Topological construction of C^*-correspondences for groupoid C^*-algebras

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प्रिय आई-बाबा आणि श्रीगुरुचरणी अर्पण

To beloved Mumma, Father and Shree Guru

यथा शिखा मयूराणां नागाणां मणयो यथा ।
तद्वद् वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम् ॥

— वेदाङ्ज्योतिषातील श्लोक ३५ मधे
बदल करून बनवलेला श्लोक

As the (beautiful) crests (on the heads) of the peacocks and the (precious) stones (on the hoods) of the (holy) cobras, Mathematics resides at the topmost position among all of the Vedangas*—the auxiliary disciplines*[1] *of Vedas.*

— A modified Verse 35 in Vedangajyotish[2]

[1]"*Vedangas*" or "*Vedangashatrani*" are the six auxiliary disciplines associated with the studies of *Vedas*.
[2] http://gretil.sub.uni-goettingen.de/gretil/1_sanskr/6_sastra/8_jyot/lagrvvju.htm

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Rohit Dilip Holkar,
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आभारप्रदर्शन

पीएच्.डी.ही पदवी ही सर्वोच्च शैक्षणिक पात्रता मानली जाते. ग्योटींगेनच्या "ग्यॉर्ग ऑगुस्ट विद्यापीठा" सारख्या प्रतिष्ठीत विद्यापीठातून ही पदवी मिळणे, हा माझ्यासाठी मोठा आनंदाचा आणि माझ्या कुटुंबियांसाठी अभिमानाचा क्षण आहे. माझ्या दृष्टीने हे केवळ माझ्या एकट्याचे कर्तृत्व नसून, अनेकांच्या प्रयत्नांचे फळ आहे. या निमित्तने त्या सर्वांचे इथे आभार मानावेसे वाटतात.

सर्वप्रथम, माझा विद्यार्थी म्हणून स्वीकार केल्याबद्दल प्रा.डॉ.राल्फ मायर यांचे मला आभार. गणिताच्या "नॉनकम्प्युटेटिव्ह जिओमेट्री" या शाखेतील अनेक विषय त्यांनी मला शिकवले. गणित शिकवण्यासठी आणि गणितविषयक चर्चा करण्यासठी प्रा.मायर कायमच तयार असत. माझ्यासाठी त्यांच्याकडे वेळ नाही असे कधीच झाले नाही. ते आजारी असतानाही, आम्ही दूरध्वनीवरुन चर्चा केल्या आहेत! ते एक अतिशय संयमी शिक्षक आहेत. एखादी सिद्धता वा संकल्पना न समजल्यास, ती त्यांना पुन्हा पुन्हा सांगायला लावली तरीही ते त्रस्त न होता समजावून सांगत. माझ्या स्वतःच्या कल्पनांवर काम करण्याचे पूर्ण स्वातंत्र्य त्यांनी मला दिले. या कल्पनांवर काम करण्यासाठी सर्वतोपरी सहाय्य त्यांनी केले. अतिशय संयमाने आणि काळजीपूर्वक त्यांनी प्रबंधाची पहिली प्रत तपासली. तीत सुधारणा करुन त्यानी ती आत्ता दिसत आहे, अशा स्वरुपात आणली. जर्मनीत म्हणतात त्याप्रमाणे, प्रा.मायर मला "डोक्टोरफाटर" म्हणजेच शब्दशः "पीएच.डी. काळातील पिता" म्हणून मिळाले, हे मी माझे नशीब समजतो. केवळ गणिती म्हणूनच नाही, तर एक उत्तम माणूस म्हणूनही ते मला अनुभवायला मिळाले. त्यांनी माझ्या समोर एक उत्तम मार्गदर्शक, शिक्षक अणि गणिती कसा असावा याचा आदर्श ठेवला आहे.

माझे द्वितीय प्रबंध मार्गदर्शक प्रा.थोमास शिक्, यांचाही मी आभारी आहे. ते अत्यंत काळजीपुर्वक माझे काम तपासत आणि त्यातील उणीवा दाखवून त्या दूर करण्यासाठी मार्गदर्शन करत. प्रबंध तपासण्याच्या काळात, त्यांच्या व्यस्त वेळापत्रकामध्ये त्यांनी माझे लिखाण तपासण्याला अग्रक्रम दिला. या काळात त्यांनी कार्यालयीन कामांतही खूप मदत केली. या सर्वांबद्दल मी त्यांचा अतिशय ऋणी आहे.

माझ्या कामावर ज्यांचा प्रभाव स्पष्टपणे जाणवतो, ते प्रा.जॉ रेनॉ. त्यांनी मला "ग्रुपॉईडस्"चे अंतरंग उलगडून दाखवले. या विषयाची मुहूर्तमेढ त्यांनी रोवली होती. अनेक महत्त्वाच्या आणि तांत्रिक बाबी त्यंनी मला शिकवल्या. चर्चा करण्यासाठी आणि गणित शिकवण्यासाठी ते कायमच तत्पर असतात. त्यांच्या सहवासातून मी कामाबाबतची आचारसंहिता म्हणजेच 'वर्क एथिक्स्' शिकलो. पुढील वाटचालीत त्याचा मला नक्कीच फार उपयोग होईल. प्रबंध समितीमधील 'रेफरी' होण्याची तयारी त्यांनी दर्शवली. या सर्वच बाबींसाठी मी त्यांचा अतिशय ऋणी आहे.

इरॅस्मस मुन्डुस च्या "युरोइन्डिया" ने दिलेली शिष्यवृत्ती, "ग्राजुएटीन कोलीग 1493" ने दिलेली शिष्यवृत्ती आणि प्रा.मायर यंनी केलेली मदत यांमुळे मला जर्मनीत राहून शिकणे, अधिवेशनांना जाणे, इतर गणित्यांना भेटणे, वाचन-साहित्य विकत घेणे इत्यादी शक्य झाले. या सर्वा संस्था आणि व्यक्तींचे त्यांनी केलेल्या आर्थिक मदतीबद्दल आभार.

माझ्या आई-बाबांनी या काळात मला अतिशय मोलाची साथ दिली. दूर असूनही, मला कधीच एकटे वाटू दिले नाही. माझा राग न् लहरीपणा त्यांनी शांतपणे सहन केला. आईने केलेल्या कष्टांमुळे मी शाळा-कॉलेजात जाऊ शकलो. बाबांनी मला चिकाटी शिकवली. या दोघांनी केलेले कष्ट आणि त्यांची शिकवण यांनी मला प्रतिकूल परीस्थितीत पुढे जात राहण्यासाठी बळ दिले. आपल्या दादाचा अभ्यास नीट व्हावा, त्याला अभ्यासाला जास्त वेळा मिळवा म्हणून घराची जबाबदारी आपल्या खांद्यावर घेणार्या माझ्या छोट्या ताईची, पूजाची उणीव या वेळी सतत जाणवते.

माझे फर्ग्युसन मधील शिक्षक डॉ.वि.वि.आचार्य यांचे वर्णन केवळ "गॉडफादर" या इंग्रजी संकल्पनेनेच करता येणे शक्य आहे. त्यांनी मला शिकवलेले गणित आणि व्यवहरज्ञान दोन्ही मला फार मोलाचे वाटते. त्यांच्या मार्गदर्शनबद्दल आणि प्रेमाबद्दल मी त्यांचा ऋणी आहे. डॉ.आचार्य आणि श्री.डी.व्ही.कुलकर्णी जर मला फर्ग्युसनमधे भेटले नसते, तर मी इथवर येणे शक्यच नव्हते! आय्.एस्.आय्.बेंगळुरू मधील प्रा.बा.सुरी आणि प्रा.अ.नवलेकर यांच्या पाठींब्याबद्दल मी त्यांचा आभारी आहे.

सहस्त्रबुद्धे काकू, काका आणि आजोबांनी माझी शालेय काळात स्वतःच्या मुलाप्रमाणे काळजी घेतली. त्यांच्याप्रती ऋण व्यक्त केल्याशिवाय हे आभारप्रदर्शन पूर्ण होणे शक्य नाही. अण्णा, दादा, संतोष, सोमा, टाकळीची आज्जी, सर्वच मावश्या, राणीअक्का यांनी मला कधी एकटे वाटू दिले नाही. आशिष, प्रशांत, नितीश, शौनक, नम्रता, अदिती, ऋजुता या सतत

जवळ राहिलेल्या मित्रांना सलाम!सुलीमान, लामा आणि त्यांची गोड मुले: हैदर नि जोसेफ या कुटूंबाने मला त्यांच्यातील एक सदस्य करुन घेतले. त्यांनी मला एकटेपणा जाणवू दिला नाही. त्यांची माया मी विसरू शकत नाही.

गणित विषयक चर्चांबद्दल मी डु ली, एल्कायुम् मोन्टाऊ, गिऑर्गी नादरेशव्हिल्ली, इन्द्रवा रॉय, लीऊ बेई, लुका जॉर्जेट्टी, रॉबिन डीले, सुलीमान अल्बंदिक अणि सुतनू रॉय यांचा आभरी आहे.

अभिजीत बेन्द्रे, अभिजीत बोरकर, अबीर, अमृता, अनुशा, अप्रमेयन, अरुण, भक्ती, भूषण, कारोलीन, गिरीशा, हेमंत, लावण्या, मेलीना, मृण्मयी, ना, निकुंज, नितीन, प्रद्युम्न, रसिका, रिहो, ऋषिकेश पंडीत, ऋषिकेष साळुंखे, युंयू, संमती, संदेश, श्रेयस आणि योगेश यांच्या सहवासात माझा कामाव्यतिरिक्तचा वेळ अतिशय आनंदात गेला.

जाता जाता गणित विभागातील सर्व शिक्षकेतर कर्मचार्यांचे: सौ.बारान, सौ.डिंगेनओट्टो, सौ. गीजकिंग, सौ. वासमुथ-क्रोकर, कु.हेत्झींश यांचे मनःपुर्वक आभार. कार्यालयीन कामांत त्यांनी फारच मदत केली.

हे लिखाण गणिताच्या दृष्टीने, व्याकरणाच्या दृष्टीने आणि भाषिकदृष्ट्या बिनचूक असावे याकरीता मी बरेच प्रयत्न केले आहेत. प्रा.मायर यांनी सातत्याने या कामी मला मदत केली आहे. तरीही लिखाणामधे काही उणिवा राहिल्यास, तो दोष माझाच समजावा.

रोहित दिलीप होळकर,
ऑगस्ट 2014, ग्योटींगेन.

Abstract

Let G and H be locally compact, Hausdorff groupoids with Haar systems. We define a topological correspondence from G to H to be a G-H bispace X carrying a G-quasi invariant and H-invariant family of measures. We show that such a correspondence gives a C^*-correspondence from $C^*(G)$ to $C^*(H)$. If the groupoids and the spaces are second countable, then this construction is functorial. We show that under a certain amenability assumption, similar results hold for the reduced C^*-algebras. We apply this theory of correspondences to study induction techniques for groupoid representations, construct morphisms of Brauer groups and produce some odd unbounded KK-cycles.

Contents

Introduction

A C* *algebraic correspondence* H from a C*-algebra A to B is an A-B-bimodule which is a Hilbert B-module and A acts on H via the adjointable operators. Let $A = \mathrm{C}^*(G,\alpha)$ and $B = \mathrm{C}^*(H,\beta)$ where the ordered pairs (G,α) and (H,β) consist of a locally compact Hausdorff groupoid and a Haar system for it. Given a G-H-bispace carrying a G-quasi-invariant and H-invariant family of measures, we show that if the H-action is proper, then $C_c(X)$ can be completed into a C*-correspondence from $\mathrm{C}^*(G,\alpha)$ to $\mathrm{C}^*(H,\beta)$.

If G is a locally compact groupoid and α is Haar system for G, we call the ordered pair (G,α) *a locally compact groupoid with a Haar system.*

Morita equivalence of C*-algebras is defined by the existence of an imprimitivity bimodule, a special kind of C*-correspondence. The starting point of my work is the well-known result that a Morita equivalence between two locally compact groupoids with Haar system induces a Morita equivalence between the groupoid C*-algebras [28]. The imprimitivity module is constructed directly from a bispace giving the Morita equivalence of the two groupoids. Which extra structure or conditions are needed for a bispace to give only a C*-algebraic correspondence instead of a Morita equivalence?

In general, we need a measure on the bispace as extra structure to get started. In the Morita equivalence case, a measure on the bispace appears automatically. The measure must be invariant for the right action and quasi-invariant for the left action. We also need that the right action is proper. Then a variant of the construction in [34] gives a C*-correspondence between the groupoid C*-algebras.

Two C*-algebraic correspondences H from A to B and K from B to C may be composed to a correspondence $H \hat{\otimes}_B K$ from A to C. In [9], Buss, Meyer and Zhu explain why the operation is associative and unital up to natural isomorphism. They prove that the C*-correspondences form a bicategory $\mathfrak{C}$. We construct the groupoid analogue of the category $\mathfrak{C}$ and call it the bicategory of topological correspondences, denoted by $\mathfrak{T}$. To construct $\mathfrak{T}$, we need to describe the process of composition of two topological correspondences. One of the most important constructions in this thesis is the construction of *a* composite of topological correspondences.

Let (X,λ) and (Y,μ) be topological correspondences from (G,α) to (H,β) and (H,β) to (K,κ), respectively. Then the construction of the composite G-K-bispace is well-known— the bispace is the quotient space $(X \times_{H^{(0)}} Y)/H$. We show how to compose the families of measures on X and Y to get a G-quasi-invariant and K-invariant family of measures on the composite bispace. However, the composite of families of measures is defined only *up to* isomorphism. This helps us to form the bicategory of topological correspondences $\mathfrak{T}$. We show that the assignment that a topological correspondence goes to a C*-correspondence is a homomorphism from $\mathfrak{T}$ to $\mathfrak{C}$.

We give many examples of topological correspondences. A continuous map $f\colon X \to Y$ between spaces gives a topological correspondence from Y to X, see Example 3.1.1. A continuous group homomorphism $\phi\colon G \to H$ gives a topological correspondences from G to H, see Example 3.1.5. These examples explain why the C*-functor is contravariant for spaces and covariant for groups. If ϕ in the above examples is proper, we get a correspondences from H to G (3.1.6).

Let E^1 and E^2 be locally compact, Hausdorff and second countable spaces and let $s, r\colon E^1 \to E^0$ be continuous maps. Let $\lambda = \{\lambda_e\}_{e\in E^0}$ be a continuous family of measures along s. Then $r\colon E^1 \to E^0$ gives a correspondence from E^0 to E^1 as in Example 3.1.1. And $s\colon E^1 \to E^0$ give a correspondence from E^1 to E^0, as in Example 3.1.2. These correspondences together produce a correspondence from E^0 to itself. In fact, just by applying the definition of a topological correspondence it is straightforward to check that s, r and λ give a topological correspondence from E^0 to itself. This correspondence is called a *topological quiver* by Muhly and Tomforde [29, Definition 3.1]. They construct a C*-correspondence associated to a topological quiver in [29, Section 3.1], and the construction in [29] is exactly the construction of a C*-correspondence from a topological correspondence. Muhly and Tomforde define the C*-algebra associated to a topological quiver ([29, Definition 3.17]) which includes a vast class of C*-algebras: graph C*-algebras, C*-algebras of topological graphs, C*-algebras of branched coverings, C*-algebras associated with topological relations are all associated to a topological quiver [29, Section 3.3]. We are thankful to Ralf Meyer for bringing it to our notice that a topological quiver is a topological correspondence.

A locally compact, Hausdorff space is a groupoid with a Haar system, and so is a locally compact Hausdorff group. A well-know fact about groupoid equivalence is that two spaces are equivalent if and only if they are homeomorphic and two groups are equivalent if and only if they are isomorphic. But since any continuous map between spaces gives a topological correspondence and so does a group homomorphism, a topological correspondence is far more general than an equivalence.

In Chapter 1, we discuss some examples which mark the difference between topological groupoids and locally compact groups. Every locally compact group has a left (or equivalently right) invariant measure— the Haar measure. Moreover, this measure is unique up to a scaling factor. However, a locally compact Hausdorff groupoid does not always come with a canonical Haar system (Example 1.3.13 and 1.3.14). Example 1.3.14 shows that *even* a compact groupoid need not have a Haar system. In Example 1.3.15, we discuss a groupoid with many Haar systems. We are thankful to Ralf Meyer for Example 1.3.15.

In [39], Seda shows that if the range map is not open, then a groupoid cannot have a Haar system. Seda gives an example of a groupoid with range map not open, so the groupoid cannot have a continuous, invariant family of measures with full support. We came across this example after formulating the counterexamples above. Dana Williams and Ralf Meyer conveyed me the paper. The groupoids in our examples also do not have open range maps. However, we must mention that we did not intend to prove a general fact as Seda. A more recent literature survey showed that Ramsay discusses Example 1.3.14 in [32].

The first nice application of the theory of topological correspondences is due to Renault [35]. Renault proves that a topological correspondence (X, λ) from (G, α) to (H, β) induces a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$ between the categories of representations of H and G.

An equivalence of groupoids is an invertible arrow in $\mathfrak{T}$. This fact along with the functoriality of our construction implies the famous result of groupoid equivalence in [28], which says that the C^*-algebras of two equivalent locally compact, Hausdorff, second countable groupoids with Haar systems are strongly Morita equivalent.

The KK-theory of Kasparov [20] has proved a valuable tool in the study of C^*-algebras. Since groupoid C^*-algebras cover a huge class of C^*-algebras, it is very natural to look for *geometrical* or *topological* flavours of KK-theory for groupoid C^*-algebras. Such attempts are made in the literature. For example, [22] develops a groupoid equivariant theory for Banach bundles to prove some cases of the Baum-Connes conjecture. Macho Stadler and O'uchi [25] give a definition of topological correspondences and show that when certain conditions are satisfied, a topological correspondence from (G, α) to (H, β) gives an element in $\mathrm{KK}(C^*(G, \alpha), C^*(H, \beta))$. Tu [42] proves a similar result for non-Hausdorff groupoids. The correspondences defined by Macho Stadler and O'uchi are special cases of the topological correspondences we define, see Example 3.1.8. Given a groupoid G with a Haar system α and a groupoid homomorphism $c\colon G \to \mathbb{R}_+^*$, we use topological correspondences to produce some unbounded KK-cycles between certain subgroupoids of G.

Let G be a groupoid endowed with a Haar system α. Given a groupoid homomorphism $G \to \mathbb{R}_+^*$, Mesland [26] produces an $\mathbb{R}$-equivariant unbounded KK-cycles from $C^*(G, \alpha)$ to $C^*(\ker(c), \kappa)$, where κ is a given Haar system on the subgroupoid $\ker(c) \subseteq G$. We generalise this result by producing a similar KK-cycle from $C^*(H, \beta)$ to $C^*(\ker(c), \kappa)$, where $H \subseteq G$ is an open subgroupoid and β is a Haar system on H.

We mention spatial hypergroupoids. Though hypergroupoids are not an application of correspondences, we came across them while studying topological correspondences. Furthermore, spatial hypergroupoids produce the compact operators on the Hilbert module that a proper H-space carrying an invariant family of measures produces, see Proposition 2.2.20.

