 2×2-block first. We calculate directly using the Fourier expansion and the functional equations worked out in this section. We start with

$$\begin{aligned} &\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} G(z,s,1,P_{1,2}) \\ G(z,s,p,P_{1,2}) \end{pmatrix} \\ &= \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} G_{0,0}\left(z,s,1,P_{1,2}\right) \\ G_{0,0}\left(z,s,p,P_{1,2}\right) \end{pmatrix} \\ &\quad + \sum_{m_2=1}^{\infty} \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} G_{0,m_2}\left(z,s,1,P_{1,2}\right) \\ G_{0,m_2}\left(z,s,p,P_{1,2}\right) \end{pmatrix} \\ &\quad + \sum_{\gamma \in P_{min} \backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} G_{m_1,0}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z,s,1,P_{1,2}\right) \\ G_{m_1,0}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z,s,p,P_{1,2}\right) \end{pmatrix}. \end{aligned}$$

Next express the Fourier coefficients through the functions defined in Definition 13.1, hence $G_{m_1,m_2}\left(z,s,f,P_{1,2}\right) = b_{m_1,m_2}(s,f) g_{m_2,m_1}(z^t,s)$ and $G_{0,0}\left(z,s,f,P_{1,2}\right) = \sum_{i=1}^{3} b_{0,0}^{(i)}(s,f) g_{0,0}^{(i)}(z^t,s)$. So the equation

$$\begin{aligned} &\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} G(z,s,1,P_{1,2}) \\ G(z,s,p,P_{1,2}) \end{pmatrix} \\ &= \sum_{i=1}^{3} g_{0,0}^{(i)}(z^t,s) \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \cdot \begin{pmatrix} b_{0,0}^{(i)}(s,1) \\ b_{0,0}^{(i)}(s,p) \end{pmatrix} \end{aligned}$$

$$+\sum_{m_2=1}^{\infty} g_{m_2,0}(z^t,s)\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}\cdot\begin{pmatrix} b_{0,m_2}(s,1) \\ b_{0,m_2}(s,p) \end{pmatrix}$$
$$+\sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})}\sum_{m_1=1}^{\infty} g_{0,m_1}\left(\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z\right)^t, s\right)\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}\cdot\begin{pmatrix} b_{m_1,0}(s,1) \\ b_{m_1,0}(s,p) \end{pmatrix}$$

holds. Apply the functional equations in Lemma 13.2 to the functions g_* and the functional equations in Lemma 13.3 to the functions b_*. This leads to

$$\begin{aligned}
&\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}\cdot\begin{pmatrix} G(z,s,1,P_{1,2}) \\ G(z,s,p,P_{1,2}) \end{pmatrix}\\
=&\sum_{i=1}^{3} g_{0,0}^{(3-i+1)}(z,1-s)\begin{pmatrix} a_{0,0}^{(3-i+1)}(1-s,1) \\ a_{0,0}^{(3-i+1)}(1-s,p) \end{pmatrix}\\
&+\sum_{m_2=1}^{\infty} g_{0,m_2}(z,1-s)\begin{pmatrix} a_{0,m_2}(1-s,1) \\ a_{0,m_2}(1-s,p) \end{pmatrix}\\
&+\sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})}\sum_{m_1=1}^{\infty} g_{m_1,0}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, 1-s\right)\begin{pmatrix} a_{m_1,0}(1-s,1) \\ a_{m_1,0}(1-s,p) \end{pmatrix}\\
=&\begin{pmatrix} G(z,1-s,1,P_{2,1}) \\ G(z,1-s,p,P_{2,1}) \end{pmatrix}.
\end{aligned}$$

Next we handle the lower left most block matrix. This can be easily solved through multiplying the above equation with the inverse matrix and making the transformation $s\to 1-s$. We start with

$$\begin{pmatrix} G(z,1-s,1,P_{1,2}) \\ G(z,1-s,p,P_{1,2}) \end{pmatrix} = \begin{pmatrix} a(1-s) & b(1-s) \\ c(1-s) & d(1-s) \end{pmatrix}^{-1}\cdot\begin{pmatrix} G(z,s,1,P_{2,1}) \\ G(z,s,p,P_{2,1}) \end{pmatrix}.$$

So it remains to calculate the inverse. The equation

$$\begin{aligned}
&\begin{pmatrix} a(1-s) & b(1-s) \\ c(1-s) & d(1-s) \end{pmatrix}^{-1}\\
=&\frac{1}{a(1-s)d(1-s)-b(1-s)c(1-s)}\begin{pmatrix} d(1-s) & -b(1-s) \\ -c(1-s) & a(1-s) \end{pmatrix}\\
=&\frac{1}{\frac{p^2-1}{p^2-p^{2-3(1-s)}}\cdot\frac{(p-1)p^{3(1-s)-1}}{p^2-p^{2-3(1-s)}}-\frac{p^2-p^{3(1-s)}}{p^2-p^{2-3(1-s)}}\cdot\frac{1-p^{3(1-s)-1}}{p^2-p^{2-3(1-s)}}}\begin{pmatrix} \frac{(p-1)p^{3(1-s)-1}}{p^2-p^{2-3(1-s)}} & -\frac{p^2-p^{3(1-s)}}{p^2-p^{2-3(1-s)}} \\ -\frac{1-p^{3(1-s)-1}}{p^2-p^{2-3(1-s)}} & \frac{p^2-1}{p^2-p^{2-3(1-s)}} \end{pmatrix}\\
=&\frac{p^2-p^{3s-1}}{p^{2-3s}-p^2+p^{5-3s}-p^{5-6s}}\begin{pmatrix} (p-1)p^{2-3s} & -p^2+p^{3-3s} \\ -1+p^{2-3s} & p^2-1 \end{pmatrix}
\end{aligned}$$

$$= \frac{\left(p^{3-3s}-1\right)p^{3s-1}}{\left(p^{3-3s}-1\right)\left(p^2-p^{2-3s}\right)} \begin{pmatrix} (p-1)p^{2-3s} & -p^2+p^{3-3s} \\ -1+p^{2-3s} & p^2-1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{p^2-p}{p^2-p^{2-3s}} & \frac{p^2-p^{3s+1}}{p^2-p^{2-3s}} \\ \frac{p-p^{3s-1}}{p^2-p^{2-3s}} & \frac{(p^2-1)p^{3s-1}}{p^2-p^{2-3s}} \end{pmatrix}$$

for the inverse holds. □

Part VI

Fourier expansion for the Eisenstein series twisted by a Maass cusp form

CHAPTER 14

Fourier expansion of the Eisenstein series $E(z, s, f, \phi, P_{2,1})$

In this chapter we want to calculate the Fourier expansion of the Eisenstein series $E\left(z, s, f, \phi, P_{2,1}\right)$ twisted by a Maass cusp form. We start with the definition of a certain family of Dirichlet series which occurs in the Fourier coefficients of these Eisenstein series.

Definition 14.1. Let N be a squarefree integer, f, M positive divisors of N, $\psi = (\psi(k))_{k\in\mathbb{N}}$ a sequence of complex numbers and n a positive divisor of (f, M). For a positive integer m_1 and a non-negative integer m_2 define the Dirichlet series $A_{m_1,m_2}(s, f, M, \psi, n, P_{2,1})$ associated to these parameters by

$$A_{m_1,m_2}\left(s, f, M, \psi, n, P_{2,1}\right) := \sum_{a\geq 1} a^{-s} \sum_{\substack{0\leq t<a(f,M)\\ \left(\frac{N}{f},(a,t)\right)=1\\ f(a,t)\mid m_1 n\frac{a}{(a,t)}}} \psi\left(\frac{m_1 n\frac{a}{(a,t)}}{f(a,t)}\right)(a,t)^{-1} e\left(m_2\frac{t}{a}\right) c_{f(a,t)}(m_1)\ .$$

The next lemma shows that the integration of a periodic function over an interval of period length is translation invariant. In later calculations this lemma is needed several times to expand the integration interval in the calculation of the Fourier coefficients of the Eisenstein series. The proof follows after a simple variable shift and is omitted.

Lemma 14.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and periodic function with period p. Integration of f over an interval of period length p has the invariance property*

$$\int_x^{x+p} f(t)\mathrm{d}t = \int_0^p f(t)\mathrm{d}t \qquad \forall x\in\mathbb{R}\ .$$

Now we are prepared to state the main result in this section.

Theorem 14.3. *Let N be a squarefree positive integer and f, M positive divisors of N. Let ϕ be a Maass cusp form for $\Gamma_0(M)$ with eigenvalue $\nu(\nu-1)$. The associated Eisenstein series $G\left(z, s, f, \phi, P_{2,1}\right)$ has the explicit Fourier expansion*

$$G\left(z, s, f, \phi, P_{2,1}\right) = \sum_{m_2=0}^{\infty} G_{0,m_2}\left(z, s, f, \phi, P_{2,1}\right) \sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty}\sum_{m_2=0}^{\infty} G_{m_1,m_2}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z, s, f, \phi, P_{2,1}\right)\ .$$

Recall Lemma 20.4 in Appendix C and introduce for the coprime pair of integers $e \mid (f, M)$ and $0 < d < \frac{(f,M)}{e}$ the following notation. Let $\mathfrak{a}_{(d,e)} := \frac{1}{h_{(d,e)}}$ denote the cusp for $\Gamma_0(M)$

associated to $h_{(d,e)} := \left(\frac{N}{f}, M\right)\left(d, \frac{(f,M)}{e}\right)$ *and* $m_{(d,e)} := \frac{M}{h_{(d,e)}} = \frac{(f,M)}{\left(d, \frac{(f,M)}{e}\right)}$ *its width. Let*

$$\sigma_{(d,e)} := \begin{pmatrix} 1 & \\ h_{(d,e)} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_{(d,e)} \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_{(d,e)}} & \\ & \sqrt{m_{(d,e)}}^{-1} \end{pmatrix}$$

be a scaling matrix associated to the cusp $\mathfrak{a}_{(d,e)}$*, where the integer* $\alpha_{(d,e)}$ *depends only on the integers* $d, e, M, \frac{N}{f}$ *modulo* $\frac{(M,f)}{\left(d, \frac{(f,M)}{e}\right)}$*. Further let* $\phi_{(d,e)} := \left(\phi_{(d,e)}(n)\right)_{n \in \mathbb{Z}}$ *denote the sequence of Fourier coefficients associated to the Fourier expansion of* ϕ *at the cusp* $\mathfrak{a}_{(d,e)}$*. Let* m_1, m_2 *be positive integers, then the following explicit formulas for the Fourier coefficients are valid.*

(1) *In the non-degenerate case we have*

$$G_{m_1,m_2}(z, s, f, \phi, P_{2,1}) = 2\frac{1}{(f,M)} f^{-3s} \sqrt{m_1 m_2} m_1^{s-\frac{3}{2}} |m_2|^{2s-\frac{5}{2}}$$

$$\sum_{e|(f,M)} \sum_{\substack{0<d\le \frac{(f,M)}{e} \\ (d,e)=1}} A_{m_1,m_2}(3s-1, f, M, \phi_{(d,e)}, m_{(d,e)}, P_{2,1}) W_{1,1}^{\left(s-\frac{\nu}{3}, \frac{2\nu}{3}\right)}\left(\begin{pmatrix} m_1|m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix} z, w_1\right) .$$

(2) *In the first partially degenerate case we have*

$$G_{m_1,0}(z, s, f, \phi, P_{2,1}) = 2\frac{1}{(f,M)} f^{-3s} m_1^{s-1} \sum_{e|(f,M)} \sum_{\substack{0<d\le \frac{(f,M)}{e} \\ (d,e)=1}} A_{m_1,0}(3s-1, f, M, \phi_{(d,e)}, m_{(d,e)}, P_{2,1})$$

$$W_{1,0}^{\left(s-\frac{\nu}{3}, \frac{2\nu}{3}\right)}\left(\begin{pmatrix} m_1 & & \\ & m_1 & \\ & & 1 \end{pmatrix} z, w_1\right) .$$

(3) *In the second partially degenerate case we have*

$$G_{0,m_2}(z, s, f, \phi, P_{2,1}) = \delta_{N,f} \sum_{e|M} \sum_{\substack{0<d\le \frac{M}{e} \\ (d,e)=1}} \left[\phi_{(d,e)}\left(m_2 m_{(d,e)}\right) + \phi_{(d,e)}\left(-m_2 m_{(d,e)}\right)\right]$$

$$|m_2|^{-s} W_{0,1}^{\left(s-\frac{\nu}{3}, \frac{2\nu}{3}\right)}\left(\begin{pmatrix} |m_2| & & \\ & 1 & \\ & & 1 \end{pmatrix} z, w_2\right) .$$

(4) *In the totally degenerate case we have*

$$G_{0,0}(z, s, f, \phi, P_{2,1}) = 0 .$$

Proof. Like in the proof of Theorem 11.3 without loss of generality it can be assumed that $z = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. We start the calculation of the Fourier coefficients dividing the summation in the Eisenstein series into several cases according to the Bruhat decomposition in Definition 3.8. Assume for the present that m_1, m_2 are arbitrary integers, then we have

$$
\begin{aligned}
&G_{m_1,m_2}(z,s,f,\phi,P_{2,1}) = \int_0^1\int_0^1\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z,s,f,\phi,P_{2,1}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{(f,M)} \sum_{k=0}^{(f,M)-1} \int_0^1\int_k^{k+1}\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z,s,f,\phi,P_{2,1}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{(f,M)} \int_0^1\int_0^{(f,M)}\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z,s,f,\phi,P_{2,1}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \sum_{i=1}^{3} \sum_{\gamma\in\Gamma_i(f,M,P_{2,1})} \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \\
&\int_0^1\int_0^{(f,M)}\int_0^1 \phi\left(\mathfrak{m}_{P_{2,1}}\left(\gamma\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z\right)\right) I_{(s,-2s)}\left(\gamma\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, P_{2,1}\right) \\
&e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&=: \sum_{i=1}^{3} G^{(i)}_{m_1,m_2}(z,s) \;.
\end{aligned}
$$

In this calculation Lemma 14.2 was applied to each integral. This is possible since the left invariance of the Eisenstein series against the matrices $\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$ gives the periodicity with period 1 in the variables x_1, x_3. Further Lemma 14.2 applied to the integral in the variable x_3 over the Eisenstein series, considered as a function in x_2, in conjunction with the invariance against $\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$, gives the periodicity with period 1 in the variable x_2.

Using the explicit description of the sets $\Gamma_i(f,M,P_{2,1})$ in Lemma 4.2 and the explicit formulas for the values of the $I_{(s,-2s)}(*,P_{2,1})$-function and for the Levi components on these sets in Lemma 8.3 and 7.4, we start calculating each of the three summands above. For further calculations note first that Lemma 4.2 gives the splitting

$$p_{(a,b,c,d,e)} = \tilde{\gamma}\begin{pmatrix} 1 & \\ h_{(d,e)} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha_{(d,e)} \\ & 1 \end{pmatrix} = \tilde{\gamma}\sigma_{(d,e)}\begin{pmatrix} m_{(d,e)}^{-1} & \\ & 1 \end{pmatrix}\sqrt{m_{(d,e)}}\,, \qquad (14.1)$$

with $\tilde{\gamma} \in \Gamma_0(M)$ and the integer $\alpha_{(d,e)} \in \mathbb{Z}$ depends only on the integers $d, e, M, \frac{N}{f}$ modulo $\frac{(M,f)}{\left(d,\frac{(f,M)}{e}\right)}$.

(1) We start with the most difficult part, the calculation of $G^{(1)}_{m_1,m_2}$. We have

$$G^{(1)}_{m_1,m_2}(z,s) = \frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|(f,M)}\sum_{\substack{0<d\le\frac{(f,M)}{e}\\(d,e)=1}}\sum_{a\neq 0}\sum_{\substack{b\in\mathbb{Z}\\ \left(\frac{N}{f},(a,b)\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\(f(a,b),c)=1}}$$
$$\int_0^1\int_0^{(f,M)}\int_0^1 \phi\left(p_{(a,b,c,d,e)}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)\left(y_1^2y_2\right)^s\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2\right]^{-\frac{3s}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,,$$

with the coefficients:

$$\begin{aligned} y =&(a,b)^2y_2\frac{\sqrt{f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + (fax_3+fbx_1+c)^2}}{a^2y_2^2+(ax_2+b)^2}\,, \\ x =&\frac{(a,b)fx_1 - rc}{\frac{a}{(a,b)}} - \frac{\frac{(a,b)^2}{a}(ax_2+b)(fax_3+fbx_1+c)}{a^2y_2^2+(ax_2+b)^2}\,. \end{aligned}$$

In the first step pass to the condition $a \ge 1$ extracting a factor 2 and substitute the splitting in (14.1) for $p_{(a,b,c,d,e)}$, note that ϕ is invariant against the center $\mathbb{R}^\times$. After that split the summation over c in this way $c = fak + s$ with $k \in \mathbb{Z}$, $0 \le s < fa$. Note that the gcd-condition transforms as follows $1 = (fa, fb, c) = (fa, fb, fak+s) = (fa, fb, s)$. This gives us

$$\begin{aligned} &G^{(1)}_{m_1,m_2}(z,s) \\ &= 2\frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|(f,M)}\sum_{\substack{0<d\le\frac{(f,M)}{e}\\(d,e)=1}}\sum_{a\ge 1}\sum_{\substack{b\in\mathbb{Z}\\ \left(\frac{N}{f},(a,b)\right)=1}}\sum_{k\in\mathbb{Z}}\sum_{\substack{0\le s<fa\\(f(a,b),s)=1}}\end{aligned}$$
$$\int_0^1\int_0^{(f,M)}\int_0^1 \phi\left(\sigma_{(d,e)}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)\left[f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + \left(fa\left(x_3+k\right)+fbx_1+s\right)^2\right]^{-\frac{3s}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,,$$

with the coefficients:

$$\begin{aligned}
y =&\frac{(a,b)^2}{m_{(d,e)}} y_2 \frac{\sqrt{f^2a^2y_1^2y_2^2 + f^2y_1^2(ax_2+b)^2 + \left(fa\left(x_3+k\right)+fbx_1+s\right)^2}}{a^2y_2^2+(ax_2+b)^2} \ , \\
x =&\frac{(a,b)fx_1 - rs}{m_{(d,e)}\frac{a}{(a,b)}} - \frac{f(a,b)rk}{m_{(d,e)}} - \frac{\frac{(a,b)^2}{m_{(d,e)}a}(ax_2+b)\left(fa\left(x_3+k\right)+fbx_1+s\right)}{a^2y_2^2+(ax_2+b)^2} \ .
\end{aligned}$$

In the second step first recall that $m_{(d,e)} \mid (f,M)$. So since $\phi(\sigma_{(d,e)}z)$ is periodic with period 1 in the real part, the integer summand $\frac{f(a,b)rk}{m_{(d,e)}}$ in the variable x can be deleted. Next pass to an infinite integral in the variable x_3 through the shift $x_3 \to x_3 + k$. Then substitute $x_3 \to x_3 + \frac{fbx_1+s}{fa}$ in the infinite integral in the variable x_3. After that split the summation over b in this way $b = a\,(f,M)\,k + t$ with $k \in \mathbb{Z}$, $0 \le t < a\,(f,M)$. Note that the two gcd-conditions transform as follows $1 = (fa, fb, s) = (fa, f\left(a\,(f,M)\,k+t\right), s) = (fa, ft, s)$ and $1 = \left(\frac{N}{f}, (a,b)\right) = \left(\frac{N}{f}, (a, a\,(f,M)\,k + t)\right) = \left(\frac{N}{f}, (a,t)\right)$. The identity

$$\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s) \\
&= 2\frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\left(y_1^2y_2\right)^s \Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \sum_{e|(f,M)} \sum_{\substack{0<d\le\frac{(f,M)}{e}\\(d,e)=1}} \sum_{a\ge1}\sum_{k\in\mathbb{Z}} \sum_{\substack{0\le t<a(f,M)\\\left(\frac{N}{f},(a,t)\right)=1}} \sum_{\substack{0\le s<fa\\(f(a,t),s)=1}} \\
&\int_{-\infty}^{\infty}\int_0^{(f,M)}\int_0^1 \phi\left(\sigma_{(d,e)}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\left[f^2a^2y_1^2y_2^2 + f^2y_1^2\left(a\left(x_2+(f,M)\,k\right)+t\right)^2 + f^2a^2x_3^2\right]^{-\frac{3s}{2}} \\
&e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \ ,
\end{aligned}$$

with the coefficients:

$$\begin{aligned}
y =&\frac{(a,t)^2}{m_{(d,e)}} y_2 \frac{\sqrt{f^2a^2y_1^2y_2^2 + f^2y_1^2\left(a\left(x_2+(f,M)\,k\right)+t\right)^2 + f^2a^2x_3^2}}{a^2y_2^2+\left(a\left(x_2+(f,M)\,k\right)+t\right)^2} \ , \\
x =&\frac{(a,t)fx_1 - rs}{m_{(d,e)}\frac{a}{(a,t)}} - \frac{\frac{f(a,t)^2}{m_{(d,e)}}\left(a\left(x_2+(f,M)\,k\right)+t\right)x_3}{a^2y_2^2+\left(a\left(x_2+(f,M)\,k\right)+t\right)^2}
\end{aligned}$$

holds.

In the third step pass to an infinite integral in the variable x_2 through the shift $x_2 \to x_2 + (f,M)\,k$. Then substitute $x_2 \to x_2 + \frac{t}{a}$ in the infinite integral in the variable x_2 and pick up an exponential $e\left(m_2\frac{t}{a}\right)$. This gives us

$$G^{(1)}_{m_1,m_2}(z,s) = 2\frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\left(y_1^2y_2\right)^s \Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)$$

$$f^{-3s}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\sum_{a\geq 1}a^{-3s}\sum_{\substack{0\leq t<a(f,M)\\\left(\frac{N}{f},(a,t)\right)=1}}e\left(m_2\frac{t}{a}\right)\sum_{\substack{0\leq s<fa\\(f(a,t),s)=1}}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1\phi\left(\sigma_{(d,e)}\begin{pmatrix}\frac{f(a,t)^2}{m_{(d,e)}a}y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2} & \frac{(a,t)fx_1-rs}{m_{(d,e)}\frac{a}{(a,t)}}-\frac{f\frac{(a,t)^2}{m_{(d,e)}a}x_2x_3}{y_2^2+x_2^2}\\ 0 & 1\end{pmatrix}\right)$$

$$\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Now Theorem 20.1 in Appendix C provides the Fourier expansion of the Maass cusp form ϕ at the cusp $\mathfrak{a}_{(d,e)}$. In the additional calculations these Fourier expansions are used to clear up the above expression. We have

$$G^{(1)}_{m_1,m_2}(z,s)=2\frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\left(y_1^2y_2\right)^s\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)$$

$$f^{-3s}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\sum_{a\geq 1}a^{-3s}\sum_{\substack{0\leq t<a(f,M)\\\left(\frac{N}{f},(a,t)\right)=1}}e\left(m_2\frac{t}{a}\right)\sum_{\substack{0\leq s<fa\\(f(a,t),s)=1}}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1\sum_{n\neq 0}2\phi_{(d,e)}(n)\sqrt{|n|\frac{f(a,t)^2}{m_{(d,e)}a}y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}}K_{\nu-\frac{1}{2}}\left(2\pi|n|\frac{f(a,t)^2}{m_{(d,e)}a}y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right)$$

$$e\left(n\frac{(a,t)fx_1-rs}{m_{(d,e)}\frac{a}{(a,t)}}-n\frac{f\frac{(a,t)^2}{m_{(d,e)}a}x_2x_3}{y_2^2+x_2^2}\right)\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Simplify the above terms and extract an exponential integral in the variable x_1. We have

$$G^{(1)}_{m_1,m_2}(z,s)=4\frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}y_1^{2s}y_2^{s+\frac{1}{2}}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)f^{\frac{1}{2}-3s}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\frac{1}{\sqrt{m_{(d,e)}}}$$

$$\sum_{a\geq 1}a^{-\frac{1}{2}-3s}\sum_{n\neq 0}\phi_{(d,e)}(n)\sqrt{|n|}\sum_{\substack{0\leq t<a(f,M)\\\left(\frac{N}{f},(a,t)\right)=1}}e\left(m_2\frac{t}{a}\right)(a,t)\sum_{\substack{0\leq s<fa\\(f(a,t),s)=1}}e\left(-\frac{nrs}{m_{(d,e)}\frac{a}{(a,t)}}\right)$$

$$\int_0^1 e\left(\left(\frac{n(a,t)f}{m_{(d,e)}\frac{a}{(a,t)}}-m_1\right)x_1\right)\mathrm{d}x_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}}\left(y_2^2+x_2^2\right)^{-\frac{1}{2}}e(-m_2x_2)$$

$$K_{\nu-\frac{1}{2}}\left(2\pi\frac{f|n|(a,t)^2}{m_{(d,e)}a}y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right)e\left(-\frac{nf(a,t)^2}{m_{(d,e)}a}\frac{x_2x_3}{y_2^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3\ . \tag{14.2}$$

Through the calculation of the series in the variable s and the integral in the variable x_1, the double integral is separated from the Dirichlet series. Begin with the calculation of the innermost sum in the variable s and extract a geometric sum. First note that Lemma 3.2 part (1a) implies that $-r$ can be absorbed into the summation. Then split the summation $s = f(a,t)k + w$ with $0 \leq k < \frac{a}{(a,t)}$ and $0 \leq w < f(a,t)$. Let $\mathbb{1}_{\mathbb{Z}}$ denote the indicator function of $\mathbb{Z}$, then the identity

$$\sum_{\substack{0\leq s<fa \\ (f(a,t),s)=1}} e\left(-\frac{nrs}{m_{(d,e)\frac{a}{(a,t)}}}\right) = \sum_{\substack{0\leq w<f(a,t) \\ (f(a,t),w)=1}} e\left(\frac{nw}{m_{(d,e)\frac{a}{(a,t)}}}\right) \sum_{0\leq k<\frac{a}{(a,t)}} e\left(\frac{nf(a,t)k}{m_{(d,e)\frac{a}{(a,t)}}}\right)$$
$$= \frac{a}{(a,t)} \mathbb{1}_{\mathbb{Z}}\left(\frac{nf(a,t)}{m_{(d,e)\frac{a}{(a,t)}}}\right) \sum_{\substack{0\leq w<f(a,t) \\ (f(a,t),w)=1}} e\left(\frac{nw}{m_{(d,e)\frac{a}{(a,t)}}}\right) \tag{14.3}$$

holds.

If $\frac{n(a,t)f}{m_{(d,e)\frac{a}{(a,t)}}}$ is an integer, which is the only relevant case since otherwise the above geometric sum vanishes, the exponential integral in the variable x_1 is non-zero if and only if $m_1 = \frac{n(a,t)f}{m_{(d,e)\frac{a}{(a,t)}}}$. Note that $n \neq 0$ implies that this equality can only hold if $m_1 \neq 0$. So one gets the formula

$$\int_0^1 e\left(\left(\frac{n(a,t)f}{m_{(d,e)\frac{a}{(a,t)}}} - m_1\right) x_1\right) \mathrm{d}x_1 = \delta_{m_1, \frac{n(a,t)f}{m_{(d,e)\frac{a}{(a,t)}}}} . \tag{14.4}$$

So put $n = \frac{m_1 m_{(d,e)} a}{f(a,t)^2}$ and the other results in (14.3), (14.4) in the formula (14.2) for $G^{(1)}_{m_1,m_2}$ and simplify the terms. This will give us the claimed separation

$$\begin{aligned}
G^{(1)}_{m_1,m_2}(z,s) =& 4(1-\delta_{0,m_1}) \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s} y_1^{2s} y_2^{s+\frac{1}{2}} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) m_1^{\frac{1}{2}} f^{-3s} \\
& \sum_{e|(f,M)} \sum_{\substack{0<d\leq\frac{(f,M)}{e} \\ (d,e)=1}} \sum_{a\geq 1} a^{1-3s} \sum_{\substack{0\leq t<a(f,M) \\ \left(\frac{N}{f},(a,t)\right)=1 \\ f(a,t)|m_1 m_{(d,e)\frac{a}{(a,t)}}}} \phi_{(d,e)}\left(\frac{m_1 m_{(d,e)\frac{a}{(a,t)}}}{f(a,t)}\right) (a,t)^{-1} e\left(m_2 \frac{t}{a}\right) \\
& \sum_{\substack{0\leq w<f(a,t) \\ (f(a,t),w)=1}} e\left(\frac{m_1 w}{f(a,t)}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_1^2x_2^2 + x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}} \left(y_2^2+x_2^2\right)^{-\frac{1}{2}} e(-m_2x_2) \\
& K_{\nu-\frac{1}{2}}\left(2\pi m_1 y_2 \frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right) e\left(-m_1\frac{x_2x_3}{y_2^2+x_2^2}\right) \mathrm{d}x_2 \mathrm{d}x_3 \\
=\;& 4(1-\delta_{0,m_1}) \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s} y_1^{2s} y_2^{s+\frac{1}{2}} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) m_1^{\frac{1}{2}} f^{-3s} \\
& \sum_{e|(f,M)} \sum_{\substack{0<d\leq\frac{(f,M)}{e} \\ (d,e)=1}} \sum_{a\geq 1} a^{1-3s} \sum_{\substack{0\leq t<a(f,M) \\ \left(\frac{N}{f},(a,t)\right)=1 \\ f(a,t)|m_1 m_{(d,e)\frac{a}{(a,t)}}}} \phi_{(d,e)}\left(\frac{m_1 m_{(d,e)\frac{a}{(a,t)}}}{f(a,t)}\right) (a,t)^{-1} e\left(m_2 \frac{t}{a}\right) c_{f(a,t)}(m_1)
\end{aligned}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}}\left(y_2^2+x_2^2\right)^{-\frac{1}{2}}e(-m_2x_2)$$
$$K_{\nu-\frac{1}{2}}\left(2\pi m_1y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right)e\left(-m_1\frac{x_2x_3}{y_2^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3$$

of the double integral and the Dirichlet series.
To complete our calculations we distinguish between the cases whether m_2 vanishes or not. In the upcoming calculations use the notation in Definition 14.1 for the Dirichlet series.

(a) 1. Case: $m_2\neq 0$. First substitute $x_2\to m_2x_2$ in the integral in the variable x_2, after that substitute $x_3\to m_1m_2x_3$ in the integral in the variable x_3. Then expand the terms with the spectral parameters. We have

$$G^{(1)}_{m_1,m_2}(z,s)$$
$$=2(1-\delta_{0,m_1})\frac{1}{(f,M)}f^{-3s}\sqrt{m_1m_2}m_1^{s-\frac{3}{2}}|m_2|^{2s-\frac{5}{2}}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}A_{m_1,m_2}(3s-1,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1})$$
$$2\pi^{-3\left(s-\frac{\nu}{3}\right)-\frac{3}{2}\left(\frac{2\nu}{3}\right)+\frac{1}{2}}\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right)\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)+3\left(\frac{2}{3}\nu\right)-1}{2}\right)(m_1y_1)^{2\left(s-\frac{\nu}{3}\right)+\frac{2\nu}{3}}\left(|m_2|\,y_2\right)^{\frac{1}{2}+\left(s-\frac{\nu}{3}\right)+\frac{1}{2}\left(\frac{2\nu}{3}\right)}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(m_1y_1)^2(|m_2|\,y_2)^2+(m_1y_1)^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3}{2}\left(s-\frac{\nu}{3}\right)-\frac{3}{4}\left(\frac{2\nu}{3}\right)}\left(\left(|m_2|\,y_2\right)^2+x_2^2\right)^{-\frac{1}{2}}e(-x_2)$$
$$K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi\left(|m_2|\,y_2\right)\frac{\sqrt{(m_1y_1)^2\left(|m_2|\,y_2\right)^2+(m_1y_1)^2x_2^2+x_3^2}}{\left(|m_2|\,y_2\right)^2+x_2^2}\right)e\left(-\frac{x_2x_3}{\left(|m_2|\,y_2\right)^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3\,.$$

Lemma 19.5 part (1) in Appendix B implies that the above double integral is a Whittaker function of the type $W_{1,1}^{(\nu_1,\nu_2)}(*,w_1)$. Finally we get

$$G^{(1)}_{m_1,m_2}(z,s)=2(1-\delta_{0,m_1})\frac{1}{(f,M)}f^{-3s}\sqrt{m_1m_2}m_1^{s-\frac{3}{2}}|m_2|^{2s-\frac{5}{2}}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}$$
$$A_{m_1,m_2}(3s-1,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1})W_{1,1}^{\left(s-\frac{\nu}{3},\frac{2\nu}{3}\right)}\left(\begin{pmatrix}m_1|m_2| & & \\ & m_1 & \\ & & 1\end{pmatrix}z,w_1\right)\,.$$

(b) 2. Case: $m_2=0$. Substitute $x_3\to m_1x_3$ in the integral in the variable x_3, then expand the terms with the spectral parameters. We have

$$G^{(1)}_{m_1,0}(z,s) = 2(1-\delta_{0,m_1})\frac{1}{(f,M)}f^{-3s}m_1^{s-1}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}A_{m_1,0}(3s-1,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1})$$

$$2\pi^{-3\left(s-\frac{\nu}{3}\right)-\frac{3}{2}\left(\frac{2\nu}{3}\right)+\frac{1}{2}}\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right)\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)+3\left(\frac{2}{3}\nu\right)-1}{2}\right)(m_1y_1)^{2\left(s-\frac{\nu}{3}\right)+\frac{2\nu}{3}}y_2^{\frac{1}{2}+\left(s-\frac{\nu}{3}\right)+\frac{1}{2}\left(\frac{2\nu}{3}\right)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(m_1y_1)^2y_2^2+(m_1y_1)^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3}{2}\left(s-\frac{\nu}{3}\right)-\frac{3}{4}\left(\frac{2\nu}{3}\right)}\left(y_2^2+x_2^2\right)^{-\frac{1}{2}}$$

$$K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi y_2\frac{\sqrt{(m_1y_1)^2y_2^2+(m_1y_1)^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right)e\left(-\frac{x_2x_3}{y_2^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3\,.$$

Lemma 19.5 part (2) in Appendix B implies that the above double integral is a Whittaker function of the type $W^{(\nu_1,\nu_2)}_{1,0}(*,w_1)$. Finally we get

$$G^{(1)}_{m_1,0}(z,s) = 2(1-\delta_{0,m_1})\frac{1}{(f,M)}f^{-3s}m_1^{s-1}\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}A_{m_1,0}(3s-1,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1})$$

$$W^{\left(s-\frac{\nu}{3},\frac{2\nu}{3}\right)}_{1,0}\left(\begin{pmatrix}m_1&&\\&m_1&\\&&1\end{pmatrix}z,w_1\right)\,.$$

(2) We proceed with the calculation of $G^{(2)}_{m_1,m_2}$. It turns out that this part of the Fourier coefficient always vanishes. We have

$$G^{(2)}_{m_1,m_2}(z,s) = \frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\sum_{\substack{b\neq0\\\left(b,\frac{N}{f}\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\(fb,c)=1}}$$

$$\int_0^1\int_0^{(f,M)}\int_0^1\phi\left(p_{(0,b,c,d,e)}\begin{pmatrix}y_2\sqrt{y_1^2b^2f^2+(fbx_1+c)^2}&-bfx_3+(fbx_1+c)x_2\\0&1\end{pmatrix}\right)$$

$$\left(y_1^2y_2\right)^s\left[f^2y_1^2b^2+(fbx_1+c)^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,.$$

Again substitute the splitting in (14.1). So we get

$$G^{(2)}_{m_1,m_2}(z,s) = \frac{1}{(f,M)}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\sum_{\substack{b\neq0\\\left(b,\frac{N}{f}\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\(fb,c)=1}}$$

$$\int_0^1\int_0^{(f,M)}\int_0^1\phi\left(\sigma_{(d,e)}\begin{pmatrix}\frac{y_2}{m_{(d,e)}}\sqrt{y_1^2b^2f^2+(fbx_1+c)^2}&-\frac{bf}{m_{(d,e)}}x_3+\frac{(fbx_1+c)x_2}{m_{(d,e)}}\\0&1\end{pmatrix}\right)$$

$$\left(y_1^2y_2\right)^s\left[f^2y_1^2b^2+(fbx_1+c)^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Now Theorem 20.1 in Appendix C provides the Fourier expansion of the cusp form ϕ at the cusp $\mathfrak{a}_{(d,e)}$. This gives us

$$G^{(2)}_{m_1,m_2}(z,s)=\frac{1}{(f,M)}\pi^{\frac12-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|(f,M)}\sum_{\substack{0<d\leq\frac{(f,M)}{e}\\(d,e)=1}}\sum_{\substack{b\neq0\\\left(b,\frac{N}{f}\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\(fb,c)=1}}$$

$$\int_0^1\int_0^{(f,M)}\int_0^1\sum_{n\neq0}2\phi_{(d,e)}(n)\sqrt{|n|\frac{y_2}{m_{(d,e)}}\sqrt{y_1^2b^2f^2+(fbx_1+c)^2}}K_{\nu-\frac12}\left(2\pi|n|\frac{y_2}{m_{(d,e)}}\sqrt{y_1^2b^2f^2+(fbx_1+c)^2}\right)$$

$$e\left(-n\frac{bf}{m_{(d,e)}}x_3+n\frac{(fbx_1+c)x_2}{m_{(d,e)}}\right)\left(y_1^2y_2\right)^s\left[f^2y_1^2b^2+(fbx_1+c)^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Note that we can extract the exponential integral in the variable x_3. Since $n,b\neq0$ and $m_{(d,e)}\mid f$ was shown in the first part, this integral vanishes

$$\int_0^1 e\left(-\frac{f}{m_{(d,e)}}bnx_3\right)\mathrm{d}x_3=0\ .$$

So this part of the Fourier coefficient always vanishes:

$$G^{(2)}_{m_1,m_2}(z,s)=0\ .$$

(3) It remains to do the easiest part, the calculation of $G^{(3)}_{m_1,m_2}(z,s)$. Again Lemma 4.2 gives a description of the set $\Gamma_3\left(f,M,P_{2,1}\right)$, which consists of two diagonal matrices at most. Then use Lemma 8.2 to calculate the values of the $I_{(s,-2s)}\left(*,P_{2,1}\right)$-function directly. We have

$$G^{(3)}_{m_1,m_2}(z,s)=\delta_{f,N}\frac{1}{(f,M)}\pi^{\frac12-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{e|M}\sum_{\substack{0<d\leq\frac{M}{e}\\(d,e)=1}}$$

$$\int_0^1\int_0^{(f,M)}\int_0^1\left[\phi\left(\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix}p_{(0,0,\pm1,d,e)}&&*\\&&*\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z\right)\right)\right.$$

$$I_{(s,-2s)}\left(\begin{pmatrix}p_{(0,0,\pm1,d,e)}&&*\\&&*\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z,P_{2,1}\right)$$

$$+\phi\left(\mathfrak{m}_{P_{2,1}}\left(\begin{pmatrix}p_{(0,0,\pm1,d,e)}&&*\\&&*\\&&1\end{pmatrix}\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z\right)\right)$$

$$I_{(s,-2s)}\left(\left(\begin{pmatrix} p_{(0,0,\pm1,d,e)} & & * \\ & & * \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, P_{2,1}\right)\right)\Bigg]$$
$$e\left(-m_1 x_1 - m_2 x_2\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \; .$$

To simplify the above terms use the left invariance of the $I_{(s,-2s)}(*, P_{2,1})$-function against the real maximal parabolic subgroup $P_{2,1}$ and the right invariance against the maximal compact subgroup O_3 in the second summand. Substituting the splitting in (14.1) for $p_{(0,0,\pm1,d,e)}$ gives the formula

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N} \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) \sum_{e|M} \sum_{\substack{0<d\leq \frac{M}{e} \\ (d,e)=1}}$$
$$\int_0^1 \int_0^{(f,M)} \int_0^1 \left[\phi\left(\sigma_{(d,e)} \begin{pmatrix} m_{(d,e)}^{-1} & \\ & 1 \end{pmatrix} \mathfrak{m}_{P_{2,1}}\left(\left(\begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}\right)\right)\right)\right.$$
$$\det\left(\left(\begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}\right)\right)^s \parallel e_3^T \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \parallel^{-3s}$$
$$+ \quad \phi\left(\sigma_{(d,e)} \begin{pmatrix} m_{(d,e)}^{-1} & \\ & 1 \end{pmatrix} \mathfrak{m}_{P_{2,1}}\left(\left(\begin{pmatrix} y_1 y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix}\right)\right)\right) \det\left(\left(\begin{pmatrix} y_1 y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix}\right)\right)^s$$
$$\left.\parallel e_3^T \begin{pmatrix} y_1 y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix} \parallel^{-3s}\right] e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \; .$$

Calculating the integrals in the variable x_1, x_3 gives us the formula

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N} \delta_{m_1,0} \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) \left(y_1^2 y_2\right)^s \sum_{e|M} \sum_{\substack{0<d\leq \frac{M}{e} \\ (d,e)=1}}$$
$$\int_0^{(f,M)} \left[\phi\left(\sigma_{(d,e)} \begin{pmatrix} \frac{y_2}{m_{(d,e)}} & \frac{x_2}{m_{(d,e)}} \\ & 1 \end{pmatrix}\right) + \phi\left(\sigma_{(d,e)} \begin{pmatrix} \frac{y_2}{m_{(d,e)}} & -\frac{x_2}{m_{(d,e)}} \\ & 1 \end{pmatrix}\right)\right] e(-m_2 x_2) \mathrm{d}x_2 \; .$$

For the calculation of the integral in the variable x_2 use the Fourier expansion of the cusp form ϕ at the cusp $\mathfrak{a}_{(d,e)}$. This gives us

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N} \delta_{m_1,0} \frac{1}{(f,M)} \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) \left(y_1^2 y_2\right)^s$$
$$\sum_{e|M} \sum_{\substack{0<d\leq \frac{M}{e} \\ (d,e)=1}} \sum_{n\neq 0} 2\phi_{(d,e)}(n) \sqrt{|n| \frac{y_2}{m_{(d,e)}}} K_{\nu-\frac{1}{2}}\left(2\pi |n| \frac{y_2}{m_{(d,e)}}\right)$$

$$\int_0^{(f,M)} e\left(\left(\frac{n}{m_{(d,e)}} - m_2\right) x_2\right) + e\left(-\left(\frac{n}{m_{(d,e)}} + m_2\right) x_2\right) \mathrm{d}x_2 \ .$$

Note that $m_{(d,e)} \mid (f,M)$ implies that the exponential integrals are only non-zero if $m_2 = \pm \frac{n}{m_{(d,e)}}$. Since $n \neq 0$ this implies immediately $m_2 \neq 0$. We have

$$\begin{aligned} G^{(3)}_{m_1,m_2}(z,s) =& \delta_{N,f}\delta_{m_1,0}(1-\delta_{m_2,0}) \sum_{e|M} \sum_{\substack{0<d\leq \frac{M}{e} \\ (d,e)=1}} \left[\phi_{(d,e)}\left(m_2 m_{(d,e)}\right) + \phi_{(d,e)}\left(-m_2 m_{(d,e)}\right)\right] |m_2|^{-s} \\ & 2\pi^{-3\left(s-\frac{\nu}{3}\right) - \frac{3\left(\frac{2\nu}{3}\right)}{2} + \frac{1}{2}} \Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right) \Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right) + 3\left(\frac{2\nu}{3}\right) - 1}{2}\right) \\ & y_1^{2\left(s-\frac{\nu}{3}\right) + \frac{2\nu}{3}} \left(|m_2| y_2\right)^{\frac{1}{2} + \left(s-\frac{\nu}{3}\right) + \frac{\frac{2\nu}{3}}{2}} K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi |m_2| y_2\right) \ . \end{aligned}$$

The explicit formula for the Whittaker function $W_{0,1}^{(\nu_1,\nu_2)}(z,w_2)$ in Lemma 19.4 in Appendix B finally gives the claimed form

$$\begin{aligned} G^{(3)}_{m_1,m_2}(z,s) =& \delta_{N,f}\delta_{m_1,0}(1-\delta_{m_2,0}) \sum_{e|M} \sum_{\substack{0<d\leq \frac{M}{e} \\ (d,e)=1}} \left[\phi_{(d,e)}\left(m_2 m_{(d,e)}\right) + \phi_{(d,e)}\left(-m_2 m_{(d,e)}\right)\right] \\ & |m_2|^{-s} W_{0,1}^{\left(s-\frac{\nu}{3}, \frac{2\nu}{3}\right)}\left(\begin{pmatrix} |m_2| & & \\ & 1 & \\ & & 1 \end{pmatrix} z, w_2\right) \ . \end{aligned}$$

□

For a general Maass cusp form the Fourier coefficients don't have to be multiplicative, so in the general case one cannot expect to be able to analyze the Dirichlet series $A_{m_1,m_2}\left(s,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1}\right)$ more closely. But since the twisting with a Maass cusp form in the Eisenstein series is linear, one can restrict to a basis of Maass cusp forms for $\Gamma_0(M)$. Atkin Lehner theory, see [**2**, thm. 5], implies that one can assume $\phi(z) = \psi(gz)$ with ψ a newform for $\Gamma_0(m)$ with $m \mid M$ and g a proper divisor of $\frac{M}{m}$. Since a newform is an eigenfunction of all Hecke operators, even at the ramified primes, the Fourier coefficients at infinity of the newform are multiplicative and fulfill the recursions in Theorem 20.2 in Appendix C. In the paper [**1**] it is proved that for a newform with squarefree level the Fourier coefficients at any cusp are identical with the Fourier coefficients at the cusp ∞ up to multiplication with a character. Note that in the case of a non squarefree level this is no longer true, so there exist cusps such that the corresponding Fourier coefficients do not satisfy any Hecke relations. These results were stated and proved for modular forms, but it can be easily seen that the same results are true for Maass forms. With this in mind we can make the assumption that the sequence $\phi_{(d,e)}$ consists of the Fourier coefficients of a newform at the cusp ∞ and begin to analyze the Dirichlet series $A_{m_1,m_2}\left(s,f,M,\phi_{(d,e)},m_{(d,e)},P_{2,1}\right)$. It turns out that this Dirichlet series can be factorized into an Euler product whose factors are the power series defined in chapter 9 and 10. We start specifying the theory of power series developed in chapter 9 and 10, so that it can be applied to the Fourier coefficients of the twisted Eisenstein series. Let ϕ be a newform of level M and let $\phi(n)$ denote the n-th Fourier coefficient of ϕ at the cusp ∞. Use the notation introduced in the part "Power series" and associate to any prime number p the power series and variables

$$X := p^{1-3s} \tag{14.5}$$

$$Y := p^{-1} \tag{14.6}$$

$$A_n := \phi\left(p^n\right) \tag{14.7}$$

$$S_p = 1 - A_1 X + Y X^2 = 1 - \phi(p)\, p^{-(3s-1)} + p^{-1} p^{-2(3s-1)} \tag{14.8}$$

$$T_p = 1 - pA_1 X + p^2 Y X^2 = 1 - \phi(p)\, p^{-(3s-2)} + p^{-1} p^{-2(3s-2)} \tag{14.9}$$

$$F_p(s,\alpha,\beta) = \sum_{n\geq 0} p^{(1-3s)n} \sum_{\substack{0\leq k\leq n\\ 2k\leq \alpha+n}} \phi\left(p^{n+\alpha-2k}\right) p^{-k} c_{p^{n-k}}(\beta) c_{p^k}(p^\alpha) \tag{14.10}$$

$$G_p(s,\alpha,\beta) = \sum_{n\geq 0} p^{(1-3s)n} \sum_{\substack{0\leq k\leq n\\ 2k+1\leq \alpha+n}} \phi\left(p^{n+\alpha-2k-1}\right) p^{-k} c_{p^{n-k}}(\beta) c_{p^{k+1}}(p^\alpha) \tag{14.11}$$

$$H_p(s,\alpha,\beta) = \sum_{n\geq 0} p^{(1-3s)n} \phi\left(p^{\alpha+n}\right) c_{p^n}(\beta) \tag{14.12}$$

$$I_p(s,\alpha,\beta) := \sum_{n\geq 0} p^{(1-3s)n} \sum_{\substack{0\leq k\leq n\\ 2k\leq \beta+n+1}} \phi\left(p^{\beta+n+1-2k}\right) p^{-k} c_{p^{n+1-k}}(\alpha) c_{p^k}\left(p^\beta\right) \tag{14.13}$$

$$J_p(s,\alpha,\beta) := \sum_{n\geq 0} p^{(1-3s)n} \sum_{\substack{0\leq k\leq n\\ 2k\leq \alpha+n}} \phi\left(p^{\alpha+n-2k}\right) p^{-k} c_{p^{n-k}}(\beta) c_{p^{k+1}}\left(p^\alpha\right) \tag{14.14}$$

$$K_p(s,\alpha,\beta) := \sum_{n\geq 0} p^{(1-3s)n} \sum_{\substack{0\leq k\leq n\\ 2k+1\leq \alpha+n}} \phi\left(p^{\alpha+n-2k-1}\right) p^{-k} c_{p^{n-k}}(\beta) c_{p^{k+1}}\left(p^\alpha\right) \tag{14.15}$$

$$L_p(s,\alpha,\beta) := \sum_{n\geq 0} p^{(1-3s)n} \phi\left(p^{\alpha+n}\right) c_{p^n}(\beta) \tag{14.16}$$

$$M_p(s,\alpha,\beta) := \sum_{n\geq 0} p^{(1-3s)n} \left(\phi\left(p^{\beta+1+n}\right) c_{p^{n+1}}(\alpha) + \sum_{\substack{1\leq k\leq n\\ 2k\leq \beta+n+1}} \phi\left(p^{\beta+n+1-2k}\right) p^{-k} c_{p^{n+1-k}}(\alpha) c_{p^k}\left(p^\beta\right)\right). \tag{14.17}$$

Note that for $(p, M) = 1$ the Hecke relations for the Fourier coefficients ϕ_{p^n} are identical with the recursion for the sequence $(A_n)_{n\in\mathbb{N}}$. Also note that the polynomials S_p, T_p are the Euler factors at the unramified primes of the L-function of ϕ. We will now use the theory developed in chapter 9 and define for each prime a holomorphic function, which occurs at the unramified primes in the Fourier coefficients of the non degenerate terms of the twisted Eisenstein series and has the right transformation behaviour.

DEFINITION 14.4. For any prime number p and nonnegative integers α,β define the meromorphic function $\mathfrak{B}_p(*,\alpha,\beta):\mathbb{C}\to\mathbb{C}$, $s\longmapsto p^{\left(s-\frac{3}{2}\right)\alpha}p^{\left(2s-\frac{5}{2}\right)\beta}\frac{F_p(s,\alpha,p^\beta)}{S_p(s)}$. The function $\mathfrak{B}_p$ is holomorphic and obeys the transformation law $\mathfrak{B}_p(1-s,\alpha,\beta)=\mathfrak{B}_p(s,\beta,\alpha)$.

PROOF. The proof follows easily from the transformation law in Lemma 9.5 and $p^{-2}X^{-1}Y^{-1}=p^{1-3(1-s)}$. We have

$$\begin{aligned}
&\mathfrak{B}_p(1-s,\alpha,\beta)\\
=&p^{\left((1-s)-\frac{3}{2}\right)\alpha}p^{\left(2(1-s)-\frac{5}{2}\right)\beta}\frac{F_p(1-s,\alpha,p^\beta)}{S_p(1-s)}\\
=&p^{\left(-\frac{1}{2}-s\right)\alpha}p^{\left(-\frac{1}{2}-2s\right)\beta}\left(pp^{1-3s}\right)^{-\alpha-\beta}\left(p^{-1}\right)^{-\beta}\frac{F_p(s,\beta,p^\alpha)}{S_p(s)}\\
=&p^{\left(s-\frac{3}{2}\right)\beta}p^{\left(2s-\frac{5}{2}\right)\alpha}\frac{F_p(s,\beta,p^\alpha)}{S_p(s)}\\
=&\mathfrak{B}_p(s,\beta,\alpha)\ .
\end{aligned}$$

□

With the same notation as in Theorem 14.3 and under the assumption that the sequence $(\phi(n))_{n\in\mathbb{N}}$ consists of the Fourier coefficients of a newform for $\Gamma_0(M)$ at the cusp ∞, the following explicit formulas for the Dirichlet series $A_{m_1,m_2}\left(s,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1}\right)$ are valid.

LEMMA 14.5. *Let $m_1=\prod\limits_p p^{\alpha_p}$, $m_2=\prod\limits_p p^{\beta_p}$ be positive integers with corresponding prime decompositions and m be a positive divisor of (f,M). The explicit formulas*

$$\begin{aligned}
&A_{m_1,m_2}\left(3s-1,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1}\right)\\
=&(f,M)\,L_\phi(3s-1)^{-1}\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\alpha_p}p^{-\left(2s-\frac{5}{2}\right)\beta_p}\mathfrak{B}_p(s,\alpha_p,\beta_p)\right)\left(\prod_{p\mid m}\frac{J_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\\
&\left(\prod_{p\mid\frac{(f,M)}{m}}\frac{K_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{f}{(f,M)}}\frac{G_p\left(s,\alpha_p,p^{\beta_p}\right)}{S_p(s)}\right)\left(\prod_{p\mid\left(\frac{N}{f},M\right)}\frac{L_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)
\end{aligned}$$

$$\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{H_p\left(s,\alpha_p,p^{\beta_p}\right)}{S_p(s)}\right)$$

and

$$\begin{aligned}&A_{m_1,0}(3s-1,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1})\\&=(f,M)\,\frac{L_\phi(3s-2)}{L_\phi(3s-1)}\left(\prod_{(p,N)=1}\phi\left(p^{\alpha_p}\right)\right)\left(\prod_{p\mid m}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)J_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\\&\quad\left(\prod_{p\mid\frac{(f,M)}{m}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)K_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{f}{(f,M)}}\frac{T_p(s)G_p\left(s,\alpha_p,0\right)}{S_p(s)}\right)\\&\quad\left(\prod_{p\mid\left(\frac{N}{f},M\right)}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)L_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{T_p(s)H_p\left(s,\alpha_p,0\right)}{S_p(s)}\right)\end{aligned}$$

for the Dirichlet series $A_{m_1,*}\left(s,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1}\right)$ in Definition 14.1 are valid.

Proof. Like in the proofs of the Lemmata 11.2, 12.2 we handle both parts together, so assume for the present that m_2 is a nonnegative integer. In the beginning split the summation in the inner sum of A_{m_1,m_2} in the way $t=al+\tilde{t}$ with $0\le l<(f,M)$ and $0\le\tilde{t}<a$. This implies $(a,t)=\left(a,\tilde{t}\right)$, also note the transformation of the gcd-condition $1=\left(\frac{N}{f},(a,t)\right)=\left(\frac{N}{f},\left(a,\tilde{t}\right)\right)$. We obtain

$$\begin{aligned}&A_{m_1,m_2}\left(s,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1}\right)\\&=\sum_{a\ge1}a^{-s}\sum_{0\le l<(f,M)}\sum_{\substack{0\le\tilde{t}<a\\\left(\frac{N}{f},(a,\tilde{t})\right)=1\\f(a,\tilde{t})\mid m_1m\frac{a}{(a,\tilde{t})}}}\phi\left(\frac{m_1m\frac{a}{(a,\tilde{t})}}{f(a,\tilde{t})}\right)(a,\tilde{t})^{-1}e\left(m_2\frac{al+\tilde{t}}{a}\right)c_{f(a,\tilde{t})}(m_1)\\&=(f,M)\sum_{a\ge1}a^{-s}\sum_{\substack{0\le\tilde{t}<a\\\left(\frac{N}{f},(a,\tilde{t})\right)=1\\f(a,\tilde{t})\mid m_1m\frac{a}{(a,\tilde{t})}}}\phi\left(\frac{m_1m\frac{a}{(a,\tilde{t})}}{f(a,\tilde{t})}\right)(a,\tilde{t})^{-1}e\left(m_2\frac{\tilde{t}}{a}\right)c_{f(a,\tilde{t})}(m_1)\end{aligned}$$

In the first step split the summation in the inner sum of A_{m_1,m_2} in the way $\tilde{t}=t_1t_2$ with $t_1 \mid a$ and $\left(t_2,\frac{a}{t_1}\right)=1$. This implies $(\tilde{t},a)=t_1$, also note the transformation of the gcd-condition $1=\left(\frac{N}{f},(a,\tilde{t})\right)=\left(\frac{N}{f},t_1\right)$. We obtain

$$\begin{aligned}
&A_{m_1,m_2}\left(s,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1}\right)\\
&=(f,M)\sum_{a\geq 1}a^{-s}\sum_{\substack{t_1|a\\ \left(\frac{N}{f},t_1\right)=1\\ ft_1|m_1m\frac{a}{t_1}}}\ \sum_{\substack{0\leq t_2<\frac{a}{t_1}\\ \left(t_2,\frac{a}{t_1}\right)=1}}\phi\left(\frac{m_1m\frac{a}{t_1}}{ft_1}\right)t_1^{-1}e\left(m_2\frac{t_2}{\frac{a}{t_1}}\right)c_{ft_1}\left(m_1\right)\\
&=(f,M)\sum_{a\geq 1}a^{-s}\sum_{\substack{t|a\\ \left(\frac{N}{f},t\right)=1\\ ft|m_1m\frac{a}{t}}}\phi\left(\frac{m_1m\frac{a}{t}}{ft}\right)t^{-1}c_{\frac{a}{t}}\left(m_2\right)c_{ft}\left(m_1\right)\ .
\end{aligned}$$

The multiplicativity of the Fourier coefficients $(\phi(n))_{n\in\mathbb{N}}$ implies that the Dirichlet series can be factorized. In order to do this decompose $m_1=m_1^{(1)}m_1^{(2)}m_1^{(3)}m_1^{(4)}m_1^{(5)}m_1^{(6)}$ and split the summation in the outer and inner sum $a=a_1a_2a_3a_4a_5a_6$ and $t=t_1t_2t_3t_4t_5t_6$ according to the divisibility conditions $\left(m_1^{(1)}a_1t_1,N\right)=1$, $m_1^{(2)}a_2t_2\mid m^\infty$, $m_1^{(3)}a_3t_3\mid\left(\frac{(f,M)}{m}\right)^\infty$, $m_1^{(4)}a_4t_4\mid\left(\frac{f}{(f,M)}\right)^\infty$, $m_1^{(5)}a_5t_5\mid\left(\frac{N}{f},M\right)^\infty$ and $m_1^{(6)}a_6t_6\mid\left(\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}\right)^\infty$. Note that the gcd-condition $1=\left(\frac{N}{f},t\right)=\left(\frac{N}{f},t_1t_2t_3t_4t_5t_6\right)=\left(\frac{N}{f},t_5t_6\right)=t_5t_6$ implies $t_5=t_6=1$. Hence the divisibility condition $ft\mid m_1m\frac{a}{t}$ is equivalent to the four conditions $t_k^2\mid m_1^{(k)}a_k$ for $k=1,2$, $\frac{(f,M)}{m}t_3^2\mid m_1^{(3)}a_3$ and $t_4^2\frac{f}{(f,M)}\mid m_1^{(4)}a_4$. We obtain

$$\begin{aligned}
&A_{m_1,m_2}(s,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1})\\
&=(f,M)\sum_{\substack{(a_1,N)=1\\ a_2|m^\infty\\ a_3|\left(\frac{(f,M)}{m}\right)^\infty\\ a_4|\left(\frac{f}{(f,M)}\right)^\infty\\ a_5|\left(\frac{N}{f},M\right)^\infty\\ a_6|\left(\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}\right)^\infty}}(a_1a_2a_3a_4a_5a_6)^{-s}\sum_{\substack{t_1|a_1\\ t_2|a_2\\ t_3|a_3\\ t_4|a_4\\ t_1^2|m_1^{(1)}a_1\\ t_2^2|m_1^{(2)}a_2\\ \frac{(f,M)}{m}t_3^2|m_1^{(3)}a_3\\ \frac{f}{(f,M)}t_4^2|m_1^{(4)}a_4}}\phi\left(\frac{m_1^{(1)}a_1}{t_1^2}\frac{m_1^{(2)}a_2}{t_2^2}\frac{m_1^{(3)}a_3}{\frac{(f,M)}{m}t_3^2}\frac{m_1^{(4)}a_4}{\frac{f}{(f,M)}t_4^2}m_1^{(5)}a_5m_1^{(6)}a_6\right)\\
&(t_1t_2t_3t_4)^{-1}c_{\frac{a_1}{t_1}\frac{a_2}{t_2}\frac{a_3}{t_3}\frac{a_4}{t_4}a_5a_6}(m_2)c_{t_1(mt_2)\left(\frac{(f,M)}{m}t_3\right)\left(\frac{f}{(f,M)}t_4\right)}\left(m_1^{(1)}m_1^{(2)}m_1^{(3)}m_1^{(4)}m_1^{(5)}m_1^{(6)}\right)
\end{aligned}$$

$$= (f, M) \left(\sum_{(a,N)=1} a^{-s} \sum_{\substack{t|a \\ t^2|m_1^{(1)}a}} \phi\left(\frac{m_1^{(1)}a}{t^2}\right) t^{-1} c_{\frac{a}{t}}(m_2) c_t\left(m_1^{(1)}\right) \right)$$

$$\left(\sum_{a|m^\infty} a^{-s} \sum_{\substack{t|a \\ t^2|m_1^{(2)}a}} \phi\left(\frac{m_1^{(2)}a}{t^2}\right) t^{-1} c_{\frac{a}{t}}(m_2) c_{mt}\left(m_1^{(2)}\right) \right)$$

$$\left(\sum_{a|\left(\frac{(f,M)}{m}\right)^\infty} a^{-s} \sum_{\substack{t|a \\ \frac{(f,M)}{m}t^2|m_1^{(3)}a}} \phi\left(\frac{m_1^{(3)}a}{\frac{(f,M)}{m}t^2}\right) t^{-1} c_{\frac{a}{t}}(m_2) c_{\frac{(f,M)}{m}t}\left(m_1^{(3)}\right) \right)$$

$$\left(\sum_{a|\left(\frac{f}{(f,M)}\right)^\infty} a^{-s} \sum_{\substack{t|a \\ \frac{f}{(f,M)}t^2|m_1^{(4)}a}} \phi\left(\frac{m_1^{(4)}a}{\frac{f}{(f,M)}t^2}\right) t^{-1} c_{\frac{a}{t}}(m_2) c_{\frac{f}{(f,M)}t}\left(m_1^{(4)}\right) \right)$$

$$\left(\sum_{a|\left(\frac{N}{f},M\right)^\infty} a^{-s} \phi\left(m_1^{(5)}a\right) c_a(m_2) \right) \left(\sum_{a|\left(\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}\right)^\infty} a^{-s} \phi\left(m_1^{(6)}a\right) c_a(m_2) \right) .$$

Next factorize each of the six Dirichlet series into an Euler product. This gives us the formula

$$A_{m_1,m_2}(s, f, M, (\phi(n))_{n\in\mathbb{N}}, m, P_{2,1})$$

$$= (f, M) \left(\prod_{(p,N)=1} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \alpha_p+n}} \phi\left(p^{\alpha_p+n-2k}\right) p^{-k} c_{p^{n-k}}(m_2) c_{p^k}\left(p^{\alpha_p}\right) \right)$$

$$\left(\prod_{p|m} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \alpha_p+n}} \phi\left(p^{\alpha_p+n-2k}\right) p^{-k} c_{p^{n-k}}(m_2) c_{p^{k+1}}\left(p^{\alpha_p}\right) \right)$$

$$\left(\prod_{p|\frac{(f,M)}{m}} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k+1\le \alpha_p+n}} \phi\left(p^{\alpha_p+n-2k-1}\right) p^{-k} c_{p^{n-k}}(m_2) c_{p^{k+1}}\left(p^{\alpha_p}\right) \right)$$

$$\left(\prod_{p\mid\frac{f}{(f,M)}}\sum_{n=0}^{\infty}p^{-ns}\sum_{\substack{0\le k\le n\\2k+1\le\alpha_p+n}}\phi\left(p^{\alpha_p+n-2k-1}\right)p^{-k}c_{p^{n-k}}(m_2)c_{p^{k+1}}\left(p^{\alpha_p}\right)\right)$$

$$\left(\prod_{p\mid\left(\frac{N}{f},M\right)}\sum_{n=0}^{\infty}p^{-ns}\phi\left(p^{\alpha_p+n}\right)c_{p^n}(m_2)\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\sum_{n=0}^{\infty}p^{-ns}\phi\left(p^{\alpha_p+n}\right)c_{p^n}(m_2)\right).$$

Finally we distinguish between the cases whether m_2 vanishes or not and use the notation for the power series introduced above to express the factors in the Euler product.

(1) 1. Case: $m_2\neq 0$. Use the notation from Definition 14.4 and note that the polynomial S_p coincides with the p-th factor in the Euler product of the L-function associated to the sequence of Fourier coefficients ϕ at the unramified primes (see Appendix C Theorem 20.2 for details). We obtain the claimed formula

$$\begin{aligned}
&A_{m_1,m_2}(3s-1,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1})\\
&=(f,M)\left(\prod_{(p,N)=1}F_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid m}J_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\frac{(f,M)}{m}}K_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\\
&\quad\left(\prod_{p\mid\frac{f}{(f,M)}}G_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\left(\frac{N}{f},M\right)}L_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}H_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\\
&=(f,M)\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\alpha_p}p^{-\left(2s-\frac{5}{2}\right)\beta_p}S_p(s)\mathfrak{B}_p(s,\alpha_p,\beta_p)\right)\\
&\quad\left(\prod_{p\mid m}J_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\frac{(f,M)}{m}}K_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\\
&\quad\left(\prod_{p\mid\frac{f}{(f,M)}}G_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\left(\frac{N}{f},M\right)}L_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}H_p\left(s,\alpha_p,p^{\beta_p}\right)\right)\\
&=(f,M)\,L_\phi(3s-1)^{-1}\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\alpha_p}p^{-\left(2s-\frac{5}{2}\right)\beta_p}\mathfrak{B}_p(s,\alpha_p,\beta_p)\right)
\end{aligned}$$

$$\left(\prod_{p|m}\frac{J_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{(f,M)}{m}}\frac{K_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)$$

$$\left(\prod_{p|\frac{f}{(f,M)}}\frac{G_p\left(s,\alpha_p,p^{\beta_p}\right)}{S_p(s)}\right)\left(\prod_{p|\left(\frac{N}{f},M\right)}\frac{L_p\left(s,\alpha_p,p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{H_p\left(s,\alpha_p,p^{\beta_p}\right)}{S_p(s)}\right).$$

(2) 2. Case: $m_2=0$. Use the explicit formula for F_p in Lemma 9.6 and extract a quotient of shifted L-functions associated to the sequence of Fourier coefficients ϕ. We obtain the claimed formula

$$\begin{aligned}
&A_{m_1,0}(3s-1,f,M,(\phi(n))_{n\in\mathbb{N}},m,P_{2,1})\\
&=(f,M)\left(\prod_{(p,N)=1}F_p\left(s,\alpha_p,0\right)\right)\left(\prod_{p|m}J_p\left(s,\alpha_p,0\right)\right)\left(\prod_{p|\frac{(f,M)}{m}}K_p\left(s,\alpha_p,0\right)\right)\\
&\quad\left(\prod_{p|\frac{f}{(f,M)}}G_p\left(s,\alpha_p,0\right)\right)\left(\prod_{p|\left(\frac{N}{f},M\right)}L_p\left(s,\alpha_p,0\right)\right)\left(\prod_{p|\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}H_p\left(s,\alpha_p,0\right)\right)\\
&=(f,M)\,\frac{L_\phi(3s-2)}{L_\phi(3s-1)}\left(\prod_{(p,N)=1}\phi\left(p^{\alpha_p}\right)\right)\left(\prod_{p|m}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)J_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\\
&\quad\left(\prod_{p|\frac{(f,M)}{m}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)K_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{f}{(f,M)}}\frac{T_p(s)G_p\left(s,\alpha_p,0\right)}{S_p(s)}\right)\\
&\quad\left(\prod_{p|\left(\frac{N}{f},M\right)}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)L_p\left(s,\alpha_p,0\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{T_p(s)H_p\left(s,\alpha_p,0\right)}{S_p(s)}\right).
\end{aligned}$$

□

CHAPTER 15

Fourier expansion of the Eisenstein series $E(z, s, f, \phi, P_{1,2})$

In this chapter we do the same calculations for the Eisenstein series $E(z, s, f, \phi, P_{1,2})$ as we did for the Eisenstein series $E(z, s, f, \phi, P_{2,1})$. Again we start with the definition of a certain family of Dirichlet series which occurs in the Fourier coefficients of the Eisenstein series.

Definition 15.1. Let N be a squarefree integer, f, M positive divisors of N and $\psi = (\psi_{k,l}(n))_{(k,l,n)\in\mathbb{N}\times\mathbb{N}_0\times\mathbb{N}}$ a sequence of complex numbers. Further let $n = (n(k,l))_{(k,l)\in\mathbb{N}\times\mathbb{N}_0}$ be a sequence of integers such that $n(k,l)$ is a divisor of $(fk,l)\left(\frac{N}{f}, M\right)$. For a positive integer m_2 and a non-negative integer m_1 define the Dirichlet series $A_{m_1,m_2}(s, f, M, \psi, n, P_{1,2})$ associated to these parameters by

$$A_{m_1,m_2}(s, f, M, \psi, n, P_{1,2}) :=$$
$$\sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s} \sum_{\substack{0\leq t<fa\\ (fa,t)\mid m_2 n(a,t)\frac{fa}{(fa,t)}}} \psi_{a,t}\left(\frac{m_2 n(a,t)\frac{fa}{(fa,t)}}{(fa,t)}\right)(fa,t)^{-1} e\left(-m_1\frac{t}{fa}\right) c_{(fa,t)}(m_2) \ .$$

Now we are prepared to state the main result in this section.