The relation between the Brauer group of a groupoid and groupoid equivalence is studied in [21] by Kumjian, Muhly, Renault and Williams. We show that a Hilsum-Skandalis morphism from a groupoid H to G induces a homomorphism $\mathrm{Br}(G) \to \mathrm{Br}(H)$.

Now we talk about the hypotheses, motivations and techniques. We work with locally compact, Hausdorff groupoids. Let (H, β) be a pair consisting of a locally compact, Hausdorff groupoid with a Haar system. The construction of a Hilbert module from a proper H-space carrying a continuous, invariant family of measures works when the space and the groupoid is locally compact and Hausdorff. However, the main result of constructing a C^*-correspondence from a topological correspondence holds for paracompact, locally compact, Hausdorff spaces, and locally compact and Hausdorff groupoids. This is because we use Lemma 1.3.28 to prove that the representation of the left groupoid on the Hilbert module is non-degenerate (Lemma 2.3.1). And Lemma 1.3.28 needs paracompactness. Since we wish to prove the functoriality of this constructions, the functoriality discussion assumes that all the groupoids and the spaces are second countable, locally compact and Hausdorff. The second countability hypothesis can be replace by paracompactness.

We also assume that the measures are positive Radon. We use the Radon-Nikodym derivatives every now and then and hence we need that all the measures are σ-finite. Many results (especially in the first chapter) are valid with fewer assumptions, hence we mention hypotheses in the beginning of the chapter or section or beginning of a discussion.

For groupoid actions we do not assume that the momentum maps are open or surjective. Neither do we demand a family of measures along a continuous open map $f\colon X \to Y$ to have full support in each fibre. Since we work with groupoids with Haar systems, most of the times the source map (equivalently the range map) of a groupoid is automatically open.

All the Hilbert spaces in this thesis are separable.

Our notion of C*-correspondence (Definition 1.7.3) is wider, in the sense that many authors demand that the Hilbert module involved in a C*-correspondence is *full*, or for some authors a C*-correspondence is what we call a *proper* correspondence (see Section 1.7.2).

The process of constructing a C*-correspondence from a topological correspondence is divided into two main parts: constructing the Hilbert module and defining the representation of the left groupoid C*-algebra on this Hilbert module. For the first part, we use the representation theory of groupoids and the transverse measure theory introduced by Renault in [34]. For the second part, our motivation and techniques come from the theory of quasi-invariant measures for locally compact groups.

Most of the examples of topological correspondences are topological analogues of standard examples of C*-correspondences.

Our main references for unbounded bivariant K-theory are the original work of Baaj and Julg [2] and Mesland's work [26].

The main reference for bicategories is Bénabou's report [3]. The relatively modern report [24], also provides a good categorical structure to our work. Readers should keep in mind that the direction of arrows in the commutative diagrams in Bénabou's book is opposite to our standard conventions.

Chapterwise description of the contents

Chapter 1: In this chapter we discuss the analysis on locally compact groupoids, proper actions of groupoids and the cohomology theory for groupoids. We discuss the preliminaries regarding topological and Borel groupoids, actions of groupoids and invariant families of measures. In the literature, the experts assume many results about proper actions without proving them. We write detailed proofs of some of these important results which are necessary for our work. We prove that the quotient of a locally compact, Hausdorff (second countable) space by a proper action inherits the *nice* topological properties, that is, the quotient is also locally compact, Hausdorff (second countable, respectively) provided that the source map of the groupoid is open.

Let G be a groupoid, $f\colon X \to Y$ a G-map and λ a G-equivariant continuous family of measures along f. We prove that λ induces a *continuous* family of measures on the quotient spaces $[f]\colon X/G \to Y/G$.

We write a brief introduction to the cohomology theory for groupoids introduced by Westman [43]. One of the main results shows that for a proper groupoid the first Borel (as well as continuous) cohomology group with real coefficients is trivial, see Proposition 1.4.10. We thank Renault for this result. Then we discuss quasi-invariant measures.

In the last part of this chapter we discuss the representation theory of locally compact groupoids with a Haar system. The fundamental work in the representation theory of groupoids is Renault's thesis [33], in which he proves the first version of his famous disintegration theorem for locally compact groupoids. Renault uses quasi-invariant measures on the space of units of the groupoid to integrate a representation of the groupoid. The disintegration of representations is concerned with proving the existence of a suitable quasi-invariant measure on the space of units. The proof of the disintegration theorem in [33] needs a technical condition, namely, if (G, α) is the groupoid, then G should have sufficiently many non-singular G-sets (see [33, Definition 1.3.27]).

Renault overcomes this technical assumption in the next work [34], where a much more general version of the disintegration theorem is proved. This version of the disintegration theorem does not need the existence of sufficiently many non-singular G-sets. Furthermore, the theorem is proved for locally Hausdorff groupoids. Renault uses the theory of transverse measures to prove this flavour of the disintegration theorem. We discuss this version of the disintegration theorem after discussing the one in [33]. Since transverse measures play an important role here, we explore the theory of transverse measures from Appendix 1 of [1]. The appendix is self-contained and complete, but, a young student like me found it very brief. Hence we take it as an exercise to write all computations involved in this appendix in detail.

A quick review of some notions of amenability of groupoids from [1] follows the discussion of representation theory. We sketch the well-known fact that the full and reduced C*-algebras of an amenable topological groupoid are isomorphic.

The chapter ends with a short list of definitions related to C*-correspondences.

Chapter 2: This chapter contains the main construction. The following is our definition of a topological correspondence.

Definition (Topological correspondence, Definition 2.1.1)**.** A *topological correspondence* from a locally compact Hausdorff groupoid with a Haar system (G, α) to a locally compact Hausdorff groupoid with a Haar system (H, β) is a pair (X, λ) where:

i) X is a locally compact, Hausdorff, second countable G-H-bispace;

ii) $\lambda = \{\lambda_u\}_{u \in H^{(0)}}$ is an H-invariant continuous family of measures along the momentum map $s_X \colon X \to H^{(0)}$;

iii) the action of H is proper;

iv) Δ is a continuous function $\Delta : G \ltimes X \to \mathbb{R}^+$ such that for each $u \in H^{(0)}$ and $F \in C_c(G \times_{s_G, G^{(0)}, r_X} X)$

$$\int_{X_u} \int_{G^{r_X(x)}} F(\gamma^{-1}, x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x) = \int_{X_u} \int_{G^{r_X(x)}} F(\gamma, \gamma^{-1}x)\,\Delta(\gamma, \gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x).$$

Immediately after the definition of topological correspondence, we discuss the role of the adjoining function. Then we write the formulae of the action of $C_c(G)$ and $C_c(H)$ on $C_c(X)$ and the formula of a $C_c(H)$-valued inner product on $C_c(X)$. Lemma 2.1.11 shows that $C_c(X)$ is a $C_c(G)$-$C_c(H)$-bimodule and that the formula for the the inner product indeed defines a $C_c(H)$-conjugate bilinear map on $C_c(X)$. Now we have to extend this setup to the C*-algebras to get the C*-correspondence.

We complete this setup to a C*-correspondence in two parts: constructing a $\mathrm{C}^*(H, \beta)$-Hilbert module $\mathcal{H}(X)$ and defining a representation of $\mathrm{C}^*(G, \alpha)$ on this Hilbert module.

When the action of H on X is free, it is not hard to construct the Hilbert module $\mathcal{H}(X)$ using the theory of groupoid equivalence. This construction does not need sophisticated machinery but only the observation that $(X \times_{H^{(0)}} X)/H$ is a groupoid with a Haar system. We first construct $\mathcal{H}(X)$ in this case.

Then we turn our attention to the general case, that is, when the action of H is not free. Using representation theory and the theory of transverse measures, we construct $\mathcal{H}(X)$. This technical construction requires the disintegration theorem.

In the latter part we define a representation of $\mathrm{C}^*(G, \alpha)$ on $\mathcal{H}(X)$ using the adjoining function Δ. To check that this representation is continuous we use the disintegration theorem.

We advise the reader to jump to Section 3.1 (Chapter 3) after the discussion that follows Definition 2.1.1 and then come back. Section 3.1 contains many examples of topological correspondences, ranging from continuous maps to generalised induction and restriction correspondences.

The notion of topological correspondence does not carry over to the reduced C^*-algebras directly. We are very thankful to Ralf Meyer for pointing out this fact and correcting it. If the action of the left groupoid is amenable, then a topological correspondence from (G, α) to (H, β) produces a C^*-correspondence from $\mathrm{C}^*_{\mathrm{r}}(G, \alpha)$ to $\mathrm{C}^*_{\mathrm{r}}(H, \beta)$. Lemma 1.3.29 shows that a proper action of a groupoid with a Haar system is amenable. Hence if the left action is proper, a topological correspondence produces a C^*-correspondence between the reduced C^*-algebras.

The middle part of Chapter 2 discusses the process of composing correspondences. We thank Renault a lot for sharing his deep insight in the theory of groupoids, which helped us to construct the family of measures on the composite correspondence.

The end of the Chapter is devoted to the bicategory of topological correspondences and the functoriality of the assignment $X \mapsto \mathcal{H}(X)$. Many results in the section are intuitively obvious, but the detailed proofs are very technical and complicated. In this document, the functoriality is the most technical part of writing, and hence for reading, too.

Chapter 3: This chapter contains many examples of topological correspondences. We mention the induction of representations of groupoids, discussed in Renault's recent work [35]. Renault discusses how the induction process for groupoids works for groups. One of the important results of ours in this chapter is the *explicit* construction of the *induction* correspondence using pull-backs of certain subsets of the space of the units along the source or the range maps which leads us to Proposition 3.2.2. This theorem relates our theory of correspondences to the classical induction process; which becomes a corollary to this Theorem. That is, we get

Corollary (Theorem 6.13, in [15]). *Suppose G is a locally compact group and H is a closed subgroup, with modular functions Δ_G and Δ_H. Let μ be a pseudomeasure of positive type on H, let σ_μ be the associated unitary representation of H, and let ν be the injection of $\sqrt{\Delta_G/\Delta_H}\,\mu$ into G, that is, the pseudomeasure on G defined by*

$$\nu(f) = \int_H \sqrt{\frac{\Delta_G(\xi)}{\Delta_H(\xi)}}\, f(\xi)\,\mathrm{d}\mu(\xi).$$

Then ν is of positive type, and the associated unitary representation π_ν of G is unitarily equivalent to the induced representation $\Pi = \mathrm{ind}_H^G(\sigma_\mu)$.

For a locally compact Hausdorff space X, Folland calls a continuous linear functional on $C_c(X)$ a *pseudomeasure*. However, because of the Riesz representation theorem we prefer calling it a measure.

In Section 3.3 we briefly introduce our work in [17] on spatial hypergroupoids. Hypergroups are well-known in analysis. There are two equivalent notions of hypergroup: Jewett's [18] definition of a hypergroup is similar to that of a group except that the product of two elements of the hypergroup is a probability measure on the set, rather than an element of the set. Equivalently, in [19] a hypergroup is defined as a certain convolution algebra of measures on a space. Renault [35] adopts the latter notion of hypergroups and defines *hypergroupoids* accordingly. He proves a disintegration theorem for representations of hypergroupoids.

The first example of a hypergroupoid that we came across is called a *spatial hypergroupoid*. It is well-known that if X is a free and proper right H-space, then $G := (X \times_{s_X, H^{(0)}, s_X} X)/H$ is a topological groupoid where $s_X : X \to H^{(0)}$ is the anchor map. Furthermore, X gives an equivalence between G and H. The fact that the action of H is free plays an important role to define the composition on G. When the action of H is not free, the product of two elements in G is not an element of the set. If X carries an H-invariant family of measures, however, then it is possible to define a *-algebra structure on $C_c(G)$. Assume that β is a Haar system on H. Then we complete the *-algebra $C_c(G)$ to a C^*-algebra $\mathrm{C}^*(G)$ using the representations of (H, β). Our construction shows that $C_c(X)$ can be completed to a $\mathrm{C}^*(G)$-$\mathrm{C}^*(H)$-Hilbert bimodule.

In this case, G is a spatial hypergroupoid. The H-invariant family of measures on X produces an invariant family of measures for G. Thus we get the first example of a hypergroupoid.

In Section 3.4 we discuss morphisms of Brauer groups. Kumjian, Muhly, Renault and Williams define the Brauer group for a locally compact Hausdorff groupoid in [21] and show that the Brauer groups of two equivalent groupoids are isomorphic. If G and H are groupoids, then we show that a Hilsum-Skandalis morphism from H to G induces a homomorphism from the Brauer group $\mathrm{Br}(G)$ to $\mathrm{Br}(H)$.

In Section 3.5 we give an application of topological correspondences in KK-theory. We extend a result of Mesland [26]. Let G be a groupoid and α a Haar system for G. Let $c : G \to \mathbb{R}_+^*$ be a homomorphism. Assume that κ is a Haar system for $\ker(c)$. Then Mesland ([26]) proves that c produces an unbounded KK-cycle from $\mathrm{C}^*(G, \alpha)$ to $\mathrm{C}^*(\ker(c), \kappa)$. We generalise this result of Mesland by replacing $\mathrm{C}^*(G, \alpha)$ by $\mathrm{C}^*(H, \beta)$, where $H \subseteq G$ is an open subgroupoid and β is a Haar system on H. At the end of the section we discuss a few examples of this result.

The thesis in a glance

Definitions

Definition (2.1.1 Topological correspondence)**.** A *topological correspondence* from a locally compact Hausdorff groupoid with a Haar system (G, α) to a locally compact Hausdorff groupoid with a Haar system (H, β) is a pair (X, λ) where:

i) X is a locally compact, Hausdorff, second countable G-H-bispace;

ii) $\lambda = \{\lambda_u\}_{u\in H^{(0)}}$ is an H-invariant continuous family of measures along the momentum map $s_X : X \to H^{(0)}$;

iii) the action of H is proper;

iv) Δ is a continuous function $\Delta : G \ltimes X \to \mathbb{R}^+$ such that for each $u \in H^{(0)}$ and $F \in C_c(G \times_{s_G, G^{(0)}, r_X} X)$

$$\int_{X_u}\int_{G^{r_X(x)}} F(\gamma^{-1}, x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x) = \int_{X_u}\int_{G^{r_X(x)}} F(\gamma, \gamma^{-1}x)\,\Delta(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x).$$

Definition (2.1.8 and 2.1.9)**.** The left and right actions, and the inner product] For $\phi \in C_c(G)$, $f \in C_c(X)$ and $\psi \in C_c(H)$, define functions ϕf, $f\psi$ on X as follows:

$$(\phi\cdot f)(x) := \int_{G^{r_X(x)}} \phi(\gamma) f(\gamma^{-1}x)\,\Delta^{1/2}(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma),$$
$$(f\cdot\psi)(x) := \int_{H^{s_X(x)}} f(x\eta)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta).$$

For $f, g \in C_c(X)$ define the function $\langle f, g\rangle$ on H by

$$\langle f, g\rangle(\eta) := \int_{X_{r_H(\eta)}} \overline{f(x)} g(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x).$$

Definition (2.4.18 Composition)**.** For correspondences $(X, \alpha) : (G_1, \lambda_1) \to (G_2, \lambda_2)$ and $(Y, \beta) : (G_2, \lambda_2) \to (G_3, \lambda_3)$ the composite correspondence $(\Omega, \mu) : (G_1, \lambda_1) \to (G_3, \lambda_3)$ is defined by:

i) the space $\Omega := (X \times_{s_X, G_2{}^{(0)}, r_Y} Y)/G_2$,

ii) a family of measures $\mu = \{\mu_u\}_{u\in G_3^{(0)}}$ on Ω that lifts to $\{b\alpha\times\beta_u\}_{u\in G^{(0)}}$ on $Z := X \times_{s_X, G_2{}^{(0)}, r_Y} Y$ for a cochain $b \in \mathrm{C}^0_{G_3}(Z \times_{\pi,\Omega,\pi} Z, \mathbb{R}^*_+)$ satisfying $d^0(b) = \Delta$. Here $\pi : Z \to \Omega$ is the quotient map, the fibre product $Z \times_{\pi,\Omega,\pi} Z$ is thought of as the groupoid of the equivalence relation induced by π on Z and Δ is a 1-cocycle $Z \times_{\pi,\Omega,\pi} Z \to \mathbb{R}^*_+$ given by

$$((x, y), (x\gamma, \gamma^{-1}y)) \mapsto \Delta_2(\gamma, \gamma^{-1}y),$$

where Δ_2 is the adjoining function for (Y, β).

We briefly describe the terms in (ii) above: $\{\alpha\times\beta_u\}_{u\in G_2{}^{(0)}}$ is a continuous family of measures on the fibre product $Z = X \times_{s_X, G_2{}^{(0)}, r_Y} Y$ (see Lemma 2.4.8). For $f \in C_c(Z)$

$$\int f\,\alpha\times\beta_u := \int_X\int_{Y_u} f(x, y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y).$$

There is a continuous 0-cocycle $b : Z \to \mathbb{R}^*_+$ such that $\Delta = \frac{b\circ\pi_2}{b\circ\pi_1} = d^0(b)$, where π_i for $i = 1, 2$ is the projection on the i^{th} factor $Z \times_{\pi,\Omega,\pi} Z \to Z$ (see Lemma 2.4.9). Remark 2.4.13 says that for $u \in G_3{}^{(0)}$ and f as above, there is measure μ_u on Ω with

$$\int\left(\int f(x\gamma, \gamma^{-1}y)\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right)\mathrm{d}\mu_u[x, y] = \int f(x, y) b(x, y)\,\mathrm{d}\alpha_{r_y(y)}(x)\,\mathrm{d}\beta_u(y).$$

The family of measures $\{\mu_u\}_{u\in G_3{}^{(0)}}$ is the required family of measures (see Proposition 2.4.14).

Definition (3.4.1, Hilsum–Skandalis morphism)**.** *A Hilsum–Skandalis morphism* from a groupoid H to a groupoid G is an H-G-bispace X such that

i) the action of G is free and proper;

ii) the left momentum map induces a bijection from X/G to $H^{(0)}$.

Results

Proposition (1.2.19)**.** *Let X be an H-space and r_H open. If H acts properly, then the quotient space, X/H, is locally compact Hausdorff.*

Proposition (1.3.27)**.** *Let X and Y be proper H-spaces, let $\pi\colon X \to Y$ be a continuous surjection and let $\lambda := \{\lambda^y\}_{y\in Y}$ be a continuous family of measures along π. Then the induced family of measures, $[\lambda] := \{[\lambda]^{[y]}\}_{[y]\in Y/H}$, is a continuous family of measures with full support along $[\pi]$.*

Proposition (1.4.10)**.** *Let G be a proper groupoid and α a Haar system on G. Then every $\mathbb{R}$-valued 1-cocycle is a coboundary, that is,* $\mathrm{H}^1(G;\mathbb{R}) = 0$.