Theorem 15.2. *Let N be a squarefree positive integer and f, M positive divisors of N. Let ϕ be a Maass cusp form for $\Gamma_0(M)$ with eigenvalue $\nu(\nu-1)$. The associated Eisenstein series $G(z, s, f, \phi, P_{1,2})$ has the explicit Fourier expansion*

$$G(z, s, f, \phi, P_{1,2}) = \sum_{m_2=0}^{\infty} G_{0,m_2}(z, s, f, \phi, P_{1,2}) + \sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty}\sum_{m_2=0}^{\infty} G_{m_1,m_2}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z, s, f, \phi, P_{1,2}\right) .$$

Recall Lemma 20.4 in Appendix C and introduce for the coprime pair of integers $e \mid \left(\frac{N}{f}, M\right)$ and $0 < d < \frac{\left(\frac{N}{f},M\right)}{e}$ and the row vector (a,b,c) with coprime integer entries the following notation. Let $\mathfrak{a}_{(a,b,d,e)} := \frac{1}{h_{(a,b,d,e)}}$ denote the cusp for $\Gamma_0(M)$ associated to $h_{(a,b,d,e)} := \left(\frac{f}{(f,b)}, M\right)\left(\frac{(f,b)a}{(fa,b)}, M, f, b\right)\left(d, \frac{\left(\frac{N}{f},M\right)}{e}\right)$ and $m_{(a,b,d,e)} := \frac{M}{h_{(a,b,d,e)}} =$

$\frac{M}{\left(\frac{f}{(f,b)},M\right)\left(\frac{(f,b)a}{(fa,b)},M,f,b\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)}$ *its width. Let*

$$\sigma_{(a,b,c,d,e)} := \begin{pmatrix} 1 & 0 \\ h_{(a,b,d,e)} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_{(a,b,c,d,e)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_{(a,b,d,e)}} & \\ & \sqrt{m_{(a,b,d,e)}}^{-1} \end{pmatrix}$$

be a scaling matrix associated to the cusp $\mathfrak{a}_{(a,b,d,e)}$, where the integer $\alpha_{(a,b,c,d,e)}$ depends only on the integers $\frac{(f,b)a}{(fa,b)}, (f,b), c, d, e, f, M$ modulo $\frac{M}{\left(\frac{f}{(f,b)},M\right)\left(\frac{(f,b)a}{(fa,b)},M,f,b\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)}$. Further let $\phi_{(a,b,d,e)} := \left(\phi_{(a,b,d,e)}(n)\right)_{n\in\mathbb{Z}}$ denote the sequence of Fourier coefficients associated to the Fourier expansion of ϕ at the cusp $\mathfrak{a}_{(a,b,d,e)}$.

Let m_1, m_2 be positive integers, then the following explicit formulas for the Fourier coefficients are valid.

(1) *In the non-degenerate case we have*

$$G_{m_1,m_2}(z,s,f,\phi,P_{1,2}) = 2f^{1-3s}\sqrt{m_1|m_2|}\,|m_2|^{s-\frac{3}{2}}\,m_1^{2s-\frac{5}{2}} \sum_{e\left|\left(\frac{N}{f},M\right)\right.} \sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} A_{m_1,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$
$$W_{1,1}^{\left(\frac{2\nu}{3},s-\frac{\nu}{3}\right)}\left(\begin{pmatrix} m_1|m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix} z, w_1\right).$$

(2) *In the first partially degenerate case we have*

$$G_{m_1,0}(z,s,f,\phi,P_{1,2}) = \delta_{N,f}\left[\phi(m_1)+\phi(-m_1)\right]|m_1|^{-s}\,W_{1,0}^{\left(\frac{2\nu}{3},s-\frac{\nu}{3}\right)}\left(\begin{pmatrix} |m_1| & & \\ & |m_1| & \\ & & 1 \end{pmatrix} z, w_3\right).$$

(3) *In the second partially degenerate case we have*

$$G_{0,m_2}(z,s,f,\phi,P_{1,2}) = 2f^{1-3s}|m_2|^{s-1} \sum_{e\left|\left(\frac{N}{f},M\right)\right.} \sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} A_{0,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$
$$W_{1,0}^{\left(s-\frac{\nu}{3},\frac{2\nu}{3}\right)}\left(\begin{pmatrix} |m_2| & & \\ & 1 & \\ & & 1 \end{pmatrix} z, w_1\right).$$

(4) *In the totally degenerate case we have*

$$G_{0,0}(z,s,f,\phi,P_{1,2}) = 0\,.$$

PROOF. Like in the proof of Theorem 11.3 without loss of generality it can be assumed that $z = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. We start the calculation of the Fourier coefficients dividing the summation in the Eisenstein series into several cases according to the Bruhat decomposition in Definition 3.8. Assume for the present that m_1, m_2 are arbitrary integers, then we have

$$\begin{aligned}
&G_{m_1,m_2}(z,s,f,\phi,P_{1,2}) = \int_0^1\int_0^1\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, s, f, \phi, P_{1,2}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \left(\frac{N}{f}, M\right)^{-1} \sum_{k=0}^{\left(\frac{N}{f},M\right)-1} \int_k^{k+1}\int_0^1\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, s, f, \phi, P_{1,2}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \left(\frac{N}{f}, M\right)^{-1} \int_0^{\left(\frac{N}{f},M\right)}\int_0^1\int_0^1 G\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, s, f, \phi, P_{1,2}\right) e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \sum_{i=1}^3 \left(\frac{N}{f}, M\right)^{-1} \sum_{\gamma\in\Gamma_i(f,M,P_{1,2})} \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \\
&\quad \int_0^{\left(\frac{N}{f},M\right)}\int_0^1\int_0^1 \phi\left(\mathfrak{m}_{P_{1,2}}\left(\gamma\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z\right)\right) I_{(2s,-s)}\left(\gamma\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} z, P_{1,2}\right) \\
&\quad e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&=: \sum_{i=1}^3 G^{(i)}_{m_1,m_2}(z,s)\ .
\end{aligned}$$

In this calculation Lemma 14.2 was applied to the integral in the variable x_3. The same argument as in Theorem 14.3 gives that the Eisenstein series is periodic in the variable x_3 with period 1. Using the explicit description of the sets $\Gamma_i(f,M,P_{1,2})$ in Lemma 5.2 and the explicit formulas for the values of the $I_{(2s,-s)}(*,P_{1,2})$-function and for the Levi components on these sets in Lemma 8.3 and 7.4, we start calculating each of the three summands above. For further calculations note first that Lemma 5.2 gives the splitting

$$q_{(a,b,c,d,e)} = \tilde{\gamma}\begin{pmatrix} 1 & 0 \\ h_{(a,b,d,e)} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha_{(a,b,c,d,e)} \\ 0 & 1 \end{pmatrix} = \tilde{\gamma}\sigma_{(a,b,c,d,e)}\begin{pmatrix} m^{-1}_{(a,b,d,e)} & \\ & 1 \end{pmatrix}\sqrt{m_{(a,b,d,e)}}, \tag{15.1}$$

with $\tilde\gamma \in \Gamma_0(M)$ and $\alpha_{(a,b,c,d,e)} \in \mathbb{Z}$ depends on $\frac{(f,b)a}{(fa,b)}, (f,b), c, d, e, f, M$ only modulo $\frac{M}{\left(\frac{f}{(f,b)},M\right)\left(\frac{(f,b)a}{(fa,b)},M,f,b\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)}$.

(1) We start with the most difficult part, the calculation of $G^{(1)}_{m_1,m_2}$. We have

$$G^{(1)}_{m_1,m_2}(z,s) = \left(\frac{N}{f},M\right)^{-1} \pi^{\frac12-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \sum_{e|\left(\frac{N}{f},M\right)} \sum_{\substack{0<d\le\frac{\left(\frac{N}{f},M\right)}{e}\\ (d,e)=1}}$$

$$\sum_{\substack{a\neq0\\ \left(a,\frac{N}{f}\right)=1}} \sum_{b\in\mathbb{Z}} \sum_{\substack{c\in\mathbb{Z}\\ (fa,b,c)=1}} \int_0^{\left(\frac{N}{f},M\right)}\int_0^1\int_0^1 \phi\left(q_{(a,b,c,d,e)}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)(y_1y_2^2)^s$$

$$\left[f^2a^2y_1^2y_2^2 + y_2^2(b-fax_1)^2 + (c-fax_3+(fax_1-b)x_2)^2\right]^{-\frac{3s}{2}} e\left(-m_1x_1-m_2x_2\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ ,$$

with the coefficients:

$$\begin{aligned} y =& y_1(fa,b)^2\frac{\sqrt{f^2a^2y_1^2y_2^2+y_2^2(fax_1-b)^2+((fax_1-b)x_2-fax_3+c)^2}}{(fax_1-b)^2+f^2a^2y_1^2}\ ,\\ x =& \frac{(fa,b)cr}{fa} + \frac{(fa,b)^2}{fa}\frac{f^2a^2y_1^2x_2+(fax_3-c)(fax_1-b)}{f^2a^2y_1^2+(fax_1-b)^2}\ . \end{aligned}$$

In the first step pass to the condition $a\ge1$, extracting a factor 2 and substitute the splitting in (15.1) for $q_{(a,b,c,d,e)}$. Note that ϕ is invariant against the center $\mathbb{R}^\times$. After that split the summation over c in this way $c=\left(\frac{N}{f},M\right)fak+s$ with $k\in\mathbb{Z}$, $0\le s<\left(\frac{N}{f},M\right)fa$. Note that the gcd-condition transforms as follows $1=(fa,b,c)=\left(fa,b,\left(\frac{N}{f},M\right)fak+s\right)=(fa,b,s)$. Further the properties of the scaling matrix imply $\sigma_{(a,b,c,d,e)}=\sigma_{(a,b,s,d,e)}$. We have

$$G^{(1)}_{m_1,m_2}(z,s) = 2\left(\frac{N}{f},M\right)^{-1}\left(y_1y_2^2\right)^s \pi^{\frac12-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \sum_{e|\left(\frac{N}{f},M\right)} \sum_{\substack{0<d\le\frac{\left(\frac{N}{f},M\right)}{e}\\ (d,e)=1}}$$

$$\sum_{\substack{a\ge1\\ \left(a,\frac{N}{f}\right)=1}} \sum_{b\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \sum_{\substack{0\le s<\left(\frac{N}{f},M\right)fa\\ (fa,b,s)=1}} \int_0^{\left(\frac{N}{f},M\right)}\int_0^1\int_0^1 \phi\left(\sigma_{(a,b,s,d,e)}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)$$

$$\left[f^2a^2y_1^2y_2^2+y_2^2(b-fax_1)^2+\left(s-fa\left(x_3-\left(\frac{N}{f},M\right)k\right)+(fax_1-b)\,x_2\right)^2\right]^{-\frac{3s}{2}}$$

$$e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \ ,$$

with the coefficients:

$$\begin{aligned} y =& y_1 \frac{(fa,b)^2}{m_{(a,b,d,e)}} \frac{\sqrt{f^2a^2y_1^2y_2^2 + y_2^2(fax_1-b)^2 + \left((fax_1-b)\, x_2 - fa\left(x_3 - \left(\frac{N}{f}, M\right)k\right) + s\right)^2}}{(fax_1-b)^2 + f^2a^2y_1^2} \ , \\ x =& \frac{(fa,b)sr}{m_{(a,b,d,e)}fa} + \frac{(fa,b)r\left(\frac{N}{f}, M\right)k}{m_{(a,b,d,e)}} + \frac{(fa,b)^2}{m_{(a,b,d,e)}fa} \frac{f^2a^2y_1^2x_2 + \left(fa\left(x_3 - \left(\frac{N}{f}, M\right)k\right) - s\right)(fax_1-b)}{f^2a^2y_1^2 + (fax_1-b)^2} \ . \end{aligned}$$

In the second step recall that $m_{(a,b,d,e)} \mid (fa,b)\left(\frac{N}{f}, M\right)$. So since $\phi(\sigma_{(a,b,s,d,e)}z)$ is periodic with period 1 in the real part, the integer summand $\frac{(fa,b)r\left(\frac{N}{f},M\right)k}{m_{(a,b,d,e)}}$ in the variable x can be deleted. Next pass to an infinite integral in the variable x_3 through the shift $x_3 \to x_3 + \left(\frac{N}{f}, M\right)k$. Then substitute $x_3 \to x_3 - \frac{(fax_1-b)x_2+s}{fa}$ in the infinite integral in the variable x_3. After that split the summation over b in this way $b = fak + t$ with $k \in \mathbb{Z}$, $0 \leq t < fa$. Note that the gcd-condition transforms as follows $1 = (fa, b, s) = (fa, fak+t, s) = (fa, t, s)$. Again the properties of the scaling matrix and the width imply the transformations $\sigma_{(a,b,s,d,e)} = \sigma_{(a,t,s,d,e)}$ and $m_{(a,b,d,e)} = m_{(a,t,d,e)}$. We obtain

$$\begin{aligned} G^{(1)}_{m_1,m_2}(z,s) =& 2\left(\frac{N}{f}, M\right)^{-1} \left(y_1y_2^2\right)^s \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \\ & \sum_{e \mid \left(\frac{N}{f},M\right)} \sum_{\substack{0<d\leq \frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} \sum_{\substack{a\geq 1 \\ \left(a,\frac{N}{f}\right)=1}} \sum_{k\in\mathbb{Z}} \sum_{0\leq t<fa} \sum_{\substack{0\leq s<\left(\frac{N}{f},M\right)fa \\ ((fa,t),s)=1}} \\ & \int_{-\infty}^{\infty}\int_0^1\int_0^1 \phi\left(\sigma_{(a,t,s,d,e)}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)\left[f^2a^2y_1^2y_2^2 + y_2^2(fa(x_1-k)-t)^2 + f^2a^2x_3^2\right]^{-\frac{3s}{2}} \\ & e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \ , \end{aligned}$$

with the coefficients:

$$\begin{aligned} y =& y_1\frac{(fa,t)^2}{m_{(a,t,d,e)}}\frac{\sqrt{f^2a^2y_1^2y_2^2 + y_2^2(fa(x_1-k)-t)^2 + f^2a^2x_3^2}}{(fa(x_1-k)-t)^2 + f^2a^2y_1^2} \ , \\ x =& \frac{(fa,t)sr}{m_{(a,t,d,e)}fa} + \frac{(fa,t)^2}{m_{(a,t,d,e)}fa}\frac{f^2a^2y_1^2x_2 + (fax_3 + (fa(x_1-k)-t)x_2)(fa(x_1-k)-t)}{f^2a^2y_1^2 + (fa(x_1-k)-t)^2} \ . \end{aligned}$$

In the third step pass to an infinite integral in the variable x_1 through the shift $x_1 \to x_1 - k$. Then substitute $x_1 \to x_1 - \frac{t}{fa}$ in the infinite integral in the variable x_1 and pick up an exponential $e\left(-m_1\frac{t}{fa}\right)$. This gives us

$$\begin{aligned} G^{(1)}_{m_1,m_2}(z,s) =& 2\left(\frac{N}{f}, M\right)^{-1} \left(y_1 y_2^2\right)^s \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) \\ & \sum_{e \mid \left(\frac{N}{f}, M\right)} \sum_{\substack{0<d\leq \frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} \sum_{\substack{a\geq 1 \\ \left(a, \frac{N}{f}\right)=1}} \sum_{0\leq t<fa} e\left(-m_1 \frac{t}{fa}\right) \sum_{\substack{0\leq s<\left(\frac{N}{f},M\right)fa \\ ((fa,t),s)=1}} \\ & \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \phi\left(\sigma_{(a,t,s,d,e)} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \left[f^2 a^2 y_1^2 y_2^2 + f^2 a^2 y_2^2 x_1^2 + f^2 a^2 x_3^2\right]^{-\frac{3s}{2}} \\ & e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ , \end{aligned}$$

with the coefficients:

$$\begin{aligned} y =& y_1 \frac{(fa,t)^2}{m_{(a,t,d,e)}} \frac{\sqrt{f^2 a^2 y_1^2 y_2^2 + f^2 a^2 y_2^2 x_1^2 + f^2 a^2 x_3^2}}{f^2 a^2 x_1^2 + f^2 a^2 y_1^2} = y_1 \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{\sqrt{y_1^2 y_2^2 + y_2^2 x_1^2 + x_3^2}}{x_1^2 + y_1^2} \ , \\ x =& \frac{(fa,t)sr}{m_{(a,t,d,e)} fa} + \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{f^2 a^2 y_1^2 x_2 + (fax_3 + fax_1 x_2)(fax_1)}{f^2 a^2 y_1^2 + f^2 a^2 x_1^2} \\ =& \frac{(fa,t)sr}{m_{(a,t,d,e)} fa} + \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} x_2 + \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{x_3 x_1}{y_1^2 + x_1^2} \ . \end{aligned}$$

Now Theorem 20.1 in Appendix C provides the Fourier expansion of the Maass cusp form ϕ at the cusp $\mathfrak{a}_{(a,t,d,e)}$. In the additional calculations these Fourier expansions are used to clear up the above expression. We have

$$\begin{aligned} & G^{(1)}_{m_1,m_2}(z,s) = 2\left(\frac{N}{f}, M\right)^{-1} \left(y_1 y_2^2\right)^s \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right) \Gamma\left(\frac{3s+\nu-1}{2}\right) f^{-3s} \\ & \sum_{e \mid \left(\frac{N}{f}, M\right)} \sum_{\substack{0<d\leq \frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} \sum_{\substack{a\geq 1 \\ \left(a, \frac{N}{f}\right)=1}} a^{-3s} \sum_{0\leq t<fa} e\left(-m_1 \frac{t}{fa}\right) \sum_{\substack{0\leq s<\left(\frac{N}{f},M\right)fa \\ ((fa,t),s)=1}} \\ & \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \sum_{n\neq 0} 2\phi_{(a,t,d,e)}(n) \sqrt{|n|\, y_1 \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{\sqrt{y_1^2 y_2^2 + y_2^2 x_1^2 + x_3^2}}{x_1^2 + y_1^2}} \\ & K_{\nu-\frac{1}{2}}\left(2\pi\, |n|\, y_1 \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{\sqrt{y_1^2 y_2^2 + y_2^2 x_1^2 + x_3^2}}{x_1^2 + y_1^2}\right) \\ & e\left(n\left(\frac{(fa,t)sr}{m_{(a,t,d,e)} fa} + \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} x_2 + \frac{(fa,t)^2}{m_{(a,t,d,e)} fa} \frac{x_3 x_1}{y_1^2 + x_1^2}\right)\right) \left[y_1^2 y_2^2 + y_2^2 x_1^2 + x_3^2\right]^{-\frac{3s}{2}} \\ & e(-m_1 x_1 - m_2 x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ . \end{aligned}$$

Simplify the above terms and extract an exponential integral in the variable x_2. This gives us

$$
\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s)\\
&= 4\left(\frac{N}{f},M\right)^{-1} y_1^{s+\frac{1}{2}} y_2^{2s} \pi^{\frac{1}{2}-3s} \Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) f^{-\frac{1}{2}-3s} \sum_{e\mid\left(\frac{N}{f},M\right)} \sum_{\substack{0<d\le\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}\\
&\sum_{\substack{a\ge1\\ \left(a,\frac{N}{f}\right)=1}} a^{-\frac{1}{2}-3s} \sum_{0\le t<fa} \frac{1}{\sqrt{m_{(a,t,d,e)}}} \sum_{n\neq0} \phi_{(a,t,d,e)}(n)\sqrt{|n|}\, e\left(-m_1\frac{t}{fa}\right)(fa,t) \sum_{\substack{0\le s<\left(\frac{N}{f},M\right)fa\\((fa,t),s)=1}} e\left(\frac{(fa,t)nrs}{m_{(a,t,d,e)}fa}\right)\\
&\int_0^1 e\left(\left(\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}-m_2\right)x_2\right)\mathrm{d}x_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2+y_2^2x_1^2+x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}} \left(x_1^2+y_1^2\right)^{-\frac{1}{2}} e(-m_1x_1)\\
&K_{\nu-\frac{1}{2}}\left(2\pi\frac{|n|\,(fa,t)^2}{m_{(a,t,d,e)}fa} y_1 \frac{\sqrt{y_1^2y_2^2+y_2^2x_1^2+x_3^2}}{x_1^2+y_1^2}\right) e\left(\frac{n(fa,t)^2}{m_{(a,t,d,e)}fa}\frac{x_3x_1}{y_1^2+x_1^2}\right)\mathrm{d}x_1\mathrm{d}x_3\,. \qquad (15.2)
\end{aligned}
$$

Through the calcualtion of the series in the variable s and the integral in the variable x_2 the double integral is separated from the Dirichlet series. Begin with the calculation of the innermost sum in the variable s and extract a geometric sum. First note that Lemma 3.4 implies that r can be absorbed into the summation. Then split the summation $s=(fa,t)k+w$ with $0\le k<\left(\frac{N}{f},M\right)\frac{fa}{(fa,t)}$ and $0\le w<(fa,t)$. We have

$$
\begin{aligned}
\sum_{\substack{0\le s<\left(\frac{N}{f},M\right)fa\\((fa,t),s)=1}} e\left(\frac{(fa,t)nrs}{m_{(a,t,d,e)}fa}\right) &= \sum_{\substack{0\le w<(fa,t)\\((fa,t),w)=1}} e\left(\frac{n(fa,t)w}{m_{(a,t,d,e)}fa}\right) \sum_{0\le k<\left(\frac{N}{f},M\right)\frac{fa}{(fa,t)}} e\left(\frac{n(fa,t)k}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}\right)\\
&= \left(\frac{N}{f},M\right)\frac{fa}{(fa,t)}\mathbb{1}_{\mathbb{Z}}\left(\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}\right) \sum_{\substack{0\le w<(fa,t)\\((fa,t),w)=1}} e\left(\frac{n(fa,t)w}{m_{(a,t,d,e)}fa}\right)\,. \qquad (15.3)
\end{aligned}
$$

If $\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}$ is an integer, which is the only relevant case since otherwise the above geometric sum vanishes, the exponential integral in the variable x_2 is nonzero if and only if $m_2=\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}$. Note that $n\neq0$ implies that this equality can only hold if $m_2\neq0$. So one gets the formula

$$
\int_0^1 e\left(\left(\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}-m_2\right)x_2\right)\mathrm{d}x_2 = \delta_{m_2,\frac{n(fa,t)}{m_{(a,t,d,e)}\frac{fa}{(fa,t)}}}\,. \qquad (15.4)
$$

So put $n=\frac{m_2m_{(a,t,d,e)}fa}{(fa,t)^2}$ and the other results in (15.3), (15.4) in the formula (15.2) for $G^{(1)}_{m_1,m_2}$ and simplify the terms. This will give us the claimed separation of the double integral and the Dirichlet series. The identity

$$
G^{(1)}_{m_1,m_2}(z,s) = 4(1-\delta_{0,m_2})\left(\frac{N}{f},M\right)^{-1} y_1^{s+\frac{1}{2}} y_2^{2s}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)
$$

$$f^{-\frac{1}{2}-3s}\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}\sum_{\substack{a\geq 1\\\left(a,\frac{N}{f}\right)=1}}a^{-\frac{1}{2}-3s}$$

$$\sum_{\substack{0\leq t<fa\\(fa,t)|m_2m_{(a,t,d,e)}\frac{fa}{(fa,t)}}}\frac{1}{\sqrt{m_{(a,t,d,e)}}}\phi_{(a,t,d,e)}\left(\frac{m_2m_{(a,t,d,e)}fa}{(fa,t)^2}\right)\sqrt{\left|\frac{m_2m_{(a,t,d,e)}fa}{(fa,t)^2}\right|}e\left(-m_1\frac{t}{fa}\right)(fa,t)$$

$$\left(\frac{N}{f},M\right)\frac{fa}{(fa,t)}\sum_{\substack{0\leq w<(fa,t)\\((fa,t),w)=1}}e\left(\frac{m_2w}{(fa,t)}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_2^2x_1^2+x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}}\left(x_1^2+y_1^2\right)^{-\frac{1}{2}}e(-m_1x_1)$$

$$K_{\nu-\frac{1}{2}}\left(2\pi\left|m_2\right|y_1\frac{\sqrt{y_1^2y_2^2+y_2^2x_1^2+x_3^2}}{x_1^2+y_1^2}\right)e\left(m_2\frac{x_3x_1}{y_1^2+x_1^2}\right)\mathrm{d}x_1\mathrm{d}x_3$$

$$= \quad 4(1-\delta_{0,m_2})y_1^{s+\frac{1}{2}}y_2^{2s}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sqrt{|m_2|}f^{1-3s}\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}$$

$$\sum_{\substack{a\geq 1\\\left(a,\frac{N}{f}\right)=1}}a^{1-3s}\sum_{\substack{0\leq t<fa\\(fa,t)|m_2m_{(a,t,d,e)}\frac{fa}{(fa,t)}}}\phi_{(a,t,d,e)}\left(\frac{m_2m_{(a,t,d,e)}\frac{fa}{(fa,t)}}{(fa,t)}\right)(fa,t)^{-1}e\left(-m_1\frac{t}{fa}\right)c_{(fa,t)}(m_2)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_2^2x_1^2+x_3^2\right]^{\frac{1}{4}-\frac{3s}{2}}\left(x_1^2+y_1^2\right)^{-\frac{1}{2}}e(-m_1x_1)$$

$$K_{\nu-\frac{1}{2}}\left(2\pi\left|m_2\right|y_1\frac{\sqrt{y_1^2y_2^2+y_2^2x_1^2+x_3^2}}{x_1^2+y_1^2}\right)e\left(m_2\frac{x_3x_1}{y_1^2+x_1^2}\right)\mathrm{d}x_1\mathrm{d}x_3$$

holds.

To complete our calculations we distinguish between the cases whether m_1 vanishes or not. In the upcoming calculations use the notation from Definition 15.1 for the Dirichlet series.

(a) 1. Case: $m_1\neq 0$. First substitute $x_1\to m_1x_1$ in the integral in the variable x_1, after that substitute $x_3\to -m_1m_2x_3$ in the integral in the variable x_3. Then expand the terms with the spectral parameters. We have

$$G^{(1)}_{m_1,m_2}(z,s)=2(1-\delta_{0,m_2})f^{1-3s}\sqrt{m_1\left|m_2\right|}\left|m_2\right|^{s-\frac{3}{2}}m_1^{2s-\frac{5}{2}}$$

$$\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}A_{m_1,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$

$$2\pi^{-3\left(s-\frac{\nu}{3}\right)-\frac{3}{2}\left(\frac{2\nu}{3}\right)+\frac{1}{2}}\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right)\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)+3\left(\frac{2\nu}{3}\right)-1}{2}\right)\left(\left|m_2\right|y_2\right)^{2\left(s-\frac{\nu}{3}\right)+\frac{2\nu}{3}}\left(m_1y_1\right)^{\frac{1}{2}+\left(s-\frac{\nu}{3}\right)+\frac{1}{2}\left(\frac{2\nu}{3}\right)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(m_1y_1)^2(|m_2|\,y_2)^2+(|m_2|\,y_2)^2\,x_1^2+x_3^2\right]^{\frac{1}{4}-\frac{3}{2}(s-\frac{\nu}{3})-\frac{3}{4}\left(\frac{2\nu}{3}\right)}\left((m_1y_1)^2+x_1^2\right)^{-\frac{1}{2}}e(-x_1)$$
$$K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi\,(m_1y_1)\,\frac{\sqrt{(m_1y_1)^2\,(|m_2|\,y_2)^2+(|m_2|\,y_2)^2\,x_1^2+x_3^2}}{(m_1y_1)^2+x_1^2}\right)e\left(-\frac{x_1x_3}{(m_1y_1)^2+x_1^2}\right)\mathrm{d}x_1\mathrm{d}x_3\,.$$

Lemma 19.5 part (1) in Appendix B implies that the above double integral is a Whittaker function of the type $W_{1,1}^{(\nu_1,\nu_2)}(*,w_1)$. So we get

$$\begin{aligned}G^{(1)}_{m_1,m_2}(z,s)=&2(1-\delta_{0,m_2})f^{1-3s}\sqrt{m_1\,|m_2|}\,|m_2|^{s-\frac{3}{2}}\,m_1^{2s-\frac{5}{2}}\\&\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}A_{m_1,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\&W_{1,1}^{\left(s-\frac{\nu}{3},\frac{2\nu}{3}\right)}\left(\begin{pmatrix}|m_2|\,m_1&&\\&|m_2|&\\&&1\end{pmatrix}\begin{pmatrix}y_2y_1&&\\&y_2&\\&&1\end{pmatrix},w_1\right)\,.\end{aligned}$$

So in the last step apply the transformation laws [**4**, (3.24),(3.16)] for the Whittaker function and use the right invariance of the Whittaker function against the maximal compact subgroup $O(3)$ and the center $\mathbb{R}^\times$. This gives us finally

$$\begin{aligned}G^{(1)}_{m_1,m_2}(z,s)=&2(1-\delta_{0,m_2})f^{1-3s}\sqrt{m_1\,|m_2|}\,|m_2|^{s-\frac{3}{2}}\,m_1^{2s-\frac{5}{2}}\\&\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}A_{m_1,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\&W_{1,1}^{\left(\frac{2\nu}{3},s-\frac{\nu}{3}\right)}\left(\begin{pmatrix}m_1\,|m_2|&&\\&m_1&\\&&1\end{pmatrix}z,w_1\right)\,.\end{aligned}$$

(b) 2. Case: $m_1=0$. Substitute $x_3\to-m_2x_3$ in the integral in the variable x_3, then expand the terms with the spectral parameters. We have

$$G^{(1)}_{0,m_2}(z,s)=2(1-\delta_{0,m_2})f^{1-3s}\,|m_2|^{s-1}$$
$$\sum_{e|\left(\frac{N}{f},M\right)}\sum_{\substack{0<d\leq\frac{\left(\frac{N}{f},M\right)}{e}\\(d,e)=1}}A_{0,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$
$$2\pi^{-3\left(s-\frac{\nu}{3}\right)-\frac{3}{2}\left(\frac{2\nu}{3}\right)+\frac{1}{2}}\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right)\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)+3\left(\frac{2}{3}\nu\right)-1}{2}\right)(|m_2|\,y_2)^{2\left(s-\frac{\nu}{3}\right)+\frac{2\nu}{3}}\,y_1^{\frac{1}{2}+\left(s-\frac{\nu}{3}\right)+\frac{1}{2}\left(\frac{2\nu}{3}\right)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2(|m_2|\, y_2)^2 + (|m_2|\, y_2)^2 x_1^2 + x_3^2\right]^{\frac{1}{4}-\frac{3}{2}(s-\frac{\nu}{3})-\frac{3}{4}\left(\frac{2\nu}{3}\right)} \left(y_1^2+x_1^2\right)^{-\frac{1}{2}}$$

$$K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi y_1 \frac{\sqrt{y_1^2(|m_2|\, y_2)^2 + (|m_2|\, y_2)^2 x_1^2 + x_3^2}}{y_1^2+x_1^2}\right) e\left(-\frac{x_1x_3}{y_1^2+x_1^2}\right) \mathrm{d}x_1 \mathrm{d}x_3 \,.$$

Lemma 19.5 part (3) in Appendix B implies that the above double integral is a Whittaker function of the type $W_{0,1}^{(\nu_1,\nu_2)}(*, w_1)$. Then apply the transformation laws [**4**, (3.24),(3.17)] for the Whittaker function and use the right invariance of the Whittaker function against the maximal compact subgroup $O(3)$ and the center $\mathbb{R}^\times$. This gives us finally

$$\begin{aligned}
G_{0,m_2}^{(1)}(z,s) =& 2(1-\delta_{0,m_2}) f^{1-3s}\, |m_2|^{s-1} \\
&\sum_{e \mid \left(\frac{N}{f},M\right)} \sum_{\substack{0<d\leq \frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} A_{0,m_2}\left(3s-1, f, M, \left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, \left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, P_{1,2}\right) \\
&W_{0,1}^{\left(\frac{2\nu}{3}, s-\frac{\nu}{3}\right)}\left(\begin{pmatrix} |m_2| & & \\ & |m_2| & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_2y_1 & & \\ & y_2 & \\ & & 1 \end{pmatrix}, w_1\right) \\
=& 2(1-\delta_{0,m_2}) f^{1-3s}\, |m_2|^{s-1} \\
&\sum_{e \mid \left(\frac{N}{f},M\right)} \sum_{\substack{0<d\leq \frac{\left(\frac{N}{f},M\right)}{e} \\ (d,e)=1}} A_{0,m_2}\left(3s-1, f, M, \left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, \left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, P_{1,2}\right) \\
&W_{1,0}^{\left(s-\frac{\nu}{3}, \frac{2\nu}{3}\right)}\left(\begin{pmatrix} |m_2| & & \\ & 1 & \\ & & 1 \end{pmatrix} z, w_1\right) \,.
\end{aligned}$$

(2) We proceed with the calculation of $G_{m_1,m_2}^{(2)}$. It turns out that this part of the Fourier coefficient always vanishes. We have

$$\begin{aligned}
G_{m_1,m_2}^{(2)}(z,s) =& \delta_{N,f}\left(\frac{N}{f}, M\right)^{-1} \pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right) \sum_{b\neq 0} \sum_{\substack{c\in\mathbb{Z} \\ (b,c)=1}} \\
&\int_0^{\left(\frac{N}{f},M\right)} \int_0^1 \int_0^1 \phi\left(\begin{pmatrix} y_1\sqrt{b^2y_2^2+(bx_2-c)^2} & cx_1 - bx_3 \\ 0 & 1 \end{pmatrix}\right) \\
&(y_1y_2^2)^s \left[y_2^2b^2 + (c-bx_2)^2\right]^{-\frac{3s}{2}} e(-m_1x_1 - m_2x_2) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \,.
\end{aligned}$$

Now Theorem 20.1 in Appendix C provides the Fourier expansion of the Maass cusp form ϕ at the cusp ∞. This gives us

$$G^{(2)}_{m_1,m_2}(z,s) = \delta_{N,f}(y_1y_2^2)^s\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\sum_{b\neq 0}\sum_{\substack{c\in\mathbb{Z}\\(b,c)=1}}$$

$$\int_0^1\int_0^1\int_0^1\sum_{n\neq 0}2\phi(n)\sqrt{|n|\,y_1\sqrt{b^2y_2^2+(bx_2-c)^2}}K_{\nu-\frac{1}{2}}\left(2\pi\,|n|\,y_1\sqrt{b^2y_2^2+(bx_2-c)^2}\right)\,.$$

$$e\left(ncx_1-nbx_3\right)\left[y_2^2b^2+(c-bx_2)^2\right]^{-\frac{3s}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\;.$$

Note that we can extract the exponential integral in the variable x_3. Since $n, b \neq 0$ this integral vanishes

$$\int_0^1 e(-bnx_3)\mathrm{d}x_3 = 0\;.$$

So this part of the Fourier coefficient always vanishes:

$$G^{(2)}_{m_1,m_2}(z,s) = 0\;.$$

(3) It remains to do the easiest part, the calculation of $G^{(3)}_{m_1,m_2}(z,s)$. Again Lemma 5.2 gives a description of the set $\Gamma_3(f, M, P_{1,2})$, which consists of two diagonal matrices at most. Then use Lemma 8.2 to calculate the values of the $I_{(2s,-s)}(*, P_{1,2})$-function directly. We have

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N}\left(\frac{N}{f},M\right)^{-1}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\int_0^{\left(\frac{N}{f},M\right)}\int_0^1\int_0^1$$

$$\left[\phi\left(\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z\right)\right)I_{(2s,-s)}\left(\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z,P_{1,2}\right)\right.$$

$$\left.+\,\phi\left(\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z\right)\right)I_{(2s,-s)}\left(\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1&x_2&x_3\\&1&x_1\\&&1\end{pmatrix}z,P_{1,2}\right)\right]$$

$$e\left(-m_1x_1-m_2x_2\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\;.$$

To simplify the above terms use the right-invariance of the $I_{(2s,-s)}(*, P_{1,2})$-function against the maximal compact subgroup O_3 in the second summand. This gives us

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{f,N}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\int_0^1\int_0^1\int_0^1$$

$$\left[\phi\left(\mathfrak{m}_{P_{1,2}}\left(\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}\right)\right)\det\left(\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}\right)^{-s}\left\|\begin{pmatrix}y_1y_2&y_1x_2&x_3\\&y_1&x_1\\&&1\end{pmatrix}^{-1}e_1\right\|^{-3s}\right.$$

$$+\phi\left(\mathfrak{m}_{P_{1,2}}\left(\left(\begin{pmatrix} y_1y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix}\right)\right)\right)\det\left(\left(\begin{pmatrix} y_1y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix}\right)\right)^{-s}$$
$$\left.\| \begin{pmatrix} y_1y_2 & y_1(-x_2) & x_3 \\ & y_1 & -x_1 \\ & & 1 \end{pmatrix}^{-1} e_1 \|^{-3s}\right] e(-m_1x_1 - m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 .$$

Calculating the integrals in the variables x_2 and x_3 gives us the formula

$$\begin{aligned} G^{(3)}_{m_1,m_2}(z,s) =& \delta_{f,N}\delta_{m_2,0}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\left(y_1^2y_2\right)^{-s}\left(y_1^{-1}y_2^{-1}\right)^{-3s} \\ & \int_0^1\left[\phi\left(\begin{pmatrix} y_1 & x_1 \\ & 1 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} y_1 & -x_1 \\ & 1 \end{pmatrix}\right)\right] e(-m_1x_1)\mathrm{d}x_1 . \end{aligned}$$

For the calculation of the integral in the variable x_1 use the Fourier expansion of the cusp form ϕ at the cusp ∞. We have

$$\begin{aligned} G^{(3)}_{m_1,m_2}(z,s) =& \delta_{f,N}\delta_{m_2,0}\pi^{\frac{1}{2}-3s}\Gamma\left(\frac{3s-\nu}{2}\right)\Gamma\left(\frac{3s+\nu-1}{2}\right)\left(y_1y_2^2\right)^{s} \\ & \sum_{n\neq 0} 2\phi(n)\sqrt{|n|\,y_1}K_{\nu-\frac{1}{2}}(2\pi\,|n|\,y_1)\int_0^1 e\left((n-m_1)x_1\right) + e\left(-(n+m_1)x_1\right)\mathrm{d}x_1 . \end{aligned}$$

Note that $n \neq 0$ implies that $m_1 \neq 0$ is a necessary condition for the exponential integral to be non-zero. We obtain

$$\begin{aligned} G^{(3)}_{m_1,m_2}(z,s) =& \delta_{N,f}\delta_{0,m_2}(1-\delta_{m_1,0})\left[\phi(m_1)+\phi(-m_1)\right]|m_1|^{-s} \\ & 2\pi^{-\frac{3\left(\frac{2\nu}{3}\right)}{2}-3\left(s-\frac{\nu}{3}\right)+\frac{1}{2}}\Gamma\left(\frac{3\left(s-\frac{\nu}{3}\right)}{2}\right)\Gamma\left(\frac{3\left(\frac{2\nu}{3}\right)+3\left(s-\frac{\nu}{3}\right)-1}{2}\right) \\ & \left(|m_1|\,y_1\right)^{\frac{1}{2}+\frac{\frac{2\nu}{3}}{2}+\left(s-\frac{\nu}{3}\right)} y_2^{\frac{2\nu}{3}+2\left(s-\frac{\nu}{3}\right)} K_{\frac{3\left(\frac{2\nu}{3}\right)-1}{2}}\left(2\pi\,|m_1|\,y_1\right) . \end{aligned}$$

The explicit formula for the Whittaker function of the type $W^{(\nu_1,\nu_2)}_{1,0}(z,w_3)$ in Lemma 19.4 in Appendix B finally gives the formula

$$G^{(3)}_{m_1,m_2}(z,s) = \delta_{N,f}\delta_{0,m_2}(1-\delta_{m_1,0})\left[\phi(m_1)+\phi(-m_1)\right]|m_1|^{-s}\,W^{\left(\frac{2\nu}{3},s-\frac{\nu}{3}\right)}_{1,0}\left(\begin{pmatrix} |m_1| & & \\ & |m_1| & \\ & & 1 \end{pmatrix} z, w_3\right) .$$

□

For the calculation of the Dirichlet series $A_{m_1,m_2}(s,f,M,\psi,n,P_{1,2})$ we make the same considerations as in the previous chapter and adopt the notation from there. Again we will restrict our calculations to Fourier coefficients from newforms.

LEMMA 15.3. *Let $m_1=\prod_p p^{\alpha_p}$, $m_2=\prod_p p^{\beta_p}$ be positive integers with corresponding prime decompositions and define for coprime integers d and e the positive divisor $n_{(d,e)}$ of $\left(\frac{N}{f},M\right)$ by*

$$n_{(d,e)}=\frac{\left(\frac{N}{f},M\right)}{\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)}\,.$$

With the same notation as in Theorem 15.2 and under the assumption, that ϕ is a newform for $\Gamma_0(M)$, the explicit formulas below for the Dirichlet series $A_{*,m_2}\left(s,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$ in Definition 15.1 are valid.

(1) In the non-degenerate case we have

$$\begin{aligned}&A_{m_1,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\&=L_\phi(3s-1)^{-1}\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\beta_p}p^{-\left(2s-\frac{5}{2}\right)\alpha_p}\mathfrak{B}_p(s,\beta_p,\alpha_p)\right)\left(\prod_{p\mid n_{(d,e)}}\frac{\phi\left(p^{\beta_p+1}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\\&\left(\prod_{p\mid\frac{\left(\frac{N}{f},M\right)}{n_{(d,e)}}}\frac{\phi\left(p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{\phi\left(p^{\beta_p}\right)}{S_p(s)}\right)\left(\prod_{p\mid(f,M)}\frac{M_p(s,p^{\alpha_p},\beta_p)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{f}{(f,M)}}\frac{I_p(s,p^{\alpha_p},\beta_p)}{S_p(s)}\right).\end{aligned}$$

(2) In the partially degenerate case we have

$$\begin{aligned}&A_{0,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\&=\frac{L_\phi(3s-2)}{L_\phi(3s-1)}\left(\prod_{(p,N)=1}\phi\left(p^{\beta_p}\right)\right)\left(\prod_{p\mid n_{(d,e)}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)\phi\left(p^{\beta_p+1}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\\&\left(\prod_{p\mid\frac{\left(\frac{N}{f},M\right)}{n_{(d,e)}}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)\phi\left(p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{T_p(s)\phi\left(p^{\beta_p}\right)}{S_p(s)}\right)\end{aligned}$$

$$\left(\prod_{p\mid(f,M)} \frac{\left(1-\phi(p)p^{-(3s-2)}\right) M_p(s,0,\beta_p)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p\mid\frac{f}{(f,M)}} \frac{T_p(s)I_p(s,0,\beta_p)}{S_p(s)}\right).$$

PROOF. We proceed analogously to the proof of Lemma 14.5 and handle both parts together, so assume for the present that m_1 is a nonnegative integer. In the first step split the summation in the inner sum of A_{m_1,m_2} in the way $t=t_1t_2$ with $t_1 \mid fa$ and $\left(t_2,\frac{fa}{t_1}\right)=1$ and note that this implies $(t,fa)=t_1$. The dependencies of $\phi_{(a,t,d,e)}$ and $m_{(a,t,d,e)}$ on a,t in Theorem 15.2 imply the transformations $\phi_{(a,t,d,e)}=\phi_{(a,t_1,d,e)}$ and $m_{(a,t,d,e)}=m_{(a,t_1,d,e)}$. We have

$$\begin{aligned}
&A_{m_1,m_2}\left(s,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\
&=\sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s}\sum_{\substack{0\leq t<fa\\ (fa,t)\mid m_2 m_{(a,t,d,e)}\frac{fa}{(fa,t)}}}\phi_{(a,t,d,e)}\left(\frac{m_2 m_{(a,t,d,e)}\frac{fa}{(fa,t)}}{(fa,t)}\right)(fa,t)^{-1}e\left(-m_1\frac{t}{fa}\right)c_{(fa,t)}\left(m_2\right)\\
&=\sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s}\sum_{\substack{t_1\mid fa\\ t_1^2\mid m_2 m_{(a,t_1,d,e)}fa}}\sum_{\substack{0\leq t_2<\frac{fa}{t_1}\\ \left(t_2,\frac{fa}{t_1}\right)=1}}\phi_{(a,t_1,d,e)}\left(\frac{m_2 m_{(a,t_1,d,e)}\frac{fa}{t_1}}{t_1}\right)t_1^{-1}e\left(-m_1\frac{t_2}{\frac{fa}{t_1}}\right)c_{t_1}\left(m_2\right)\\
&=\sum_{\substack{a\geq 1\\ \left(a,\frac{N}{f}\right)=1}} a^{-s}\sum_{\substack{t\mid fa\\ t^2\mid m_2 m_{(a,t,d,e)}fa}}\phi_{(a,t,d,e)}\left(\frac{m_2 m_{(a,t,d,e)}fa}{t^2}\right)t^{-1}c_t\left(m_2\right)c_{\frac{fa}{t}}\left(m_1\right).
\end{aligned}$$

The multiplicativity of the Fourier coefficients $\left(\phi_{(a,t,d,e)}(n)\right)_{n\in\mathbb{N}}$ implies that the Dirichlet series can be factorized. In order to do this decompose $m_2=m_2^{(1)}m_2^{(2)}m_2^{(3)}m_2^{(4)}m_2^{(5)}$ and split the summation in the outer and inner sum $a=a_1a_2a_3a_4a_5$ and $t=t_1t_2t_3t_4t_5$ according to the divisibility conditions $\left(m_2^{(1)}a_1t_1,N\right)=1$, $m_2^{(2)}a_2t_2 \mid \left(\frac{f}{(f,M)}\right)^\infty$, $m_2^{(3)}a_3t_3 \mid (f,M)^\infty$, $m_2^{(4)}a_4t_4 \mid \left(\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}\right)^\infty$ and $m_2^{(5)}a_5t_5 \mid \left(\frac{N}{f},M\right)^\infty$. Note that the gcd-condition $\left(a,\frac{N}{f}\right)=1$ implies that $a_4,a_5,t_4,t_5=1$. Next we examine the behaviour of the cusp and its width under the above splitting. The identity

$$h_{(a,t,d,e)}=\left(\frac{f}{(f,t_1t_2t_3)},M\right)\left(\frac{(f,t_1t_2t_3)a_1a_2a_3}{(fa_1a_2a_3,t_1t_2t_3)},M,f,t_1t_2t_3\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)$$

$$
\begin{aligned}
&=\left(\frac{f}{(f,t_2)(f,t_3)},M\right)\left(\frac{a_1a_2(f,t_2)a_3(f,t_3)}{t_1t_2t_3},M,f,t_3\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)\\
&=\frac{(f,M)}{(f,M,t_3)}\left(\frac{a_3(f,t_3)}{t_3},M,f,t_3\right)\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)\\
&=h_{(a_3,t_3,d,e)}
\end{aligned}
\tag{15.5}
$$

holds. So we get the transformations $\phi_{(a,t,d,e)} = \phi_{(a_3,t_3,d,e)}$ and $m_{(a,t,d,e)} = m_{(a_3,t_3,d,e)}$. Further (15.5) delivers us precise formulas for the gcd's

$$
\left(f,M,m_{(a,t,d,e)}\right)=\left(f,M,\frac{M}{h_{(a,t,d,e)}}\right)=\frac{(f,M,t_3)}{\left(\frac{a_3(f,t_3)}{t},M,f,t_3\right)}\,, \tag{15.6}
$$

$$
\left(\frac{N}{f},M,m_{(a,t,d,e)}\right)=\frac{\left(\frac{N}{f},M\right)}{\left(d,\frac{\left(\frac{N}{f},M\right)}{e}\right)}=n_{(d,e)}\,. \tag{15.7}
$$

Recall that $m_{(a_3,t_3,d,e)} \mid M$ holds, hence the divisibility condition $t^2 \mid m_2 m_{(a_3,t_3,d,e)} f a$ is equivalent to the three conditions $t_1^2 \mid m_2^{(1)} a_1$, $t_2^2 \mid \frac{f}{(f,M)} m_2^{(2)} a_2$ and $t_3^2 \mid (f,M)\left(f,M,m_{(a_3,t_3,d,e)}\right) m_2^{(3)} a_3$. Using (15.6) and (15.7) we obtain

$$
\begin{aligned}
&A_{m_1,m_2}\left(s,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)\\
=&\sum_{\substack{(a_1,N)=1\\ a_2\mid\left(\frac{f}{(f,M)}\right)^\infty\\ a_3\mid(f,M)^\infty}}(a_1a_2a_3)^{-s}\sum_{\substack{t_1\mid a_1\\ t_2\mid\frac{f}{(f,M)}a_2\\ t_3\mid(f,M)a_3\\ t_1^2\mid m_2^{(1)}a_1\\ t_2^2\mid m_2^{(2)}\frac{f}{(f,M)}a_2\\ t_3^2\mid m_2^{(3)}(f,M,m_{(a_3,t_3,d,e)})(f,M)a_3}}\\
&\phi_{(a_3,t_3,d,e)}\left(\frac{m_2^{(1)}a_1}{t_1^2}\cdot\frac{m_2^{(2)}\frac{f}{(f,M)}a_2}{t_2^2}\cdot\frac{m_2^{(3)}(f,M)\left(f,M,m_{(a_3,t_3,d,e)}\right)a_3}{t_3^2}\cdot m_2^{(4)}\cdot\left(m_2^{(5)}\left(\frac{N}{f},M,m_{(a_3,t_3,d,e)}\right)\right)\right)\\
&(t_1t_2t_3)^{-1}c_{t_1t_2t_3}\left(m_2^{(1)}m_2^{(2)}m_2^{(3)}m_2^{(4)}m_2^{(5)}\right)c_{\frac{a_1}{t_1}\frac{\frac{f}{(f,M)}a_2}{t_2}\frac{(f,M)a_3}{t_3}}(m_1)
\end{aligned}
$$

$$= \sum_{\substack{(a_1,N)=1\\ a_2|\left(\frac{f}{(f,M)}\right)^\infty\\ a_3|(f,M)^\infty}} (a_1a_2a_3)^{-s} \sum_{\substack{t_1|a_1\\ t_2|\frac{f}{(f,M)}a_2\\ t_3|(f,M)a_3\\ t_1^2|m_2^{(1)}a_1\\ t_2^2|m_2^{(2)}\frac{f}{(f,M)}a_2\\ t_3^2|m_2^{(3)}\frac{(f,M,t_3)}{\left(\frac{a_3(f,t_3)}{t_3},M,f,t_3\right)}(f,M)a_3}}$$

$$\phi_{(a_3,t_3,d,e)}\left(\frac{m_2^{(1)}a_1}{t_1^2}\cdot\frac{m_2^{(2)}\frac{f}{(f,M)}a_2}{t_2^2}\cdot\frac{m_2^{(3)}(f,M)\frac{(f,M,t_3)}{\left(\frac{a_3(f,t_3)}{t_3},M,f,t_3\right)}}{t_3^2}\cdot m_2^{(4)}\cdot\left(m_2^{(5)}n_{(d,e)}\right)\right)$$

$$(t_1t_2t_3)^{-1}c_{t_1t_2t_3}\left(m_2^{(1)}m_2^{(2)}m_2^{(3)}m_2^{(4)}m_2^{(5)}\right)c_{\frac{a_1}{t_1}\frac{\frac{f}{(f,M)}a_2}{t_2}\frac{(f,M)a_3}{t_3}}(m_1)\ . \tag{15.8}$$

In the paper [**1**, Thm. 1.1, Thm. 1.2] it is proved that for a newform with squarefree level the Fourier coefficients at any cusp are identical with the Fourier coefficients at the cusp ∞ up to multiplication with a character. So we can replace w.l.o.g. $\phi_{(a_3,t_3,d,e)}(n)$ by $\phi(n)$. This allows us to factorize the Dirichlet series in (15.8). We have

$$A_{m_1,m_2}\left(s,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$

$$=\phi\left(m_2^{(4)}\right)\phi\left(m_2^{(5)}n_{(d,e)}\right)\left(\sum_{(a,N)=1}a^{-s}\sum_{\substack{t|a\\ t^2|m_2^{(1)}a}}\phi\left(\frac{m_2^{(1)}a}{t^2}\right)t^{-1}c_t\left(m_2^{(1)}\right)c_{\frac{a}{t}}(m_1)\right)$$

$$\left(\sum_{a|\left(\frac{f}{(f,M)}\right)^\infty}a^{-s}\sum_{\substack{t|a\\ t^2|m_2^{(2)}\frac{f}{(f,M)}a}}\phi\left(\frac{m_2^{(2)}\frac{f}{(f,M)}a}{t^2}\right)t^{-1}c_t\left(m_2^{(2)}\right)c_{\frac{\frac{f}{(f,M)}a}{t}}(m_1)\right)$$

$$\left(\sum_{a|(f,M)^\infty}a^{-s}\sum_{\substack{t|a\\ t^2|m_2^{(3)}(f,M)\frac{(f,M,t)}{\left(\frac{a(f,t)}{t},M,f,t\right)}a}}\phi\left(\frac{m_2^{(3)}(f,M)\frac{(f,M,t)}{\left(\frac{a(f,t)}{t},M,f,t\right)}a}{t^2}\right)t^{-1}c_t\left(m_2^{(3)}\right)c_{\frac{(f,M)a}{t}}(m_1)\right).$$

Next factorize each of the three Dirichlet series into an Euler product. This gives us

$$A_{m_1,m_2}\left(s,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$

$$= \prod_{p|n_{(d,e)}} \phi\left(p^{\beta_p+1}\right) \prod_{p|\frac{N}{fn_{(d,e)}}} \phi\left(p^{\beta_p}\right)$$

$$\left(\prod_{(p,N)=1} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \beta_p+n}} \phi\left(p^{\beta_p+n-2k}\right) p^{-k} c_{p^{n-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)$$

$$\left(\prod_{p|\frac{f}{(f,M)}} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \beta_p+n+1}} \phi\left(p^{\beta_p+n+1-2k}\right) p^{-k} c_{p^{n+1-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)$$

$$\left(\prod_{p|(f,M)} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \beta_p+1+\min(1,k)-\min(1,k,n-k+\min(1,k))+n}}\right.$$

$$\left.\phi\left(p^{\beta_p+1+\min(1,k)-\min(1,k,n-k+\min(1,k))+n-2k}\right) p^{-k} c_{p^{n+1-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)$$

$$= \prod_{p|n_{(d,e)}} \phi\left(p^{\beta_p+1}\right) \prod_{p|\frac{N}{fn_{(d,e)}}} \phi\left(p^{\beta_p}\right)$$

$$\left(\prod_{(p,N)=1} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \beta_p+n}} \phi\left(p^{\beta_p+n-2k}\right) p^{-k} c_{p^{n-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)$$

$$\left(\prod_{p|\frac{f}{(f,M)}} \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{0\le k\le n \\ 2k\le \beta_p+n+1}} \phi\left(p^{\beta_p+n+1-2k}\right) p^{-k} c_{p^{n+1-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)$$

$$\left(\prod_{p|(f,M)} \sum_{n=0}^{\infty} p^{-ns} \left(\phi\left(p^{\beta_p+1+n}\right) c_{p^{n+1}}(m_1) + \sum_{\substack{1\le k\le n \\ 2k\le \beta_p+n+1}} \phi\left(p^{\beta_p+n+1-2k}\right) p^{-k} c_{p^{n+1-k}}(m_1) c_{p^k}\left(p^{\beta_p}\right)\right)\right).$$

Finally we distinguish between the cases whether m_2 vanishes or not and use the notation for the power series introduced in the previous chapter to express the factors in the Euler product.

(1) 1. Case: $m_1 \neq 0$. Use the notation from Definition 14.4 and note, that the polynomial S_p coincides with the p-th factor in the Euler product of the L-function associated to the sequence of Fourier coefficients $\phi(n)$ at the unramified primes (see Appendix C Theorem 20.2 for details). We have

$$A_{m_1,m_2}\left(3s-1, f, M, \left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, \left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0}, P_{1,2}\right)$$

$$=\prod_{p|n_{(d,e)}}\phi\left(p^{\beta_p+1}\right)\prod_{p|\frac{N}{fn_{(d,e)}}}\phi\left(p^{\beta_p}\right)\left(\prod_{(p,N)=1}F_p\left(s,\beta_p,p^{\alpha_p}\right)\right)\left(\prod_{p|\frac{f}{(f,M)}}I_p(s,p^{\alpha_p},\beta_p)\right)\left(\prod_{p|(f,M)}M_p(s,p^{\alpha_p},\beta_p)\right)$$

$$=\prod_{p|n_{(d,e)}}\phi\left(p^{\beta_p+1}\right)\prod_{p|\frac{N}{fn_{(d,e)}}}\phi\left(p^{\beta_p}\right)\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\beta_p}p^{-\left(2s-\frac{5}{2}\right)\alpha_p}S_p(s)\mathfrak{B}_p(s,\beta_p,\alpha_p)\right)$$

$$\left(\prod_{p|\frac{f}{(f,M)}}I_p(s,p^{\alpha_p},\beta_p)\right)\left(\prod_{p|(f,M)}M_p(s,p^{\alpha_p},\beta_p)\right)$$

$$=L_\phi(3s-1)^{-1}\left(\prod_{(p,N)=1}p^{-\left(s-\frac{3}{2}\right)\beta_p}p^{-\left(2s-\frac{5}{2}\right)\alpha_p}\mathfrak{B}_p(s,\beta_p,\alpha_p)\right)\left(\prod_{p|n_{(d,e)}}\frac{\phi\left(p^{\beta_p+1}\right)}{1-\phi(p)p^{-(3s-1)}}\right)$$

$$\left(\prod_{p|\frac{\left(\frac{N}{f},M\right)}{n_{(d,e)}}}\frac{\phi\left(p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{\phi\left(p^{\beta_p}\right)}{S_p(s)}\right)\left(\prod_{p|(f,M)}\frac{M_p(s,p^{\alpha_p},\beta_p)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{f}{(f,M)}}\frac{I_p(s,p^{\alpha_p},\beta_p)}{S_p(s)}\right).$$

(2) 2. Case: $m_1=0$. Use the explicit formula for F_p in Lemma 9.6 and extract a quotient of shifted L-functions associated to the sequence of Fourier coefficients ϕ. We have

$$A_{0,m_2}\left(3s-1,f,M,\left(\phi_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},\left(m_{(a,t,d,e)}\right)_{(a,t)\in\mathbb{N}\times\mathbb{N}_0},P_{1,2}\right)$$

$$=\prod_{p|n_{(d,e)}}\phi\left(p^{\beta_p+1}\right)\prod_{p|\frac{N}{fn_{(d,e)}}}\phi\left(p^{\beta_p}\right)\left(\prod_{(p,N)=1}F_p\left(s,\beta_p,0\right)\right)\left(\prod_{p|\frac{f}{(f,M)}}I_p(s,0,\beta_p)\right)\left(\prod_{p|(f,M)}M_p(s,0,\beta_p)\right)$$

$$=\frac{L_\phi(3s-2)}{L_\phi(3s-1)}\left(\prod_{(p,N)=1}\phi\left(p^{\beta_p}\right)\right)\left(\prod_{p|n_{(d,e)}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)\phi\left(p^{\beta_p+1}\right)}{1-\phi(p)p^{-(3s-1)}}\right)$$

$$\left(\prod_{p|\frac{\left(\frac{N}{f},M\right)}{n_{(d,e)}}}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)\phi\left(p^{\beta_p}\right)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{\frac{N}{f}}{\left(\frac{N}{f},M\right)}}\frac{T_p(s)\phi\left(p^{\beta_p}\right)}{S_p(s)}\right)$$

$$\left(\prod_{p|(f,M)}\frac{\left(1-\phi(p)p^{-(3s-2)}\right)M_p(s,0,\beta_p)}{1-\phi(p)p^{-(3s-1)}}\right)\left(\prod_{p|\frac{f}{(f,M)}}\frac{T_p(s)I_p(s,0,\beta_p)}{S_p(s)}\right).$$

□

Part VII

Fourier expansion for the minimal Eisenstein series

CHAPTER 16

Dirichlet series associated to the minimal Eisenstein series

In the case of the minimal Eisenstein series there are several Dirichlet series of different complexity occuring in the Fourier coefficients. This chapter is concerned with the definition and evaluation of these Dirichlet series. We start specifying the theory of power series developed in chapter 9 and 10 so that it can be applied to the Fourier coefficients of the minimal Eisenstein series. Use the notation introduced in the part "Power series" and associate to any prime number p the power series and variables

$$X := p^{-3s_2} \tag{16.1}$$

$$Y := p^{1-3s_1} \tag{16.2}$$

$$A_n := \sigma_{1-3s_1}(p^n) \tag{16.3}$$

$$S_p = 1 - A_1 X + Y X^2 \tag{16.4}$$

$$T_p = 1 - pA_1 X + p^2 Y X^2 \tag{16.5}$$

$$F_p(s_1, s_2, \alpha, \beta) = \sum_{n\geq 0} p^{-3s_2 n} \sum_{\substack{0\leq k\leq n \\ 2k\leq \alpha+n}} c_{p^{n-k}}(\beta) c_{p^k}(p^\alpha) \sigma_{1-3s_1}\left(p^{n+\alpha-2k}\right) p^{(1-3s_1)k} \tag{16.6}$$

$$G_p(s_1, s_2, \alpha, \beta) = \sum_{n\geq 0} p^{-3s_2 n} \sum_{\substack{0\leq k\leq n \\ 2k+1\leq \alpha+n}} c_{p^{n-k}}(\beta) c_{p^{k+1}}(p^\alpha) \sigma_{1-3s_1}\left(p^{n+\alpha-2k-1}\right) p^{(1-3s_1)k} . \tag{16.7}$$

Subsequently the polynomials S_p and T_p are calculated and it is checked that the sequence $(A_n)_{n\in\mathbb{N}}$ fulfills the necessary recursion. The identities

$$\begin{aligned} S_p =& 1 - A_1 X + Y X^2 = 1 - \sigma_{1-3s_1}(p)\, p^{-3s_2} + p^{1-3s_1}\left(p^{-3s_2}\right)^2 \\ =& 1 - \left(1 + p^{1-3s_1}\right) p^{-3s_2} + p^{1-6s_2-3s_1} = \left(1 - p^{-3s_2}\right)\left(1 - p^{1-3s_1-3s_2}\right) , \end{aligned}$$

$$\begin{aligned} T_p =& 1 - pA_1 X + p^2 Y X^2 = 1 - p\sigma_{1-3s_1}(p)\, p^{-3s_2} + p^2 p^{1-3s_1}\left(p^{-3s_2}\right)^2 \\ =& 1 - \left(1 + p^{1-3s_1}\right) p^{1-3s_2} + p^{3-6s_2-3s_1} = \left(1 - p^{1-3s_2}\right)\left(1 - p^{2-3s_2-3s_1}\right) \end{aligned}$$

and

$$\begin{aligned} A_1A_{n+1} - YA_n =& \sigma_{1-3s_1}(p)\,\sigma_{1-3s_1}\left(p^{n+1}\right) - p^{1-3s_1}\sigma_{1-3s_1}(p^n) \\ =& \sum_{k=0}^{1} p^{(1-3s_1)k} \sum_{k=0}^{n+1} p^{(1-3s_1)k} - p^{1-3s_1} \sum_{k=0}^{n} p^{(1-3s_1)k} \\ =& \sum_{k=0}^{n+2} p^{(1-3s_1)k} = A_{n+2} \end{aligned}$$

hold.

The situation here is similar to the situation of the Eisenstein series twisted by a Maass cusp form. First note that the polynomials S_p, T_p are the Euler factors of certain shifted Riemann zeta functions. Further the theory developed in chapter 9 is used to define a holomorphic function for each prime number, which describes the unramified parts of the Fourier coefficients in the non degenerate terms. It turns out that these holomorphic functions have the right transformation behaviour.

Definition 16.1. For any prime number p and nonnegative integers α, β define the meromorphic function $\mathfrak{A}_p(*, \alpha, \beta) : \mathbb{C}^2 \to \mathbb{C}$, $(s_1, s_2) \longmapsto p^{(2s_1+s_2-1)\alpha} p^{(s_1+2s_2-1)\beta} \frac{F_p(s_1,s_2,\alpha,p^\beta)}{S_p(s_1,s_2)}$.

The theory developed for F_p in chapter 9 implies the holomorphicity of the function $\mathfrak{A}_p(*, \alpha, \beta)$ and after a short calculation also the invariance against the action of the Weyl-group, see [**4**, (2.5)] for details. In order to execute this calculation a trivial polynomial identity is needed, which is stated in the lemma below.

Lemma 16.2. *Let α, β be nonnegative integers, then in the polynomial ring $\mathbb{C}[X, Y]$ the identity*

$$\sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=0}^{\alpha+\beta-k-l} X^{k+j} Y^{l+j} = \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} X^{k+l} Y^l \sum_{j=0}^{\alpha+k-l} Y^j \,,$$

holds.

Proof. Split the innermost sum and divide the whole sum into two parts. After that reorder the Cauchy product of the two innermost sums in the first part and consecutively make the index shifts $j \to j - (\alpha - l + 1)$ and $\alpha \to \alpha - l$ in the second part. We have

$$\begin{aligned} \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=0}^{\alpha+\beta-k-l} X^{k+j} Y^{l+j} =& \sum_{k=0}^{\beta} \sum_{l=0}^{\alpha} \sum_{j=0}^{\alpha-l} X^{k+j} Y^{l+j} + \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=\alpha-l+1}^{\alpha+\beta-k-l} X^{k+j} Y^{l+j} \\ =& \sum_{k=0}^{\beta} \sum_{m=0}^{\alpha} X^{k+m} \sum_{h=0}^{\alpha-m} Y^{h+m} + \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=0}^{\beta-k-1} X^{k+(\alpha-l+1)+j} Y^{l+(\alpha-l+1)+j} \end{aligned}$$

$$= \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} X^{k+l} \sum_{j=0}^{\alpha-l} Y^{j+l} + \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=0}^{\beta-k-1} X^{(k+j+1)+l} Y^{\alpha+j+1} .$$

Next reorder the Cauchy product of the two innermost sums in the second sum. So we get

$$\begin{aligned}
\sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} \sum_{j=0}^{\alpha+\beta-k-l} X^{k+j} Y^{l+j} &= \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} X^{k+l} \sum_{j=0}^{\alpha-l} Y^{j+l} + \sum_{l=0}^{\alpha} \sum_{n=0}^{\beta} X^{n+l} \sum_{h=0}^{n-1} Y^{\alpha+n-h} \\
&= \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} X^{k+l} \left[\sum_{j=0}^{\alpha-l} Y^{j+l} + \sum_{h=0}^{k-1} Y^{\alpha+k-h} \right] \\
&= \sum_{l=0}^{\alpha} \sum_{k=0}^{\beta} X^{k+l} Y^{l} \sum_{j=0}^{\alpha+k-l} Y^{j} .
\end{aligned}$$

□

LEMMA 16.3. *The function $\mathfrak{A}_p(*, \alpha, \beta)$ is holomorphic and invariant against the action of the Weyl group.*

PROOF. The explicit formula for F_p in Lemma 9.4 implies immediately the holomorphicity of the function $\mathfrak{A}_p(*, \alpha, \beta)$. To show the invariance against the action of the Weyl group, it is sufficient to check the invariance for the elements w_2, w_3, since theses two elements generate the Weyl group, see [**4**, (2.5)] for the definition of this action.