Theorem (2.2.19)**.** *Let (H,β) be a Hausdorff, locally compact groupoid with a Haar system and let X be a locally compact, Hausdorff, proper right H-space carrying an H-invariant continuous family of measures λ. Then using Formulae* (2.1.8) *and* (2.1.9) *the right $C_c(H)$-module $C_c(X)$ can be completed to a* $\mathrm{C}^*(H)$*-Hilbert module $\mathcal{H}(X)$.*

Proposition (2.2.20)**.** *Let (H,β) be a Hausdorff, locally compact groupoid with a Haar system and let X be a locally compact, Hausdorff proper right H-space carrying an H-invariant continuous family of measures λ. Then using Formulae* (2.1.8) *and* (2.1.9) *the right $C_c(H)$-module $C_c(X)$ can be completed to a* $\mathrm{C}^*_\mathrm{r}(H)$*-Hilbert module $\mathcal{H}_\mathrm{r}(X)$.*

Lemma (2.3.1)**.** *Let (X,λ) be a topological correspondence from (G,α) to (H,β), where the topologies on the groupoids are locally compact and Hausdorff. Then the action of $C_c(G)$ on $C_c(X)$ defined by Definition 2.1.8 extends to an action of* $\mathrm{C}^*(G)$ *on the* $\mathrm{C}^*(H)$*-Hilbert module $\mathcal{H}(X)$ by adjointable operators.*

Lemma (2.3.2)**.** *In the situation of the lemma above, assume, in addition, that the transformation groupoid $G \ltimes X$ is amenable, that is, the action of G on X is amenable. Then the action of $C_c(G)$ on $C_c(X)$ defined by Definition 2.1.8 extends to an action of* $\mathrm{C}^*_\mathrm{r}(G)$ *on the* $\mathrm{C}^*_\mathrm{r}(H)$*-Hilbert module $\mathcal{H}_\mathrm{r}(X)$ by adjointable operators.*

Theorem (2.3.3)**.** *Let (G,α) and (H,β) be locally compact, Hausdorff groupoids with Haar systems. If (X,λ) is a correspondence from (G,α) to (H,β) then using the family of measures λ the space $C_c(X)$ can be completed to a* C^**-correspondence $\mathcal{H}(X)$ from* $\mathrm{C}^*(G)$ *to* $\mathrm{C}^*(H)$.

Proposition (2.3.4)**.** *Let (G,α) and (H,β) be locally compact, Hausdorff groupoids with Haar systems. Let (X,λ) be a correspondence from (G,α) to (H,β). If the action of G on X is amenable, then using the family of measures λ the space $C_c(X)$ can be completed to a* C^**-correspondence $\mathcal{H}_\mathrm{r}(X)$ from* $\mathrm{C}^*_\mathrm{r}(G)$ *to* $\mathrm{C}^*_\mathrm{r}(H)$.

Corollary (2.3.5). *Assume the same hypotheses as in Theorem 2.3.3. If the action of G is proper, then* $C_c(X)$ *can be completed to a* C^**-correspondence* $\mathcal{H}_{\mathrm{r}}(X)$ *from* $\mathrm{C}^*_{\mathrm{r}}(G)$ *to* $\mathrm{C}^*_{\mathrm{r}}(H)$.

Theorem (2.4.19). *Let* $(X,\alpha)\colon (G_1,\lambda_1)\to(G_2,\lambda_2)$ *and* $(Y,\beta) : (G_2,\lambda_2)\to(G_3,\lambda_3)$ *be topological correspondences between locally compact, Hausdorff groupoids. Let* $(\Omega,\mu)\colon(G_1,\lambda_1)\to(G_3,\lambda_3)$ *be a composite of the correspondence. Then* $\mathcal{H}(\Omega)$ *and* $\mathcal{H}(X)\hat{\otimes}_{\mathrm{C}^*(G_2)}\mathcal{H}(Y)$ *are isomorphic correspondences from* $\mathrm{C}^*(G_1,\lambda_1)$ *to* $\mathrm{C}^*(G_3,\lambda_3)$.

Proposition (2.5.13). *Topological correspondences form a bicategory with the composition from Theorem 2.4.19 and some (obvious) associativity and identity isomorphisms. The groupoids are assumed to be locally compact, Hausdorff, second countable groupoids with Haar systems.*

Theorem (2.5.19). *The assignment* $X\mapsto\mathcal{H}(X)$ *is a bifunctor from the bicategory of topological correspondences* $\mathfrak{T}$ *to the bicategory of* C^**-correspondences* $\mathfrak{C}$.

Proposition (3.2.2). *Let* (G,α) *be a locally compact, Hausdorff groupoid with a Haar system and* $H\subseteq G$ *a closed subgroupoid. Let* β *be a Haar system for* H*. Then the* G*-*H*-bispace* $G_{H^{(0)}}$ *gives a topological correspondence from* (G,α) *to* (H,β)*. Here* $G_{H^{(0)}}=s_G^{-1}(H^{(0)})\subseteq G$ *with a measure family induced by the Haar system of* G *as in Example 3.1.8.*

Proposition (3.3.6). *Let* X *be a locally compact, Hausdorff proper* H*-space for a locally compact, Hausdorff groupoid with a Haar system* (H,β)*. Let* λ *be an invariant family of measures on* X*. Let* $\mathrm{C}^*(X*X/H)$ *be the completion of the *-algebra* $C_c((X*X)/H)$ *as in Theorem 3.3.3. Then* $\mathrm{C}^*(X*X/H)\simeq\mathbb{K}(\mathcal{H}(X,\lambda))$.

Theorem (3.4.15). *A Hilsum–Skandalis morphism from a groupoid* H *to a groupoid* G *induces a homomorphism from* $\mathrm{Br}(G)$ *to* $\mathrm{Br}(H)$.

Theorem (3.5.10). *Let* (G,λ) *be a second countable locally compact Hausdorff groupoid with a Haar system, let* c *be a real exact cocycle on* G *and let* H *be an open subgroupoid of* G *such that* $H^{(0)}=G^{(0)}$*. Let* α *be a Haar system for* H*. If for each* $e\in G^{(0)}$*, the measure* λ_e^{-1} *is* (H,α)*-quasi-invariant, then the operator* D *in Proposition 3.5.9 makes the* $\mathbb{R}$*-equivariant correspondence* $(\mathcal{H}(G),D)$ *into an odd* $\mathbb{R}$*-equivariant unbounded KK-bimodule from* $\mathrm{C}^*(H)$ *to* $\mathrm{C}^*(K)$.

Proposition (3.5.11). *Assume that we have the same data as in Theorem 3.5.10 and the same hypotheses. If the left action is amenable, then a similar result as in Theorem 3.5.10 holds for* $(\mathcal{H}_{\mathrm{r}}(G),D_{\mathrm{r}})$ *from* $\mathrm{C}^*_{\mathrm{r}}(H)$ *to* $\mathrm{C}^*_{\mathrm{r}}(K)$.

Chapter 1

Locally compact Hausdorff groupoids

In this chapter we shall discuss basic notions and notation about topological groupoids. The chapter discusses three main topics: actions of groupoids, some measure theoretic prerequisites, and a few other basic notions which we need for our work.

The discussion of groupoid actions is concerned with proper actions of groupoids and the topology on the corresponding quotient spaces. The measure theory part is concerned with spaces with groupoid invariant families of measures and the behaviour of the families of measures under proper actions. And the third part discusses various topics in the theory of groupoids, which include the representation theory of groupoids and groupoid C^*-algebras, groupoid cohomology, a brief survey of definitions of amenability for groupoids, and a very short collection of definitions regarding C^*-algebraic correspondences.

We prove most of the claims in the first and the second part, namely, in Section 1.2, Proper actions and quotients and Section 1.3, Proper actions and families of measures. There are many facts about proper actions and invariant families of measures which are used in the literature very often, however the proofs are left as an exercise most of the times. Or the experts assume that readers are familiar with the proofs. We take this as an exercise and write down the proofs which we could not find explicitly written in the literature. If a claim is proved already, we cite the corresponding literature.

In the last part, which discusses various topics in the theory of groupoids, all the material is well-known and well-written. Hence we merely cite the main literature, most of which is the work of J. Renault. The only differently written section is the Subsection 1.6.2, where we discuss transverse measures. This is based on Appendix I in [1]. The appendix is short and contains many ideas. Being a beginner, we take this also as an exercise and write down thorough proofs of the claims in [1, Appendix I].

In the last section, we choose our definitions for a C^*-correspondence and its morphisms. The main two reasons to write these well-known definitions are: (i) some authors do not differentiate between *proper* C^*-correspondences and C^*-correspondences and (ii) some authors assume that the Hilbert module involved in a C^*-correspondence is full.

1.1 Basics, notation and conventions

Notation and general conventions for all of the text: The symbols $\approx$ and $\simeq$ stand for homeomorphism and isomorphism, respectively. Let X be a Borel space, then $\mathcal{B}(X)$ and $\mathcal{B}^+(X)$ denote the sets of Borel functions and positive Borel functions on X, respectively. Due to the Riesz representation theorems ([38, Theorem 2.14, Theorem 6.19]), we abuse the notation for a measure in the following fashion: if μ is a Borel measure on X, then for $f \in \mathcal{B}(X)$ we write $\mu(f)$ as well as $\int f \,\mathrm{d}\mu$ to denote the integral of f with respect to μ. Let X be a topological space, then $C(X)$, $C_c(X)$ and $C_0(X)$ denote the sets of complex valued continuous function, continuous functions with compact support and continuous functions vanishing at infinity defined on X, respectively. When the sets $C_c(X)$ and $C_0(X)$ are discussed as topological spaces, we assume $C_c(X)$ is bestowed with the inductive limit topology ([15, Beginning of Section 6.3] or [4, Proposition 5, No. 4, §4, II]) and $C_0(X)$ is bestowed with the $\| \; \|_\infty$-topology. Let G be a group(oid), X a set. Let G act on X from the left (or right), see Definition 1.2.1. Then $G\backslash X$ (respectively, X/G) denotes the quotient space for the action, except in Section 1.6.2, where we write X/G for the quotient by a left action.

Definition 1.1.1. A groupoid is a small category in which every arrow is invertible.

For a groupoid G, we denote the set of objects by $G^{(0)}$ and call it the *base* of G or the *set of units* of G. For a topological groupoid G, the word *set* will be replaced by *space*. The set of arrows is denoted by $G^{(1)}$. A popular convention that we adopt is to write G itself for the set of arrows.

Each $\gamma \in G = G^{(1)}$ has a domain (or source) and a range, which we denote $s_G(\gamma)$ and $r_G(\gamma)$, respectively. If $\gamma \in G^{(0)}$, then $s_G(\gamma) = r_G(\gamma)$, which can be identified with γ itself. The set of units of G sits inside G via the unit map $\mathrm{Ut}_G \colon G^{(0)} \hookrightarrow G$.

An arrow γ goes from its source $s_G(\gamma)$ to its range $r_G(\gamma)$. By definition, γ is invertible. Denote the inverse of γ by $\mathrm{inv}_G(\gamma)$. The map $\gamma \mapsto \mathrm{inv}_G(\gamma)$ is a bijection from G to itself. Using the definition of an invertible arrow in a category, it is easy to see that $\mathrm{inv}_G(\mathrm{inv}_G(\gamma)) = \gamma$. A nicer way to denote the inverse of γ is γ^{-1}. As an element and its inverse are composable, we have $s_G(\gamma) = r_G(\gamma^{-1})$ and $r_G(\gamma) = s_G(\gamma^{-1})$. By definition, $\gamma\gamma^{-1} = s_G(\gamma^{-1}) = r_G(\gamma)$ and $\gamma^{-1}\gamma = r_G(\gamma^{-1}) = s_G(\gamma)$.

It is clear from the definition that, in general, two arrows in G need not be composable. Two arrows γ and γ' are composable if and only if $s_G(\gamma) = r_G(\gamma')$. The set of composable arrows is denoted by $G^{(2)}$ or $G * G$. In particular, $r_G(\gamma)$ and γ are composable, and so are γ and $s_G(\gamma)$. Furthermore, $r_G(\gamma)\gamma = \gamma s_G(\gamma) = \gamma$.

Below is the list of important maps which are related to a groupoid:

- *the inverse map:* $\mathrm{inv}_G \colon G \to G$, this is a bijection with $\mathrm{inv}_G^2 = \mathrm{id}_G$;
- *the range map:* $r_G \colon G \to G^{(0)}$, this is a surjection;
- *the source map:* $s_G \colon G \to G^{(0)}$, this is a surjection;
- *the unit map:* $\mathrm{Ut}_G \colon G^{(0)} \to G$, this is an injection;
- *the multiplication map:* $m \colon G^{(2)} \to G$ sending (γ, γ') to their product $\gamma\gamma'$.

The relations between s_G, r_G and inv_G are:

i) $r_G = s_G \circ \mathrm{inv}_G$,

ii) $s_G = r_G \circ \mathrm{inv}_G$.

A notation: Let G be a groupoid. For $A, B \subseteq G^{(0)}$ define

$$\begin{aligned} G^A &= \{\gamma \in G : r_G(\gamma) \in A\} = r_G^{-1}(A), \\ G_A &= \{\gamma \in G : s_G(\gamma) \in A\} = s_G^{-1}(A), \\ G_B^A &= G^A \cap G_B = \{\gamma \in G : r_G(\gamma) \in A \text{ and } s_G(\gamma) \in B\}, \\ G|_A &= G^A \cap G_A. \end{aligned}$$

Note that $G|_A$ is a groupoid. We call it the restriction of G to A. When $A = \{u\}$ and $B = \{v\}$ are singletons, we write G^u, G_v and G_v^u instead of $G^{\{u\}}$, $G_{\{v\}}$ and $G_{\{v\}}^{\{u\}}$, respectively. For $u \in G^{(0)}$, G_u^u is a group. It is called the *isotropy* group at u.

Definition 1.1.2. The following are the definitions of topological and Borel groupoids:

1. a *topological groupoid* is a groupoid G with topologies on the sets G and $G^{(0)}$ such that all the maps listed above are continuous maps;

2. a *Borel groupoid* is a groupoid G with Borel structures on the sets G and $G^{(0)}$ such that all the maps described above are Borel maps.

Since $G^{(0)} \hookrightarrow G$ is a topological (or Borel) embedding, we view $G^{(0)}$ as a topological (or Borel) subspace of G. We give a few examples of topological groupoids. With some obvious changes, they can be used as examples of Borel groupoids.

Example 1.1.3. Let G be a topological group and let $e \in G$ denote the identity element in G. The group G is a topological groupoid. For this groupoid, $G^{(0)} = \{e\}$, $G^{(1)} = G$ and the composition of arrows is the group multiplication.

Example 1.1.4. A space X is a groupoid. In this groupoid, $X^{(0)} = X$ and if $x, y \in X^{(0)}$ are units then there is no arrow from x to y if $x \neq y$. The only arrows are the identity arrows.

Example 1.1.5 (Example 1.2.a in [33]). For a right action of a topological group G on a space X, the *transformation group* $X \rtimes G$ is a topological groupoid. It is also called a *transformation groupoid* for a group action. In $X \rtimes G$, the space of units is $(X \rtimes G)^{(0)} = X \times \{e\}$. Very often $X \times \{e\}$ is identified with X. We also do so. Two arrows $(x, \gamma), (y, \eta) \in X \times G$ are composable if and only if $x\gamma = y$, and then $(x, \gamma)(x\gamma, \eta) = (x, \gamma\eta)$. The inverse of (x, γ) is $(x\gamma, \gamma^{-1})$, and $s_{X \rtimes G}((x, \gamma)) = (x\gamma, e)$, $r_{X \rtimes G}((x, \gamma)) = (x, e)$.

Example 1.1.5 can be modified for a left G-space X to get $G \ltimes X$.

1.2 Proper actions and quotients

Let X, Y and Z be spaces and let $f\colon X \to Z$ and $g\colon Y \to Z$ be maps. The fibre product of X and Y on Z along the maps f and g is the set $\{(x,y) \in X \times Y : f(x) = g(y)\}$, which is denoted by $X \times_{f,Z,g} Y$ or $X \times_{f,g} Y$, when the space Z is clear from the context.

Definition 1.2.1 (Left action of a groupoid on a set)**.** Let G be a groupoid and let X be a set. A left action of G on X is given by a pair (r_X, a), where $r_X\colon X \to G^{(0)}$ is a map and $a\colon G \times_{s_G,r_X} X \to X$ is a map from the fibre product over $G^{(0)}$ for the source map s_G and r_X such that

i) $a(\gamma, x) = x$ for all $\gamma \in G^{(0)}$;

ii) if $(\gamma, \gamma') \in G^{(2)}$ and $(s_G(\gamma'), r_X(x)) \in G \times_{s_G,r_X} X$, then $(\gamma, a(\gamma', x)) \in G \times_{s_G,r_X} X$ and
$$a(\gamma, a(\gamma', x)) = a(\gamma\gamma', x).$$

The map r_X is called the *momentum map* or the *anchor map* for the action. When $(\gamma, x) \in G \times_{s_G,r_X} X$ we call γ and x *composable.*

When G is a topological (or Borel) groupoid, X is a topological (respectively, Borel) space and r_X and a are continuous (respectively, Borel) maps, the action is called a continuous (respectively Borel) action and we say that X is a left G-space. A right (set-theoretic, continuous or Borel) action of G on X can be defined similarly, and then we call X a right G-space.

Our work uses continuous actions most of the time. Hence the word *action* will stand for a *continuous action* from now on. We shall explicitly mention when the action is not continuous.

Remark 1.2.2. In the literature, the momentum map for a continuous action is often assumed to be surjective or open. We ask for none of these conditions.

A convention: Since we shall not come across any case where there are more than one different left (right) action of a groupoid G on a space X, we denote the momentum map by r_X (respectively, s_X). When we write 'X is a left (right) G-space' without specifying the momentum map, the above convention will be tacitly assumed and then in such instances, the momentum map is r_X (respectively, s_X).

A notation: Let X, Y and Z be spaces and let $f\colon X \to Z$ and $g\colon Y \to Z$ be maps. When the maps f and g are obvious from the context, we denote the fibre product $X \times_{f,Z,g} Y$ either by $X *_Z Y$ or by $X * Y$. The sets $G \times_{s_G,r_X} X$ and $X \times_{s_X,r_G} G$ in the discussion above will be written as $G * X$ and $X * G$, respectively.

A notation: Let X be a left G-space and let $A \subseteq G^{(0)}$. Define
$$X^A = \{x \in X : r_X(x) \in A\} = r_X^{-1}(A).$$
If X is a right G-space, then we define
$$X_A = \{x \in X : s_X(x) \in A\} = s_X^{-1}(A).$$
If $A = \{u\}$ is singleton, then we write X^u and X_u for $X^{\{u\}}$ and $X_{\{u\}}$, respectively.

A notation: Let X be a left G-space. For subsets $K \subseteq G$ and $A \subseteq X$ with $s_G(K) \cap r_X(A) \neq \emptyset$, define $KA = \{\gamma x : \gamma \in K, x \in A \text{ and } (\gamma, x) \in G * X\}$. If $s_G(K) \cap r_X(A) = \emptyset$, then $AK = \emptyset$. By an abuse of notation, for $x \in X$ we write Kx instead of $K\{x\}$. The meaning of γA for $\gamma \in G$ is similar. For a right action, we define AK, xK, $A\gamma$ similarly.

Definition 1.2.3. Let G be a groupoid, X and Y left G-spaces and $\pi\colon X \to Y$ a map. We call π a G-map or a G-equivariant map if, for all $(\gamma, x) \in G * X$, $(\gamma, \pi(x)) \in G * Y$ and $\pi(\gamma x) = \gamma\pi(x)$.

Recall that a map $f\colon X \to Y$ is proper if the inverse image of every compact set in Y under f is a compact set in X.

Definition 1.2.4. Let X be a left G-space.

i) The action of G on X is *proper* if the map $\Psi\colon G * X \to X \times X$, $(\gamma, x) \mapsto (\gamma x, x)$, is proper.

ii) The action is *free* if the map Ψ above is injective.

When the action of G on X is proper, we call X a proper G-space.

Definition 1.2.5 (Bispace)**.** Let G and H be groupoids. A G-H-bispace is a space X with a left action of G and a right action of H such that for all $\gamma \in G$, $x \in X$ and $\eta \in H$ with $s_G(\gamma) = r_X(x)$ and $s_X(x) = r_H(\eta)$ we have $s_X(\gamma x) = s_X(x)$, $r_X(x\eta) = r_X(x)$, and

$$(\gamma x)\eta = \gamma(x\eta).$$

Note that in Definition 1.2.5, the momentum maps r_X and s_X are part of the data. These momentum maps come with the actions of G and H.

Example 1.2.6. Let G be a group and let X be a space. If G is given its groupoid structure as in Example 1.1.3, then Definition 1.2.1 reduces to the usual definition of an action of the group G on X. Conversely, assume that a group G acts on X from the left. Let $e \in G$ be the identity element. Take the constant map from X to e as the momentum map and define $a\colon G \times X \to X$ by $a(\gamma, x) = \gamma \cdot x$. This data satisfies the conditions in Definition 1.2.1. Thus for a group G, an action of G as a *group* and *groupoid* means the same.

Example 1.2.7. If Y and X are spaces and Y is thought of as a groupoid as in Example 1.1.4, then an action of Y on X is a continuous map $s_X\colon X \to Y$ with an action map a. As Y has only the identity arrows, the only choice for $a\colon Y * X \to X$ is $a(y, x) = x$. Now an easy claim to prove is: an action of the groupoid Y on X is same as a continuous map from X to Y.

Example 1.2.8. Let G be a groupoid. Then the multiplication from the left by $\gamma \in G$ on $G^{s_G(\gamma)}$ is an action of G on itself. Clearly, the momentum map for this action is r_G. This is called the *left multiplication action*. The *right multiplication* action is defined similarly. The space G equipped with the left and right actions is a G-G-bispace.

Example 1.2.9. G acts on $G^{(0)}$ from the right (and left). The momentum map for the action is the identity map on $G^{(0)}$. The action is $r_G(\gamma)\gamma = s_G(\gamma)$ (or $\gamma s_G(\gamma) = r_G(\gamma)$, respectively). However, $G^{(0)}$ is not a G-G-bispace with these actions.

Example 1.2.10. If G is a groupoid and X is a right G-space, then similar to the group case as in Example 1.1.5, one can construct the transformation groupoid $X \rtimes G$. The base space of this groupoid is $X * G^{(0)}$, which is homeomorphic to X via the map $(x, s_X(x)) \mapsto x$. The arrow space is $X * G$. Two arrows $(x, \gamma), (y, \eta) \in X \times G$ are composable if and only if $x\gamma = y$, and $(x, \gamma)(x\gamma, \eta) = (x, \gamma\eta)$. The inverse of (x, γ) is $(x\gamma, \gamma^{-1})$. It can be checked now that $s_{X \rtimes G}((x, \gamma)) = (x\gamma, s_G(\gamma))$ and $r_{X \rtimes G}((x, \gamma)) = (x, r_G(\gamma))$. Example 1.1.5 is a special case of this construction.

Let H be a topological groupoid and X a left H-space. When the action of H is proper, the quotient space X/H inherits many *good* topological properties from X. We discuss this inheritance of properties in the rest of the section. The quotient map $X \to X/H$ will be denoted by p_X.

A subset of a topological space is *relatively compact* if the closure is compact.

Hypothesis: For the rest of the section, all the groupoids are topological groupoids and actions are continuous actions. Furthermore all topological spaces, including topological groupoids, are locally compact and Hausdorff, unless stated otherwise.

Lemma 1.2.11. *Let X, Y be spaces and $f\colon X \to Y$ an open surjection.*

i) For a compact $K \subseteq Y$, there is a compact $K' \subseteq X$ with $f(K') = K$.

ii) Let $\{y_i\}_{i\in I}$ be a net in Y with $y_i \to f(x)$. Then there is a subnet $\{y_{T_m}\}_{m\in M}$ and a net $\{x_m\}_{m\in M}$ indexed by the same set which converges to x in X, and which satisfies $f(x_m) = y_{T_m}$.

Proof. (i). Let $\{U_\alpha\}$ be a covering of $f^{-1}(K)$, where each U_α is a relatively compact open set. Then $\{f(U_\alpha)\}$ is an open cover of K. Let $f(U_{\alpha_i}), \dots, f(U_{\alpha_n})$ be a finite cover of K. Then $\left(\bigcup_{i=1}^n \overline{U}_{\alpha_i}\right) \cap f^{-1}(K)$ is the required compact set K'.

(ii). See [44, Proposition 1.15]. The proposition is a stronger result which states that if (ii) holds for a map $f\colon X \to Y$, then f is open. □

Let X be a right H-space. Define $\Psi\colon X * H \to X \times X$ by $\Psi(x, \eta) = (x, x\eta)$.

Remark 1.2.12. For $A \subseteq X$, AH is called the *saturation* of A for the action of H. If $B \subseteq X$ and $B = BH$, then B is called a *saturated subset.* Note that $p_X^{-1}(p_X(A)) = AH$. Also, by the definition of the quotient topology, $p_X(A) \subseteq X/H$ is open (or closed) if and only if $AH \subseteq X$ is open (or closed, respectively).

Lemma 1.2.13. *Let H be a groupoid. Then the range map $r_H\colon H \to H^{(0)}$ is open if and only if for every right H-space X the quotient map $p_X\colon X \to X/H$ is open.*

Proof. Let $U \subseteq X$ be open. We show that $X - UH$ is closed. If $X - UH = \emptyset$, we are done. So assume $X - UH \neq \emptyset$ and let $\{x_i\}_{i\in I}$ be a convergent net in $X - UH$. We show that the net cannot converge to a point in UH.

Assume the contrary, that is, say, $x_i \to x\eta$ and $x\eta \in UH$. Due to the continuity of the momentum map, $s_X(x_i) \to s_X(x\eta) = s_G(\eta)$. Since s_G is an open surjection, there is a convergent net $\{\eta_j\}_{j\in J}$ with $s_G(\eta_j) = s_X(x_j)$ for a subnet $\{x_j\}$ of $\{x_i\}$ and $\eta_j \to \eta$. Then by Lemma 1.2.11, $x_j\eta_j^{-1} \to x \in U$. As U is open, $\{x_j\eta_j^{-1}\}_{j\in J}$ is ultimately in U. By an abuse of notation, instead of taking a tail of

J, we write that $\{x_j\eta_j^{-1}\}_{j\in J}$ is in U. But then $\{(x_j\eta_j^{-1})\eta_j = x_j(\eta_j^{-1}\eta_j)\}_{j\in J}$ is a subnet of $\{x_i\}_{i\in I}$, which is in UH. This contradicts our hypothesis. Hence the limit of $\{x_i\}_{i\in I}$ must be in $X - UH$.

Conversely, let $p_X\colon X \to X/H$ be open for any right H-space X. Specifically for the right multiplication action of H on itself, $p_H\colon H \to H/H$ is open. The quotient H/H here is obtained from the equivalence relation $x \sim y$ if and only if $x\eta = y$ for some $\eta \in H$.

The range map, $r_H\colon H \to H^{(0)}$ induces an equivalence relation: $x \sim' y$ if and only if $r_H(x) = r_H(y)$. If $x \sim y$, then $x \sim' y$. Conversely, if $x \sim' y$, then $x(x^{-1}y) = y$ and hence $x \sim y$. Hence the quotient map induced by $\sim$ and $\sim'$ is the same map, namely, p_H.

Let $[r_H]\colon H/H \to H^{(0)}$ be the (continuous) bijection induced by r_H. We claim that $[r_H]$ is a homeomorphism, which will imply that it is an open map. To see this, we observe that the unit map $\mathrm{Ut}_H\colon H^{(0)} \to H$ is continuous and $p_H \circ \mathrm{Ut}_H$ is the inverse of $[r_H]$, which is clearly continuous. But $[r_H] \circ p_H = r_H$. Hence r_H is open. □

Remark 1.2.14. The proof of Lemma 1.2.13 does not need that the momentum map s_X is a surjection.

Remark 1.2.15. Let H be a groupoid and β a Haar system on H (see Definition 1.3.3), then Corollary 1.3.5 says that the range and source maps for H are open. Thus the hypothesis for Lemma 1.2.13 is satisfied. Hence if X is an H-space, the quotient map $X \to X/H$ is open.

Lemma 1.2.16. *Let X be an H-space $\Psi\colon X*H \to X\times X$ be the map $(x,\eta) \mapsto (x, x\eta)$. The conditions (i)–(iv) are equivalent and (i) implies (v), where:*

(i) the action is proper;

(ii) the transformation groupoid $X \rtimes H$ is a proper groupoid;

(iii) given a compact subset $K \subseteq X$, the set $\Psi_2^{-1}(K) = \{\eta \in H : K\cdot\eta \cap K \neq \emptyset\}$ is compact;

(iv) for all compact sets $K \subseteq X$, $\Psi_2^{-1}(K)$ is relatively compact in H;

(v) Ψ is a closed map.

Proof. (i) $\iff$ (ii) is a direct consequence of the definition of a proper action.

(i) $\implies$ (iii): We note that $\Psi_2^{-1}(K) = \Psi^{-1}(K\times K)$, hence it is compact.

(iii) $\implies$ (i): Let $K \subseteq X$ be compact and let p_i be projections on the i-th factor of $X\times X$ for $i = 1,2$. Due to the continuity, $\Psi^{-1}(K)$ is closed and $\Psi^{-1}(K) \subseteq \Psi_2^{-1}(p_1(K)\cup p_2(K))$, which is compact.

(iii) $\implies$ (iv) is obvious.

(iv) $\implies$ (iii): Let $\Psi_2^{-1}(K)$ be relatively compact for $K \subset X$ compact. Due to the continuity of Ψ, $\Psi_2^{-1}(K)$ is closed. Hence $\Psi_2^{-1}(K) = \overline{\Psi_2^{-1}(K)}$ is compact.

This proves that the conditions (i)–(iv) are equivalent.

(i) $\implies$ (v): Let $F \subseteq X*H$ be a closed set and π_i the projection on the i-th factor from $X*H$ for $i = 1,2$. Let $a \in X\times X \setminus \Psi(F)$. We want to find a neighbourhood of a disjoint from $\Psi(F)$. Let K be an open neighbourhood of a with $\overline{K}$ compact. Such K exists because $X\times X$ is locally compact.

Then $\Psi^{-1}(\overline{K})$ is compact, so $F \cap \Psi^{-1}(\overline{K}) \subseteq X * H$ is a compact set. Now $\Psi(F \cap \Psi^{-1}(\overline{K})) \subseteq X \times X$ is compact and hence closed. But $\Psi(F \cap \Psi^{-1}(\overline{K})) = \Psi(F) \cap \overline{K}$. Hence $K \setminus (\Psi(F) \cap \overline{K})$ is an open neighbourhood of a. □

Remark 1.2.17. (ii) of Lemma 1.2.16 gives that G is proper if and only if the action of G on $G^{(0)}$ from Example 1.2.9 is proper.

Lemma 1.2.18. *If X is a proper H-space, then $KH \subseteq X$ is closed for all $K \subseteq X$ compact.*

Proof. Let $K \subseteq X$ compact and let $x \in X - KH$. Let $V \subseteq X$ be an open neighbourhood of x with $\overline{V}$ compact; such a neighbourhood exists since X is locally compact and Hausdorff. Let $\Psi\colon X*H \to X \times X$ be as in Lemma 1.2.16. Since the action of H on X is proper, $\Psi^{-1}(K \times \overline{V}) \subseteq X*H$ is compact. $\Psi^{-1}(K \times \overline{V}) = \emptyset$ implies that $KH \cap \overline{V} = \emptyset$, hence V is an open neighbourhood of x which is disjoint from KH.

Now assume that $\Psi^{-1}(K \times \overline{V}) \neq \emptyset$. Let $\pi_H\colon X * H \to H$ be the projection. Then $\pi_H(\Psi^{-1}(K \times \overline{V})) \subseteq H$ is compact, since the projection π_H is continuous and Ψ is closed (Lemma 1.2.16). Now the continuity of the action gives that $K \cdot \pi_H^{-1}(\Psi^{-1}(K \times \overline{V})) \subseteq X$ is compact and hence closed. Thus $x \in V - K \cdot \pi_H^{-1}(\Psi^{-1}(K \times \overline{V})) \neq \emptyset$ is an open neighbourhood of x which is disjoint from KH. □

Proposition 1.2.19. *Let X be an H-space and r_H open. If H acts properly, then the quotient space, X/H, is locally compact Hausdorff.*

Proof. First we show that X/H is Hausdorff. Let $[x], [y] \in X/H$ be distinct points. Choose representatives $x, y \in X$ of these points, respectively. Let $U', V' \subseteq X$ be open and disjoint neighbourhood of x and y, respectively. As xH and yH are closed in X (see Lemma 1.2.18), we can replace U' by $U' - yH$ and V' by $V' - xH$ and assume that U' does not intersect the orbit of y and V' does not intersect the orbit of x. Let $U \subseteq U'$ and $V \subseteq V'$ be open and relatively compact neighbourhoods of x and y, respectively, with $\overline{U} \subseteq U'$ and $\overline{V} \subseteq V'$. Then $\overline{U}$ and $\overline{V}$ are disjoint compact sets.

Note that $x \notin \overline{V}H$, because if it is, then $(xH \cap \overline{V}) \neq \emptyset$. But $(xH \cap \overline{V}) \subseteq (xH \cap V') = \emptyset$. Define $A = (UH - \overline{V}H)$. Then $x \in A \neq \emptyset$, and A is open by Lemma 1.2.18. Using a similar argument we can see that $B := (VH - \overline{U}H)$ is an open neighbourhood of y.

$A \cap \overline{V}H = \emptyset$ by definition, and $B \subseteq VH$, so $A \cap B = \emptyset$.

We have proved that $AH \cap BH = \emptyset$. Lemma 1.2.13 implies that $p_X(A)$ and $p_X(B)$ are open neighbourhoods of $[x]$ and $[y]$, and $AH \cap BH = \emptyset$ is equivalent to $p_X(A) \cap p_X(B) = \emptyset$.

Now we prove that X/H is locally compact. For a given $[x] \in X/H$ we produce an open neighbourhood of $[x]$ whose closure is compact. Let $U \subseteq X$ be a relatively compact open neighbourhood of $x \in X$. Then $p_X(U)$ is an open neighbourhood of $[x]$. We prove that $\overline{p_X(U)} = p_X(\overline{U})$, where the latter set is compact. Thus $p_X(U)$ is the required neighbourhood of $[x]$.

The continuity of p_X gives $p_X(\overline{U}) \subseteq \overline{p_X(U)}$. We prove the converse inclusion. Due to Lemma 1.2.18, $p_X^{-1}(p_X(\overline{U})) = \overline{U}H \subseteq X$ is closed, hence $p_X(\overline{U})$ is closed, by the definition of a closed set in X/H. But $p_X(\overline{U}) \supseteq p_X(U)$, hence $p_X(\overline{U}) \supseteq \overline{p_X(U)}$. □

Corollary 1.2.20. *If X is second countable, then under the same hypotheses as in Proposition 1.2.19, X/H is second countable and paracompact.*

Proof. The open image of a second countable set is second countable. And every second countable locally compact Hausdorff space is paracompact. Here is a proof for the claim that every second countable locally compact Hausdorff space is paracompact: Let X be a locally compact Hausdorff second countable space. Then X is regular. To see this, let $x \in X$ be a point and let $C \subseteq X$ be a closed set that does not contain x. Then $X - C$ is an open neighbourhood of x. Since X is locally compact, Hausdorff there is an open neighbourhood $U \subseteq X - C$ of x with $\overline{U}$ compact. Thus U and $X - \overline{U}$ are open neighbourhoods of x and C which are disjoint.

Urysohn metrization theorem ([30, Theorem 34.1]) says that every regular space with countable basis is metrizable, hence X is metrizable. By the theorem of Stone ([40, Corollary 1]) we conclude that X is paracompact. □

1.3 Proper actions and families of measures

The analogue of the Haar measure on a locally compact group is given by a *Haar system* in the theory of groupoids. A Haar system on a groupoid is a special type of *continuous family of measures* along the range map. We discuss continuous families of measures in the beginning of this section. The latter part of the section deals with the behaviour of families of measures under quotients by proper actions. If X and Y are proper H-spaces and λ is a continuous family of measures along a continuous map $\pi\colon X \to Y$, which need not be surjective, then we show that λ induces a continuous family of measures along the map $[\pi]\colon X/H \to Y/H$. This result is proved for a free and proper action in [34].

In the last part, given a proper groupoid with a Haar system, we construct an invariant continuous family of probability measures along the range map, but this family need not have *full support*. A group G is a proper groupoid if and only if G is compact. Hence the Haar measure on G may be modified to a probability measure. The result we prove is an analogue of this fact.

In the literature, *invariance* of families of measures means *left invariance*. However, our main theorems are concerned with right invariance. Hence we discuss right invariant families of measures. Indeed, similar results hold for left invariant families of measures. We shall use the left invariant analogues of our definitions while discussing representation theory.

A hypothesis and a convention: All the measures we deal with are assumed to be σ-finite positive Radon measures. We do not differentiate between a measure on a space X and the corresponding Riesz functional on $C_c(X)$. We use the same notation for both.

Definition 1.3.1 (Invariant continuous family of measures)**.** Let X and Y be right H-spaces for a groupoid H and let $\pi\colon X \to Y$ be an H-equivariant continuous map. An *H-invariant continuous family of measures* along π is a family of Radon measures $\lambda = \{\lambda_y\}_{y \in Y}$ such that[1]:

i) each λ_y is defined on $\pi^{-1}(y)$;

ii) (invariance) for all composable pairs $(y, \eta) \in Y * H$, the condition $\lambda_y \eta = \lambda_{y\eta}$ holds;

iii) (continuity condition) for $f \in C_c(X)$ the function $\Lambda(f)(y) := \int_{\pi^{-1}(y)} f \,\mathrm{d}\lambda_y$ on Y is continuous.