(1) We begin with the invariance for w_2. Use the explicit formula in Lemma 9.4 and the identity in Lemma 16.2 and calculate directly

$$\begin{aligned}
\mathfrak{A}_p(w_2(s_1, s_2), \alpha, \beta) =& \mathfrak{A}_p\left(s_1 + s_2 - \frac{1}{3}, \frac{2}{3} - s_2, \alpha, \beta\right) \\
=& \quad p^{\left(2\left(s_1+s_2-\frac{1}{3}\right)+\left(\frac{2}{3}-s_2\right)-1\right)\alpha} p^{\left(\left(s_1+s_2-\frac{1}{3}\right)+2\left(\frac{2}{3}-s_2\right)-1\right)\beta} \\
& \sum_{k=0}^{\beta} \sum_{l=0}^{\alpha} p^{k+l} \sigma_{1-3\left(s_1+s_2-\frac{1}{3}\right)} \left(p^{\alpha+k-l}\right) \left(p^{-3\left(\frac{2}{3}-s_2\right)}\right)^{k+l} \left(p^{1-3\left(s_1+s_2-\frac{1}{3}\right)}\right)^{l} \\
=& p^{(2s_1+s_2-1)\alpha} p^{(s_1-s_2)\beta} \sum_{k=0}^{\beta} \sum_{l=0}^{\alpha} p^{k+l} \sigma_{2-3s_1-3s_2} \left(p^{\alpha+k-l}\right) \left(p^{3s_2-2}\right)^{k+l} \left(p^{2-3s_1-3s_2}\right)^{l} .
\end{aligned}$$

Make the index shift $k \to \beta - k$ and expand the divisor function. This gives us

$$\mathfrak{A}_p(w_2(s_1, s_2), \alpha, \beta) = p^{(2s_1+s_2-1)\alpha} p^{(s_1-s_2)\beta} \sum_{k=0}^{\beta} \sum_{l=0}^{\alpha} p^{(\beta-k)+l} \left(p^{3s_2-2}\right)^{\beta-k} \left(p^{-3s_1}\right)^{l} \sum_{j=0}^{\alpha+(\beta-k)-l} p^{(2-3s_1-3s_2)j}$$

$$=p^{(2s_1+s_2-1)\alpha}p^{(s_1+2s_2-1)\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}\sum_{j=0}^{\alpha+\beta-k-l}p^{k+j}\left(p^{-3s_2}\right)^{k+j}\left(p^{1-3s_1}\right)^{l+j} .$$

To get the invariance apply the identity in Lemma 16.2 to the sum. We have

$$\begin{aligned}\mathfrak{A}_p(w_2(s_1,s_2),\alpha,\beta) =&p^{(2s_1+s_2-1)\alpha}p^{(s_1+2s_2-1)\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}\left(p^{-3s_2}\right)^{k+l}\left(p^{1-3s_1}\right)^{l}\sum_{j=0}^{\alpha+k-l}\left(p^{1-3s_1}\right)^{j}\\ =&p^{(2s_1+s_2-1)\alpha}p^{(s_1+2s_2-1)\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}\left(p^{-3s_2}\right)^{k+l}\left(p^{1-3s_1}\right)^{l}\sigma_{1-3s_1}\left(p^{\alpha+k-l}\right)\\ =&\mathfrak{A}_p(s_1,s_2,\alpha,\beta) .\end{aligned}$$

(2) Next the invariance for w_3 is proved. We use Lemma 9.4 and the functional equation of the divisor function and then calculate directly

$$\begin{aligned}&\mathfrak{A}_p(w_3(s_1,s_2),\alpha,\beta) = \mathfrak{A}_p\left(\frac{2}{3}-s_1, s_1+s_2-\frac{1}{3},\alpha,\beta\right)\\ =&p^{\left(2\left(\frac{2}{3}-s_1\right)+\left(s_1+s_2-\frac{1}{3}\right)-1\right)\alpha}p^{\left(\left(\frac{2}{3}-s_1\right)+2\left(s_1+s_2-\frac{1}{3}\right)-1\right)\beta}\\ &\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}\sigma_{1-3\left(\frac{2}{3}-s_1\right)}\left(p^{\alpha+k-l}\right)\left(p^{-3\left(s_1+s_2-\frac{1}{3}\right)}\right)^{k+l}\left(p^{1-3\left(\frac{2}{3}-s_1\right)}\right)^{l}\\ =&p^{(s_2-s_1)\alpha}p^{(s_1+2s_2-1)\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}p^{(3s_1-1)(\alpha+k-l)}\sigma_{1-3s_1}\left(p^{\alpha+k-l}\right)\left(p^{1-3s_1-3s_2}\right)^{k+l}\left(p^{3s_1-1}\right)^{l}\\ =&p^{(2s_1+s_2-1)\alpha}p^{(s_1+2s_2-1)\beta}\sum_{k=0}^{\beta}\sum_{l=0}^{\alpha}p^{k+l}\sigma_{1-3s_1}\left(p^{\alpha+k-l}\right)\left(p^{-3s_2}\right)^{k+l}\left(p^{1-3s_1}\right)^{l}\\ =&\mathfrak{A}_p(s_1,s_2,\alpha,\beta) .\end{aligned}$$

□

After these preparations we define and evaluate the Dirichlet series corresponding to the minimal Eisenstein series.

Definition 16.4. Let N be a squarefree integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. For nonnegative integers m_1, m_2 define the following Dirichlet series associated to these parameters.

(1) The Dirichlet series A_{m_1,m_2} is defined by

$$A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) := \sum_{a\geq 1} a^{-3s_2} \sum_{\substack{d\geq 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}} (a,b)^{-3s_1} e\left(m_2\frac{b}{a}\right)$$
$$\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}} e\left(m_1\frac{c}{f(a,b)}\right) \sum_{\substack{e \bmod dh(a,b)N\\ (e,dh)=1}} e\left(m_1\frac{ae}{dhf(a,b)^2}\right) .$$

(2) The Dirichlet series B_{m_2} is defined by

$$B_{m_2}(s_1,s_2,f,P_{min}) := \sum_{a\geq 1} a^{-3s_1-3s_2} \sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}} (a,b)^{3s_1} e\left(m_2\frac{b}{a}\right) \sum_{\substack{c \bmod Na\\ (f(a,b),c)=1}} 1 .$$

(3) The Dirichlet series C_{m_1} is defined by

$$C_{m_1}(s_1,s_2,f,h,P_{min}) := \sum_{\substack{b\geq 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_1-3s_2} \sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right) \sum_{\substack{d\geq 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{e \bmod dhNb\\ (dh,e)=1}} 1 .$$

(4) The Dirichlet series D_{m_1} is defined by

$$D_{m_1}(s_2,f,P_{min}) := \sum_{\substack{b\geq 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_2} \sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right) .$$

LEMMA 16.5. *Let $m_1 = \prod_p p^{\alpha_p}$, $m_2 = \prod_p p^{\beta_p}$ be positive integers in prime factor decomposition. The explicit formulas below for the Dirichlet series in Definition 16.4 are valid.*

(1) The formula for A_{m_1,m_2} is

$$\begin{aligned}
&A_{m_1,m_2}(s_1,s_2,f,h,P_{min})\\
=&\frac{N^3}{f^2} m_1^{1-2s_1-s_2} m_2^{1-s_1-2s_2} L_{\chi_{\frac{N}{f}}}(3s_1)^{-1} L_{\chi_{fh}}(3s_2)^{-1} L_{\chi_N}(3s_1+3s_2-1)^{-1} \prod_{(p,N)=1} \mathfrak{A}_p(s_1,s_2,\alpha_p,\beta_p)\\
&\prod_{p|h} \frac{(p-1)\left(1-p^{-3s_2}\right)\sigma_{1-3s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2-3s_1}\right)p^{(1-3s_1)\alpha_p}\sigma_{2-3s_1-3s_2}\left(p^{\beta_p}\right)}{1-p^{1-3s_1}}\\
&\prod_{p|f} G_p\left(s_1,s_2,\alpha_p,p^{\beta_p}\right) \prod_{p|N} p^{(2s_1+s_2-1)\alpha_p} p^{(s_1+2s_2-1)\beta_p} \sigma_{1-3s_2}\left(\prod_{p|\frac{N}{fh}} p^{\beta_p}\right) .
\end{aligned}$$

(2) The formula for $A_{m_1,0}$ is

$$\begin{aligned}
&A_{m_1,0}(s_1,s_2,f,h,P_{min})\\
=&\frac{N^3}{f^2}\sigma_{1-3s_1}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)L_{\chi_{\frac{N}{fh}}}(3s_1+3s_2-2)\zeta(3s_2-1)L_{\chi_{\frac{N}{f}}}(3s_1)^{-1}L_{\chi_{fh}}(3s_2)^{-1}L_{\chi_N}(3s_1+3s_2-1)^{-1}\\
&\prod_{p|h}\frac{(p-1)\left(1-p^{-3s_2}\right)\left(1-p^{2-3s_1-3s_2}\right)-p\left(1-p^{1-3s_1-3s_2}\right)\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2}\right)p^{(1-3s_1)\alpha_p}}{1-p^{1-3s_1}}\\
&\prod_{p|f}(p-1)\left(\sigma_{1-3s_1}\left(p^{\alpha_p-1}\right)-\sigma_{1-3s_1}\left(p^{\alpha_p}\right)p^{-3s_2}\right)\ .
\end{aligned}$$

(3) The formula for A_{0,m_2} is

$$A_{0,m_2}(s_1,s_2,f,h,P_{min})=\frac{N^3}{f^2}\prod_{p|fh}(p-1)\,\sigma_{1-3s_2}(m_2)\frac{L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{\zeta(3s_2)L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}\ .$$

(4) The formula for $A_{0,0}$ is

$$A_{0,0}(s_1,s_2,f,h,P_{min})=\frac{N^3}{f^2}\prod_{p|fh}(p-1)\,\frac{\zeta(3s_2-1)L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{\zeta(3s_2)L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}\ .$$

(5) The formula for B_{m_2} is

$$B_{m_2}(s_1,s_2,f,P_{min})=\left(\frac{N}{f}\right)^2\prod_{p|f}(p-1)\,\sigma_{2-3s_1-3s_2}(m_2)\frac{L_{\chi_{\frac{N}{f}}}(3s_2-1)}{\zeta(3s_1+3s_2-1)L_{\chi_N}(3s_2)}\ .$$

(6) The formula for B_0 is

$$B_0(s_1,s_2,f,P_{min})=\left(\frac{N}{f}\right)^2\prod_{p|f}(p-1)\,\frac{\zeta(3s_1+3s_2-2)L_{\chi_{\frac{N}{f}}}(3s_2-1)}{\zeta(3s_1+3s_2-1)L_{\chi_N}(3s_2)}\ .$$

(7) The formula for C_{m_1} is

$$\begin{aligned}
C_{m_1}(s_1,s_2,f,h,P_{min})=&\frac{N^2}{f}\prod_{p|h}(p-1)\prod_{p|f}\left((p-1)\sigma_{2-3s_1-3s_2}\left(p^{\alpha_p}\right)-p^{1+(2-3s_1-3s_2)\alpha_p}\right)\\
&\sigma_{2-3s_1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)\frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}\ .
\end{aligned}$$

(8) The formula for C_0 is

$$C_0(s_1,s_2,f,h,P_{min})=\frac{N^2}{f}\prod_{p|hf}(p-1)\,\frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}\ .$$

(9) The formula for D_{m_1} is

$$D_{m_1}(s_2,f,P_{min}) = \frac{N}{f}\prod_{p|f}\left((p-1)\sigma_{1-3s_2}(p^{\alpha_p}) - p^{1+(1-3s_2)\alpha_p}\right)\sigma_{1-3s_2}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_N}(3s_2)^{-1} .$$

(10) The formula for D_0 is

$$D_0(s_2,f,P_{min}) = \frac{N}{f}\prod_{p|f}(p-1)\,\frac{L_{\chi_{\frac{N}{f}}}(3s_2-1)}{L_{\chi_N}(3s_2)} .$$

Proof. In all parts we handle the cases whether m_1, m_2 vanish or do not vanish together, assuming for the present that m_1, m_2 are nonnegative integers.

(1) First split the summation over b in this way $b = b_1 b_2$ with $b_1 \mid a$ and $\left(b_2, \frac{a}{b_1}\right) = 1$. We have

$$\begin{aligned} A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) = &\sum_{a\ge 1} a^{-3s_2} \sum_{\substack{d\ge 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{b_1|a\\ \left(b_1,\frac{N}{f}\right)=1}} \sum_{\substack{b_2 \bmod \frac{N}{f}\frac{a}{b_1}\\ \left(b_2,\frac{a}{b_1}\right)=1}} (a,b_1b_2)^{-3s_1}\, e\left(m_2\frac{b_1b_2}{a}\right) \\ &\sum_{\substack{c \bmod N(a,b_1b_2)\\ (f(a,b_1b_2),c)=1}} e\left(m_1\frac{c}{f(a,b_1b_2)}\right) \sum_{\substack{e \bmod dh(a,b_1b_2)N\\ (e,dh)=1}} e\left(m_1\frac{ae}{dhf(a,b_1b_2)^2}\right) \\ = &\sum_{a\ge 1} a^{-3s_2} \sum_{\substack{d\ge 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{b|a\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_1} \sum_{\substack{b_2 \bmod \frac{N}{f}\frac{a}{b}\\ \left(b_2,\frac{a}{b}\right)=1}} e\left(m_2\frac{b_2}{\frac{a}{b}}\right) \\ &\sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right) \sum_{\substack{e \bmod dhbN\\ (e,dh)=1}} e\left(m_1\frac{ae}{dhfb^2}\right) . \end{aligned}$$

Next split the summation over c in this way $c = fbk + c$ with $0 \le k < \frac{N}{f}$, $0 \le c < fb$, split the summation over b_2 in this way $b_2 = \frac{a}{b}l + g$ with $0 \le l < \frac{N}{f}$, $0 \le g < \frac{a}{b}$ and split the summation over e in this way $e = dhj + e$ with $0 \le j < bN$, $0 \le e < dh$. This gives us

$$\begin{aligned} A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) = &\sum_{a\ge 1} a^{-3s_2} \sum_{\substack{d\ge 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{b|a\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_1} \sum_{0\le l<\frac{N}{f}} \sum_{\substack{g \bmod \frac{a}{b}\\ \left(g,\frac{a}{b}\right)=1}} e\left(m_2\frac{g}{\frac{a}{b}}\right) \\ &\sum_{0\le k<\frac{N}{f}} \sum_{\substack{c \bmod fb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right) \sum_{j \bmod bN} e\left(m_1\frac{aj}{fb^2}\right) \sum_{\substack{e \bmod dh\\ (e,dh)=1}} e\left(m_1\frac{ae}{dhfb^2}\right) \end{aligned}$$

$$= \left(\frac{N}{f}\right)^2 \sum_{a \geq 1} a^{-3s_2} \sum_{\substack{d \geq 1 \\ \left(d, \frac{N}{fh}\right)=1}} d^{-3s_1} \sum_{\substack{b|a \\ \left(b, \frac{N}{f}\right)=1}} b^{-3s_1} c_{\frac{a}{b}}(m_2)\, c_{fb}(m_1)$$

$$\sum_{j \bmod bN} e\left(m_1 \frac{\frac{a}{b} j}{fb}\right) \sum_{\substack{e \bmod dh \\ (e,dh)=1}} e\left(m_1 \frac{\frac{a}{b} e}{dhfb}\right) .$$

Extract the geometric series in the variable j and evaluate it. We obtain

$$\sum_{j \bmod bN} e\left(m_1 \frac{\frac{a}{b} j}{fb}\right) = \begin{cases} bN & \text{, if } fb \mid m_1 \frac{a}{b} \text{ ,} \\ 0 & \text{, otherwise .} \end{cases}$$

Substituting this result back into the above equation and splitting the summation over d in this way $d = eg$ with $e \mid h^\infty$ and $(g, h) = 1$, yields the formula

$$A_{m_1,m_2}(s_1, s_2, f, h, P_{min}) = \frac{N^3}{f^2} \sum_{a \geq 1} a^{-3s_2} \sum_{\substack{b|a \\ \left(b, \frac{N}{f}\right)=1 \\ fb^2|m_1 a}} b^{1-3s_1} c_{\frac{a}{b}}(m_2)\, c_{fb}(m_1) \sum_{\substack{d \geq 1 \\ \left(d, \frac{N}{fh}\right)=1}} d^{-3s_1} c_{dh}\left(\frac{m_1 a}{fb^2}\right)$$

$$= \frac{N^3}{f^2} \sum_{a \geq 1} a^{-3s_2} \sum_{\substack{b|a \\ \left(b, \frac{N}{f}\right)=1 \\ fb^2|m_1 a}} b^{1-3s_1} c_{\frac{a}{b}}(m_2)\, c_{fb}(m_1) \sum_{\left(g, \frac{N}{f}\right)=1} g^{-3s_1} c_g\left(\frac{m_1 a}{fb^2}\right) \sum_{e|h^\infty} e^{-3s_1} c_{eh}\left(\frac{m_1 a}{fb^2}\right) .$$

The next step is to factorize the Dirichlet series. In order to do this, decompose the summation in the outer and inner sum in the way $a = a_1 a_2 a_3 a_4$ and $b = b_1 b_2 b_3 b_4$ according to the divisibility conditions $(a_1 b_1, N) = 1$, $a_2 b_2 \mid h^\infty$ and $a_3 b_3 \mid f^\infty$ and $a_4 b_4 \mid \left(\frac{N}{fh}\right)^\infty$. Note that the gcd-condition $\left(b, \frac{N}{f}\right) = 1$ is equivalent to $b_2, b_4 = 1$. So we get

$$A_{m_1,m_2}(s_1, s_2, f, h, P_{min}) = \frac{N^3}{f^2} \sum_{(a_1,N)=1} \sum_{a_2|h^\infty} \sum_{a_3|f^\infty} \sum_{a_4|\left(\frac{N}{fh}\right)^\infty} (a_1 a_2 a_3 a_4)^{-3s_2}$$

$$\sum_{\substack{b_1|a_1 \\ b_3|a_3 \\ b_1^2|m_1 a_1 \\ fb_3^2|m_1 a_3}} (b_1 b_3)^{1-3s_1} c_{\frac{a_1 a_2 a_3 a_4}{b_1 b_3}}(m_2)\, c_{b_1(fb_3)}(m_1) \sum_{\left(g, \frac{N}{f}\right)=1} g^{-3s_1} c_g\left(\frac{m_1 a_1 a_2 a_3 a_4}{f b_1^2 b_3^2}\right)$$

$$\sum_{e|h^\infty} e^{-3s_1} c_{eh}\left(\frac{m_1 a_1 a_2 a_3 a_4}{f b_1^2 b_3^2}\right) .$$

Using the multiplicativity and the other properties of the Ramanujan sums the above Dirichlet series can be factorized in a product of four Dirichlet series. We have

$$
\begin{aligned}
A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) =&\frac{N^3}{f^2} \sum_{a\left|\left(\frac{N}{fh}\right)^\infty\right.} a^{-3s_2} c_a\left(m_2\right) \\
&\sum_{a|h^\infty} a^{-3s_2} c_a\left(m_2\right) \sum_{e|h^\infty} e^{-3s_1} c_{eh}\left(m_1 a\right) \\
&\sum_{a|f^\infty} a^{-3s_2} \sum_{\substack{b|a\\ fb^2|m_1 a}} b^{1-3s_1} c_{\frac{a}{b}}\left(m_2\right) c_{fb}\left(m_1\right) \sum_{g|f^\infty} g^{-3s_1} c_g\left(\frac{m_1 a}{fb^2}\right) \\
&\sum_{(a,N)=1} a^{-3s_2} \sum_{\substack{b|a\\ b^2|m_1 a}} b^{1-3s_1} c_{\frac{a}{b}}\left(m_2\right) c_b\left(m_1\right) \sum_{(g,N)=1} g^{-3s_1} c_g\left(\frac{m_1 a}{b^2}\right) .
\end{aligned}
$$

Now we can start with the evaluation of each Dirichlet series dividing between the cases whether m_1, m_2 vanish or not.

(a) 1. Case: $m_1 = 0$. Note that in this case the divisibility conditions are always satisfied. We have

$$
\begin{aligned}
A_{0,m_2}(s_1,s_2,f,h,P_{min}) =&\frac{N^3}{f^2} \left(\sum_{a\left|\left(\frac{N}{fh}\right)^\infty\right.} a^{-3s_2} c_a\left(m_2\right)\right) \left(\sum_{a|h^\infty} a^{-3s_2} c_a\left(m_2\right) \sum_{e|h^\infty} e^{-3s_1} \phi(eh)\right) \\
&\left(\sum_{a|f^\infty} a^{-3s_2} \sum_{b|a} b^{1-3s_1} c_{\frac{a}{b}}\left(m_2\right) \phi(fb) \sum_{g|f^\infty} g^{-3s_1} \phi(g)\right) \\
&\left(\sum_{(a,N)=1} a^{-3s_2} \sum_{b|a} b^{1-3s_1} c_{\frac{a}{b}}\left(m_2\right) \phi(b) \sum_{(g,N)=1} g^{-3s_1} \phi(g)\right) .
\end{aligned}
$$

Our next goal is to factotrize the product of the Dirichlet series further. First pull out the Dirichlet series in the variables g, e, then factorize the convolution of Dirichlet series in the third and fourth Dirichlet series. So we get

$$
A_{0,m_2}(s_1,s_2,f,h,P_{min}) =\frac{N^3}{f^2} \left(\sum_{a\geq 1} a^{-3s_2} c_a(m_2)\right) \left(\sum_{\left(a,\frac{N}{f}\right)=1} a^{-3s_1} \phi(a)\right)
$$

$$\left(\sum_{(a,N)=1} a^{1-3s_1-3s_2}\phi(a)\right)\left(\sum_{a|f^\infty} a^{1-3s_1-3s_2}\phi(fa)\right)\left(\sum_{a|h^\infty} a^{-3s_1}\phi(ha)\right).$$

Use Lemma 18.4 in Appendix A for the evaluation of the second and third Dirichlet series and use Lemma 18.5 for the fourth and fifth Dirichlet series. We obtain

$$\begin{aligned}
&A_{0,m_2}(s_1,s_2,f,h,P_{min})\\
=&\frac{N^3}{f^2}\left(\sum_{a\geq 1} a^{-3s_2}c_a(m_2)\right)\frac{L_{\chi_{\frac{N}{f}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)}\frac{L_{\chi_N}(3s_1+3s_2-2)}{L_{\chi_N}(3s_1+3s_2-1)}\left(\prod_{p|f}\frac{p-1}{1-p^{2-3s_1-3s_2}}\right)\left(\prod_{p|h}\frac{p-1}{1-p^{1-3s_1}}\right)\\
=&\frac{N^3}{f^2}\prod_{p|fh}(p-1)\left(\sum_{a\geq 1} a^{-3s_2}c_a(m_2)\right)\frac{L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}.
\end{aligned}$$

Finally apply Lemma 18.4 to the first Dirichlet series distinguishing between the cases m_2 vanishes or not. So the two formulas

$$A_{0,m_2}(s_1,s_2,f,h,P_{min})=\frac{N^3}{f^2}\prod_{p|fh}(p-1)\,\sigma_{1-3s_2}(m_2)\frac{L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{\zeta(3s_2)L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}$$

and

$$A_{0,0}(s_1,s_2,f,h,P_{min})=\frac{N^3}{f^2}\prod_{p|fh}(p-1)\,\frac{\zeta(3s_2-1)L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{\zeta(3s_2)L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}$$

hold.

(b) 2. Case: $m_1\neq 0$. Split $m_1=m_1^{(N)}m_1^{(f)}m_1^{(h)}m_1^{\left(\frac{N}{fh}\right)}$ according to the divisibility conditions $\left(m_1^{(N)},N\right)=1$, $m_1^{(f)}\mid f^\infty$, $m_1^{(h)}\mid h^\infty$ and $m_1^{\left(\frac{N}{fh}\right)}\mid\left(\frac{N}{fh}\right)^\infty$. Further use Lemma 18.4 to evaluate the Dirichlet series in the variable g. We have

$$\begin{aligned}
A_{m_1,m_2}(s_1,s_2,f,h,P_{min})=&\frac{N^3}{f^2}\sum_{a|\left(\frac{N}{fh}\right)^\infty} a^{-3s_2}c_a(m_2)\\
&\sum_{a|h^\infty} a^{-3s_2}c_a(m_2)\sum_{e|h^\infty} e^{-3s_1}c_{eh}(m_1a)\\
&\sum_{a|f^\infty} a^{-3s_2}\sum_{\substack{b|a\\ fb^2|m_1a}} b^{1-3s_1}c_{\frac{a}{b}}(m_2)\,c_{fb}(m_1)\,\sigma_{1-3s_1}\left(\frac{m_1^{(f)}a}{fb^2}\right)\prod_{p|f}(1-p^{-3s_1})
\end{aligned}$$

$$\sum_{(a,N)=1} a^{-3s_2} \sum_{\substack{b|a \\ b^2|m_1 a}} b^{1-3s_1} c_{\frac{a}{b}}(m_2)\, c_b(m_1)\, \sigma_{1-3s_1}\left(\frac{m_1^{(N)} a}{b^2}\right) L_{\chi_N}(3s_1)^{-1} .$$

Next we factorize the last two Dirichlet series and use the notation for the Dirichlet series F_p and G_p. This gives us

$$A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) = \frac{N^3}{f^2} L_{\chi_{\frac{N}{f}}}(3s_1)^{-1} \sum_{a|\left(\frac{N}{fh}\right)^{\infty}} a^{-3s_2} c_a(m_2)$$

$$\left(\sum_{a|h^{\infty}} a^{-3s_2} c_a(m_2) \sum_{e|h^{\infty}} e^{-3s_1} c_{eh}(m_1 a)\right) \prod_{p|f} G_p(s_1,s_2,\alpha_p,m_2) \prod_{(p,N)=1} F_p(s_1,s_2,\alpha_p,m_2) .$$

Finally distinguish between the cases whether m_2 vanishes or not.

(i) 1. Case: $m_2 \neq 0$. Use Lemma 18.4 and Lemma 18.6 to evaluate the two Dirichlet series and use the notation in Definition 16.1. We obtain

$$\begin{aligned}
&A_{m_1,m_2}(s_1,s_2,f,h,P_{min}) \\
=&\frac{N^3}{f^2} L_{\chi_{\frac{N}{f}}}(3s_1)^{-1} \left(\sigma_{1-3s_2}\left(\prod_{p|\frac{N}{fh}} p^{\beta_p}\right) \prod_{p|\frac{N}{fh}} \left(1-p^{-3s_2}\right)\right) \prod_{p|f} G_p\left(s_1,s_2,\alpha_p,p^{\beta_p}\right) \\
&\prod_{p|h} \frac{(p-1)\left(1-p^{-3s_2}\right)\sigma_{1-3s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2-3s_1}\right) p^{(1-3s_1)\alpha_p} \sigma_{2-3s_1-3s_2}\left(p^{\beta_p}\right)}{1-p^{1-3s_1}} \\
&L_{\chi_N}(3s_2)^{-1} L_{\chi_N}(3s_1+3s_2-1)^{-1} \prod_{(p,N)=1} p^{-(2s_1+s_2-1)\alpha_p} p^{-(s_1+2s_2-1)\beta_p} \mathfrak{A}_p(s_1,s_2,\alpha_p,\beta_p) \\
=&\frac{N^3}{f^2} L_{\chi_{\frac{N}{f}}}(3s_1)^{-1} L_{\chi_{fh}}(3s_2)^{-1} L_{\chi_N}(3s_1+3s_2-1)^{-1} \sigma_{1-3s_2}\left(\prod_{p|\frac{N}{fh}} p^{\beta_p}\right) \\
&\prod_{p|h} \frac{(p-1)\left(1-p^{-3s_2}\right)\sigma_{1-3s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2-3s_1}\right) p^{(1-3s_1)\alpha_p} \sigma_{2-3s_1-3s_2}\left(p^{\beta_p}\right)}{1-p^{1-3s_1}} \\
&\prod_{p|f} G_p\left(s_1,s_2,\alpha_p,p^{\beta_p}\right) \prod_{p|N} p^{(2s_1+s_2-1)\alpha_p} p^{(s_1+2s_2-1)\beta_p} m_1^{1-2s_1-s_2} m_2^{1-s_1-2s_2} \prod_{(p,N)=1} \mathfrak{A}_p(s_1,s_2,\alpha_p,\beta_p) .
\end{aligned}$$

(ii) 2. Case: $m_2 = 0$. Again evaluate the Dirichlet series and use the explicit formulas in Lemma 9.6 and Lemma 10.2 for F_p and G_p. We obtain

$$A_{m_1,0}(s_1,s_2,f,h,P_{min}) = \frac{N^3}{f^2} L_{\chi_{\frac{N}{f}}}(3s_1)^{-1} \prod_{p|\frac{N}{fh}} \frac{1-p^{-3s_2}}{1-p^{1-3s_2}}$$

$$\prod_{p|h} \frac{(p-1)\left(1-p^{-3s_2}\right)\left(1-p^{2-3s_1-3s_2}\right) - p\left(1-p^{1-3s_1-3s_2}\right)\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2}\right)p^{(1-3s_1)\alpha_p}}{\left(1-p^{1-3s_1}\right)\left(1-p^{2-3s_1-3s_2}\right)\left(1-p^{1-3s_2}\right)}$$

$$\prod_{p|f}(p-1)\frac{\sigma_{1-3s_1}\left(p^{\alpha_p-1}\right)-\sigma_{1-3s_1}\left(p^{\alpha_p}\right)p^{-3s_2}}{\left(1-p^{1-3s_2}\right)\left(1-p^{2-3s_2-3s_1}\right)} \prod_{(p,N)=1} \frac{\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}{\left(1-p^{1-3s_2}\right)\left(1-p^{2-3s_2-3s_1}\right)}\sigma_{1-3s_1}\left(p^{\alpha_p}\right)$$

$$= \frac{N^3}{f^2}\sigma_{1-3s_1}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_{\frac{N}{fh}}}(3s_1+3s_2-2)\zeta(3s_2-1)L_{\chi_{\frac{N}{f}}}(3s_1)^{-1}L_{\chi_{fh}}(3s_2)^{-1}L_{\chi_N}(3s_1+3s_2-1)^{-1}$$

$$\prod_{p|h} \frac{(p-1)\left(1-p^{-3s_2}\right)\left(1-p^{2-3s_1-3s_2}\right) - p\left(1-p^{1-3s_1-3s_2}\right)\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2}\right)p^{(1-3s_1)\alpha_p}}{1-p^{1-3s_1}}$$

$$\prod_{p|f}(p-1)\left(\sigma_{1-3s_1}\left(p^{\alpha_p-1}\right)-\sigma_{1-3s_1}\left(p^{\alpha_p}\right)p^{-3s_2}\right) .$$

(2) First split the summation over c in this way $c = f(a,b)k + c$ with $0 \le k < \frac{N}{f}\frac{a}{(a,b)}$, $0 \le c < f(a,b)$, then split the summation over b in this way $b = de$ with $d \mid a$ and $\left(e, \frac{a}{d}\right) = 1$. We have

$$B_{m_2}(s_1,s_2,f,P_{min}) = \sum_{a\ge1} a^{-3s_1-3s_2} \sum_{\substack{d|a \\ \left(d,\frac{N}{f}\right)=1}} \sum_{\substack{0\le e<\frac{N}{f}\frac{a}{d} \\ \left(e,\frac{a}{d}\right)=1}} (a,de)^{3s_1} e\left(m_2\frac{de}{a}\right) \sum_{0\le k<\frac{N}{f}\frac{a}{(a,de)}} \sum_{\substack{c \bmod f(a,de) \\ (f(a,de),c)=1}} 1$$

$$= \frac{N}{f}\sum_{a\ge1} a^{1-3s_1-3s_2} \sum_{\substack{d|a \\ \left(d,\frac{N}{f}\right)=1}} d^{3s_1-1}\phi(fd) \sum_{\substack{0\le e<\frac{N}{f}\frac{a}{d} \\ \left(e,\frac{a}{d}\right)=1}} e\left(m_2\frac{e}{\frac{a}{d}}\right) .$$

Next split the summation over e in this way $e = k\frac{a}{d} + e$ with $0 \le k < \frac{N}{f}$, $0 \le e < \frac{a}{d}$. So we get

$$B_{m_2}(s_1,s_2,f,P_{min}) = \left(\frac{N}{f}\right)^2 \sum_{a\ge1} a^{1-3s_1-3s_2} \sum_{\substack{d|a \\ \left(d,\frac{N}{f}\right)=1}} d^{3s_1-1}\phi(fd)c_{\frac{a}{d}}(m_2) .$$

The next step is to factorize the Dirichlet series. In order to do this, decompose the summation in the outer and inner sum in the way $a = a_1a_2a_3$ and $d = d_1d_2d_3$ according to the divisibility conditions $(a_1d_1, N) = 1$, $a_2d_2 \mid \left(\frac{N}{f}\right)^\infty$ and $a_3d_3 \mid f^\infty$. Note that the gcd-condition $\left(d, \frac{N}{f}\right) = 1$ implies $d_2 = 1$. We obtain

$$B_{m_2}(s_1,s_2,f,P_{min}) = \left(\frac{N}{f}\right)^2 \sum_{(a_1,N)=1} \sum_{a_2|\left(\frac{N}{f}\right)^\infty} \sum_{a_3|f^\infty} (a_1a_2a_3)^{1-3s_1-3s_2}$$

$$\sum_{d_1|a_1} \sum_{d_3|a_3} (d_1d_3)^{3s_1-1}\phi(fd_1d_3)c_{\frac{a_1a_2a_3}{d_1d_3}}(m_2)$$

$$= \left(\frac{N}{f}\right)^2 \left[\sum_{(a_1,N)=1} a_1^{1-3s_1-3s_2} \sum_{d_1|a_1} d_1^{3s_1-1}\phi(d_1)c_{\frac{a_1}{d_1}}(m_2)\right]$$

$$\left[\sum_{a_2|\left(\frac{N}{f}\right)^\infty} a_2^{1-3s_1-3s_2}c_{a_2}(m_2)\right]\left[\sum_{a_3|f^\infty} a_3^{1-3s_1-3s_2} \sum_{d_3|a_3} d_3^{3s_1-1}\phi(fd_3)c_{\frac{a_3}{d_3}}(m_2)\right].$$

Split the Dirichlet convolutions in the first and third Dirichlet series and factorize both. This gives us

$$B_{m_2}(s_1,s_2,f,P_{min}) = \left(\frac{N}{f}\right)^2 \left[\sum_{(a,N)=1} a^{1-3s_1-3s_2}a^{3s_1-1}\phi(a)\right]$$

$$\left[\sum_{(a,N)=1} a^{1-3s_1-3s_2}c_a(m_2)\right]\left[\sum_{a|\left(\frac{N}{f}\right)^\infty} a^{1-3s_1-3s_2}c_a(m_2)\right]$$

$$\left[\sum_{a|f^\infty} a^{1-3s_1-3s_2}a^{3s_1-1}\phi(fa)\right]\left[\sum_{a|f^\infty} a^{1-3s_1-3s_2}c_a(m_2)\right]$$

$$= \left(\frac{N}{f}\right)^2 \left[\sum_{a\geq 1} a^{1-3s_1-3s_2}c_a(m_2)\right]\left[\sum_{a|f^\infty} a^{-3s_2}\phi(fa)\right]\left[\sum_{(a,N)=1} a^{-3s_2}\phi(a)\right].$$

Use Lemma 18.4 in Appendix A to evaluate the third Dirichlet series and Lemma 18.5 for the evaluation of the second Dirichlet series. We obtain

$$B_{m_2}(s_1,s_2,f,P_{min}) = \left(\frac{N}{f}\right)^2 \left[\sum_{a\geq 1} a^{1-3s_1-3s_2}c_a(m_2)\right] \prod_{p|f} \frac{p-1}{1-p^{1-3s_2}} \frac{L_{\chi_N}(3s_2-1)}{L_{\chi_N}(3s_2)}.$$

Finally use Lemma 18.4 in Appendix A to evaluate the first Dirichlet series distinguishing between the cases whether m_2 vanishes or not. We have

$$B_{m_2}(s_1,s_2,f,P_{min}) = \left(\frac{N}{f}\right)^2 \prod_{p|f}(p-1)\left[\sum_{a\geq 1} a^{1-3s_1-3s_2}c_a(m_2)\right] \frac{L_{\chi_{\frac{N}{f}}}(3s_2-1)}{L_{\chi_N}(3s_2)}$$

$$= \begin{cases} \left(\frac{N}{f}\right)^2 \prod_{p|f} (p-1)\, \sigma_{2-3s_1-3s_2}(m_2) \frac{L_{\chi_{\frac{N}{f}}}(3s_2-1)}{\zeta(3s_1+3s_2-1) L_{\chi_N}(3s_2)} & \text{, if } m_2 \neq 0 \text{ ,} \\ \left(\frac{N}{f}\right)^2 \prod_{p|f} (p-1)\, \frac{\zeta(3s_1+3s_2-2) L_{\chi_{\frac{N}{f}}}(3s_2-1)}{\zeta(3s_1+3s_2-1) L_{\chi_N}(3s_2)} & \text{, if } m_2 = 0 \text{ .} \end{cases}$$

(3) Split the summation over e in this way $e = dhk + l$ with $0 \leq k < Nb$, $0 \leq l < dh$ and note that the gcd-condition transforms as follows $1 = (dh, e) = (dh, l)$. Further split the summation over c in this way $c = fbj + i$ with $0 \leq j < \frac{N}{f}$, $0 \leq i < fb$ and note that the gcd-condition transforms as follows $1 = (fb, c) = (fb, i)$. We have

$$\begin{aligned} C_{m_1}(s_1, s_2, f, h, P_{min}) = & \sum_{\left(b, \frac{N}{f}\right)=1} b^{-3s_1-3s_2} \sum_{0 \leq j < \frac{N}{f}} \sum_{\substack{i \bmod fb \\ (fb,i)=1}} e\left(m_1 \frac{i}{fb}\right) \sum_{\left(d, \frac{N}{fh}\right)=1} d^{-3s_1} \sum_{0 \leq k < Nb} \sum_{\substack{l \bmod dh \\ (dh,l)=1}} 1 \\ = & \frac{N^2}{f} \left(\sum_{\left(b, \frac{N}{f}\right)=1} b^{1-3s_1-3s_2} c_{fb}(m_1) \right) \left(\sum_{\left(d, \frac{N}{fh}\right)=1} d^{-3s_1} \phi(dh) \right) . \end{aligned}$$

Split the summation in the first Dirichlet series $b = b_1 b_2$ with $(b_1, N) = 1$ and $b_2 \mid f^\infty$ and in the second Dirichlet series $d = d_1 d_2$ with $\left(d_1, \frac{N}{f}\right) = 1$ and $d_2 \mid h^\infty$. Then factorize the Dirichlet series. So we get

$$\begin{aligned} C_{m_1}(s_1, s_2, f, h, P_{min}) = & \frac{N^2}{f} \left(\sum_{(b,N)=1} b^{1-3s_1-3s_2} c_b(m_1) \right) \left(\sum_{b|f^\infty} b^{1-3s_1-3s_2} c_{fb}(m_1) \right) \\ & \left(\sum_{\left(d, \frac{N}{f}\right)=1} d^{-3s_1} \phi(d) \right) \left(\sum_{d|h^\infty} d^{-3s_1} \phi(dh) \right) . \end{aligned}$$

Use Lemma 18.4 in Appendix A to evaluate the third Dirichlet series and Lemma 18.5 for the fourth one. This gives us

$$\begin{aligned} & C_{m_1}(s_1, s_2, f, h, P_{min}) \\ = & \frac{N^2}{f} \left(\sum_{(b,N)=1} b^{1-3s_1-3s_2} c_b(m_1) \right) \left(\sum_{b|f^\infty} b^{1-3s_1-3s_2} c_{fb}(m_1) \right) \frac{L_{\chi_{\frac{N}{f}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)} \prod_{p|h} \frac{p-1}{1-p^{1-3s_1}} \\ = & \frac{N^2}{f} \prod_{p|h} (p-1) \left(\sum_{(b,N)=1} b^{1-3s_1-3s_2} c_b(m_1) \right) \left(\sum_{b|f^\infty} b^{1-3s_1-3s_2} c_{fb}(m_1) \right) \frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)} . \end{aligned}$$

Distinguish between the cases whether m_1 vanishes or not and use Lemma 18.4 in Appendix A to evaluate the first Dirichlet series and Lemma 18.5 for the second one. We obtain

$$\begin{aligned} C_{m_1}(s_1,s_2,f,h,P_{min}) =& \frac{N^2}{f}\prod_{p|h}(p-1)\prod_{p|f}\left((p-1)\sigma_{2-3s_1-3s_2}\left(p^{\alpha_p}\right)-p^{1+(2-3s_1-3s_2)\alpha_p}\right) \\ & \sigma_{2-3s_1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)\frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)} \end{aligned}$$

and

$$\begin{aligned} C_0(s_1,s_2,f,h,P_{min}) =& \frac{N^2}{f}\prod_{p|h}(p-1)\frac{L_{\chi_N}(3s_1+3s_2-2)}{L_{\chi_N}(3s_1+3s_2-1)}\prod_{p|f}\frac{p-1}{1-p^{2-3s_1-3s_2}}\frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)}{L_{\chi_{\frac{N}{f}}}(3s_1)} \\ =& \frac{N^2}{f}\prod_{p|hf}(p-1)\frac{L_{\chi_{\frac{N}{fh}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)}{L_{\chi_{\frac{N}{f}}}(3s_1)L_{\chi_N}(3s_1+3s_2-1)}\ . \end{aligned}$$

(4) Split the summation over c in this way $c=fbk+l$ with $0\le k<\frac{N}{f}$, $0\le l<fb$. We have

$$D_{m_1}(s_2,f,P_{min}) = \sum_{\substack{b\ge 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_2}\sum_{0\le k<\frac{N}{f}}\sum_{\substack{l \bmod fb\\ (fb,l)=1}} e\left(m_1\frac{l}{fb}\right) = \frac{N}{f}\sum_{\substack{b\ge 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_2}c_{fb}\left(m_1\right)\ .$$

Next split the summation over b in this way $b=b_1b_2$ with $(b_1,N)=1$ and $b_2\mid f^\infty$. We obtain

$$D_{m_1}(s_2,f,P_{min}) = \frac{N}{f}\left(\sum_{(b,N)=1} b^{-3s_2}c_b\left(m_1\right)\right)\left(\sum_{b|f^\infty} b^{-3s_2}c_{fb}\left(m_1\right)\right)\ .$$

Distinguish between the cases whether m_1 vanishes or not and use Lemma 18.4 in Appendix A to evaluate the first Dirichlet series and Lemma 18.5 for the second one. This gives us finally

$$D_{m_1}(s_2,f,P_{min}) = \frac{N}{f}\prod_{p|f}\left((p-1)\sigma_{1-3s_2}\left(p^{\alpha_p}\right)-p^{1+(1-3s_2)\alpha_p}\right)\sigma_{1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)L_{\chi_N}(3s_2)^{-1}$$

and

$$D_0(s_2, f, P_{min}) = \frac{N}{f} \frac{L_{\chi_N}(3s_2 - 1)}{L_{\chi_N}(3s_2)} \prod_{p|f} \frac{p-1}{1 - p^{1-3s_2}} = \frac{N}{f} \prod_{p|f} (p-1) \, \frac{L_{\chi_{\frac{N}{f}}}(3s_2 - 1)}{L_{\chi_N}(3s_2)} \ .$$

□

CHAPTER 17

Fourier expansion for the minimal Eisenstein series

In this chapter the Fourier expansion of the Eisenstein series $E(z, s_1, s_2, f, h, P_{min})$ is calculated. We begin with a lemma, which is needed to do the technical steps in the calculation of the Fourier coefficients.

LEMMA 17.1. *Let N be a squarefree positive integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. Let a, d, b, r, z be integers, where a and d are non-zero and these integers satisfy the gcd-conditions $\left(a, b, \frac{N}{f}\right) = 1$, $\left(d, \frac{N}{fh}\right) = 1$ and the equation $-r\frac{b}{(a,b)} = 1 + z\frac{a}{(a,b)}$. Then the map*

$$\begin{aligned}\phi : \left\{ (af, bf, c, dh, e, k, l) \in \mathbb{Z}^7 \middle| \substack{(af,bf,c)=1 \\ (dh,e)=1 \\ k,l\in\mathbb{Z} \\ c \bmod N(a,b) \\ e \bmod dhN(a,b)} \right\} &\longrightarrow \left\{ (af, bf, c, dh, e) \in \mathbb{Z}^5 \middle| \substack{(af,bf,c)=1 \\ (dh,e)=1} \right\} \\ (af, bf, c, dh, e, k, l) &\longmapsto (af, bf, Nak + Nbl + c, dh, dh(a, b)N(kr - lz) + e)\end{aligned}$$

is a bijection.

PROOF. (1) The first thing to do is to check that ϕ is welldefined. This is done quickly, since only two gcd conditions

$$(af, bf, Nak + Nbl + c) = (af, bf, c) = 1$$

and

$$(dh, dh(a, b)N(kr - lz) + e) = (dh, e) = 1$$

have to be verified.

(2) Next we show that ϕ is injective. In order to do this, suppose that two elements have the same image under ϕ, precisely:

$$\begin{aligned}&\phi((af, bf, c_1, dh, e_1, k_1, l_1)) = \phi((af, bf, c_2, dh, e_2, k_2, l_2)) \\ \Longleftrightarrow &(af, bf, Nak_1 + Nbl_1 + c_1, dh, dh(a, b)N(k_1r - l_1z) + e_1) \\ &\quad = (af, bf, Nak_2 + Nbl_2 + c_2, dh, dh(a, b)N(k_2r - l_2z) + e_2)\ .\end{aligned}$$

We start looking at the equation induced through comparison of the last entries. We have

$$dh(a,b)N(k_1r - l_1z) + e_1 = dh(a,b)N(k_2r - l_2z) + e_2 \; . \tag{17.1}$$

Reducing both sides in (17.1) modulo $dh(a,b)N$ gives $e_1 = e_2$. Since d is non-zero this implies the equation

$$\begin{aligned} & k_1r - l_1z = k_2r - l_2z \\ \Longleftrightarrow \quad & (k_1 - k_2)r = (l_1 - l_2)z \; . \end{aligned}$$

Since $(r,z) = 1$ we get $k_1 = k_2 + xz$ and $l_1 = l_2 + xr$ with a suitable integer x. In the last step we look at the equation induced through comparison of the entries at the third place. We have

$$\begin{aligned} & Nak_1 + Nbl_1 + c_1 = Nak_2 + Nbl_2 + c_2 \\ \Longleftrightarrow \quad & N(a,b)x\left(r\frac{b}{(a,b)} + z\frac{a}{(a,b)}\right) = c_2 - c_1 \\ \Longleftrightarrow \quad & -N(a,b)x = c_2 - c_1 \; . \end{aligned}$$

Reducing the last equation modulo $N(a,b)$ implies $c_1 = c_2$ and $x = 0$. So $k_1 = k_2$ and $l_1 = l_2$.

(3) It remains to show that ϕ is surjective. Let $(af, bf, c, dh, e) \in \mathbb{Z}^5$ be a row satisfying the gcd conditions $(af, bf, c) = 1$ and $(dh, e) = 1$. Split $e = dhN(a,b)x + e_1$ with e_1 mod $dhN(a,b)$, $x \in \mathbb{Z}$ and split $c = N(a,b)y + c_1$ with c_1 mod $N(a,b)$, $y \in \mathbb{Z}$. The above gcd conditions imply $(dh, e_1) = 1$ and $(af, bf, c_1) = 1$. To show that the row (af, bf, c, dh, e) lies in the image of ϕ, it is sufficient to solve the linear system in the integral variables k, l given by

$$\begin{aligned} & x = kr - lz \quad \wedge \quad y = \frac{a}{(a,b)}k + \frac{b}{(a,b)}l \\ \Longleftrightarrow \quad & \begin{pmatrix} r & -z \\ \frac{a}{(a,b)} & \frac{b}{(a,b)} \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \; . \end{aligned}$$

Since $\det\left(\begin{pmatrix} r & -z \\ \frac{a}{(a,b)} & \frac{b}{(a,b)} \end{pmatrix}\right) = r\frac{b}{(a,b)} + z\frac{a}{(a,b)} = -1$ this system has an integer solution (k_1, l_1). So $(af, bf, c_1, k_1, l_1, dh, e_1)$ is a preimage of (af, bf, c, dh, e) under the map ϕ.

□

After these preparations we can state the main theorem concerning the Fourier expansion of the completed minimal Eisenstein series.

THEOREM 17.2. *Let N be a squarefree positive integer, f a positive divisor of N and h a positive divisor of $\frac{N}{f}$. The completed minimal Eisenstein series $G(z,s_1,s_2,f,h,P_{min})$ has the explicit Fourier expansion*

$$G(z,s_1,s_2,f,h,P_{min}) = \sum_{m_2=0}^{\infty} G_{0,m_2}(z,s_1,s_2,f,h,P_{min}) + \sum_{\gamma\in P_{min}\backslash GL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty}\sum_{m_2=0}^{\infty} G_{m_1,m_2}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z, s_1,s_2,f,h,P_{min}\right).$$

Let m_1, m_2 be positive integers with prime decompositions $m_1 = \prod_p p^{\alpha_p}$ and $m_2 = \prod_p p^{\beta_p}$. Then the following explicit formulas for the Fourier coefficients are valid.

(1) In the non-degenerate case we have

$$G_{m_1,m_2}(z,s_1,s_2,f,h,P_{min}) = \prod_{(p,N)=1} \mathfrak{A}_p(s_1,s_2,\alpha_p,\beta_p)\,(m_1m_2)^{-1}\, W_{1,1}^{(s_2,s_1)}\left(\begin{pmatrix} m_1m_2 & & \\ & m_1 & \\ & & 1\end{pmatrix} z, w_1\right)$$
$$h^{-3s_1}\prod_{p|h}\frac{(p-1)\left(1-p^{-3s_2}\right)\sigma_{1-3s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2-3s_1}\right)p^{(1-3s_1)\alpha_p}\sigma_{2-3s_1-3s_2}\left(p^{\beta_p}\right)}{\left(1-p^{1-3s_1}\right)\left(1-p^{-3s_1}\right)\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$
$$\prod_{p|h} p^{(2s_1+s_2-1)\alpha_p} p^{(s_1+2s_2-1)\beta_p}$$
$$f^{1-3s_1-3s_2}\prod_{p|f}\frac{G_p\left(s_1,s_2,\alpha_p,p^{\beta_p}\right)p^{(2s_1+s_2-1)\alpha_p}p^{(s_1+2s_2-1)\beta_p}}{\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}\prod_{p|\frac{N}{fh}}\frac{p^{(2s_1+s_2-1)\alpha_p}p^{(s_1+2s_2-1)\beta_p}\sigma_{1-3s_2}\left(p^{\beta_p}\right)}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}.$$

(2) In the first partially degenerate case we have

$$G_{m_1,0}(z,s_1,s_2,f,h,P_{min})$$
$$= h^{-3s_1}\prod_{p|h}\frac{(p-1)\left(1-p^{-3s_2}\right)\left(1-p^{2-3s_1-3s_2}\right) - p\left(1-p^{1-3s_1-3s_2}\right)\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2}\right)p^{(1-3s_1)\alpha_p}}{\left(1-p^{1-3s_1}\right)\left(1-p^{-3s_1}\right)\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$
$$f^{1-3s_1-3s_2}\prod_{p|f}(p-1)\frac{\sigma_{1-3s_1}\left(p^{\alpha_p-1}\right)-\sigma_{1-3s_1}\left(p^{\alpha_p}\right)p^{-3s_2}}{\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}\prod_{p|\frac{N}{fh}}\frac{1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$
$$L_{\chi_{\frac{N}{fh}}}(3s_1+3s_2-2)\zeta(3s_2-1)\sigma_{1-3s_1}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)W_{m_1,0}^{(s_2,s_1)}(z,w_1)$$

$$+f^{1-3s_1-3s_2}\prod_{p|f}\frac{(p-1)\sigma_{2-3s_1-3s_2}(p^{\alpha_p})-p^{1+(2-3s_1-3s_2)\alpha_p}}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{(1-p^{-3s_1})(1-p^{-3s_1-3s_2+1})}$$

$$h^{-3s_1}\prod_{p|h}(p-1)\,\sigma_{2-3s_1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)L_{\chi_{\frac{N}{fh}}}(3s_1-1)\zeta(3s_2)\,W^{(s_2,s_1)}_{m_1,0}(z,w_5)$$

$$+\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\prod_{p|f}\frac{(p-1)\sigma_{1-3s_2}(p^{\alpha_p})-p^{1+(1-3s_2)\alpha_p}}{1-p^{-3s_2}}$$

$$\sigma_{1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)\zeta(3s_1)\,\zeta(3s_1+3s_2-1)\,W^{(s_2,s_1)}_{m_1,0}(z,w_3)\ .$$

(3) In the second partially degenerate case we have

$$G_{0,m_2}(z,s_1,s_2,f,h,P_{min})$$

$$=h^{-3s_1}\prod_{p|h}\frac{p-1}{(1-p^{-3s_1})(1-p^{1-3s_1-3s_2})}f^{1-3s_1-3s_2}\prod_{p|f}\frac{p-1}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{fh}}\frac{1}{(1-p^{-3s_1})(1-p^{1-3s_1-3s_2})}$$

$$L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)\sigma_{1-3s_2}(m_2)W^{(s_2,s_1)}_{0,m_2}(z,w_1)$$

$$+\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p|f}\frac{p-1}{1-p^{-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\sigma_{2-3s_1-3s_2}(m_2)\zeta(3s_1)\,L_{\chi_{\frac{N}{f}}}(3s_2-1)W^{(s_2,s_1)}_{0,m_2}(z,w_4)$$

$$+\delta_{f,N}\zeta(3s_1)\,\zeta(3s_1+3s_2-1)\,\sigma_{1-3s_2}(m_2)W^{(s_2,s_1)}_{0,m_2}(z,w_2)\ .$$

(4) In the totally degenerate case we have

$$G_{0,0}(z,s_1,s_2,f,h,P_{min})$$

$$=h^{-3s_1}\prod_{p|h}\frac{p-1}{(1-p^{-3s_1})(1-p^{1-3s_1-3s_2})}f^{1-3s_1-3s_2}\prod_{p|f}\frac{p-1}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{fh}}\frac{1}{(1-p^{-3s_1})(1-p^{1-3s_1-3s_2})}$$

$$\zeta(3s_2-1)L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)W^{(s_2,s_1)}_{0,0}(z,w_1)$$

$$+\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p|f}\frac{p-1}{1-p^{-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\zeta(3s_1)\,\zeta(3s_1+3s_2-2)L_{\chi_{\frac{N}{f}}}(3s_2-1)W^{(s_2,s_1)}_{0,m_2}(z,w_4)$$

$$+f^{1-3s_1-3s_2}h^{-3s_1}\prod_{p|h}(p-1)\prod_{p|f}\frac{p-1}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

$$\zeta\left(3s_2\right)L_{\chi_{\frac{N}{fh}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)W_{0,0}^{(s_2,s_1)}(z,w_5)$$

$$+\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p|f}\frac{p-1}{1-p^{-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\zeta\left(3s_1\right)L_{\chi_{\frac{N}{f}}}(3s_2-1)\zeta\left(3s_1+3s_2-1\right)W_{0,0}^{(s_2,s_1)}(z,w_3)$$

$$+\delta_{f,N}\zeta\left(3s_1\right)\zeta\left(3s_2-1\right)\zeta\left(3s_1+3s_2-1\right)W_{0,0}^{(s_2,s_1)}(z,w_2)$$

$$+\delta_{f,N}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)W_{0,0}^{(s_2,s_1)}(z,w_0)\ .$$

Proof. Again without loss of generality it can be assumed that $z=\begin{pmatrix}y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\in\mathfrak{h}^3$. We start the calculation of the Fourier coefficients dividing the summation in the Eisenstein series into several cases according to the Bruhat decomposition in Definition 3.8. Assume for the present that m_1, m_2 are arbitrary integers, then we have

$$\begin{aligned}
&G_{m_1,m_2}\left(z,s_1,s_2,f,h,P_{min}\right)\\
=&\int_0^1\int_0^1\int_0^1 G\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}z,s_1,s_2,f,h,P_{min}\right)e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\left(\frac{N}{f}\right)^{-3}\sum_{k_1,k_2,k_3=0}^{\frac{N}{f}-1}\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_{k_1}^{k_1+1}G\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}z,s_1,s_2,f,h,P_{min}\right)\\
&e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\left(\frac{N}{f}\right)^{-3}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}G\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}z,s_1,s_2,f,h,P_{min}\right)e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\sum_{i=1}^{6}\sum_{\gamma\in\Gamma_i(f,h,P_{min})}\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\\
&\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}I_{(s_1,s_2)}\left(\gamma z\right)e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3
\end{aligned}$$

$$=:\sum_{i=1}^{6} G^{(i)}_{m_1,m_2}(z,s_1,s_2)\ .$$

With the same argument as in Theorem 14.3 we applied Lemma 14.2 to all three integrals. Using the explicit description of the sets $\Gamma_i(f,h,P_{min})$ in Lemma 6.1 and the explicit formula for the values of the $I_{(s_1,s_2)}$-function in Lemma 8.6, we start calculating each of the six summands above.

(1) We first handle the most difficult term $G^{(1)}_{m_1,m_2}(z,s_1,s_2)$. We have

$$\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&\sum_{a\neq 0}\ \sum_{\substack{b\in\mathbb{Z}\\ \left(a,b,\frac{N}{f}\right)=1}}\ \sum_{\substack{c\in\mathbb{Z}\\ (f(a,b),c)=1}}\ \sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}\ \sum_{\substack{e\in\mathbb{Z}\\ (e,dh)=1}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\left(\frac{|a|}{h\,|d|\,(a,b)}\right)^{3s_1} y_1^{2s_2+s_1}y_2^{2s_1+s_2}\\
&\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\\
&\Bigg[f^2a^2y_1^2y_2^2+a^2y_2^2\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)^2\\
+&\left((fax_3+fbx_1+c)-\left(fx_1-\frac{rc}{(a,b)}+\frac{ea}{hd(a,b)^2}\right)(ax_2+b)\right)^2\Bigg]^{-\frac{3s_1}{2}} e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .
\end{aligned}$$

Using the definition of the integer r in Lemma 3.1 one can choose an integer z such that $-r\frac{b}{(a,b)}=1+z\frac{a}{(a,b)}$ and apply Lemma 17.1. This gives us

$$\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&y_1^{2s_2+s_1}y_2^{2s_1+s_2}h^{-3s_1}\sum_{a\neq 0}|a|^{3s_1}\sum_{\substack{b\in\mathbb{Z}\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{-3s_1}\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}}\ \sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}|d|^{-3s_1}\sum_{\substack{e \bmod dhN(a,b)\\ (e,dh)=1}}\ \sum_{k,l\in\mathbb{Z}}\\
&\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+(Nak+Nbl+c))^2\right]^{-\frac{3s_2}{2}}\\
&\Bigg[f^2a^2y_1^2y_2^2+a^2y_2^2\left(fx_1-\frac{r(Nak+Nbl+c)}{(a,b)}+\frac{(dhN(a,b)(kr-lz)+e)a}{hd(a,b)^2}\right)^2\\
+&((fax_3+fbx_1+(Nak+Nbl+c))-\left(fx_1-\frac{r(Nak+Nbl+c)}{(a,b)}+\frac{(dhN(a,b)(kr-lz)+e)a}{hd(a,b)^2}\right)
\end{aligned}$$

$$(ax_2+b))^2\Big]^{-\frac{3s_1}{2}} e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$=\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)$$
$$y_1^{2s_2+s_1}y_2^{2s_1+s_2}h^{-3s_1}\sum_{a\neq 0}|a|^{3s_1}\sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}|d|^{-3s_1}\sum_{\substack{b\in\mathbb{Z}\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{-3s_1}\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}}\sum_{\substack{e \bmod dhN(a,b)\\ (e,dh)=1}}\sum_{k,l\in\mathbb{Z}}$$
$$\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+\left(fa\left(x_3+\frac{N}{f}k\right)+fb\left(x_1+\frac{N}{f}l\right)+c\right)^2\right]^{-\frac{3s_2}{2}}$$
$$\left[f^2a^2y_1^2y_2^2+a^2y_2^2\left(f\left(x_1+\frac{N}{f}l\right)+\frac{ae}{dh(a,b)^2}-\frac{rc}{(a,b)}\right)^2\right.$$
$$\left.+\left(\left(fa\left(x_3+\frac{N}{f}k\right)+fb\left(x_1+\frac{N}{f}l\right)+c\right)-\left(f\left(x_1+\frac{N}{f}l\right)+\frac{ae}{dh(a,b)^2}-\frac{rc}{(a,b)}\right)(ax_2+b)\right)^2\right]^{-\frac{3s_1}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

First pass to infinite integrals in the variables x_1, x_3 by the shifts $x_1\to x_1+\frac{N}{f}l$, $x_3\to x_3+\frac{N}{f}k$, then shift $x_3\to x_3+\frac{b}{a}x_1+\frac{c}{fa}$. After that split the summation over b in this way $b=k\frac{N}{f}a+b$ with $k\in\mathbb{Z}$, $0\le b<\frac{N}{f}a$ and note that the gcd-condition transform as follows $1=(a,b)=\left(a,k\frac{N}{f}a+b\right)=(a,b)$. We obtain

$$G^{(1)}_{m_1,m_2}(z,s_1,s_2)$$
$$=\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)$$
$$y_1^{2s_2+s_1}y_2^{2s_1+s_2}h^{-3s_1}\sum_{a\neq 0}|a|^{3s_1}\sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}|d|^{-3s_1}\sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{-3s_1}\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}}\sum_{\substack{e \bmod dhN(a,b)\\ (e,dh)=1}}\sum_{k\in\mathbb{Z}}$$
$$\int_{-\infty}^{\infty}\int_0^{\frac{N}{f}}\int_{-\infty}^{\infty}\left[f^2a^2y_1^2y_2^2+f^2y_1^2\left(a\left(x_2+\frac{N}{f}k\right)+b\right)^2+f^2a^2x_3^2\right]^{-\frac{3s_2}{2}}$$
$$\left[f^2a^2y_1^2y_2^2+a^2y_2^2\left(fx_1+\frac{ae}{dh(a,b)^2}-\frac{rc}{(a,b)}\right)^2\right.$$
$$\left.+\left(fax_3-\left(fx_1+\frac{ae}{dh(a,b)^2}-\frac{rc}{(a,b)}\right)\left(a\left(x_2+\frac{N}{f}k\right)+b\right)\right)^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Pass to an infinite integral in the variable x_2 by the shift $x_2 \to x_2 + \frac{N}{f}k$, then shift $x_2 \to x_2 + \frac{b}{a}$ picking up the exponential $e\left(m_2\frac{b}{a}\right)$. After that shift $x_1 \to x_1 + \frac{ae}{dhf(a,b)^2} - \frac{rc}{f(a,b)}$ picking up the exponential $e\left(m_1\frac{ae}{dhf(a,b)^2} - m_1\frac{rc}{f(a,b)}\right)$. We have

$$\begin{aligned}
&G^{(1)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&y_1^{2s_2+s_1}y_2^{2s_1+s_2}h^{-3s_1}\sum_{a\neq 0}|a|^{3s_1}\sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}|d|^{-3s_1}\sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{-3s_1}e\left(m_2\frac{b}{a}\right)\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}}\\
&\sum_{\substack{e \bmod dhN(a,b)\\ (e,dh)=1}}e\left(m_1\frac{ae}{dhf(a,b)^2}-m_1\frac{rc}{f(a,b)}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f^2a^2y_1^2y_2^2+f^2a^2y_1^2x_2^2+f^2a^2x_3^2\right]^{-\frac{3s_2}{2}}\\
&\left[f^2a^2y_1^2y_2^2+f^2a^2y_2^2x_1^2+(fax_3-fax_1x_2)^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .
\end{aligned}$$

Finally we do some additional simplifications. Pass in the summation over a, d to positive values collecting two factors 2, note that this is possible since the minus sign can be absorbed in the other summations. Next absorb $-r$ into the summation over c, note that this is also possible since Lemma 3.2 gives that r is coprime to $N(a,b)$. Use the formula in [**4**, (3.11)] and the notation in Definition 16.4 to extract the Whittaker function and the Dirichlet series $A_{m_1,m_2}(s_1,s_2,f,h,P_{min})$. So we get

$$\begin{aligned}
G^{(1)}_{m_1,m_2}(z,s_1,s_2) =& \left(\frac{N}{f}\right)^{-3}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)h^{-3s_1}f^{-3s_1-3s_2}\\
&\sum_{a\geq 1}a^{-3s_2}\sum_{\substack{d\geq 1\\ \left(d,\frac{N}{fh}\right)=1}}d^{-3s_1}\sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{-3s_1}e\left(m_2\frac{b}{a}\right)\sum_{\substack{c \bmod N(a,b)\\ (f(a,b),c)=1}}e\left(m_1\frac{c}{f(a,b)}\right)\\
&\sum_{\substack{e \bmod dh(a,b)N\\ (e,dh)=1}}e\left(m_1\frac{ae}{dhf(a,b)^2}\right)W^{(s_2,s_1)}_{m_1,m_2}(z,w_1)\\
=&\left(\frac{N}{f}\right)^{-3}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)h^{-3s_1}f^{-3s_1-3s_2}\\
&A_{m_1,m_2}\left(s_1,s_2,f,h,P_{min}\right)W^{(s_2,s_1)}_{m_1,m_2}(z,w_1)\ .
\end{aligned}$$

Finally use Lemma 16.5 and evaluate the Dirichlet series $A_{m_1,m_2}\left(s_1,s_2,f,h,P_{min}\right)$. Further use the transformation law for Jacquet's Whittaker function in [**4**, (3.16)] in the non degenerate case. So we obtain the formulas

$$G^{(1)}_{m_1,m_2}(z,s_1,s_2) = \prod_{(p,N)=1} \mathfrak{A}_p(s_1,s_2,\alpha_p,\beta_p)\,(m_1m_2)^{-1}\, W^{(s_2,s_1)}_{1,1}\left(\begin{pmatrix} m_1m_2 & & \\ & m_1 & \\ & & 1\end{pmatrix} z, w_1\right)$$

$$h^{-3s_1}\prod_{p|h}\frac{(p-1)\left(1-p^{-3s_2}\right)\sigma_{1-3s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2-3s_1}\right)p^{(1-3s_1)\alpha_p}\sigma_{2-3s_1-3s_2}\left(p^{\beta_p}\right)}{\left(1-p^{1-3s_1}\right)\left(1-p^{-3s_1}\right)\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

$$\prod_{p|h} p^{(2s_1+s_2-1)\alpha_p}p^{(s_1+2s_2-1)\beta_p}$$

$$f^{1-3s_1-3s_2}\prod_{p|f}\frac{G_p\left(s_1,s_2,\alpha_p,p^{\beta_p}\right)p^{(2s_1+s_2-1)\alpha_p}p^{(s_1+2s_2-1)\beta_p}}{\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}\prod_{p|\frac{N}{fh}}\frac{p^{(2s_1+s_2-1)\alpha_p}p^{(s_1+2s_2-1)\beta_p}\sigma_{1-3s_2}\left(p^{\beta_p}\right)}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

and

$$G^{(1)}_{m_1,0}(z,s_1,s_2) = L_{\chi_{\frac{N}{fh}}}(3s_1+3s_2-2)\zeta(3s_2-1)\sigma_{1-3s_1}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)W^{(s_2,s_1)}_{m_1,0}(z,w_1)$$

$$h^{-3s_1}\prod_{p|h}\frac{(p-1)\left(1-p^{-3s_2}\right)\left(1-p^{2-3s_1-3s_2}\right) - p\left(1-p^{1-3s_1-3s_2}\right)\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_2}\right)p^{(1-3s_1)\alpha_p}}{\left(1-p^{1-3s_1}\right)\left(1-p^{-3s_1}\right)\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

$$f^{1-3s_1-3s_2}\prod_{p|f}(p-1)\frac{\sigma_{1-3s_1}\left(p^{\alpha_p-1}\right)-\sigma_{1-3s_1}\left(p^{\alpha_p}\right)p^{-3s_2}}{\left(1-p^{-3s_2}\right)\left(1-p^{1-3s_1-3s_2}\right)}\prod_{p|\frac{N}{fh}}\frac{1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

and

$$G^{(1)}_{0,m_2}(z,s_1,s_2) = L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)\sigma_{1-3s_2}(m_2)W^{(s_2,s_1)}_{0,m_2}(z,w_1)$$

$$h^{-3s_1}\prod_{p|h}\frac{p-1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}f^{1-3s_1-3s_2}\prod_{p|f}\frac{p-1}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{fh}}\frac{1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}$$

and

$$G^{(1)}_{0,0}(z,s_1,s_2) = \zeta(3s_2-1)L_{\chi_{\frac{N}{hf}}}(3s_1-1)L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2)W^{(s_2,s_1)}_{0,0}(z,w_1)$$

$$h^{-3s_1}\prod_{p|h}\frac{p-1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}f^{1-3s_1-3s_2}\prod_{p|f}\frac{p-1}{1-p^{1-3s_1-3s_2}}\prod_{p|\frac{N}{fh}}\frac{1}{\left(1-p^{-3s_1}\right)\left(1-p^{1-3s_1-3s_2}\right)}\,.$$

(2) We continue with the calculation of $G^{(2)}_{m_1,m_2}(z,s_1,s_2)$. Since many steps are similar to the calculations done for the other Eisenstein series we abbreviate some arguments. Note that we collect an additional factor 2 caused by the sign in $\tau_{(0,\pm1)}$. We have

$$G^{(2)}_{m_1,m_2}(z,s_1,s_2) = 2\frac{1}{4}\delta_{h,\frac{N}{f}}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$
$$\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\sum_{a\neq 0}\sum_{\substack{b\in\mathbb{Z}\\ \left(a,b,\frac{N}{f}\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\ (f(a,b),c)=1}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}(a,b)^{3s_1}y_1^{2s_2+s_1}y_2^{2s_1+s_2}$$
$$\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+(fax_3+fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\left[a^2y_2^2+(ax_2+b)^2\right]^{-\frac{3s_1}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Split the summation over c in this way $c = kNa + c$ with $k \in \mathbb{Z}$, $0 \leq c < Na$ and note, that the corresponding gcd-condition transforms in the right way. Then pass to an infinite integral in the variable x_3 and shift $x_3 \to x_3 + \frac{fbx_1+c}{fa}$. We obtain

$$G^{(2)}_{m_1,m_2}(z,s_1,s_2) = \frac{1}{2}\delta_{h,\frac{N}{f}}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$
$$\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)y_1^{2s_2+s_1}y_2^{2s_1+s_2}\sum_{a\neq 0}\sum_{\substack{b\in\mathbb{Z}\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{3s_1}\sum_{\substack{c \bmod Na\\ (f(a,b),c)=1}}$$
$$\int_{-\infty}^{\infty}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\left[f^2a^2y_1^2y_2^2+f^2y_1^2(ax_2+b)^2+f^2a^2x_3^2\right]^{-\frac{3s_2}{2}}\left[a^2y_2^2+(ax_2+b)^2\right]^{-\frac{3s_1}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Split the summation over b in this way $b = k\frac{N}{f}a + b$ with $k \in \mathbb{Z}$, $0 \leq b < \frac{N}{f}a$ and note that the corresponding gcd-condition holds again for b. Then pass to an infinite integral in the variable x_2 and shift $x_2 \to x_2 + \frac{b}{a}$, picking up an exponential $e\left(m_2\frac{b}{a}\right)$. So we get

$$G^{(2)}_{m_1,m_2}(z,s_1,s_2) = \frac{1}{2}\delta_{h,\frac{N}{f}}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$
$$\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)y_1^{2s_2+s_1}y_2^{2s_1+s_2}\sum_{a\neq 0}\sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{3s_1}e\left(m_2\frac{b}{a}\right)\sum_{\substack{c \bmod Na\\ (f(a,b),c)=1}}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\frac{N}{f}}\left[f^2a^2y_1^2y_2^2+f^2a^2y_1^2x_2^2+f^2a^2x_3^2\right]^{-\frac{3s_2}{2}}\left[a^2y_2^2+a^2x_2^2\right]^{-\frac{3s_1}{2}}$$
$$e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .$$