[1]Indeed, for $X_y = \emptyset$ we assume that λ_y is the empty measure.

We clarify that in the above definition the measure $\lambda_y\eta$ is given by $\int f \,\mathrm{d}\lambda_y\eta = \int f(x\eta)\,\mathrm{d}\lambda_y(x)$ for $f \in \mathcal{B}_+(X)$.

Let X and Y be Borel spaces, let H be a Borel groupoid, let the actions be Borel and let π be a Borel map. Then λ is called an H-invariant *Borel* family of measures if the continuity condition above is replaced by the condition

iii') for every compactly supported $f \in \mathcal{B}_+(X)$, the function $\Lambda(f)$ is Borel.

(For the continuous case as well as the Borel case) if for each $y \in Y$, $\mathrm{supp}(\lambda_y) = \pi^{-1}(y)$, we say the family of measures λ has *full support*. Depending on the case, if there is a continuous or Borel function f on X with $\Lambda(f) = 1$ on $\pi(X)$, we say that λ is *proper*. Lemma 1.1.2 in [1] says that in the continuous case λ is proper if and only if $\lambda_y \neq 0$ for all $y \in Y$. Hence if λ is continuous and has full support, then λ is proper.

In the whole document we assume that given a family of measures $\lambda = \{\lambda_u\}$, each $\lambda_u \neq 0$. In the continuous case this means that λ is proper. Some of the results in this chapter hold without this assumption. But we do not assume that they have full support.

Let Pt be the trivial point group(oid). If X and Y are spaces and $\pi\colon X \to Y$ is a continuous map, then π is a Pt-equivariant map between Pt-spaces. A continuous Pt-invariant family of measures along π is simply called a *continuous family of measures along* π. The nomenclature for the Borel case is analogous.

Most of the families of measures we come across are continuous. Hence we drop the word continuous and simply say that λ is an H-invariant family of measures. We shall write it explicitly when a family of measures is Borel.

If X and Y are left H-spaces and π is an H-equivariant map from X to Y, then we can define an H-equivariant family of measures $\{\lambda^y\}_{y\in Y}$ in a similar fashion.

Remark 1.3.2. When π is a continuous surjection and λ has full support, some of the definitions of a continuous family of measures in the literature demand that $\Lambda\colon C_c(X) \to C_c(Y)$ is a surjection. This assumption is redundant because of Lemma 1.3.16 below.

A convention: We denote families of measures by small Greek letters. For a given family of measures, the corresponding integration function that appears in the continuity condition in Definition 1.3.1 will be denoted by the Greek upper case letter used to denote the family of measures. For α, β and μ it will be A, B and M, respectively. Proposition 2.4.14 is the only exception to this convention, where (by mistake) we have (ended up in denoting) two families of measures by m and μ and we write M and μ for the corresponding functions induced between the function spaces.

Definition 1.3.3. 1. Let H be a groupoid, X a left H-space. An H-invariant continuous family of measures along the momentum map r_X is called a *left H-invariant continuous family of measures on X*. A *right H-invariant continuous family of measures on X* is defined analogously.

2. For a groupoid H, a *Haar system* on H is a left H-invariant continuous family of measures with full support on H for the *left multiplication* action of H on itself.

Unlike the group case, a second countable, locally compact, Hausdorff groupoid need not carry a Haar system, and Haar systems are usually not unique (see Examples 1.3.13 and 1.3.14).

Nearly a year after formulating Definition 1.3.1, we came across Renault's paper [34], where he defines the same notion. Renault calls it 'π-*système*'.

If β is a Haar system on H, we call the pair (H, β) 'a groupoid with Haar system'. We shall be working with groupoids with Haar systems most of the time.

Lemma 1.3.4. *Let H be a groupoid, let $\pi\colon X \to Y$ be a continuous H-map between the H-spaces X and Y and let λ be a continuous family of measures along π. If λ_y has full support for all $y \in \pi(X)$, then π is an open map onto its image.*

Proof. Consider the map $\pi\colon X \to \pi(X)$ and then the proof is same as the proof of Proposition 2.2.1 in [31]. □

Corollary 1.3.5. *If (H, β) is a groupoid with a Haar system, then the range and source maps are open.*

Proof. Lemma 1.3.4 implies that the range map r_H is open. Since $s_H = r_H \circ \mathrm{inv}_H$ and inv_H is a homeomorphism, s_H is open. □

We give a few examples of groupoids with Haar systems and invariant families of measures.

Example 1.3.6. Let G be a locally compact group. Then a Haar measure on G is a Haar system on G.

Example 1.3.7. If a space X is thought of as a groupoid, then the set of Dirac delta measures at each point $\{\delta_x\}_{x\in X}$ is a Haar system for the groupoid X.

Example 1.3.8. For a group H and an H-space X, any H-invariant measure on X is an H-invariant family of measures on X.

Example 1.3.9. Here is a special case of the previous example: let (X, λ) be a measure space. Then λ is a system of measures for the action of the point groupoid $\{\mathrm{Pt}\}$ on X.

Example 1.3.10. Let H be a locally compact group with a Haar measure β, let X be a right H-space and let $X \rtimes H$ be the corresponding transformation groupoid discussed in Example 1.1.5. Then $(X \rtimes H)^x = H$ for all $x \in (X \rtimes H)^{(0)}$ and the measure β along each fibre is a Haar system for $X \rtimes H$.

Example 1.3.11. Let (G, α) be a pair consisting of a groupoid and a Haar system for it. Let X be a right G-space. Let $X \rtimes G$ be the transformation groupoid as in Example 1.2.10. For $x \in X = (X \rtimes G)^{(0)}$ define the measure $\bar{\alpha}^x$ on $(X \rtimes G)^x = G^{s_X(x)}$ by

$$\int_{(X\rtimes G)^x} f \,\mathrm{d}\bar{\alpha}^x = \int_{G^{s_X(x)}} f(x,\gamma)\,\mathrm{d}\alpha^{s_X(x)}(\gamma)$$

for $f \in C_c(X \rtimes G)$. Then $\bar{\alpha} = \{\bar{\alpha}^x\}_{x\in X}$ is a Haar system for $X \rtimes G$.

Example 1.3.12. Let G be a groupoid and α a Haar system on G. We get a *right* invariant family of measures on G using α. This family is denoted by α^{-1}. For $u \in G^{(0)}$ and $f \in C_c(G)$, α_u^{-1} is defined as follows:

$$\int_{G_u} f \,\mathrm{d}\alpha_u^{-1} = \int f \circ \mathrm{inv}_G \,\mathrm{d}\alpha^u = \int_{G^u} f(\gamma^{-1})\,\mathrm{d}\alpha^u(\gamma).$$

When G is a group, we have $\alpha^{-1} = \Delta_G\,\alpha$, where Δ_G is the modular function of G.

Example 1.3.13 (A groupoid that does not have a Haar system). Let $G = [0, 1/2] \times \mathbb{R} \cup [1/2, 1] \times \{0\} \subseteq [0, 1] \times \mathbb{R}$. We equip G with the subspace topology from $\mathbb{R}^2$ and define the groupoid structure on G by:

i) $G^{(0)} = [0, 1] \times \{0\} \approx [0, 1]$;

ii) $r_G = s_G = \pi_1$, where $\pi_1 \colon G \to [0, 1]$ is the projection map (note that for $u \in [0, 1]$, $r_G^{-1}(u) = s_G^{-1}(u)$ is either $\mathbb{R}$ or $\{0\}$);

iii) for $(u, x), (u, y) \in G$, define $(u, x)(u, y) = (u, x + y)$.

Then G is a topological groupoid.

Let $L = \{(x, x) : x \in \mathbb{R}\} \cap G$. Denote the Euclidean metric on $G \subseteq \mathbb{R}^2$ by d and let $\mathrm{B}(L, 1/10) = \{\gamma \in G : \mathrm{d}(\gamma, L) < 1/10\}$. Being a closed subset of the normal space $\mathbb{R}^2$, G is normal. Using this normality, extend the constant function 1 on L to a non-negative function f in $C_c(G)$ with $f = 0$ outside $\mathrm{B}(L, 1/8)$.

Let $\lambda = \{\lambda^u\}_{u\in[0,1]}$ be any family of measures along r_G with *full* support. Then

$$\Lambda(f)(u) \begin{cases} > 0 & \text{if } u \leq 1/2 \\ = 0 & \text{if } u > 1/2. \end{cases}$$

Thus $\lambda(f)$ is not continuous at $u = 1/2$, at least, since $\lim_{u\to 1/2^-} \Lambda(f) = 0 \neq \Lambda(f)(1/2) > 0$. Here $\lim_{u\to 1/2^-} \lambda(f)$ stands for the limit of $\lambda(f)$ from the right.

Example 1.3.14 (A compact groupoid that does not have a Haar system). Let $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ be the cyclic group of order 2. Let $G = [0, 1/2] \times \mathbb{Z}/2\mathbb{Z} \cup [1/2, 1] \times \{0\} \subseteq \mathbb{R}^2$ be subspace. We make G into a groupoid using the following data and operations

i) $G^{(0)} = [0, 1] \times \{0\} \approx [0, 1]$;

ii) $r_G = s_G = \pi_1$, where $\pi_1 \colon G \to [0, 1]$ is the projection map;

iii) for $u \in [0, 1]$, $r_G^{-1}(u) = s_G^{-1}(u)$ is either $\mathbb{Z}/2\mathbb{Z}$ or $\{0\}$. Using the group structure on the fibres for $(u, x), (u, y) \in G$, define $(u, x)(u, y) = (u, x + y)$.

This is a bundle of groups with fibre either $\mathbb{Z}/2\mathbb{Z}$ or the trivial group. Then G is a compact topological groupoid.

Let λ be any invariant continuous family of measures with *full* support. Then λ^u is a Haar measure on $r_G^{-1}(u)$, hence λ^u is the discrete measure with the weight $\Lambda(c)(u) \neq 0$ where c is the constant function 1. The continuity of λ gives that $\Lambda(c)$ is continuous. Thus $\Lambda(c)$ is a continuous positive function of G.

Since G is compact, $C_c(G) = C(G)$. Let $A = [0, 1] \times \{0\}$ and $B = [0, 1/2] \times \{1\}$. Then χ_A and χ_B which are the characteristic function of A and B, respectively, are continuous on G. Furthermore, $\chi_A + \chi_B = \chi_G$. Due to the continuity of λ, $\Lambda(\chi_A), \Lambda(\chi_B)$ and $\Lambda(\chi_G)$ are continuous functions on $[0, 1] = G^{(0)}$ with $\Lambda(\chi_G)(u) > 0$ for all $u \in [0, 1]$. It can be checked that

$$\frac{1}{2}\Lambda(\chi_G)(u) = \Lambda(\chi_A)(u) = \Lambda(\chi_B)(u) \text{ for } u \in [0, 1/2], \text{ and}$$

$$\frac{1}{2}\Lambda(\chi_G)(u) = \Lambda(\chi_A)(u) \text{ and } \Lambda(\chi_B)(u) = 0 \text{ for } u \in (1/2, 1].$$

But now $\Lambda(\chi_B)$ is not continuous at $1/2$, since $\lim_{x\to 1/2^-}(\Lambda(\chi_B)) = 0 \neq \Lambda(\chi_B(1/2)) > 0$. This is a contradiction which arose because we assumed that λ is an invariant continuous family of measures with full support.

Example 1.3.15 (A groupoid with many Haar systems). Let X be a space and let μ and ν be two non-equivalent Radon measures on X. Construct the groupoid G of the trivial equivalence relation on X. Then $G^{(0)} = X$, $G^{(1)} = X \times X$, $s_G = \pi_2$ and $r_G = \pi_1$, where π_1 and π_2 are the projection maps from $X \times X$ to X on the first and the second factors, respectively. The arrows $(x, y), (w, z) \in G$ are composable if and only if $y = w$ and $(x, y)(y, z) = (x, z)$. For $(x, y) \in G$, $(x, y)^{-1} = (y, x)$.

For $u \in G^{(0)}$, $r_G^{-1}(u) = X$. For each $u \in X$ put $\lambda_1^u = \mu$ and let $\lambda_1 = \{\lambda_1^u\}_{u\in X}$. Then (X, λ) is a groupoid with Haar system. Similarly (X, λ_2) is a groupoid with Haar system, where $\lambda_2^u = \nu$ for all $u \in X$. For no $x \in G^{(0)}$, $\lambda_1^u \sim \lambda_2^u$.

Let X and Y be right H-spaces and $\pi\colon X \to Y$ an H-equivariant map. As before, we denote the quotient of X by the action of H by X/H. For $x \in X$ the equivalence class of x in X/H is denoted by $[x]$. The map π induces a map from X/H to Y/H, which we denote by $[\pi]$.

Lemma 1.3.16 (Lemma 1.1, [34]). *Let X and Y be spaces, let $\pi\colon X \to Y$ be an open surjection and let λ be a family of measures with full support along π. For every open $U \subseteq X$ and for a non-negative function $g \in C_c(\pi(U))$, there is a non-negative function $f \in C_c(U)$ with $\Lambda(f) = g$.*

Lemma 1.3.17 (Lemma 1.2, [34]). *Let X, Y and Z be spaces, let π and τ be open surjections from X and Y to Z, respectively. Let π_2 denote the projection from the fibre product $X * Y$ onto the second factor Y. Assume that for each $z \in Z$, there is a measure λ_z on $\pi^{-1}(z)$. For each $y \in Y$ define the measure $\lambda_{2y} = \lambda_{\tau(y)} \times \delta_y$, where δ_y is the point-mass at y. Then λ is continuous if and only if λ_2 is continuous.*

Lemma 1.3.18 (Lemma 1.3, [34]). *Let X and Y be right H-spaces, let both actions of H be free and proper and let $\pi\colon X \to Y$ be an open surjection.*

i) An H-invariant continuous family of measures λ along π induces a continuous family of measures $[\lambda]$ along the induced map $[\pi]\colon X/H \to Y/H$, where $[\lambda]$ is given by the formula

$$\int f \,\mathrm{d}[\lambda]^{[y]} = \int f([x])\,\mathrm{d}\lambda^y(x).$$

Let $[\Lambda]$ denote the corresponding integration function.

ii) Conversely, given a continuous family of measures τ along $[\pi]$, there is a unique H-invariant continuous family of measures λ along π with $\tau = [\lambda]$.

One of the goals of this section is to prove (i) of Lemma 1.3.18, when the action is proper but not free and π is not an open surjection.

Let (H, β) be a groupoid with a Haar system and X a right H-space. For $x \in X$ define the measure $\tilde{\beta}_X^x$ on $X \times H^{s_X(x)}$ by

$$\int_{X\times H^{s_X(x)}} f(v, \eta)\,\mathrm{d}\tilde{\beta}_X^x(v, \eta) = \int_{H^{s_X(x)}} f(v, \eta)\,\mathrm{d}\beta^{s_X(x)}(\eta), \tag{1.3.19}$$

for $f \in C_c(X * H)$. This is a special case of the family of measures on the fibre product $X * Y$ in Lemma 1.3.17.

Lemma 1.3.20. *Let (H, β) and X be as above.*

(i) For $F \in C_c(X * H)$, the function $B_X(F)\colon x \mapsto \int F(x, \gamma)\, \mathrm{d}\beta^{s_X(x)}(\gamma)$ is in $C_c(X)$.

(ii) Let $\pi_X : X * H \to X$ be the projection on X. Then the family of measures $\tilde{\beta}_X := \{\tilde{\beta}_X^x\}_{x \in X}$ along π_X is continuous.

(iii) If X is a proper H-space and $f \in C_c(X)$, then the function $x \mapsto \int f(x\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta)$ is in $C_c(X/H)$.

Proof. (i): We observe that due to the Stone-Weierstraß Theorem, the subalgebra of $C_c(X * H)$ generated by the set $D := \{f \cdot g : f \in C_c(X), g \in C_c(H)\}$ is dense in $C_c(X * H)$ in the inductive limit topology. Hence it is sufficient to check the claim for a function in D. Let $f \cdot g = F \in D$. Let $h \in C_c(X)$ be a function with $h|_{\mathrm{supp}(f)} = 1$. Then $hF \in C_c(X * H)$ and

$$\mathcal{B}_X(hF)(x) = \int f(x) g(\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta) = f(x) B(g)(s_X(x)),$$

where $B(g) \in C_c(H^{(0)})$ by the continuity of β. Since $f \in C_c(X)$ and $B(g) \circ s_X \in C(X)$, the product is in $C_c(X)$. Here $B\colon C_c(H) \to C_c(H^{(0)})$ is the integration map in the continuity condition in Definition 1.3.1.

(ii): This is a consequence of (i) above.

(iii): Given $f \in C_c(X)$, define $F(x, \gamma) = f(x\gamma)$. Since the action is proper, (iii) of Lemma 1.2.16 says that the set $\Psi_2^{-1}(\mathrm{supp}(f)) \subseteq H$ is compact. But $\mathrm{supp}(F) \subseteq \left(\mathrm{supp}(f) \times \Psi_2^{-1}(\mathrm{supp}(f))\right)$, and the latter set is compact. Hence $F \in C_c(X * H)$. Now we apply (i) of this lemma to F to see that the function $h\colon x \mapsto \int f(x\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta)$ is continuous on X. It is not hard to see that $h(x) = h(x\eta)$, due to the invariance of the family β. If $p\colon X \to X/H$ is the quotient map, then $h \circ p$ is continuous on X/H. Also $\mathrm{supp}(h \circ p) \subseteq p(\mathrm{supp}(h))$, and $p(\mathrm{supp}(h)) \subseteq X/H$ is compact. □

Proposition 1.3.21. *Let (H, β) be a groupoid with a Haar system and X a proper right H-space. For $[x] \in X/H$ define a measure $\beta_X^{[x]}$ on $xH \subseteq X$ by*

$$\int f\, \mathrm{d}\beta_X^{[x]} = \int f(x\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta). \tag{1.3.22}$$

Then $\beta_X := \{\beta_X^{[x]}\}_{[x] \in X/H}$ is a well-defined continuous family of measures with full support along the quotient map $p_X : X \to X/H$.

Proof. Let $x\gamma$ be a representative in the orbit of x. Then

$$\int f((x\gamma)\eta)\, \mathrm{d}\beta^{s_X(x\gamma) = s_G(\gamma)}(\eta) = \int f(x\eta)\, \mathrm{d}\beta^{r_G(\gamma) = s_X(x)}(\eta)$$

due to the invariance of β. Hence $[\tilde{\beta}_X]^{[x]}$ is well-defined for each $[x] \in X/H$.

The continuity of β_X follows from Lemma 1.3.20.

Now we check that the support of $\beta_X^{[x]}$ is exactly the orbit of x. We use the contra-positive of Lemma 1.3.23 below. For every open neighbourhood $x \in V \subseteq xH$, we show that $\beta_X^{[x]}(V) > 0$. Let $W \subseteq X$ be open with $V = W \cap xH$. Let $f \in C_c(W)$ be non-negative with $f(x) > 0$. Extend f by zero outside W, so $f \in C_c(X)$. Due to the properness of the action, the function $\phi\colon H^{s_X(x)} \to \mathbb{C}$ defined by $\phi(\eta) = f(x\eta)$ lies in $C_c(H^{s_X(x)})$. Note that ϕ is non-zero because $\phi(s_X(x)) = f(x) > 0$. Now

$$\int_V f \,\mathrm{d}\beta^{[x]} = \int f \,\mathrm{d}\beta^{[x]} = \int_{H^{s_X(x)}} \phi \,\mathrm{d}\beta^{s_X(x)} > 0.$$

The first equality is because $f = 0$ outside W. The last inequality is due to the full support of $\beta^{s_X(x)}$. □

Lemma 1.3.23 (Characterisation of the support of a measure; Proposition 8, §2.3 Chapter III in [6]). *Let μ be a measure on a locally compact (Hausdorff) space X. For every function $f \in C_c(X)$ that is zero on* $\operatorname{supp}(\mu)$, $\mu(f) = 0$.

Let (H, β) be a groupoid with a Haar system, X and Y proper H-spaces, $\pi\colon X \to Y$ a continuous map which need not be surjective. Let $\lambda := \{\lambda^y\}_{y\in Y}$ be a continuous family of measures along π. We list the information we have:

i) a continuous family of measures along π, namely, $\lambda := \{\lambda^y\}_{y\in Y}$.

ii) Due to Lemma 1.3.20, we have the families of measures $\tilde{\beta}_X$ and $\tilde{\beta}_Y$ along the projections $\pi_X : X * H \to X$ and $\pi_Y : Y * H \to Y$, respectively.

iii) The H-invariant map π induces an obvious H-map $\pi \times I_H$ between the fibre products $X * H \to Y * H$, $(\pi \times I_H)(x, \eta) = (\pi(x), \eta)$. This map carries a family of measures $\lambda \times \delta = \{\lambda^y \times \delta_\eta\}_{(y,\eta)\in Y*H}$, where

$$\int f \,\mathrm{d}(\lambda^y \times \delta_\eta) = \int f(x, \eta) \,\mathrm{d}\lambda^y(x)$$

for $f \in C_c(X * H)$. A density argument as in Lemma 1.3.20 can be used to see that this is a continuous family of measures. Let $\Lambda \times \Delta$ denote the integration function $C_c(X * H) \to C_c(Y * H)$ induced by this family.

All this data is put in the diagram in Figure 1.1. In this diagram, the symbols below the function arrows stand for families of measures and the symbols on the top indicate the function.

Lemma 1.3.24. *The diagram in Figure 1.1 commutes at the level of measures, that is, for $f \in C_c(X*H)$,*

$$\tilde{B}_Y(\Lambda \times \Delta(f)) = \Lambda(\tilde{B}_X(f)).$$

Proof. The proof is a direct calculation and uses Fubini's Theorem.

$$\begin{aligned}\tilde{B}_Y(\Lambda \times \Delta(f))(y) &= \int \Lambda \times \Delta(f)(x, \gamma) \,\mathrm{d}\beta^{s_Y(y)=r_H(\gamma)}(\gamma) \\ &= \iint f(x, \gamma) \,\mathrm{d}\lambda^y(x) \,\mathrm{d}\beta^{s_X(x)=s_Y(y)=r_H(\gamma)}(\gamma).\end{aligned}$$

$$\begin{array}{ccc} X * H & \xrightarrow[\lambda\times\delta]{\pi\times I_H} & Y * H \\ {\scriptstyle \pi_X}\big\downarrow{\scriptstyle \tilde{\beta}_X} & & {\scriptstyle \pi_Y}\big\downarrow{\scriptstyle \tilde{\beta}_Y} \\ X & \xrightarrow[\lambda]{\pi} & Y. \end{array}$$

Figure 1.1

Applying Fubini's Theorem, the last term becomes

$$\iint f(x,\gamma)\,\mathrm{d}\beta^{s_X(x)=r_H(\gamma)}(\gamma)\,\mathrm{d}\lambda^y(x) = \int \tilde{B}_X(f)(x)\,\mathrm{d}\lambda^y(x)$$
$$= \Lambda(\tilde{B}_X(f))(y). \qquad \square$$

We take the quotient by the H-action of each space in the commutative square in Figure 1.1 and the corresponding induced maps. We analyse the quotient spaces, maps and families of measures below.

Bottom horizontal arrow: In Figure 1.1, the bottom horizontal arrow of the square induces the map $[\pi]\colon X/H \to Y/H$. The family of measures λ induces the family of measures $[\lambda] = \{[\lambda]^{[y]}\}_{[y]\in Y/H}$, where

$$\int f\,\mathrm{d}[\lambda]^{[y]} := \int f([x])\,\mathrm{d}\lambda^y(x). \tag{1.3.25}$$

We check that the integral on the left is well defined. Take $y\eta \in [y]$, then the invariance of λ gives $\lambda^{y\eta} = \lambda^y$. If $x\eta \in [x]$, then using the H-invariance of λ again, we get

$$\int f\,\mathrm{d}\lambda^{y\eta} = \int f([x]\eta)\,\mathrm{d}\lambda^y(x) = \int f([x\eta])\,\mathrm{d}\lambda^y(x) = \int f([x])\,\mathrm{d}\lambda^y(x) = \int f\,\mathrm{d}\lambda^y.$$

Left vertical arrow: The function $[x,\eta] \mapsto x\eta$ induces a homeomorphism between $(X * H)/H$ and X. The inverse of this map is $x \mapsto [x, s_X(x)]$. Thus after taking the quotient by the H-action, we get a map $[\pi_X]\colon X \to X/H$. With this identification, for $x \in X$ we get

$$[\pi_X](x) = [\pi_X]([x, s_X(x)]) = [\pi_X]([x\eta, s_H(\eta)]) = [x\eta] = [x] = p_X(x).$$

Thus $[\pi_X] = p_X$, the quotient map.

As discussed earlier, the family of measures $\tilde{\beta}_X$ induces a family of measures $\{[\tilde{\beta}_X]^{[x]}\}_{[x]\in X/H}$, which we denote by $[\tilde{\beta}_X]$, along $[\pi_X] = p_X$. We check that this family is exactly β_X. Here β_X is the family of measures along the quotient map $X \to X/H$ defined in Proposition 1.3.21. For $f \in C_c(X)$,

$$\int f([x\eta, s_H(\eta)])\,\mathrm{d}[\tilde{\beta}_X]^{[x]}([x\eta, s_H(\eta)]) = \int f([x\eta, s_H(\eta)])\,\mathrm{d}\beta^{s_X(x)}(\eta)$$
$$= \int f(x\eta)\,\mathrm{d}\beta^{s_X(x)}(\eta) = \int f\,\mathrm{d}\beta_X^{[x]}. \tag{1.3.26}$$

From Proposition 1.3.21, we know that $[\tilde{\beta}_X] = \beta_X$ is a continuous family of measures. Since p_X is open and surjective (Remark 1.2.15) and β_X has full support (Proposition 1.3.21), Lemma 1.3.16 shows that the integration map $[\tilde{B}_X] = B_X : C_c(X) \to C_c(X/H)$ is surjective.

Right vertical arrow: Due to the similarity with the left vertical arrow, analogous results hold for the right vertical arrow in Figure 1.1 and we get

i) $[\pi_Y] = p_Y$,

ii) $[\tilde{\beta}_Y] = \beta_Y$,

iii) $[\tilde{B}_Y] = B_Y : C_c(Y) \to C_c(Y/H)$ is surjective.

Top vertical arrow: We quotient the top horizontal arrow of the square and identify $(X*H)/H \approx X$, $(Y*H)/H \approx Y$ as mentioned earlier. This gives $[\pi \times I_H] = \pi : X \to Y$ and $[\lambda \times \delta] = \lambda$.

These computations give us Figure 1.2, which is obtained by taking the quotients of all spaces, maps and families of measures in Figure 1.1.

$$\begin{array}{ccc} X & \xrightarrow[\lambda]{\pi} & Y \\ {\scriptstyle p_X=[\pi_X]}\big\downarrow {\scriptstyle \beta_X} & & {\scriptstyle p_Y=[\pi_Y]}\big\downarrow {\scriptstyle \beta_Y} \\ X/H & \xrightarrow[{[\lambda]}]{[\pi]} & Y/H. \end{array}$$

Figure 1.2

Proposition 1.3.27. *Let X and Y be proper H-spaces, let $\pi : X \to Y$ be a continuous surjection and let $\lambda := \{\lambda^y\}_{y\in Y}$ be a continuous family of measures along π. Then the induced family of measures, $[\lambda] := \{[\lambda]^{[y]}\}_{[y]\in Y/H}$, is a continuous family of measures. If λ has full support, then so does $[\lambda]$.*

Proof. From the previous discussion, it is clear that $[\lambda] := \{[\lambda]^{[y]}\}_{[y]\in Y/H}$ is a well-defined family of measures. We need to check the continuity. That is, for $f \in C_c(X/H)$, the function $[\Lambda](f) \in C_c(Y/H)$ is continuous.

Let $f \in C_c(X/H)$ and let $F \in C_c(X)$ be a function with $[B_X](F) = f$.

Then
$$[\Lambda](f)([y]) = [\Lambda]([B_X](F))([y]) = \iint F(z)\, \mathrm{d}[\tilde{\beta}_X]^{[x]}(z)\, \mathrm{d}\lambda^{[y]}([x]).$$

A careful computation yields $[\tilde{\beta}_Y \circ (\lambda \times \delta)] = [\tilde{\beta}_Y] \circ [\lambda \circ \delta]$ and $[\lambda \circ \tilde{\beta}_X] = [\lambda] \circ [\tilde{\beta}_X]$. Using this fact with the commutativity of the measures in Lemma 1.3.24, we get

$$[\Lambda]([\tilde{B}_X](F)) = [\tilde{B}_Y](\Lambda(F)) = B_Y(\Lambda(F)).$$

The last one clearly is continuous, as both λ and β_Y are continuous.

It is not hard to see that $[\lambda]$ has full support if λ has. This uses the commutativity of Figure 1.2 and the fact that β_X and β_Y both have full support. □

Here is one more result that tells us how to reduce a Haar system on a proper groupoid to a family of probability measures. But the families of probability measures which we get using this method need not have full support.

Lemma 1.3.28 (Lemma 1, Appendix I in [7]). *Let X be a locally compact space, R an open equivalence relation in X, such that the quotient space X/R is paracompact; let π be the canonical mapping of X onto X/R. There is a continuous real-valued function $F \geq 0$ on X such that:*

i) F is not identically zero on any equivalence class with respect to R;

ii) for every compact subset K of X/R, the intersection of $\pi^{-1}(K)$ with $\mathrm{supp}(F)$ is compact.

Lemma 1.3.29. *If (G, α) is a Hausdorff, locally compact, second countable proper groupoid with a Haar system, then there is a left invariant continuous family of probability measures on G (which need not have full support).*

Proof. Since G has a Haar system, the range map of G is open. Lemma 1.2.13 shows that the quotient map $\pi\colon G^{(0)} \to G\backslash G^{(0)}$ is open. Since G is proper, $G\backslash G^{(0)}$ is paracompact by Corollary 1.2.20. This satisfies the hypotheses for Lemma 1.3.28, and gives us a function F on $G^{(0)}$ such that F is not identically zero on any G-orbit in $G^{(0)}$ and, for every compact $K \subseteq G\backslash G^{(0)}$ the intersection $\mathrm{supp}(F) \cap \pi^{-1}(K)$ is compact. Define $h\colon G^{(0)} \to \mathbb{R}^+$ by,

$$h(u) = \int F \circ s_G(\gamma)\, \mathrm{d}\alpha^u(\gamma).$$

Property (ii) of F from Lemma 1.3.28 and the full support condition of α^u give $h(u) > 0$. To see that $h(u) < \infty$, notice that $\mathrm{supp}(F \circ s_G) \cap G^u \subseteq G$ is compact:

$$\gamma \in \mathrm{supp}(F \circ s_G) \cap G^u \iff \gamma \in \mathrm{supp}(F \circ s_G) \text{ and } \gamma \in G^u \implies s_G(\gamma) \in \mathrm{supp}(F) \text{ and } r_G(\gamma) = u.$$

Thus if $\tilde{u}$ denotes the orbit of $u \in G^{(0)}$, then $\mathrm{supp}(F \circ s_G) \cap G^u \subseteq (s_G^{-1} \times r_G^{-1})(\mathrm{supp}(F|_{\tilde{u}}) \times \{u\})$. Property (ii) of F from Lemma 1.3.28 says that $\mathrm{supp}F|_{\tilde{u}}$ is compact. As G is a proper groupoid, the set $(s_G^{-1} \times r_G^{-1})(\mathrm{supp}(F|_{\tilde{u}}) \times \{u\})$ is compact. Hence $\mathrm{supp}(F \circ s_G) \cap G^u$ is compact.

The function h is constant on the orbits of $G^{(0)}$. Put $F' = F/h$, then

$$\int F' \circ s_G(\gamma)\, \mathrm{d}\alpha^u(\gamma) = 1. \tag{1.3.30}$$

Denote $(F' \circ s_G)\, \alpha^u$ by p^u, then $p := \{p^u\}_{u \in G^{(0)}}$ is a family of probability measures on G. For $f \in C_c(G)$ define

$$\int_{G^u} f\, \mathrm{d}p^u = \int_{G^u} f(\gamma)\, F' \circ s_G(\gamma)\, \mathrm{d}\alpha^u(\gamma).$$

The invariance of α makes p a G-invariant family of measures. Let $\eta \in G$, then a change of variables shows that

$$\int f(\eta\gamma)\,\mathrm{d}p^{s_G(\eta)}(\gamma) = \int f(\eta\gamma)\,F'\circ s_G(\gamma)\,\mathrm{d}\alpha^{s_G(\eta)}(\gamma) = \int f(\gamma)\,F'\circ s_G(\eta^{-1}\gamma)\,\mathrm{d}\alpha^{r_G(\eta)}(\gamma) = \int f(\gamma)\,F'\circ s_G(\gamma)\,\mathrm{d}\alpha^{r_G(\eta)}(\gamma) = \int f(\gamma)\,\mathrm{d}p^{r_G(\eta)}(\gamma)$$

for $f \in C_c(G)$ because $s_G(\eta^{-1}\gamma) = s_G(\gamma)$. □

Remark 1.3.31. Lemma 1.3.29 implies that every proper groupoid with a Haar system is topologically amenable. See Section 1.7.1 for the discussion. [1, Proposition 2.2.5] implies this lemma. But the proposition is a much more general statement than the lemma, and both proofs are very different.

1.4 Cohomology for groupoids

A notion of cohomology for Borel groupoids is introduced in [43]. In [33], a continuous version of the same cohomology is discussed. For our purposes, we need an equivariant continuous version of this cohomology. In the present section, we develop *equivariant* cohomology for Borel and continuous groupoids. However, in the following discussion the groupoids are assumed to be Borel groupoids and the maps are Borel maps. The whole discussion goes through when the Borel properties are replaced by the continuous properties. Which means, the discussion makes sense when the groupoids are topological groupoids and all the maps involved are continuous.

Definition 1.4.1 (Action of a groupoid on another groupoid)**.** A left action of a groupoid G on another groupoid H is given by maps $r_{H,G}\colon H \to G^{(0)}$ and $a\colon G * H \to H$ which satisfy the following conditions:

i) if $\eta, \eta' \in H$ are composable, $\gamma \in G$ with $s_G(\gamma) = r_{H,G}(\eta) = r_{H,G}(\eta)$, then $a(\gamma,\eta), a(\gamma,\eta') \in H$ are composable and
$$a(\gamma,\eta\eta') = a(\gamma,\eta)a(\gamma,\eta');$$

ii) if $u \in G^{(0)}$, then $a(u,\eta) = \eta$ for all $\eta \in H$;

iii) if $\gamma, \gamma' \in G$ are composable, then $(\gamma, a(\gamma',\eta)) \in G * H$ and
$$a(\gamma\gamma',\eta) = a(\gamma, a(\gamma',\eta)).$$

To simplify the notation, we write $\gamma\cdot\eta$ or simply $\gamma\eta$ for $a(\gamma,\eta)$. Then (i) and (ii) above read $\gamma\cdot(\eta\eta') = (\gamma\cdot\eta)(\gamma\cdot\eta')$ and $(\gamma\gamma')\cdot\eta = \gamma\cdot(\gamma'\cdot\eta)$, respectively. We call the map $r_{H,G}$ the momentum map for the action and a the action map. When the momentum map and the action map are continuous (or Borel) the action is called continuous (or Borel, respectively).

As a subgroupoid $H^{(0)} \subseteq H$ is nothing but a space. And it is not hard to see that G acts on $H^{(0)}$, in the sense of Definition 1.2.1. The momentum map in this case is $r_{H,G}|_{H^{(0)}}$ and the action map is $a|_{G*H^{(0)}}$. It is then clear that $GH^{(0)} \subseteq H^{(0)}$ because an element u in a groupoid is a unit if and only if u is composable with itself and $u^2 = u$. Hence if $u \in H^{(0)}$, then $\gamma \cdot u = \gamma \cdot (uu) = (\gamma \cdot u)(\gamma \cdot u)$. Thus $\gamma \cdot u \in H^{(0)}$.

When G is a group, our definition matches Definition 1.7 in [33, Chapter 1], which is the action of a group on a groupoid by invertible functors. A proof of this fact is below.

Lemma 1.4.2. *When G is a group, an action of G on H as in Definition 1.4.1 above is the same as the action in [33, Definition 1.7, Chapter 1], that there is a homomorphism $\phi\colon G \to \mathrm{Aut}(H)$ which gives the actions where $\mathrm{Aut}(H)$ is the set of all invertible functors from H to itself.*

Proof. For $\gamma \in G$ define $\phi(\gamma)(\eta) = \gamma \cdot \eta$.

We first prove that each $\phi(\gamma)$ is a functor from H to itself.

Note that an element u in a groupoid is a unit if and only if u is composable with itself and $u^2 = u$. If $u \in G^{(0)}$, then $\phi(\gamma)(u) = \phi(\gamma)(uu) = \phi(\gamma)(u)\phi(\gamma)(u) = (\phi(\gamma)(u))^2$. Hence for each unit $u \in H^{(0)}$, $\phi(\gamma)(u) \in H$ is a unit. (i) of Definition 1.4.1 gives that for each $\gamma \in G$, $\phi(\gamma)(\eta\eta') = \phi(\gamma)(\eta)\phi(\gamma)(\eta')$. This proves that $\phi(\gamma)$ is functor for each $\gamma \in G$.