Extract and evaluate the exponential integral $\int_0^{\frac{N}{f}} e(-m_1x_1)\mathrm{d}x_1 = \frac{N}{f}\delta_{m_1,0}$. Pass in the summation over a to positive values collecting a factor 2, note that this is possible since the minus sign can be absorbed in the other summations. Use the formula in [**4**, (3.14)] and the notation in Definition 16.4 to extract the Whittaker function and the Dirichlet series $B_{m_2}(s_1, s_2, f, P_{min})$. This gives us

$$\begin{aligned}
&G^{(2)}_{m_1,m_2}(z, s_1, s_2) = \delta_{h,\frac{N}{f}}\delta_{m_1,0}\left(\frac{N}{f}\right)^{-2}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&f^{-3s_2}\sum_{a\geq 1}a^{-3s_1-3s_2}\sum_{\substack{b \bmod \frac{N}{f}a\\ \left(a,b,\frac{N}{f}\right)=1}}(a,b)^{3s_1}e\left(m_2\frac{b}{a}\right)\sum_{\substack{c \bmod Na\\ (f(a,b),c)=1}}W^{(s_2,s_1)}_{0,m_2}(z,w_4)\\
=&\delta_{h,\frac{N}{f}}\delta_{m_1,0}\left(\frac{N}{f}\right)^{-2}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)f^{-3s_2}B_{m_2}(s_1,s_2,f,P_{min})W^{(s_2,s_1)}_{0,m_2}(z,w_4)\ .
\end{aligned}$$

Finally use Lemma 16.5 and evaluate the Dirichlet series $B_{m_2}(s_1, s_2, f, P_{min})$. We have

$$\begin{aligned}
G^{(2)}_{m_1,m_2}(z, s_1, s_2) =&\delta_{h,\frac{N}{f}}\delta_{m_1,0}f^{-3s_2}\prod_{p|f}\frac{p-1}{1-p^{-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\\
&\sigma_{2-3s_1-3s_2}(m_2)\zeta\left(3s_1\right)L_{\chi_{\frac{N}{f}}}\left(3s_2-1\right)W^{(s_2,s_1)}_{0,m_2}(z,w_4)
\end{aligned}$$

and

$$\begin{aligned}
G^{(2)}_{m_1,0}(z, s_1, s_2) =&\delta_{h,\frac{N}{f}}\delta_{m_1,0}f^{-3s_2}\prod_{p|f}\frac{p-1}{1-p^{-3s_2}}\prod_{p|\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\\
&\zeta\left(3s_1\right)\zeta(3s_1+3s_2-2)L_{\chi_{\frac{N}{f}}}\left(3s_2-1\right)W^{(s_2,s_1)}_{0,m_2}(z,w_4)\ .
\end{aligned}$$

(3) Next we handle the calculation of $G^{(3)}_{m_1,m_2}(z, s_1, s_2)$. We have

$$\begin{aligned}
&G^{(3)}_{m_1,m_2}(z, s_1, s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&\sum_{\substack{b\neq 0\\ \left(b,\frac{N}{f}\right)=1}}\sum_{\substack{c\in\mathbb{Z}\\ (fb,c)=1}}\sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}\sum_{\substack{e\in\mathbb{Z}\\ (dh,e)=1}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}y_1^{2s_2+s_1}y_2^{2s_1+s_2}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\\
&\left[h^2f^2d^2b^2y_1^2y_2^2+h^2d^2y_2^2(fbx_1+c)^2+(e-hfdbx_3+hd(fbx_1+c)x_2)^2\right]^{-\frac{3s_1}{2}}\\
&e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .
\end{aligned}$$

Split the summation over e in this way $e = dhNbk + e$ with $k \in \mathbb{Z}$, $0 \leq e < dhNb$ and note that the corresponding gcd-condition holds again for e. Then pass to an infinite integral in the variable x_3 and shift $x_3 \to x_3 - \frac{e+dhx_2(fbx_1+c)}{dhfb}$. We obtain

$$
\begin{aligned}
&G^{(3)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)\\
&y_1^{2s_2+s_1}y_2^{2s_1+s_2}\sum_{\substack{b\neq 0\\ \left(b,\frac{N}{f}\right)=1}}\ \sum_{\substack{c\in\mathbb{Z}\\ (fb,c)=1}}\ \sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}\ \sum_{\substack{e \bmod dhNb\\ (dh,e)=1}}\int_{-\infty}^{\infty}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\left[f^2b^2y_1^2+(fbx_1+c)^2\right]^{-\frac{3s_2}{2}}\\
&\left[h^2f^2d^2b^2y_1^2y_2^2+h^2d^2y_2^2(fbx_1+c)^2+h^2f^2d^2b^2x_3^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,.
\end{aligned}
$$

Split the summation over c in this way $c = Nbk + c$ with $k \in \mathbb{Z}$, $0 \leq c < Nb$ and note that the corresponding gcd-condition holds again for c. Then pass to an infinite integral in the variable x_1 and shift $x_1 \to x_1 + \frac{c}{fb}$, picking up an exponential $e\left(m_1\frac{c}{fb}\right)$. This gives us

$$
\begin{aligned}
&G^{(3)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)\\
&y_1^{2s_2+s_1}y_2^{2s_1+s_2}\sum_{\substack{b\neq 0\\ \left(b,\frac{N}{f}\right)=1}}\ \sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right)\sum_{\substack{d\neq 0\\ \left(d,\frac{N}{fh}\right)=1}}\ \sum_{\substack{e \bmod dhNb\\ (dh,e)=1}}\int_{-\infty}^{\infty}\int_0^{\frac{N}{f}}\int_{-\infty}^{\infty}\left[f^2b^2y_1^2+f^2b^2x_1^2\right]^{-\frac{3s_2}{2}}\\
&\left[h^2f^2d^2b^2y_1^2y_2^2+h^2d^2f^2b^2x_1^2y_2^2+h^2f^2d^2b^2x_3^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,.
\end{aligned}
$$

Extract and evaluate the exponential integral $\int_0^{\frac{N}{f}} e(-m_2x_2)\mathrm{d}x_2 = \frac{N}{f}\delta_{m_2,0}$ and pass to positive values in the summation over b, d, collecting two factors 2. Use the formula in [**4**, (3.15)] to extract the Whittaker function and the notation in Definition 16.4 for the Dirichlet series $C_{m_1}(s_1, s_2, f, h, P_{min})$, note that here we have $x_3 = \xi_4$. So we get

$$
\begin{aligned}
&G^{(3)}_{m_1,m_2}(z,s_1,s_2) = \left(\frac{N}{f}\right)^{-2}\delta_{m_2,0}\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)\\
&f^{-3s_1-3s_2}h^{-3s_1}\sum_{\substack{b\geq 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_1-3s_2}\sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right)\sum_{\substack{d\geq 1\\ \left(d,\frac{N}{fh}\right)=1}} d^{-3s_1}\sum_{\substack{e \bmod dhNb\\ (dh,e)=1}} W^{(s_2,s_1)}_{m_1,0}(z,w_5)\\
=&\left(\frac{N}{f}\right)^{-2}\delta_{m_2,0}\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)f^{-3s_1-3s_2}h^{-3s_1}C_{m_1}(s_1,s_2,f,h,P_{min})W^{(s_2,s_1)}_{m_1,0}(z,w_5)\,.
\end{aligned}
$$

Finally use Lemma 16.5 and evaluate the Dirichlet series $C_{m_1}(s_1, s_2, f, h, P_{min})$. So the formulas

$$G^{(3)}_{m_1,m_2}(z, s_1, s_2) = \delta_{m_2,0} f^{1-3s_1-3s_2} h^{-3s_1} \prod_{p|h} (p-1) \prod_{p|\frac{N}{f}} \frac{1}{(1-p^{-3s_1})(1-p^{-3s_1-3s_2+1})}$$
$$\prod_{p|f} \frac{(p-1)\sigma_{2-3s_1-3s_2}(p^{\alpha_p}) - p^{1+(2-3s_1-3s_2)\alpha_p}}{1-p^{1-3s_1-3s_2}}$$
$$\sigma_{2-3s_1-3s_2}\left(\prod_{(p,N)=1} p^{\alpha_p}\right) L_{\chi_{\frac{N}{fh}}}(3s_1-1)\zeta(3s_2) W^{(s_2,s_1)}_{m_1,0}(z, w_5)$$

and

$$G^{(3)}_{0,m_2}(z, s_1, s_2) = \delta_{m_2,0} f^{1-3s_1-3s_2} h^{-3s_1} \prod_{p|h} (p-1) \prod_{p|f} \frac{p-1}{1-p^{1-3s_1-3s_2}} \prod_{p|\frac{N}{f}} \frac{1}{(1-p^{-3s_1})(1-p^{1-3s_1-3s_2})}$$
$$\zeta(3s_2) L_{\chi_{\frac{N}{fh}}}(3s_1-1) L_{\chi_{\frac{N}{f}}}(3s_1+3s_2-2) W^{(s_2,s_1)}_{0,0}(z, w_5)$$

hold.

(4) Next we handle the calculation of $G^{(4)}_{m_1,m_2}(z, s_1, s_2)$. Note that we collect an additional factor 2 caused by the sign in $\tau_{(0,\pm 1)}$. We have

$$G^{(4)}_{m_1,m_2}(z, s_1, s_2) = 2\frac{1}{4}\left(\frac{N}{f}\right)^{-3} \delta_{h,\frac{N}{f}} \pi^{-3s_1-3s_2+\frac{1}{2}} \Gamma\left(\frac{3s_1}{2}\right) \Gamma\left(\frac{3s_2}{2}\right) \Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$
$$\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1) \sum_{\substack{b\neq 0 \\ \left(b,\frac{N}{f}\right)=1}} \sum_{\substack{c\in\mathbb{Z} \\ (fb,c)=1}}$$
$$\int_0^{\frac{N}{f}} \int_0^{\frac{N}{f}} \int_0^{\frac{N}{f}} y_1^{s_1+2s_2} y_2^{s_2+2s_1} \left[f^2b^2y_1^2 + (fbx_1+c)^2\right]^{-\frac{3s_2}{2}} e(-m_1x_1 - m_2x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, .$$

Pass in the summation over b to positive values collecting a factor 2. Split the summation over c in this way $c = Nbk + c$ with $k \in \mathbb{Z}$, $0 \leq c < Nb$ and note that the corresponding gcd-condition holds again for c. Then pass to an infinite integral in the variable x_1 and shift $x_1 \to x_1 + \frac{c}{fb}$, picking up an exponential $e\left(m_1\frac{c}{fb}\right)$. We obtain

$$G^{(4)}_{m_1,m_2}(z, s_1, s_2) = \left(\frac{N}{f}\right)^{-3} \delta_{h,\frac{N}{f}} \pi^{-3s_1-3s_2+\frac{1}{2}} \Gamma\left(\frac{3s_1}{2}\right) \Gamma\left(\frac{3s_2}{2}\right) \Gamma\left(\frac{3s_1+3s_2-1}{2}\right)$$

$$\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)\,y_1^{s_1+2s_2}y_2^{s_2+2s_1}\sum_{\substack{b\geq 1\\ \left(b,\frac{N}{f}\right)=1}}\ \sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right)$$
$$\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_{-\infty}^{\infty}\left[f^2b^2y_1^2+f^2b^2x_1^2\right]^{-\frac{3s_2}{2}} e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,.$$

Extract and evaluate the exponential integral $\int_0^{\frac{N}{f}} e(-m_2x_2)\mathrm{d}x_2 = \frac{N}{f}\delta_{m_2,0}$ and the integral $\int_0^{\frac{N}{f}}\mathrm{d}x_3 = \frac{N}{f}$. Then use the formula in [**4**, (3.13)] and the notation in definition 16.4 to extract the Whittaker function and the Dirichlet series $D_{m_1}(s_2,f,P_{min})$. This gives us

$$G^{(4)}_{m_1,m_2}(z,s_1,s_2)=\left(\frac{N}{f}\right)^{-1}\delta_{m_2,0}\delta_{h,\frac{N}{f}}\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)$$
$$f^{-3s_2}\sum_{\substack{b\geq 1\\ \left(b,\frac{N}{f}\right)=1}} b^{-3s_2}\sum_{\substack{c \bmod Nb\\ (fb,c)=1}} e\left(m_1\frac{c}{fb}\right)W^{(s_2,s_1)}_{m_1,0}(z,w_3)$$
$$=\left(\frac{N}{f}\right)^{-1}\delta_{m_2,0}\delta_{h,\frac{N}{f}}\zeta(3s_1)\zeta(3s_2)\zeta(3s_1+3s_2-1)\,f^{-3s_2}D_{m_1}(s_2,f,P_{min})W^{(s_2,s_1)}_{m_1,0}(z,w_3)\,.$$

Finally use Lemma 16.5 and evaluate the Dirichlet series $D_{m_1}(s_2,f,P_{min})$. So the formulas

$$G^{(4)}_{m_1,m_2}(z,s_1,s_2)=\delta_{m_2,0}\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p\mid\frac{N}{f}}\frac{1}{1-p^{-3s_2}}\prod_{p\mid f}\frac{(p-1)\sigma_{1-3s_2}(p^{\alpha_p})-p^{1+(1-3s_2)\alpha_p}}{1-p^{-3s_2}}$$
$$\sigma_{1-3s_2}\left(\prod_{(p,N)=1}p^{\alpha_p}\right)\zeta(3s_1)\zeta(3s_1+3s_2-1)\,W^{(s_2,s_1)}_{m_1,0}(z,w_3)$$

and

$$G^{(4)}_{0,m_2}(z,s_1,s_2)=\delta_{m_2,0}\delta_{h,\frac{N}{f}}f^{-3s_2}\prod_{p\mid f}\frac{p-1}{1-p^{-3s_2}}\prod_{p\mid\frac{N}{f}}\frac{1}{1-p^{-3s_2}}$$
$$\zeta(3s_1)L_{\chi_{\frac{N}{f}}}(3s_2-1)\zeta(3s_1+3s_2-1)\,W^{(s_2,s_1)}_{0,0}(z,w_3)$$

hold.

(5) Finally we treat the two easiest calculations, beginning with $G^{(5)}_{m_1,m_2}(z,s_1,s_2)$. We have

$$
\begin{aligned}
&G^{(5)}_{m_1,m_2}(z,s_1,s_2)\\
=&\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\delta_{f,N}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\\
&\sum_{d\neq 0}\sum_{\substack{e\in\mathbb{Z}\\(d,e)=1}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}} y_1^{s_1+2s_2}y_2^{s_2+2s_1}\left[d^2y_2^2+(x_2d+e)^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
&+\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}} y_1^{s_1+2s_2}y_2^{s_2+2s_1}\left[d^2y_2^2+(x_2d-e)^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .
\end{aligned}
$$

Next pass in the summation over d to positive values collecting a factor 2. Split the summation over e in this way $e = dk + e$ with $k \in \mathbb{Z}$, $0 \le e < d$ and note that the corresponding gcd-condition holds again for e. Then pass to an infinite integral in the variable x_2 and shift $x_2 \to x_2 \pm \frac{e}{d}$, picking up an exponential $e\left(\pm m_2\frac{e}{d}\right)$. So we get

$$
\begin{aligned}
&G^{(5)}_{m_1,m_2}(z,s_1,s_2) = \frac{1}{2}\delta_{f,N}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\\
&\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)y_1^{s_1+2s_2}y_2^{s_2+2s_1}\sum_{d\geq 1}\sum_{\substack{e \bmod d\\(d,e)=1}}\left[e\left(m_2\frac{e}{d}\right)+e\left(-m_2\frac{e}{d}\right)\right]\\
&\int_0^1\int_{-\infty}^{\infty}\int_0^1\left[d^2y_2^2+d^2x_2^2\right]^{-\frac{3s_1}{2}}e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ .
\end{aligned}
$$

Extract and evaluate the exponential integral $\int_0^1 e(-m_1x_1)\mathrm{d}x_1 = \delta_{m_1,0}$ and the integral $\int_0^1 \mathrm{d}x_3 = 1$. Absorb the minus sign in the second sum over e into the summation. Use the formula in [**4**, (3.12)] and extract the Whittaker function. We obtain

$$
G^{(5)}_{m_1,m_2}(z,s_1,s_2) = \delta_{m_1,0}\delta_{f,N}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)\sum_{d\geq 1}d^{-3s_2}c_d\left(m_2\right)W^{(s_2,s_1)}_{0,m_2}(z,w_2)\ .
$$

Using Lemma 18.4 in Appendix A and distinguishing between the cases whether m_2 vanishes or not the final result is

$$
G^{(5)}_{m_1,m_2}(z,s_1,s_2) = \begin{cases}\delta_{m_1,0}\delta_{f,N}\zeta\left(3s_1\right)\zeta\left(3s_1+3s_2-1\right)\sigma_{1-3s_2}(m_2)W^{(s_2,s_1)}_{0,m_2}(z,w_2) & \text{, if } m_2\neq 0\ ,\\ \delta_{m_1,0}\delta_{f,N}\zeta\left(3s_1\right)\zeta\left(3s_2-1\right)\zeta\left(3s_1+3s_2-1\right)W^{(s_2,s_1)}_{0,0}(z,w_2) & \text{, if } m_2 = 0\ .\end{cases}
$$

(6) In the last calculation we treat $G^{(6)}_{m_1,m_2}(z,s_1,s_2)$. Note that the set $\Gamma_6(f,h,P_{min})$ contains at most four elements. We have

$$
G^{(6)}_{m_1,m_2}(z,s_1,s_2) =\frac{1}{4}\left(\frac{N}{f}\right)^{-3}\delta_{f,N}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)
$$

$$
\begin{aligned}
&\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)4\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}}\int_0^{\frac{N}{f}} y_1^{s_1+2s_2}y_2^{s_2+2s_1}\\
&e(-m_1x_1-m_2x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\delta_{f,N}\delta_{m_1,0}\delta_{m_2,0}\pi^{-3s_1-3s_2+\frac{1}{2}}\Gamma\left(\frac{3s_1}{2}\right)\Gamma\left(\frac{3s_2}{2}\right)\Gamma\left(\frac{3s_1+3s_2-1}{2}\right)\\
&\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)y_1^{s_1+2s_2}y_2^{s_2+2s_1}\\
=&\delta_{f,N}\delta_{m_1,0}\delta_{m_2,0}\zeta\left(3s_1\right)\zeta\left(3s_2\right)\zeta\left(3s_1+3s_2-1\right)W_{0,0}^{(s_2,s_1)}(z,w_0)\ .
\end{aligned}
$$

In the last equation we used the formula in [**4**, (3.10)] and extracted the Whittaker function.

□

In the classical case of $SL_3(\mathbb{Z})$ one gets the well known Fourier expansion and functional equation of the minimal GL_3 Eisenstein series, which was explicitly calculated in [**4**, Thm. 7.2] and [**27**]. In the case of a level $N>1$ one has to proceed similarly as in chapter 13 and show that there exists a certain scattering matrix, such that the Eisenstein vector fulfills a functional equation. Note that unlike as in the $SL_3(\mathbb{Z})$ case, for a general level the Eisenstein series are not invariant under the involution $z\longmapsto z^t$, because the congruence subgroup $\Gamma_0(N)$ is not fixed. So in order to obtain functional equations in the general case, one probably also has to consider the Eisenstein series $G\left(z^t,s_1,s_2,f,h,P_{min}\right)$.

Part VIII

Appendix

CHAPTER 18

Appendix A: Ramanujan sums and associated L-functions

In this appendix we recall some basic facts about Ramanujan sums and divisor sums. Further we evaluate certain associated L-functions. All the results are well known but for easier readability of the main text it is advantageous to collect these facts. We start with the definition of Ramanujan sums and divisor sums.

Definition 18.1. Let n be a positive integer and m an integer. The associated Ramanujan sum $c_n(m)$ is defined through the formula

$$c_n(m) := \sum_{\substack{x \bmod n \\ (x,n)=1}} e\left(\frac{mx}{n}\right) .$$

Definition 18.2. Let n be a positive integer and s a complex number, then the associated divisor sum $\sigma_s(n)$ is defined through the formula

$$\sigma_s(n) := \sum_{d|n} d^s .$$

Next some well known properties of these Ramanujan sums and divisor sums are stated.

Lemma 18.3. *Let n_1, n_2, h, q be positive integers with $(n_1, n_2) = 1$, k, l nonnegative integers and n an arbitrary integer. Further let p be a prime number and m an integer with $(p, m) = 1$. Ramanujan sums fulfill the following properties.*

(1) Ramanujan sums are multiplicative:

$$c_{n_1 n_2}(m) = c_{n_1}(m) c_{n_2}(m) .$$

(2) Ramanujan sums satisfy a cancellation property:

$$c_{p^k}\left(p^l m\right) = c_{p^k}\left(p^l\right) .$$

(3) The following explicit formula holds:

$$c_{p^k}\left(p^l\right) = \begin{cases} 1, & \text{if} \quad k = 0 , \\ p^{k-1}(p-1), & \text{if} \quad 0 < k \le l , \\ -p^l, & \text{if} \quad k = l+1 , \\ 0, & \text{if} \quad k > l+1 . \end{cases}$$

(4) *Ramanujan sums satisfy reduction formulas:*
(a)

$$c_{p^{h+1}}\left(p^{l+1}\right) = pc_{p^h}\left(p^l\right) \ ,$$

(b)

$$c_{p^{h+1}}(0) = pc_{p^h}(0) \ .$$

(5) *The explicit formula*

$$\sum_{d|q} c_d(n) = \begin{cases} q, & \text{if} \quad q \mid n \ , \\ 0, & \text{otherwise} \end{cases}$$

for divisor sums of Ramanujan sums holds.

(6) *The divisor function satisfies the following transformation rule:*

$$\sigma_{-s}(|n|) = |n|^{-s} \, \sigma_s(|n|) \ .$$

Proof. See [**10**] for details.

□

The main application of the previous results concerning Ramanujan sums is the calculation of certain associated L-functions and finite Euler products. These functions occur in the Fourier coefficients of the Eisenstein series.

Lemma 18.4. *Let N, m be positive integers, where $m = m_1 m_2$ with $m_1 \mid N^\infty$ and $(m_2, N) = 1$. For the L-functions below the explicit formulas*

(1)

$$\sum_{a|N^\infty} a^{-s} c_a(m) = \sigma_{1-s}(m_1) \prod_{p|N} (1 - p^{-s}) \ ,$$

(2)

$$\sum_{(a,N)=1} a^{-s} c_a(m) = \sigma_{1-s}(m_2) L_{\chi_N}(s)^{-1} \ ,$$

(3)

$$\sum_{a|N^\infty} a^{-s} \phi(a) = \prod_{p|N} \frac{1 - p^{-s}}{1 - p^{1-s}} \ ,$$

(4)

$$\sum_{(a,N)=1} a^{-s} \phi(a) = \frac{L_{\chi_N}(s-1)}{L_{\chi_N}(s)} \ ,$$

are valid.

Proof. For now let m be an arbitrary integer.

(1) Since Euler's ϕ-function is just the Ramanujan sum $c_*(0)$ the parts (1) and (3) can be handled together. Use the formula for divisor sums in Lemma 18.3 part (5) and calculate directly

$$\prod_{p|N} \frac{1}{1-p^{-s}} \sum_{a|N^\infty} a^{-s} c_a(m) = \sum_{n|N^\infty} n^{-s} \sum_{a|N^\infty} a^{-s} c_a(m) = \sum_{q|N^\infty} q^{-s} \left(\sum_{d|q} c_d(m) \right) = \sum_{\substack{q|N^\infty \\ q|m}} q^{1-s}$$

$$= \begin{cases} \sum\limits_{q|N^\infty} q^{1-s}, & \text{if } m = 0\ , \\ \sum\limits_{q|m_1} q^{1-s}, & \text{if } m \neq 0\ . \end{cases}$$

$$= \begin{cases} \prod\limits_{p|N} \frac{1}{1-p^{1-s}}, & \text{if } m = 0\ , \\ \sigma_{1-s}(m_1), & \text{if } m \neq 0\ . \end{cases}$$

(2) With the same method as above the parts (2) and (4) are handled. Again we calculate directly

$$L_{\chi_N}(s) \sum_{(a,N)=1} a^{-s} c_a(m) = \sum_{(n,N)=1} n^{-s} \sum_{(a,N)=1} a^{-s} c_a(m)$$

$$= \sum_{(q,N)=1} q^{-s} \left(\sum_{d|q} c_d(m) \right) = \sum_{\substack{(q,N)=1 \\ q|m}} q^{1-s}$$

$$= \begin{cases} \sum\limits_{(q,N)=1} q^{1-s} & \text{if } m = 0\ , \\ \sum\limits_{q|m_2} q^{1-s} & \text{if } m \neq 0\ . \end{cases}$$

$$= \begin{cases} L_{\chi_N}(s-1) & \text{if } m = 0\ , \\ \sigma_{1-s}(m_2) & \text{if } m \neq 0\ . \end{cases}$$

□

LEMMA 18.5. *Let N be a squarefree positive integer and $m = \prod_p p^{\alpha_p}$ a positive integer with corresponding prime factor decomposition. The factorizations*

(1)

$$\sum_{a|N^\infty} a^{-s} c_{aN}(m) = \prod_{p|N} \left((p-1)\sigma_{1-s}\left(p^{\alpha_p}\right) - p^{1+(1-s)\alpha_p} \right)\ ,$$

(2)

$$\sum_{a|N^\infty} a^{-s} \phi(aN) = \prod_{p|N} \frac{p-1}{1-p^{1-s}}\ ,$$

of the above Dirichlet series into finite Euler products are valid.

PROOF. We handle both parts together assuming for the moment that m is any non-negative integer. The multiplicativity of the Ramanujan sums gives the factorization

$$\sum_{a|N^\infty} a^{-s} c_{aN}(m) = \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks} c_{p^k p}(m)$$

of the above Dirichlet sum in a finite Euler product. Note that in the right-hand side of the equation the squarefreeness of N was used. Now we distinguish between the cases whether m vanishes or not. For m non-zero use the cancellation property of Ramanujan sums in Lemma 18.3 part (2). This gives us

$$\sum_{a|N^\infty} a^{-s} c_{aN}(m) = \begin{cases} \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks} c_{p^{k+1}}\left(p^{\alpha_p}\right), & \text{if } m \neq 0\ , \\ \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks} \phi\left(p^{k+1}\right), & \text{if } m = 0\ . \end{cases}$$

Then apply the explicit formula in Lemma 18.3 part (3). This gives the formula

$$\begin{aligned} \sum_{a|N^\infty} a^{-s} c_{aN}(m) &= \begin{cases} \prod_{p|N} \left(\sum_{k=0}^{\alpha_p - 1} p^{-ks} c_{p^{k+1}}\left(p^{\alpha_p}\right) + p^{-\alpha_p s} c_{p^{\alpha_p+1}}\left(p^{\alpha_p}\right) \right), & \text{if } m \neq 0\ , \\ \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks} p^k (p-1), & \text{if } m = 0\ . \end{cases} \\ &= \begin{cases} \prod_{p|N} \left(\sum_{k=0}^{\alpha_p - 1} p^{-ks} p^k (p-1) + p^{-\alpha_p s}\left(-p^{\alpha_p}\right) \right), & \text{if } m \neq 0\ , \\ \prod_{p|N} (p-1) \sum_{k=0}^{\infty} p^{(1-s)k}, & \text{if } m = 0\ . \end{cases} \end{aligned}$$

Now evaluate the geometric sums. So we have

$$\sum_{a|N^\infty} a^{-s} c_{aN}(m) = \begin{cases} \prod_{p|N} \left((p-1)\sigma_{1-s}\left(p^{\alpha_p}\right) - p^{1+(1-s)\alpha_p} \right), & \text{if } m \neq 0\ , \\ \prod_{p|N} \frac{p-1}{1-p^{1-s}}, & \text{if } m = 0\,. \end{cases}$$

□

Last but not least explicit formulas for certain convoluted Dirichlet series are stated. The Dirichlet series in the first lemma will occur in the Fourier coefficients of the minimal Eisenstein series, the one in the second lemma in the Fourier coefficients of the Eisenstein series twisted by a constant Maass form.

LEMMA 18.6. *Let N be a squarefree positive integer and $m_1 = \prod_p p^{\alpha_p}$ and $m_2 = \prod_p p^{\beta_p}$ positive integers with corresponding prime factor decomposition. Then the explicit formulas*

$$\sum_{a|N^\infty} a^{-s_2} c_a(m_2) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a)$$
$$= \prod_{p|N} \frac{(p-1)\left(1-p^{-s_2}\right)\sigma_{1-s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-s_1}\right)\left(1-p^{1-s_2-s_1}\right)p^{(1-s_1)\alpha_p}\sigma_{2-s_2-s_1}\left(p^{\beta_p}\right)}{1-p^{1-s_1}}$$

and

$$\sum_{a|N^\infty} a^{-s_2}\phi(a) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a)$$
$$= \prod_{p|N} \frac{(p-1)\left(1-p^{-s_2}\right)\left(1-p^{2-s_2-s_1}\right) - p\left(1-p^{1-s_2-s_1}\right)\left(1-p^{-s_1}\right)\left(1-p^{1-s_2}\right)p^{(1-s_1)\alpha_p}}{\left(1-p^{1-s_1}\right)\left(1-p^{2-s_2-s_1}\right)\left(1-p^{1-s_2}\right)}$$

hold.

PROOF. We handle both cases together assuming for the present that m_2 is an arbitrary integer. The squarefreeness of N and the cancellation property of the Ramanujan sums imply that the Dirichlet series below can be factorized into a finite Euler product. We have

$$\sum_{a|N^\infty} a^{-s_2} c_a(m_2) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a) = \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks_2} c_{p^k}(m_2) \sum_{l=0}^{\infty} p^{-ls_1} c_{p^l p}\left(p^{\alpha_p+k}\right) .$$

Next use the explicit formula for the inner Dirichlet series in Lemma 18.5. A direct calculation gives

$$\sum_{a|N^\infty} a^{-s_2} c_a(m_2) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a)$$
$$= \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks_2} c_{p^k}(m_2) \left((p-1)\sigma_{1-s_1}\left(p^{\alpha_p+k}\right) - p^{1+(1-s_1)(\alpha_p+k)}\right)$$
$$= \prod_{p|N} \sum_{k=0}^{\infty} p^{-ks_2} c_{p^k}(m_2) \left(\frac{p-1+\left(p^{1-s_1}-p\right)p^{(1-s_1)(\alpha_p+k)}}{1-p^{1-s_1}}\right)$$
$$= \prod_{p|N} \frac{1}{1-p^{1-s_1}} \left((p^{1-s_1}-p)p^{(1-s_1)\alpha_p} \sum_{k\ge 0} p^{-(s_2+s_1-1)k} c_{p^k}(m_2) + (p-1)\sum_{k\ge 0} p^{-s_2 k} c_{p^k}(m_2)\right) .$$

Now we divide between the cases whether m_2 vanishes or not and use Lemma 18.3 to calculate the two Dirichlet series.

(1) 1. Case: $m_2 \neq 0$. In this case the cancellation property of the Ramanujan sums gives the result

$$\sum_{a|N^\infty} a^{-s_2} c_a(m_2) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a)$$
$$= \prod_{p|N} \frac{(p-1)\left(1-p^{-s_2}\right)\sigma_{1-s_2}\left(p^{\beta_p}\right) - p\left(1-p^{-s_1}\right)\left(1-p^{1-s_2-s_1}\right)p^{(1-s_1)\alpha_p}\sigma_{2-s_2-s_1}\left(p^{\beta_p}\right)}{1-p^{1-s_1}} .$$

(2) 2. Case: $m_2 = 0$. Since Euler's ϕ-function is just the Ramanujan sum $c_*(0)$ we can apply the identity

$$\sum_{n=1}^{\infty} \phi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}$$

to evaluate the above Dirichlet series. This gives us the result

$$\sum_{a|N^\infty} a^{-s_2} \phi(a) \sum_{b|N^\infty} b^{-s_1} c_{bN}(m_1 a)$$
$$= \prod_{p|N} \frac{(p-1)\left(1-p^{-s_2}\right)\left(1-p^{2-s_2-s_1}\right) - p\left(1-p^{1-s_2-s_1}\right)\left(1-p^{-s_1}\right)\left(1-p^{1-s_2}\right)p^{(1-s_1)\alpha_p}}{\left(1-p^{1-s_1}\right)\left(1-p^{2-s_2-s_1}\right)\left(1-p^{1-s_2}\right)} .$$

□

LEMMA 18.7. *Let f be a squarefree positive integer and $m = \prod_p p^{\alpha_p}$ be a positive integer with corresponding prime factor decomposition. Then the formulas*

(1)

$$\sum_{a|f^\infty} a^{-s} \sum_{t|fa} c_{\frac{fa}{t}}(m) \frac{fa}{t} \phi(t) = \prod_{p|f} \left((p^2-1)\sigma_{2-s}\left(p^{\alpha_p}\right) - p^{2+(2-s)\alpha_p} \right) ,$$

(2)

$$\sum_{a|f^\infty} a^{-s} \sum_{t|fa} \phi\left(\frac{fa}{t}\right) \frac{fa}{t} \phi(t) = \prod_{p|f} \frac{p^2-1}{1-p^{2-s}}$$

hold.

PROOF. We handle both cases together, in order to do this assume for the present that m is an arbitrary integer. The multiplicativity of the Ramanujan sums implies the factorization of the Dirichlet series in a finite Euler product. We have

$$\sum_{a|f^\infty} a^{-s} \sum_{t|fa} c_{\frac{fa}{t}}(m) \frac{fa}{t} \phi(t) = \prod_{p|f} \left(\sum_{k=0}^{\infty} p^{-ks} \sum_{l=0}^{k+1} c_{p^{k+1-l}}(m) p^{k+1-l} \phi\left(p^l\right) \right) .$$

Make an index shift in the inner sum and seperate the zeroth summand. This comes to

$$\begin{aligned}\sum_{a|f^\infty} a^{-s} \sum_{t|fa} c_{\frac{fa}{t}}(m) \frac{fa}{t} \phi(t) &= \prod_{p|f} \sum_{k=0}^{\infty} p^{-ks} \left(\sum_{l=0}^{k} c_{p^l}(m) p^l \phi\left(p^{k+1-l}\right) + c_{p^{k+1}}(m) p^{k+1} \phi\left(p^{k+1-(k+1)}\right) \right) \\ &= \prod_{p|f} \left((p-1) \sum_{k=0}^{\infty} p^{-(s-1)k} \sum_{l=0}^{k} c_{p^l}(m) + p \sum_{k=0}^{\infty} p^{-(s-1)k} c_{p^{k+1}}(m) \right) .\end{aligned}$$

Factorize the first Dirichlet series, which is a convolution of two Dirichlet series. This comes to

$$\begin{aligned}\sum_{a|f^\infty} a^{-s} \sum_{t|fa} c_{\frac{fa}{t}}(m) \frac{fa}{t} \phi(t) &= \prod_{p|f} \left((p-1) \sum_{k=0}^{\infty} p^{-(s-1)k} \sum_{k=0}^{\infty} p^{-(s-1)k} c_{p^k}(m) + p \sum_{k=0}^{\infty} p^{-(s-1)k} c_{p^{k+1}}(m) \right) \\ &= \prod_{p|f} \left(\frac{p-1}{1-p^{1-s}} \sum_{k=0}^{\infty} p^{-(s-1)k} c_{p^k}(m) + p \sum_{k=0}^{\infty} p^{-(s-1)k} c_{p^{k+1}}(m) \right) .\end{aligned}$$

In the case $m \neq 0$ use Lemma 18.4 part (1) to evaluate the first Dirichlet series and Lemma 18.5 part (1) for the second one. In the case $m = 0$ use Lemma 18.4 part (3) to evaluate the first Dirichlet series and Lemma 18.5 part (2) for the second one. So finally we get

$$\begin{aligned}&\sum_{a|f^\infty} a^{-s} \sum_{t|fa} c_{\frac{fa}{t}}(m) \frac{fa}{t} \phi(t) \\ &= \begin{cases} \prod_{p|f} \left(\frac{p-1}{1-p^{1-s}} \sigma_{1-(s-1)}\left(p^{\alpha_p}\right) \left(1 - p^{-(s-1)}\right) + p\left((p-1)\sigma_{1-(s-1)}\left(p^{\alpha_p}\right) - p^{1+(1-(s-1))\alpha_p}\right) \right), & \text{if } m \neq 0 , \\ \prod_{p|f} \left(\frac{p-1}{1-p^{1-s}} \frac{1-p^{-(s-1)}}{1-p^{1-(s-1)}} + p \frac{p-1}{1-p^{1-(s-1)}} \right), & \text{if } m = 0 . \end{cases} \\ &= \begin{cases} \prod_{p|f} \left((p^2-1)\sigma_{2-s}\left(p^{\alpha_p}\right) - p^{2+(2-s)\alpha_p} \right), & \text{if } m \neq 0 , \\ \prod_{p|f} \frac{p^2-1}{1-p^{2-s}}, & \text{if } m = 0 . \end{cases}\end{aligned}$$

□

CHAPTER 19

Appendix B: K-Bessel function and GL_3-Whittaker functions

In this appendix basic facts about the well known K-Bessel function, which is up to the root of the y-coordinate Jacquet's Whittaker function for GL_2, are summarized. The connection between the K-Bessel function and Jacquet's Whittaker function for GL_3 are worked out. These integral formulas are crucial in the calculation of the Fourier expansion for the twisted Eisenstein series.

Definition 19.1. The K-Bessel function $K_\nu : \mathbb{R}_+ \longrightarrow \mathbb{C}$ is a C^∞-differentiable function parametrized through a complex index ν defined through the formula

$$K_\nu(z) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{z}{2}\left(t+\frac{1}{t}\right)\right) t^\nu \frac{\mathrm{d}t}{t} .$$

Here is a summary of a few properties of the K-Bessel function K_ν.

Lemma 19.2. *Let A, B, C be real numbers with $A > 0$ and $4AC - B^2 > 0$. The K-Bessel function K_ν has the following properties.*

(1) The K-Bessel function satisfies the functional equation $K_\nu = K_{-\nu}$.
(2) For $\mathfrak{Re}(\nu) > \frac{1}{2}$ the K-Bessel function satisfies the integral representation

$$\int_{-\infty}^\infty (Ax^2+Bx+C)^{-\nu} e(-x)\mathrm{d}x = \frac{\pi^\nu}{\Gamma(\nu)} e^{\frac{\pi i B}{A}} \left(4AC-B^2\right)^{\frac{1}{4}-\frac{\nu}{2}} 2^{\nu+\frac{1}{2}} A^{-\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\pi\frac{\sqrt{4AC-B^2}}{A}\right) .$$

(3) For $\mathfrak{Re}(\nu) > \frac{1}{2}$ the integral formula

$$\int_{-\infty}^\infty (Ax^2+Bx+C)^{-\nu}\mathrm{d}x = \sqrt{\pi}2^{2\nu-1}\left(4AC-B^2\right)^{\frac{1}{2}-\nu} A^{\nu-1}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}$$

holds.
(4) The formula

$$\int_0^\infty K_{\nu-\frac{1}{2}}(y) y^{s+\frac{1}{2}}\frac{\mathrm{d}y}{y} = 2^{s-\frac{3}{2}}\Gamma\left(\frac{1+s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)$$

for the Mellin transform of the K-Bessel function is valid for $\mathfrak{Re}(s) + \frac{1}{2} > \left|\mathfrak{Re}(\nu) - \frac{1}{2}\right|$.

PROOF. Part (1) can be seen immediately through the substitution $t \mapsto t^{-1}$ in the integral representation of the K-bessel function. Part (2) and (3) are proved in [**4**, (3.54),(3.55)] referring to [**30**]. Part (4) can also be found in [**30**].

□

Next some important integral representations of the K-Bessel function and Jacquet's Whittaker functions are derived. Starting with integral representations, which are needed in the calculation of the Fourier expansion for the Eisenstein twisted by a constant Maass form.

LEMMA 19.3. *Let m be a non-zero integer and y_1, y_2 positive real numbers. For $\mathfrak{Re}(\nu) > \frac{1}{2}$ the four integral representations*

(1)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_1^2x_1^2 + x_2^2\right]^{-\nu} e(-mx_1)\mathrm{d}x_1\mathrm{d}x_2 = 2\pi^{\nu}y_1^{1-2\nu}y_2^{1-\nu}\,|m|^{\nu-1}\,\Gamma(\nu)^{-1}K_{\nu-1}(2\pi\,|m|\,y_2)\ ,$$

(2)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_1^2x_1^2 + x_2^2\right]^{-\nu} \mathrm{d}x_1\mathrm{d}x_2 = \pi y_1^{1-2\nu}y_2^{2-2\nu}\frac{\Gamma(\nu-1)}{\Gamma(\nu)}\ ,$$

(3)

$$\int_{-\infty}^{\infty} \left[y_1^2 + x_1^2\right]^{-\nu} e(-mx_1)\mathrm{d}x_1 = 2\pi^{\nu}\,|m|^{\nu-\frac{1}{2}}\,y_1^{\frac{1}{2}-\nu}\Gamma(\nu)^{-1}K_{\nu-\frac{1}{2}}\left(2\pi\,|m|\,y_1\right)\ ,$$

(4)

$$\int_{-\infty}^{\infty} \left[y_1^2 + x_1^2\right]^{-\nu} \mathrm{d}x_1 = \sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\ ,$$

are valid.

PROOF. (1) Apply Lemma 19.2 part (3) with $A = 1$, $B = 0$ and $C = y_1^2y_2^2 + y_1^2x_1^2$ to the integral in the variable x_2. We have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_1^2x_1^2 + x_2^2\right]^{-\nu} e(-mx_1)\mathrm{d}x_1\mathrm{d}x_2\\ =&\int_{-\infty}^{\infty} \sqrt{\pi}2^{2\nu-1}\left(4(y_1^2y_2^2 + y_1^2x_1^2)\right)^{\frac{1}{2}-\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}e(-mx_1)\mathrm{d}x_1\ .\end{aligned}$$

Make the substitution $mx_1 \longrightarrow x_1$ and simplify the terms. We have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_1^2y_2^2 + y_1^2x_1^2 + x_2^2\right]^{-\nu} e(-mx_1)\mathrm{d}x_1\mathrm{d}x_2\\ =&\sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\,|m|^{2\nu-2}\int_{-\infty}^{\infty}\left[m^2y_2^2 + x_1^2\right]^{\frac{1}{2}-\nu} e(-x_1)\mathrm{d}x_1\ .\end{aligned}$$

Next apply Lemma 19.2 part (2) with $A = 1$, $B = 0$ and $C = m^2y_2^2$ and evaluate the integral in the variable x_1. This gives us

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_1^2+x_2^2\right]^{-\nu}e(-mx_1)\mathrm{d}x_1\mathrm{d}x_2\\
=&\sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\left|m\right|^{2\nu-2}\frac{\pi^{\nu-\frac{1}{2}}}{\Gamma\left(\nu-\frac{1}{2}\right)}\left(4(m^2y_2^2)\right)^{\frac{1}{4}-\frac{\nu-\frac{1}{2}}{2}}2^{\left(\nu-\frac{1}{2}\right)+\frac{1}{2}}K_{\left(\nu-\frac{1}{2}\right)-\frac{1}{2}}\left(\pi\sqrt{4(m^2y_2^2)}\right)\\
=&2\pi^{\nu}y_1^{1-2\nu}y_2^{1-\nu}\left|m\right|^{\nu-1}\Gamma(\nu)^{-1}K_{\nu-1}\left(2\pi\left|m\right|y_2\right)\ .
\end{aligned}$$

(2) Note that the first step in the proof of part (1) is also valid for $m = 0$, so the identity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_1^2+x_2^2\right]^{-\nu}\mathrm{d}x_1\mathrm{d}x_2=\sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\int_{-\infty}^{\infty}\left[y_2^2+x_1^2\right]^{\frac{1}{2}-\nu}\mathrm{d}x_1$$

holds. So it remains to apply Lemma 19.2 part (3) with $A = 1$, $B = 0$ and $C = y_2^2$ to evaluate the last integral. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_1^2+x_2^2\right]^{-\nu}\mathrm{d}x_1\mathrm{d}x_2\\
=&\sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\sqrt{\pi}2^{2\left(\nu-\frac{1}{2}\right)-1}\left(4y_2^2\right)^{\frac{1}{2}-\left(\nu-\frac{1}{2}\right)}\frac{\Gamma\left(\left(\nu-\frac{1}{2}\right)-\frac{1}{2}\right)}{\Gamma\left(\nu-\frac{1}{2}\right)}\\
=&\pi y_1^{1-2\nu}y_2^{2-2\nu}\frac{\Gamma(\nu-1)}{\Gamma(\nu)}\ .
\end{aligned}$$

(3) Make the substitution $mx_1 \longrightarrow x_1$ and simplify the terms. The identity

$$\int_{-\infty}^{\infty}\left[y_1^2+x_1^2\right]^{-\nu}e(-mx_1)\mathrm{d}x_1=\left|m\right|^{2\nu-1}\int_{-\infty}^{\infty}\left[m^2y_1^2+x_1^2\right]^{-\nu}e(-x_1)\mathrm{d}x_1$$

holds. Apply Lemma 19.2 part (2) with $A = 1$, $B = 0$ and $C = y_1^2m^2$. This gives us

$$\begin{aligned}
\int_{-\infty}^{\infty}\left[y_1^2+x_1^2\right]^{-\nu}e(-mx_1)\mathrm{d}x_1=&\left|m\right|^{2\nu-1}\frac{\pi^{\nu}}{\Gamma(\nu)}\left(4y_1^2m^2\right)^{\frac{1}{4}-\frac{\nu}{2}}2^{\nu+\frac{1}{2}}K_{\nu-\frac{1}{2}}\left(\pi\sqrt{4y_1^2m^2}\right)\\
=&\ 2\pi^{\nu}\left|m\right|^{\nu-\frac{1}{2}}y_1^{\frac{1}{2}-\nu}\Gamma(\nu)^{-1}K_{\nu-\frac{1}{2}}\left(2\pi\left|m\right|y_1\right)\ .
\end{aligned}$$

(4) Apply Lemma 19.2 part (3) with $A = 1$, $B = 0$ and $C = y_1^2$. We have

$$\int_{-\infty}^{\infty}\left[y_1^2+x_1^2\right]^{-\nu}\mathrm{d}x_1=\sqrt{\pi}2^{2\nu-1}\left(4y_1^2\right)^{\frac{1}{2}-\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}=\sqrt{\pi}y_1^{1-2\nu}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma(\nu)}\ .$$

□

The following lemmata establish a connection between the K-Bessel function and Jacquet's Whittaker function for GL_3. Note that we use the notation in [**4**] and start with the degenerate case, whose proof is similar to that of Lemma 19.3. Note that the results of the following lemma can be found in [**4**, (3.40),(3.45)].

LEMMA 19.4. *Let* $z = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. *Then the degenerate Whittaker functions* $W_{0,1}^{(\nu_1,\nu_2)}(z,w_2)$ *and* $W_{1,0}^{(\nu_1,\nu_2)}(z,w_3)$ *have the representations through the standard K-Bessel function given by the formulas*

(1)

$$W_{0,1}^{(\nu_1,\nu_2)}(z,w_2) = 2\pi^{-3\nu_1-\frac{3\nu_2}{2}+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\frac{1}{2}+\nu_1+\frac{\nu_2}{2}} K_{\frac{3\nu_2-1}{2}}(2\pi y_2)\,,$$

(2)

$$W_{1,0}^{(\nu_1,\nu_2)}(z,w_3) = 2\pi^{-\frac{3\nu_1}{2}-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{\frac{1}{2}+\frac{\nu_1}{2}+\nu_2} y_2^{\nu_1+2\nu_2} K_{\frac{3\nu_1-1}{2}}(2\pi y_1)\,.$$

PROOF. First we assume $\mathfrak{Re}(\nu_1), \mathfrak{Re}(\nu_2) > \frac{1}{3}$, so that we can apply the integral formulas developed before. The general result follows through meromorphic continuation.

(1) The definition of $W_{0,1}^{(v_1,v_2)}(z,w_2)$ in [**4**, (3.12)] gives the formula

$$\begin{aligned} &W_{0,1}^{(\nu_1,\nu_2)}(z,w_2) \\ =&\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \int_{-\infty}^{\infty} \left[\xi_2^2+y_2^2\right]^{-\frac{3\nu_2}{2}} e(-\xi_2)\mathrm{d}\xi_2\,. \end{aligned}$$

Apply Lemma 19.2 part (2) with $A = 1$, $B = 0$ and $C = y_2^2$ and $\nu = \frac{3\nu_2}{2}$ to the integral in the variable ξ_2. This gives us

$$\begin{aligned} W_{0,1}^{(\nu_1,\nu_2)}(z,w_2) =&\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \\ &\frac{\pi^{\frac{3\nu_2}{2}}}{\Gamma\left(\frac{3\nu_2}{2}\right)} \left(4y_2^2\right)^{\frac{1}{4}-\frac{\frac{3\nu_2}{2}}{2}} 2^{\frac{3\nu_2}{2}+\frac{1}{2}} K_{\frac{3\nu_2}{2}-\frac{1}{2}}\left(\pi\sqrt{4y_2^2}\right) \\ =&2\pi^{-3\nu_1-\frac{3\nu_2}{2}+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\frac{1}{2}+\nu_1+\frac{\nu_2}{2}} K_{\frac{3\nu_2-1}{2}}(2\pi y_2)\,. \end{aligned}$$

(2) The definition of $W_{1,0}^{(\nu_1,\nu_2)}(z,w_3)$ in [**4**, (3.13)] gives the formula

$$\begin{aligned} &W_{1,0}^{(\nu_1,\nu_2)}(z,w_3) \\ =&\pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \int_{-\infty}^{\infty} \left[\xi_1^2+y_1^2\right]^{-\frac{3\nu_1}{2}} e(-\xi_1)\mathrm{d}\xi_1\,. \end{aligned}$$

Apply Lemma 19.2 part (2) with $A = 1$, $B = 0$ and $C = y_1^2$ and $\nu = \frac{3\nu_1}{2}$ to the integral in the variable ξ_1. This gives us

$$\begin{aligned} W_{1,0}^{(\nu_1,\nu_2)}(z,w_3) =& \pi^{-3\nu_1-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)y_1^{2\nu_1+\nu_2}y_2^{\nu_1+2\nu_2} \\ & \frac{\pi^{\frac{3\nu_1}{2}}}{\Gamma\left(\frac{3\nu_1}{2}\right)}\left(4y_1^2\right)^{\frac{1}{4}-\frac{\frac{3\nu_1}{2}}{2}}2^{\frac{3\nu_1}{2}+\frac{1}{2}}K_{\frac{3\nu_1}{2}-\frac{1}{2}}\left(\pi\sqrt{4y_1^2}\right) \\ =& 2\pi^{-\frac{3\nu_1}{2}-3\nu_2+\frac{1}{2}}\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)y_1^{\frac{1}{2}+\frac{\nu_1}{2}+\nu_2}y_2^{\nu_1+2\nu_2}K_{\frac{3\nu_1-1}{2}}(2\pi y_1)\ . \end{aligned}$$

□

Finally we state a result from [**4**, (3.56)], which expresses Jacquet's Whittaker function for GL_3 as a double integral of the K-Bessel function. This integral formula is needed to incorporate the Fourier expansion of a GL_2 Maass cusp form into the Fourier expansion of the Eisenstein series twisted by a Maass cusp form.

LEMMA 19.5. *Let* $z = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3$. *Then the following representations of Jacquet's Whittaker functions through a double integral over the standard K-Bessel function hold.*

(1) The representation for $W_{1,1}^{(\nu_1,\nu_2)}(z,w_1)$ *reads as follows*

$$\begin{aligned} W_{1,1}^{(\nu_1,\nu_2)}(z,w_1) = & 2\pi^{-3\nu_1-\frac{3\nu_2}{2}+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)y_1^{2\nu_1+\nu_2}y_2^{\frac{1}{2}+\nu_1+\frac{\nu_2}{2}} \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3\nu_1}{2}-\frac{3\nu_2}{4}}(y_2^2+x_2^2)^{-\frac{1}{2}}e(-x_2)K_{\frac{3\nu_2-1}{2}}\left(2\pi y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right) \\ & e\left(-\frac{x_2x_3}{y_2^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3\ . \end{aligned}$$

(2) The representation for $W_{1,0}^{(\nu_1,\nu_2)}(z,w_1)$ *reads as follows*

$$\begin{aligned} W_{1,0}^{(\nu_1,\nu_2)}(z,w_1) = & 2\pi^{-3\nu_1-\frac{3\nu_2}{2}+\frac{1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)y_1^{2\nu_1+\nu_2}y_2^{\frac{1}{2}+\nu_1+\frac{\nu_2}{2}} \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_1^2y_2^2+y_1^2x_2^2+x_3^2\right]^{\frac{1}{4}-\frac{3\nu_1}{2}-\frac{3\nu_2}{4}}(y_2^2+x_2^2)^{-\frac{1}{2}}K_{\frac{3\nu_2-1}{2}}\left(2\pi y_2\frac{\sqrt{y_1^2y_2^2+y_1^2x_2^2+x_3^2}}{y_2^2+x_2^2}\right) \\ & e\left(-\frac{x_2x_3}{y_2^2+x_2^2}\right)\mathrm{d}x_2\mathrm{d}x_3\ . \end{aligned}$$

(3) The representation for $W_{0,1}^{(\nu_1,\nu_2)}(z,w_1)$ *reads as follows*

$$W_{0,1}^{(\nu_1,\nu_2)}(z,w_1) = 2\pi^{-3\nu_2-\frac{3\nu_1}{2}+\frac{1}{2}}\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_2+3\nu_1-1}{2}\right)y_2^{2\nu_2+\nu_1}y_1^{\frac{1}{2}+\nu_2+\frac{\nu_1}{2}}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y_2^2y_1^2+y_2^2x_1^2+x_3^2\right]^{\frac{1}{4}-\frac{3\nu_2}{2}-\frac{3\nu_1}{4}}(y_1^2+x_1^2)^{-\frac{1}{2}}K_{\frac{3\nu_1-1}{2}}\left(2\pi y_1\frac{\sqrt{y_2^2y_1^2+y_2^2x_1^2+x_3^2}}{y_1^2+x_1^2}\right)$$
$$e\left(-\frac{x_1x_3}{y_1^2+x_1^2}\right)\mathrm{d}x_1\mathrm{d}x_3\ .$$

PROOF. The first part is proved in [**4**, (3.56)]. For the proof of the second part recall the definition of the Whittaker function $W_{1,0}^{(\nu_1,\nu_2)}(z,w_1)$ in [**4**, (3.11)] and use the same argument as in the first part with the difference that the exponential factor $e(-x_2)$ is absent. The correctness of the last part is easily seen through applying the transformation laws in [**4**, (3.17),(3.24)] to the Whittaker function in the second part.

□

CHAPTER 20

Appendix C: Automorphic forms on GL_2

In this appendix the facts, which are needed in this thesis, about the Fourier expansion and L-series associated to an automorphic form for the group GL_2 are summarized. For brevity the Atkin Lehner theory is omitted. As a reference we refer to the original paper [**2**]. We start with the main tool, the Fourier expansion and the L-series of an automorphic form, for a reference see [**15**, thm 3.1], [**14**] and [**7**, ch. 3.5, 3.13].

THEOREM 20.1. *Let ϕ be a Maass cusp form with eigenvalue $\nu(\nu-1)$ for the congruence subgroup $\Gamma_0(N)$. At any cusp $\mathfrak{a}$ the cusp form ϕ possesses a Fourier expansion*

$$\phi(\sigma_{\mathfrak{a}} z) = \sum_{n \neq 0} 2\phi_{\mathfrak{a}}(n)\sqrt{|n|\, y}K_{\nu-\frac{1}{2}}(2\pi\,|n|\, y)e(nx)\ .$$

Denote the Fourier coefficients at infinity with ϕ_n. For $\mathfrak{Re}(s) > 1$ the L-function

$$L_\phi(s) := \sum_{n=1}^{\infty} \phi_n n^{-s}$$

converges absolutely.

Now we state the main results about the Fourier coefficients and the functional equation of a newform. Details about newforms, Hecke operators, the Fricke involution and functional equations can be found in [**2**], [**14**, ch. 6] or [**22**]. Note that we use the normalization of the Hecke operators in [**14**, ch. 6].

THEOREM 20.2. *Let ϕ be a newform with eigenvalue $\nu(\nu-1)$ for the congruence subgroup $\Gamma_0(N)$. Then the Fourier coefficients of ϕ have the following properties and the L-series of ϕ possesses an Euler product.*

(1) *The Fourier coefficients $(\phi_n)_{n\in\mathbb{N}}$ are normalized, multiplicative and identical to the Hecke eigenvalues, hence $T_n\phi = \phi_n\phi$ with the Hecke relations $\phi_{p^{n+1}} = \phi_p\phi_{p^n}$ for primes $p \mid N$ and $\phi_{p^{n+2}} = \phi_p\phi_{p^{n+1}} - p^{-1}\phi_{p^n}$ for primes $(p,N) = 1$.*
(2) *For $\mathfrak{Re}(s) > 1$ the L-series L_ϕ has an Euler product*

$$L_\phi(s) = \prod_{p|N} \frac{1}{1-\phi_p p^{-s}} \prod_{(p,N)=1} \frac{1}{1-\phi_p p^{-s} + p^{-1}p^{-2s}}\ .$$

Next we state the functional equation of the L-series of a Maass newform. This result is well known, but since most proofs in the literature are abbreviated or given for modular forms, a complete proof will be given here. We will generalize the proof in [**7**, thm 3.13.5] to newforms of level N.

THEOREM 20.3. *Let ϕ be a newform with eigenvalue $\nu(\nu-1)$ for the congruence subgroup $\Gamma_0(N)$. Let $a = \pm 1$ be the eigenvalue of the Fricke involution for the eigenform ϕ, hence $\phi\left(-\frac{1}{Nz}\right) = a\phi(z)$. Further assume that ϕ is an odd or even cusp form, hence $\phi\left(\begin{pmatrix} y & -x \\ & 1 \end{pmatrix}\right) = (-1)^\epsilon \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right)$ with $\epsilon = 0$ if ϕ is even and $\epsilon = 1$ if ϕ is odd. Then the completed L-series $\mathfrak{L}_\phi(s) := (2i)^\epsilon \pi^{-s-\frac{1}{2}} N^{\frac{s}{2}} \Gamma\left(\frac{s+1+\epsilon-\nu}{2}\right)\Gamma\left(\frac{s+\epsilon+\nu}{2}\right) L_\phi(s)$ satisfies the functional equation*

$$\mathfrak{L}_\phi(s) = (-1)^\epsilon a \mathfrak{L}_\phi(-s) \ .$$

PROOF. (1) 1. Case: ϕ is even. For $\mathfrak{Re}(s) > 1$ the cuspidality of ϕ implies that the Mellin transform

$$N^{\frac{s}{2}} \int_0^\infty \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y} = N^{\frac{s}{2}} \int_0^\infty \sum_{n\neq 0} 2\phi_n \sqrt{|n|\,y} K_{\nu-\frac{1}{2}}(2\pi\,|n|\,y) y^s \frac{\mathrm{d}y}{y}$$

converges absolutely. Substitute $y \to 2\pi n y$ and note that since ϕ is even, the Fourier coefficients satisfy $\phi_{-n} = \phi_n$. We have

$$\begin{aligned} N^{\frac{s}{2}} \int_0^\infty \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y} =& 4\,(2\pi)^{-s-\frac{1}{2}} N^{\frac{s}{2}} \sum_{n=1}^\infty \phi_n n^{-s} \int_0^\infty K_{\nu-\frac{1}{2}}(y) y^{s+\frac{1}{2}} \frac{\mathrm{d}y}{y} \\ =& \pi^{-s-\frac{1}{2}} N^{\frac{s}{2}} \Gamma\left(\frac{1+s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right) L_\phi(s) \\ =& (2i)^0 \pi^{-s-\frac{1}{2}} N^{\frac{s}{2}} \Gamma\left(\frac{s+1+0-\nu}{2}\right)\Gamma\left(\frac{s+0+\nu}{2}\right) L_\phi(s) \\ =& \mathfrak{L}_\phi(s) \ . \end{aligned} \tag{20.1}$$

In the last equation the formula for the Mellin transform of the K-Bessel function in Lemma 19.2 was applied. Now we calculate the above Mellin transform using Riemann's trick and breaking the integral into two parts. We have

$$N^{\frac{s}{2}} \int_0^\infty \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y} = N^{\frac{s}{2}} \int_0^{\frac{1}{\sqrt{N}}} \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y} + N^{\frac{s}{2}} \int_{\frac{1}{\sqrt{N}}}^\infty \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y} \ .$$

Apply the fact that ϕ is an eigenform of the Fricke involution with eigenvalue a in the first integral. After that substitute $y \to (Ny)^{-1}$ in the first integral. This gives us

$$N^{\frac{s}{2}} \int_0^\infty \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^s \frac{\mathrm{d}y}{y}$$

$$
\begin{aligned}
=&N^{\frac{s}{2}}\int_0^{\frac{1}{\sqrt{N}}} a\phi\left(\begin{pmatrix}\frac{y}{Ny^2} & \\ & 1\end{pmatrix}\right) y^s\frac{\mathrm{d}y}{y}+\int_{\frac{1}{\sqrt{N}}}^{\infty}\phi\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\left(\sqrt{N}y\right)^s\frac{\mathrm{d}y}{y}\\
=&-aN^{\frac{s}{2}}N^{-s}\int_{\infty}^{N\frac{1}{\sqrt{N}}}\phi\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right) y^{-s}\frac{\mathrm{d}y}{y}+\int_{\frac{1}{\sqrt{N}}}^{\infty}\phi\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\left(\sqrt{N}y\right)^s\frac{\mathrm{d}y}{y}\\
=&\int_{\frac{1}{\sqrt{N}}}^{\infty}\phi\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\left[a\left(\sqrt{N}y\right)^{-s}+\left(\sqrt{N}y\right)^{s}\right]\frac{\mathrm{d}y}{y}\ .
\end{aligned}
$$

The cuspidality of ϕ implies rapid decay in the cusps in the variable y, hence the above integral is an entire function on $\mathbb{C}$. Further $a = \pm 1$ implies that the above integral is invariant up to the constant factor $a = (-1)^{\epsilon}a$ under the transformation $s \to -s$. Comparison with (20.1) gives the functional equation for the even newform ϕ.

(2) 2. Case: ϕ is odd. In the case of an odd newform the antisymmetry of the Fourier coefficients would imply that the above Mellin transform vanishes, so our approach is to take the derivative of ϕ. Again the cuspidality implies the absolute convergence of the Mellin transform

$$
\begin{aligned}
&N^{\frac{s}{2}}\int_0^{\infty}\frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix}y & x\\ & 1\end{pmatrix}\right)\right)\Bigg|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\
=&N^{\frac{s}{2}}\int_0^{\infty}\frac{\partial}{\partial x}\left(\sum_{n\neq 0}2\phi_n\sqrt{|n|\,y}K_{\nu-\frac{1}{2}}(2\pi\,|n|\,y)e(nx)\right)\Bigg|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\
=&N^{\frac{s}{2}}\int_0^{\infty}\left(\sum_{n\neq 0}2\phi_n\sqrt{|n|\,y}K_{\nu-\frac{1}{2}}(2\pi\,|n|\,y)2\pi i n\right) y^{s+1}\frac{\mathrm{d}y}{y}
\end{aligned}
$$

of the derivative of ϕ. Substitute $y \to 2\pi n y$ and note that since ϕ is odd the antisymmetry of the Fourier coefficients implies $\phi_{-n}(-n) = \phi_n n$. We have

$$
\begin{aligned}
&N^{\frac{s}{2}}\int_0^{\infty}\frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix}y & x\\ & 1\end{pmatrix}\right)\right)\Bigg|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\
=&8\pi i\,(2\pi)^{-s-\frac{3}{2}}N^{\frac{s}{2}}\sum_{n=1}^{\infty}\phi_n n^{-s}\int_0^{\infty}K_{\nu-\frac{1}{2}}(y)y^{s+\frac{3}{2}}\frac{\mathrm{d}y}{y}\\
=&2i\pi^{-s-\frac{1}{2}}N^{\frac{s}{2}}L_{\phi}(s)\Gamma\left(\frac{2+s-\nu}{2}\right)\Gamma\left(\frac{s+1+\nu}{2}\right)\\
=&(2i)^1\pi^{-s-\frac{1}{2}}N^{\frac{s}{2}}\Gamma\left(\frac{s+1+1-\nu}{2}\right)\Gamma\left(\frac{s+1+\nu}{2}\right)L_{\phi}(s)\\
=&\mathfrak{L}_{\phi}(s)\ . \qquad (20.2)
\end{aligned}
$$

In the last equation the formula for the Mellin transform of the K-Bessel function in Lemma 19.2 was applied. Now we calculate the above Mellin transform through breaking the integral into two parts and differentiating the Fricke involution in the first integral. This gives us

$$\begin{aligned}&N^{\frac{s}{2}}\int_0^\infty \frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ =&N^{\frac{s}{2}}\int_0^{\frac{1}{\sqrt{N}}} \frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}+N^{\frac{s}{2}}\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\,.\end{aligned}$$

Apply the fact that ϕ is an eigenform of the Fricke involution with eigenvalue a in the first integral and then differentiate the integrand functions using the chain rule. Note that the variables x and y are interchanged. We have

$$\begin{aligned}&N^{\frac{s}{2}}\int_0^\infty \frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ =&N^{\frac{s}{2}}\int_0^{\frac{1}{\sqrt{N}}} \frac{\partial}{\partial x}\left(a\phi\left(\begin{pmatrix} \frac{y}{N(y^2+x^2)} & -\frac{x}{N(y^2+x^2)}\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ &+N^{\frac{s}{2}}\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ =&N^{\frac{s}{2}}a\int_0^{\frac{1}{\sqrt{N}}} \nabla\phi\left(\begin{pmatrix} \frac{y}{N(y^2+x^2)} & -\frac{x}{N(y^2+x^2)}\\ & 1\end{pmatrix}\right)\begin{pmatrix}\frac{x^2-y^2}{N(x^2+y^2)^2} & \frac{-2xy}{N(x^2+y^2)^2}\\ \frac{2xy}{N(x^2+y^2)^2} & \frac{x^2-y^2}{N(x^2+y^2)^2}\end{pmatrix} e_2\Bigg|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ &+N^{\frac{s}{2}}\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) y^{s+1}\frac{\mathrm{d}y}{y}\\ =&aN^{\frac{s}{2}}\int_0^{\frac{1}{\sqrt{N}}} \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} \frac{1}{Ny} & \\ & 1\end{pmatrix}\right)\frac{(-1)}{Ny^2} y^{s+1}\frac{\mathrm{d}y}{y}+N^{\frac{s}{2}}\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) y^{s+1}\frac{\mathrm{d}y}{y}\,.\end{aligned}$$

Now substitute $y \to (Ny)^{-1}$ in the first integral. This gives us finally

$$\begin{aligned}&N^{\frac{s}{2}}\int_0^\infty \frac{\partial}{\partial x}\left(\phi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)\right)\Big|_{x=0} y^{s+1}\frac{\mathrm{d}y}{y}\\ =&aN^{\frac{s}{2}}\int_\infty^{\frac{1}{N\frac{1}{\sqrt{N}}}} N^{-s}\frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) y^{-s+1}\frac{\mathrm{d}y}{y}+N^{\frac{s}{2}}\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) y^{s+1}\frac{\mathrm{d}y}{y}\\ =&\int_{\frac{1}{\sqrt{N}}}^\infty \frac{\partial \phi}{\partial x}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\left[-a\left(\sqrt{N}y\right)^{-s}+\left(\sqrt{N}y\right)^{s}\right]\mathrm{d}y\,.\end{aligned}$$

Again this is an entire function on $\mathbb{C}$, which is up to the constant factor $-a = (-1)^\epsilon a$ invariant under the transformation $s \to -s$. Comparison with (20.2) gives the functional equation for the odd newform ϕ.

□

Finally an explicit description of the cusps and their width for the congruence subgroup $\Gamma_0(N)$ in the case of a squarefree N is given. The following lemma gives an explicit description of the inequivalent cusps, their width, scaling matrices and a set of right coset representatives for $\Gamma_0(N)$ in the case of a squarefree N.

Lemma 20.4. *Let N be a positive squarefree integer.*

(1) A set of coset representatives for $\Gamma_0(N) \setminus SL_2(\mathbb{Z})$ is given by

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| (c,d) = 1, d \mid N, 0 < c \leq \frac{N}{d} \right\} ,$$

where for each c, d we choose a, b so that $ad - bc = 1$.

(2) A set of inequivalent cusps for $\Gamma_0(N)$ is given by $\left\{ \frac{1}{h} \middle| h \mid N \right\}$.

(3) The width m_h of the cusp $\frac{1}{h}$ satisfies the formula $m_h = \frac{N}{h}$. A scaling matrix is given through $\sigma_h := \begin{pmatrix} 1 & \\ h & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_h} & \\ & \sqrt{m_h}^{-1} \end{pmatrix}$.

(4) The double coset decomposition

$$\Gamma_0(N) \setminus SL_2(\mathbb{Z}) / P_{min} = \dot{\bigcup_{h \mid N}} \Gamma_0(N) \begin{pmatrix} 1 & \\ h & 1 \end{pmatrix} P_{min}$$

is valid.

Proof. Part (1) and (2) is the version stated in [**8**, prop. 3.3.7, prop. 3.3.8], the proof for an arbitrary level can be found in [**25**]. In [**5**, Ch. 2] the explicit formulas for the width, the scaling matrices in part (3) and the double coset decomposition in part (4) can be found. Further in [**5**] the question how scaling matrices to different representatives of the same cusp are related is discussed in much greater detail and for arbitrary level.

□

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