Now we show that each of the $\phi(\gamma)$ is invertible. (ii) of Definition 1.4.1 gives that $\gamma \mapsto \phi(\gamma)$ is a homomorphism. Use (iii) of Definition 1.4.1 to see that $\phi(\gamma)$ is invertible:

$$\phi(\gamma)\phi(\gamma^{-1})(\eta) = \phi(\gamma\gamma^{-1})(\eta) = \phi(r_G(\gamma))(\eta) = \eta = \mathrm{Id}_H(\eta).$$

Similarly, $\phi(\gamma^{-1})\phi(\gamma) = \mathrm{Id}_H$. Thus $\phi(\gamma)^{-1} = \phi(\gamma^{-1})$. Hence $\phi(\gamma) \in \mathrm{Aut}(H)$.

It is routine to prove the converse, that is, an action of the group G on the groupoid H as in [33, Definition 1.7, Chapter 1] satisfies Definition 1.4.1. □

A continuous (and Borel) version of Lemma 1.4.2 can be proved along similar lines, merely by adding continuity (or Borelness) of the action map and the momentum map and the continuity (Borelness) of the group homomorphism ϕ.

Example 1.4.3. Let G be a groupoid and H a space. Then as action of G on H is the same as an action of G on H viewed as a groupoid. In this case, condition (i) in Definition 1.4.1 is irrelevant and Definition 1.2.1 and Definition 1.4.1 match.

Example 1.4.4. Let $\pi\colon V \to X$ be a group bundle. The bundle can be viewed as a topological groupoid H as follows: $H^{(0)} := X$, $H^{(1)} := V$ and the source and range maps are π. For $x \in X$, $r_H^{-1}(x) = s_X^{-1}(x) = \pi^{-1}(x)$ is a group. For $v, v' \in \pi^{-1}(x)$ define the inverse of v to be v^{-1} and the composition of v and v' to be $v.v'$. Also, the unit element section embeds $H^{(0)}$ into $H^{(1)}$.

Example 1.4.5. Since a vector bundle $\pi\colon V \to X$ is a group bundle , the previous example implies that a vector bundle is a groupoid. If X is a G-space, then the statement that the vector bundle $\pi\colon V \to X$ is G-equivariant is equivalent to the statement that the groupoid G acts on groupoid of the vector bundle.

Example 1.4.6. Let G and H be groupoids and let X be a G-H-bispace. Define an action of G on the transformation groupoid $X \rtimes H$ by $\gamma(x, \eta) := (\gamma x, \eta)$. The momentum map for this action is $(x, \eta) \mapsto r_X(x) \in G^{(0)}$. Let $(x\eta, \eta'), (x, \eta) \in X \rtimes H$ be composable elements then $(\gamma x\eta, \eta')(\gamma x, \eta)$ are composable and

$$\gamma \cdot (x, \eta) \cdot \gamma\,(x\eta, \eta') = (\gamma x, \eta)(\gamma x\eta, \eta') = (\gamma x, \eta\eta') = \gamma(x, \eta\eta').$$

This verifies (i) of Definition 1.4.1. The other conditions are easy to check. Thus H acts on the groupoid $G \ltimes X$ in our sense. This is the main example for us.

Let H be a Borel groupoid and assume that H acts on a Borel groupoid G. Let $G^{(0)}$ and $G^{(1)}$ have the usual meaning. For $n = 2, 3, \ldots$ define

$$G^{(n)} = \{(\gamma_0, \ldots, \gamma_{n-1}) \in \underbrace{G \times G \times \cdots \times G}_{n\text{-times}} : s(\gamma_i) = r(\gamma_{i+1}) \text{ for } 0 \leq i < n-1\}.$$

Definition 1.4.7. Let G, H be Borel groupoids, let A be an abelian Borel group and let H act on G. The A-valued H-invariant Borel cochain complex $(B\mathcal{C}^{\bullet}_H(G; A), d^{\bullet})$ is defined as follows:

i) The abelian groups BC^n_H are:

(a) $BC^0_H(G; A) := \{\, f : G^{(0)} \to A$: f is an H-invariant Borel map$\}$;

(b) for $n > 0$ $BC^n_H(G; A) := \{f \colon G^{(n)} \to A : f$ is an H-invariant Borel map and $f(\gamma_0, \ldots, \gamma_{n-1}) = 0$ if $\gamma_i \in G^{(0)}$ for some $0 \leq i < n-1\}$,

ii) the coboundary map d is:

(a) $d^0 \colon BC^0_H(G; A) \to BC^1_H(G; A)$ is $d^0(f)(\gamma) = f(s_G(\gamma)) - f(r_G(\gamma))$,

(b) for $n > 0$, $d^n \colon BC^n_H(G; A) \to BC^{n+1}_H(G; A)$ is

$$d^n(f)((\gamma_0, \ldots, \gamma_n)) = f(\gamma_1, \ldots, \gamma_n) + \sum_{i=1}^{n} (-1)^i f(\gamma_0, \ldots, \gamma_{i-1}\gamma_i, \ldots, \gamma_n) + (-1)^{n+1} f(\gamma_0, \ldots, \gamma_{n-1}).$$

The n-th cohomology group of this complex is the n-th H-invariant *Borel cohomology* of G for $n \geq 0$, and it is denoted by $\mathrm{H}^n_{\mathrm{Bor},H}(G; A)$. By adding the action of H to all the maps and spaces, the machinery and the results in [43, §1 and §2] can be generalised to our setting.

Remark 1.4.8. Any H-invariant Borel function f on $G^{(0)}$ is a 0-cochain. A cochain $f \in BC^0_H(G; A)$ is a cocycle *iff* $d^0(f) = 0$ which is true *iff* f is constant on the orbits of $G^{(0)}$. A cochain $k \in BC^1_H(G; A)$ is a cocycle *iff* $k(\gamma_0) - k(\gamma_0\gamma_1) + k(\gamma_1) = 0$ for all composable γ_0 and γ_1, which means that k is an H-invariant Borel groupoid homomorphism.

We drop the suffixes B and Bor and write merely $\mathrm{C}^0_H(G; A)$ and $\mathrm{H}^n_H(G; A)$.

Remark 1.4.9. Let $b, b' \in \mathrm{C}^0_H(G; A)$ and $\Delta \in \mathrm{C}^1_H(G; A)$ with $d^0(b) = d^0(b') = \Delta$. Put $c = b - b'$. Then for all $\gamma \in G$,

$$\begin{aligned} d^0(b)(\gamma) &= d^0(b')(\gamma) \\ b(s_G(\gamma)) - b(r_G(\gamma)) &= b'(s_G(\gamma)) - b'(r_G(\gamma)) \\ b(s_G(\gamma)) - b'(s_G(\gamma)) &= b(r_G(\gamma)) - b'(r_G(\gamma)) \\ c(s_G(\gamma)) &= c(r_G(\gamma)) \end{aligned}$$

Thus $b - b'$ is a function on $G^{(0)}/G$.

Proposition 1.4.10. *Let G be a proper groupoid and α a Haar system on G. Then every $\mathbb{R}$-valued 1-cocycle is a coboundary, that is,* $\mathrm{H}^1(G;\mathbb{R}) = 0$.

Proof. Since G is a proper groupoid with a Haar system, Lemma 1.3.29 gives a family of probability measures $p = \{p^u\}_{u\in G^{(0)}}$. For a 1-cocycle $c\colon G \to \mathbb{R}$ the function

$$\underline{b}(u) = \int c(\gamma)\,\mathrm{d}p^u(\gamma), \quad u \in G^{(0)},$$

satisfies $c = \underline{b} \circ r - \underline{b} \circ s$. To see this, let $\eta \in G$ and compute:

$$\begin{aligned}
(\underline{b} \circ r - \underline{b} \circ s)(\eta) &= \int c(\gamma)\,\mathrm{d}p^{r(\eta)}(\gamma) - \int c(\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma) \\
&= \int c(\eta\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma) - \int c(\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma) \\
&= \int (c(\eta\gamma) - c(\gamma))\,\mathrm{d}p^{s(\eta)}(\gamma) \\
&= \int (c(\eta) + c(\gamma) - c(\gamma))\,\mathrm{d}p^{s(\eta)}(\gamma) \\
&= c(\eta) \int \mathrm{d}p^{s(\eta)}(\gamma) \\
&= c(\eta).
\end{aligned}$$

□

Remark 1.4.11. Since p is a continuous family of measures, the proof works for both Borel and continuous cohomology.

1.5 Quasi-invariant measures

Let (G, α) be a (locally compact Hausdorff) groupoid with a Haar system. Then α is an invariant family of measures along the range map. Using α we get a right invariant family of measures α^{-1} along the source map by $\int f(\gamma)\,\mathrm{d}\alpha_u^{-1}(\gamma) = \int f(\gamma^{-1})\,\mathrm{d}\alpha^u(\gamma)$ for all $f \in C_c(G)$. Let X be a left G-space and let μ be a measure on X. We define a measure $\mu \circ \alpha^{-1}$ on the space $G * X$ by

$$\int_{G*X} f\,\mathrm{d}(\mu \circ \alpha^{-1}) = \int_X \int_{G^{r_X(x)}} f(\gamma^{-1}, x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\mu(x)$$

for $f \in C_c(G * X)$ Due to the Riesz representation theorem, we also write $\mu \circ \alpha^{-1}(f)$ for the integral $\int_{G*X} f\,\mathrm{d}(\mu \circ \alpha)$.

Definition 1.5.1 (Quasi-invariant measure). Let (G, α) be a groupoid with a Haar system and X a G-space. A measure μ on X is called (G, α)-*quasi-invariant* if $\mu \circ \alpha$ and $(\mu \circ \alpha) \circ \mathrm{inv}_{G\ltimes X}$ are equivalent.

In the above definition, $\mathrm{inv}_{G\ltimes X}$ is the inverse function on the transformation groupoid $G \ltimes X$. Thus for $f \in C_c(G * X)$,

$$(\mu \circ \alpha) \circ \mathrm{inv}(f) = \int_X \int_{G^{r_X(x)}} f(\gamma, \gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\mathrm{d}\mu(x).$$

Remark 1.5.2. It is a somewhat technical fact that a (G,α)-quasi-invariant measure μ is G-invariant if and only if the Radon-Nikodym derivative $\mathrm{d}(\mu\circ\alpha)/\mathrm{d}(\mu\circ\alpha^{-1})=1$ $\mu\circ\alpha$-almost everywhere on $G*X$. A proof can be written using a density argument as in Lemma 1.3.20 and using Lemma 1.3.16.

Remark 1.5.3. Let $u\in H^{(0)}$ and let $x\in X$ be such that $s(x)=u$. As in [1] or [8], it can be shown that the Radon-Nikodym derivative $\mathrm{d}(\mu\circ\alpha)/\mathrm{d}(\mu\circ\alpha^{-1})$ is a $\mu\circ\alpha$-almost everywhere a groupoid homomorphism from the transformation groupoid $G\ltimes X$ to $\mathbb{R}_+^*$.

Let μ be a measure on $G^{(0)}$. Then for the left action of G on $G^{(0)}$ we get the measure $\mu\circ\alpha$ on the *whole* of G:

$$\mu\circ\alpha(f)=\int_{G^{(0)}}\int_{G^u} f(\gamma)\,\mathrm{d}\alpha^u(\gamma)\,\mathrm{d}\mu(u).$$

When the groupoid with the Haar measure (G,α) in the discussion is fixed, the standard convention is that the phrase 'μ is a quasi-invariant measure' means that μ is a (G,α)-quasi-invariant measure on $G^{(0)}$. When the Haar system on G is fixed, saying that μ is a G-quasi-invariant measure has the same meaning.

Definition 1.5.4 (A measured groupoid)**.** *A measured groupoid* is a triple (G,α,μ) where G is a groupoid, α is a Haar system on G and μ is a quasi-invariant measure on $G^{(0)}$.

Definition 1.5.5 (Modular function)**.** The *modular function* of a measured groupoid (G,α,μ) is the Radon-Nikodym derivative $\mathrm{d}(\mu\circ\alpha)/\mathrm{d}(\mu\circ\alpha^{-1})$.

The reason to use the article *the* for the modular function is that if Δ and Δ' are two modular functions on G, then $\Delta=\Delta'$ $\mu\circ\alpha$-almost everywhere. Due to Remark 1.5.3 the modular function is $\mu\circ\alpha$-almost everywhere homomorphism of groupoids from G to $\mathbb{R}_+^*$. Let $\Delta_{G,\mu}$ denote the modular function for a measured groupoid (G,α,μ).

Remark 1.5.6. Remark 1.5.3 says that $\Delta_{G,\mu}$ is a $\mu\circ\alpha$-almost everywhere homomorphism on $G*X$. Furthermore, there is a Borel set $U\subseteq G^{(0)}$ with $\mu(G^{(0)}-U)=0$ such that $\Delta_{G,\nu}|_{G_U^U}$ is a strict Borel homomorphism, see [1, Appendix I.b].

Lemma 1.5.7. *Let μ be a quasi-invariant measure on $G^{(0)}$. If $\mu'\sim\mu$, then μ' is also quasi-invariant and*

$$\Delta_{G,\mu'}=\left(\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}\circ r_G\right)\cdot\Delta_{G,\mu}\cdot\left(\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}\circ s_G\right)^{-1}.$$

Proof. For the above value of $\Delta_{G,\mu'}$ one can directly compute that $\int f\,\mathrm{d}(\mu'\circ\alpha^{-1})=\int f\,\Delta_{G,\mu'}\mathrm{d}(\mu'\circ\alpha)$. □

1.6 Representations of groupoids and groupoid C*-algebras

Definition 1.6.1 (Borel Hilbert bundle)**.** Let X be a space, $\mathcal{H}=\{\mathcal{H}_x\}_{x\in X}$ a family of separable Hilbert spaces and $p\colon\mathcal{H}\to X$ the projection map. We call $(\mathcal{H},\pi)$ a *Borel Hilbert bundle* over X if

i) $\mathcal{H}$ carries a Borel structure and the projection map p is a Borel map,

ii) there is a sequence of sections $\{\zeta_n\}$ such that

a) the map $\bar{\zeta_n}\colon \mathcal{H} \to \mathbb{C}$ sending $(x,h) \mapsto \langle \zeta_n(x)\,,h\rangle$ is Borel for each n;

b) the map $x \mapsto \langle \zeta_n(x)\,,\zeta_m(x)\rangle$ is Borel for each n and m;

c) the functions $\bar{\zeta_n}$ and π separate points of $\mathcal{H}$.

The sequence $\{\zeta_n\}$ is called a *fundamental sequence* for the bundle $(\mathcal{H},p)$. Two Borel Hilbert bundles $(\mathcal{H},p)$ and $(\mathcal{H}',p')$ are said to be isomorphic if there is a Borel isomorphism $\phi'\colon \mathcal{H} \to \mathcal{H}'$ such that for each $x \in X$, $\phi'\colon \mathcal{H}_x \to \mathcal{H}'_x$ is a unitary operator.

A section ζ is called *Borel* if the function $x \mapsto \langle \zeta_n(x)\,,\zeta(x)\rangle$ is Borel for all n. Let μ be a (Borel) measure on X. Then the set

$$\mathcal{L}^2(X,\mu\,;\mathcal{H}) = \{\zeta \in \Gamma(X;\mathcal{H}) : \zeta \text{ is measurable and } \mu(\langle\zeta\,,\zeta\rangle) < \infty\}$$

is a Hilbert space under the obvious operations. We call this space *the Hilbert space of μ-square-integrable sections of* $(\mathcal{H},p)$. An element of $\mathcal{L}^2(X,\mu\,;\mathcal{H})$ is called a μ-square-integrable section.

Let (G,α) be a groupoid with Haar system. Denote $G * G = G \times_{s_G,r_G} G$ and let α^2 denote the family of product measures $\{\alpha_u^{-1} \times \alpha^u\}_{u\in G^{(0)}}$ on $G * G$.

Definition 1.6.2 (Borel G-Hilbert bundle, Definition 1.6 Chapter II [33]). For a groupoid with a Haar system (G,α) a Borel G-Hilbert bundle is a triple $(\mathcal{H},p,\pi)$, where $(\mathcal{H},p)$ is a Borel Hilbert bundle over $G^{(0)}$ and π is an assignment $\pi\colon \gamma \mapsto \pi(\gamma) \in \mathcal{U}(\mathcal{H}_{s(\gamma)},\mathcal{H}_{r(\gamma)})$ satisfying the following conditions:

i) for $u \in G^{(0)}$, $\pi(u) = \mathrm{Id}_{\mathcal{H}_u}$;

ii) if γ and γ' are composable then $\pi(\gamma\gamma') = \pi(\gamma)\pi(\gamma')\ \alpha^2$;

iii) $\pi(\gamma)^* = \pi(\gamma^{-1})\ \alpha$;

iv) $\gamma \mapsto \langle \pi(\gamma)(\zeta \circ s(\gamma))\,,\eta \circ r(\gamma)\rangle$ is Bore for every pair of Borel sections ζ and η.

Now let μ be a quasi-invariant measure on $G^{(0)}$. Then we define μ-measurable Hilbert bundles over $G^{(0)}$ and μ-measurable G-Hilbert bundles in a similar fashion; we replace "Borel" by "μ-measurable" everywhere in Definitions 1.6.1 and 1.6.2 and require the identities in Definition 1.6.2 to only hold almost everywhere with respect to the appropriate measure μ, $\mu \circ \alpha^{-1}$, or $\mu \circ (\alpha^{-1} * \alpha)$, where $\alpha^{-1} * \alpha$ is the family of measures $\alpha_u^{-1} \times \alpha^u$ for $u \in G^{(0)}$ along the map $G * G \to G^{(0)}$, $(\gamma,\eta) \mapsto s_G(\gamma) = r_G(\eta)$. Moreover, an isomorphism between two μ-measurable G-Hilbert bundles is only required to be defined and well-behaved almost everywhere.

We call two μ-measurable G-Hilbert bundles $(\mathcal{H},p,\pi)$ and $(\mathcal{H}',p',\pi')$ isomorphic if there is an isomorphism $\phi\colon \mathcal{H} \to \mathcal{H}'$ such that $\pi'(\gamma) \cdot \phi \circ s(\gamma) = \phi \circ r(\gamma) \cdot \pi(\gamma)\ \mu \circ \alpha$-almost everywhere.

Now we discuss the representation theory for groupoids. More precisely, the representation theory for locally compact groupoids with Haar system. For this discussion, our main references are [33] and [34]. Though in both of them Renault proves a disintegration theorem for groupoids, the topological hypotheses are drastically different. In [33], the topology on the groupoids is Hausdorff, locally compact ([33, Page 16, Chapter 1, Section 2] and second countable ([33, Page

64, Chapter 2, Section 1]). There is a very technical assumption about the Borel structure of the groupoids. We discuss this theory in Subsection 1.6.1.

In the later work [34], Renault gets rid of the technical assumption, the second countability and he worked with locally Hausdorff spaces. Thus this is a very general setting and it is a strong result. We discuss this theory in Subsection 1.6.3.

We use the latter form [34] of the disintegration theorem. Hence the reader might find some hypotheses and cross references in Subsection 1.6.1 not matching with the original reference, namely, [33]. But still all the results are valid.

1.6.1 The representation theory-I

Definition 1.6.3 (Representation of a groupoid, Definition 2.1.6 [33]). *A representation* of a groupoid with Haar system (G, α) is a quadruple $(\mu, \mathcal{H}, p, \pi)$ where μ is a (G, α)-quasi-invariant measure on $G^{(0)}$ and $(\mathcal{H}, p, \pi)$ is a μ-measurable G-Hilbert bundle.

Two representations $(\mu, \mathcal{H}, p, \pi)$ and $(\mu', \mathcal{H}', p', \pi')$ are equivalent if μ is equivalent to μ' and $(\mathcal{H}, p, \pi)$ and $(\mathcal{H}', p', \pi')$ are isomorphic G-Hilbert bundles μ (and hence μ')-almost everywhere. If ϕ implements the isomorphism between $(\mathcal{H}, p, \pi)$ and $(\mathcal{H}', p', \pi')$, then there is an isomorphism $\Phi\colon \mathcal{L}^2(G^{(0)}, \mu\,; \mathcal{H}) \to \mathcal{L}^2(G^{(0)}, \mu'\,; \mathcal{H}')$ given by

$$\Phi(\zeta)(u) = \phi(u)(\zeta(u)) \sqrt{\frac{\mu}{\mu'}(u)}. \tag{1.6.4}$$

When G is a group thought of as a groupoid, we have $G^{(0)} = \{e\}$, here e is the identity in G. Hence a measurable G-Hilbert bundle is a unitary representation of G. In this case, the point mass δ_e at the unit e of G is a (G, α)-quasi-invariant measure and the modular function of the measured groupoid (G, α, δ_e) is exactly the modular function of G. In this case, Definition 1.6.3 gives us a measurable unitary representation of G.

A convention: Most of the time, we drop the projection map of the G-Hilbert bundle and write '$(\mu, \mathcal{H}, \pi)$ is a representation of (G, α)', instead of '$(\mu, \mathcal{H}, p, \pi)$ is a representation of (G, α)'.

One of the most important features in the theory of representations of groups is that given a unitary representation of G there is a non-degenerate *-representation of the convolution algebra $C_c(G)$. This is one of the most important facts in the study of C*-algebras as well, since it allows to study the representations of G in terms of those of $C_c(G)$ and vice versa. We establish an analogue of the same fact in groupoid representation theory. This lets us define the C*-algebra of G in a fashion similar to the group case. For this purpose one needs to define the convolution algebra $C_c(G)$. We do it below.

Let (G, α) be a groupoid with a Haar system. Define a convolution and an involution on $C_c(G)$ as follows:

$$f * g(\gamma') = \int_{G^{r(\gamma')}} f(\gamma) g(\gamma^{-1}\gamma')\, \mathrm{d}\alpha^{r_G(\gamma')}(\gamma)\,, \tag{1.6.5}$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}. \tag{1.6.6}$$

For $f \in C_c(G)$ define the I-norm $\|\cdot\|_I$ using the left norm $\|\cdot\|_l$ and the right norm $\|\cdot\|_r$:

$$\begin{aligned}
\|f\|_r &:= \sup_{u \in G^{(0)}} \left\{ \int_{G^u} |f| \, \mathrm{d}\alpha^u \right\}, \\
\|f\|_l &:= \sup_{u \in G^{(0)}} \left\{ \int_{G_u} |f| \, \mathrm{d}(\alpha^{-1})_u \right\}, \\
\|f\|_I &:= \max \{ \|f\|_r, \|f\|_l \}.
\end{aligned}$$

Using the continuity of the Haar system, it can be shown that all the terms above are finite. We wish to make two important remarks regarding the involution and the I-norm.

Remark 1.6.7. The formula for the involution (1.6.6) for groupoids differs from the usual formula for the involution for groups, in which a power of the modular function appears. Assume G is a group and denote the *-algebra of G thought of as a groupoid by $C_c(G)'$. Let $C_c(G)$ denote the *-algebra of G as a group as in [11, Chapter VII]. The map $f \mapsto f\sqrt{\Delta_{G,\alpha}}$ induces an isomorphism of *-algebras $C_c(G)' \to C_c(G)$. A similar claim is true for the reduced C*-algebras.

Remark 1.6.8. The topology defined by the I-norm on $C_c(G)$ is coarser than the inductive limit topology on $C_c(G)$, see [31, Prposition 2.2.2]. Hence if $U \subseteq C_c(G)$ is open in the topology induced by the I-norm, then U is open in the inductive limit topology also. As a consequence of which, if $\pi\colon C_c(G) \to X$ is a continuous map when $C_c(G)$ has the the topology induced by the I-norm, then π is also continuous when $C_c(G)$ has the inductive limit topology. This fact is used extensively while studying the representations of $C_c(G)$.

Proposition 1.6.9 (Theorem 2.2.1 [31]). *Let (G, α) be a locally compact Hausdorff groupoid with a Haar system. Then $C_c(G)$ is a separable, normed *-algebra under the convolution product and the I-norm, and the involution is an isometry.*

A non-degenerate *-representation of the *-algebra $C_c(G)$ is a *-homomorphism $\pi\colon C_c(G) \to \mathbb{B}(H)$ for a separable Hilbert space H such that the set $\{\pi(f)\zeta : f \in C_c(G),\ \zeta \in H\} \subseteq H$ is dense. We call π continuous if it is continuous when $C_c(G)$ is given the inductive limit topology and $\mathbb{B}(H)$ has the weak operator topology. Two representations $\pi\colon C_c(G) \to H$ and $\pi'\colon C_c(G) \to H'$ of $C_c(G)$ are equivalent if there is a unitary operator $\phi\colon H \to H'$ that intertwines π and π'.

The relation between the unitary representations of (G, α) (Definition 1.6.3) and the non-degenerate representations of $C_c(G)$ is given by the following two theorems.

Theorem 1.6.10. (Integration of a representation) *Let (G, α) be a locally compact, Hausdorff groupoid with a Haar system.*

i) A representation $(\mu, \mathcal{H}, \pi)$ of (G, α) induces a non-degenerate *-representation of $C_c(G)$ on the Hilbert space $\mathcal{L}^2(G^{(0)}, \mu\,; \mathcal{H})$. This representation is continuous in the inductive limit topology and is bounded in the I-norm.

ii) If two representations of (G, α) are equivalent, then the representations of $C_c(G)$ which they induce are also equivalent.

Proof. See Proposition 2.1.7 of [33]. □

The representation that $(\mu, \mathcal{H}, \pi)$ induces on $C_c(G)$ is called the *integrated form* of $(\mu, \mathcal{H}, \pi)$. If $(\mu, \mathcal{H}, \pi)$ is a representation of G, the operator $\bar{\pi}_\mu(f)$ for $f \in C_c(G)$ on $\mathcal{L}^2(G^{(0)}, \mu\,; \mathcal{H})$ is defined by

$$\langle \bar{\pi}_\mu(f)\zeta\,, \psi\rangle = \int_G f(\gamma)\,\langle \pi(\gamma)(\zeta \circ s)(\gamma)\,, \psi \circ r(\gamma)\rangle\,\sqrt{\frac{\mathrm{d}(\mu \circ \alpha^{-1})}{\mathrm{d}(\mu \circ \alpha)}(\gamma)}\,\mathrm{d}(\mu \circ \alpha)(\gamma). \tag{1.6.11}$$

When G is a group, this formula matches the usual formula for the integrated representation with the exception of the factor of the modular function. Remark 1.6.7 explains how both the formulae give isomorphic C*-algebras.

Theorem 1.6.12 (Disintegration theorem)**.** *Let (G, α) be a locally compact, Hausdorff, second countable groupoid with a Haar system.*

*i) Every non-degenerate continuous representation of the convolution *-algebra $C_c(G)$ into a separable Hilbert space H is the integrated form of a representation of the groupoid (G, α).*

ii) The process of integration establishes an equivalence between the category of unitary representations of (G, α) on Hilbert bundles with separable fibres and the category of continuous, non-degenerate representations of $C_c(G)$ on separable Hilbert spaces.

Proof. For the proof of (i), take σ to be the cocycle $\sigma(\gamma, \gamma') = 1$ in [33, Theorem 2.1.21]. And (ii) is Corollary 2.1.23 in [33]. □

Remark 1.6.13. The original proof of Theorem 2.1.21 in [33] needs the existence of sufficiently many non-singular G-sets. But as we mentioned in the introduction, Renault removed this requirement in [34], which allows us to write the theorem.

Corollary 1.6.14 (Corollary 2.1.22, [33])**.** *If (G, α) satisfies the assumptions for Theorem 1.6.12, then every continuous non-degenerate representation of $C_c(G)$ on a separable Hilbert space is bounded for the I-norm.*

Here $C_c(G)$ has the inductive limit topology and if H is the Hilbert space in the corollary then $\mathbb{B}(H)$ has the weak operator topology.

The process of obtaining a representation of (G, α) from one of $C_c(G)$ is called *disintegration* of the representation.

Due to the integration and the disintegration theorems, we can use the term '*a representation of G* for a representation of G on a measurable G-Hilbert bundle or a *-representation of $C_c(G)$. Corollary 1.6.14 allows us to define the C*-norm on $C_c(G)$.

Definition 1.6.15 (C*-algebra of a groupoid)**.** For a groupoid with a Haar system (G, α) define the universal C*-norm $\|\cdot\|$ on $C_c(G)$ as

$$\|f\| := \sup\{\|L(f)\| : L \text{ is a non-degenerate continuous } {}^*\text{-representation of } C_c(G)\}.$$

The completion of $C_c(G)$ in this norm is a C*-algebra, which is called the C*-algebra of (G, α).

Note that the C*-algebra of G depends on the Haar system α. Hence we denote this C*-algebra by $C^*(G, \alpha)$.

For a groupoid with a Haar measure (G, α), as in the group case, one can take the *left regular representation* of $C_c(G)$ and complete $C_c(G)$ with respect to the norm of the left regular representation. This gives the reduced C*-algebra of (G, α), which we denote by $C^*_r(G, \alpha)$ or $C^*_r(G)$. For more details, Chapter 2 of [33] is a good reference.

Example 1.6.16. If X is a space with delta Dirac measures as Haar system, then its groupoid C*-algebra is $C_0(X)$.

Example 1.6.17. If G is a locally compact group then its groupoid C*-algebra is the group C*-algebra. See Remark 1.6.7.

Example 1.6.18. If G is a group acting on a space X, then the C*-algebra of the transformation groupoid $G \ltimes X$ is isomorphic to the crossed product $G \ltimes C_0(X)$.

1.6.2 Transverse measures

Following A. Connes' idea of non-commutative integration in [10], Renault develops the general theory of transverse measures, discussed in [34]. The motivation, important results and examples of the theory are discussed in Appendix I of [1]. Let X be a proper (G, α)-space with a family of measures. The theory of transverse measures gives a process of inducing an equivalence class of measures on the quotient space X/G. Each induced equivalence class of measures is 'symmetric' in a certain sense, see Proposition 1.6.33. When $X = G$ and the family of measures is α itself, this process of inducing classes of measures gives *distinct* equivalence classes of *quasi-invariant* measures on $G/G = G^{(0)}$.

Use of the transverse measure theory allows us to write many results in a compact fashion. We use the transverse measure theory in Chapter 2 to construct a Hilbert C*-module and the composition of topological correspondences. In Chapter 3, we use this theory to discuss the example of a spatial hypergroupoid. Use of transverse measure classes reduces the pains of checking and writing lots of small isomorphism results.

The present subsection is based on Appendix I of [1]. All the definitions and results are copied from the same source. But we elaborate many constructions and give details of most of the arguments there.

In the following discussion (G, α) is a pair consisting of a locally compact, Hausdorff groupoid and a Haar system on it. We assume that G acts on X (or Y) from the left, as mentioned earlier r_X (r_Y, respectively) are the momentum maps, and the actions are *continuous*. The momentum maps for the actions are continuous, but need not be surjective or open[2]. Only in this section, we write the quotient by a left or right action as X/G instead of $G\backslash X$ or X/G, respectively.

In this section a 'G-invariant family of measures' on X means a 'G-invariant continuous proper family of measures along the momentum map r_X'. For a Borel space X, let $\mathcal{B}(X)$ and $\mathcal{B}_+(X)$ denote the sets of Borel and non-negative Borel functions on X, respectively.

[2]If the space carries a continuous family of measures with full support, then the momentum maps are open.

Let $\pi\colon X \to Y$ be a continuous G-map between G-spaces and let λ be a G-invariant family of measures along π. Recall that λ is *proper* if there is $e \in \mathcal{B}_+(X)$ with $\lambda(e) = 1$ on $\pi(X)$. In the continuous case , a family of measures is proper if and only if each measure in the family is non-zero by [1, Lemma 1.1.2].

Remark 1.6.19. Our hypotheses, proofs and computations of results [1, Appendix 1] are milder than the original ones, though, the definitions are the same. In the original work G and X are Borel spaces, the invariant families of measures are Borel but the actions are proper actions. The main result, namely, Proposition 1.6.33 (which is [Proposition A.1.20 [1]]) is stated for *proper* invariant families of measures. However, the discussion below goes through when the continuity is replaced by Borelness. Since we have been working in the continuous setting and we plan to do so even later, we write the proof in the continuous setting. This will also allow us to redirect the reader in latter chapters to this section for some computations.

Definition 1.6.20 (Definition A.1.13, [1]). Let $\Delta \to X$ be a Borel $\mathbb{R}_+^*$-principal bundle. We define a Δ-measure as a map μ from the Borel sections of Δ to $[0, \infty]$ such that, for some Borel section σ, there is a measure μ_σ on X with $\mu(f\sigma) = \mu_\sigma(f)$ for all positive Borel functions f on X.

The measure μ_σ for a section σ gives a measure μ_τ for every section τ. For a section τ and a non-negative Borel function f on X we have,

$$\mu_\tau(f) = \mu(f\tau) = \mu\left(f\,\frac{\tau}{\sigma}\cdot\sigma\right) = \mu_\sigma(f\frac{\tau}{\sigma}).$$

Hence $\mu_\tau = \frac{\tau}{\sigma}\mu_\sigma$. Thus a Δ-measure μ gives a family of measures $\{\mu_\sigma\}_{\sigma\in\Gamma(X;\Delta)}$, such that any two measures μ_σ and μ_τ are equivalent and $\mathrm{d}\mu_\tau/\mathrm{d}\mu_\sigma = \tau/\sigma$. Conversely, if ν is a measure equivalent to μ_σ and f is a positive function, then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\sigma}\sigma$ is a section of $\Delta \to X$ and

$$\nu(f) = \mu_\sigma(f\,\mathrm{d}\nu/\mathrm{d}\mu_\sigma) = \mu_{\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\sigma}\sigma}(f).$$

Thus a Δ-measure defines the equivalence class of a measure.

Given a measure ν on a space X, let $\Delta = X \times \mathbb{R}_+^*$ be the trivial $\mathbb{R}_+^*$-principal Borel bundle. We construct a Δ-measure μ on X as follows: Let $\mu\colon \Gamma(X;\Delta) \to [0,\infty]$ be $\mu(\sigma) = \int_X \sigma\,\mathrm{d}\nu$ for a section σ. Then for a non-negative Borel function f on X, we get $\mu(f\sigma) = \int f\sigma\,\mathrm{d}\nu$. In this case, for each section σ the measure μ_σ is $\sigma\nu$.

Lemma 1.6.21. *Let X be a space.*

i) Any measure ν on X comes from an $\mathbb{R}_+^*$-principal bundle $\Delta \to X$ and a Δ-measure.

ii) Given a Δ-measure μ on X and two sections σ, σ' of $\Delta \to X$, the measures $\mu_\sigma, \mu_{\sigma'}$ are equivalent.

For $\sigma \in \Gamma(X;\Delta)$, μ_σ is called *the measure defined by* σ for μ or simply the measure defined by σ. We call the measure class $\{\mu_\sigma\}_{\sigma\in\Gamma(X;\Delta)}$ the measure class defined by μ.

By Proposition 1.3.27, if X, Y are proper G-spaces, $\pi\colon Y \to X$ is a G-map and β is a continuous invariant family of measures along π, then β induces a family of continuous measures $[\beta]$ along

$[\pi]\colon Y/G \to X/G$. In Proposition 1.3.27, if the continuity properties are replaced by Borel properties, a similar claim holds. The proof can be written along the same lines as the proof of Proposition 1.3.27.

Let $\Delta \to G^{(0)}$ be a G-equivariant Borel $\mathbb{R}_+^*$-principal bundle. Given a proper G-space X, we have the G-equivariant $\mathbb{R}_+^*$-principal pull-back bundle $r_X^*(\Delta) \to X$. For $\gamma \in G$ and $(x,t) \in r_X^*(\Delta)$, $\gamma(x,t) = (\gamma x, \gamma t)$. Let $\Delta_X = (r_X^*(\Delta))/G$, which is a bundle over X/G. Note that $\Delta_G = \Delta$.

Definition 1.6.22 (Transverse measure, Definition A.1.15 in [1]). Let G be an analytic Borel groupoid and let $\Delta \to G^{(0)}$ be a G-equivariant Borel $\mathbb{R}_+^*$-principal bundle. A Δ-transverse measure T is a coherent assignment $(X,\lambda) \mapsto \mathsf{T}(\lambda)$ where

i) (X,λ) is a proper (left) G-space with an invariant family of measures,

ii) $\mathsf{T}(\lambda)$ is a Δ_X-measure on X/G,

iii) coherence means that for every invariant Borel G-map $\pi\colon Y \to X$, where X, Y are proper analytic Borel G-spaces, every invariant Borel system of measures β along π and every invariant Borel system λ for r_X we have $\mathsf{T}(\lambda \circ \beta) = \mathsf{T}(\lambda) \circ [\beta]$.

We recall from the introduction of this section that in this section all the actions are left actions but we denote the quotient by X/G. When Δ is the graph of a homomorphism $\delta\colon G \to \mathbb{R}_+^*$, a Δ-transverse measure is also called a transverse measure of module δ.

We discuss an example of a transverse measure. Given a quasi-invariant measure ν on $G^{(0)}$, we construct a transverse measure. We know that $\Delta_{G,\nu}\colon G \to \mathbb{R}_+^*$ is a $\nu \circ \alpha$-almost everywhere homomorphism. Chose a ν-conull Borel set $U \subseteq G^{(0)}$ such that $\Delta_{G,\nu}|_{G_U^U}$ is a strict homomorphism. Let Δ be the Borel $\mathbb{R}_+^*$-bundle associated with the graph of $\Delta_{G,\nu}|_{G_U^U}$. As a space, Δ is merely $U \times \mathbb{R}_+^*$. Define an action of $\mathbb{R}_+^*$ on Δ as $(u,t)t' = (u,tt')$. Then $\Delta \to G^{(0)}$ is a Borel $\mathbb{R}_+^*$-principal bundle. Define an action of G_U^U on this bundle by $\gamma(s_G(\gamma),t) = (r_G(\gamma), \Delta_{G,\nu}(\gamma)t)$. This makes Δ a Borel G-bundle. Thus we have a G-equivariant Borel $\mathbb{R}_+^*$-bundle over $G^{(0)}$.

Let α_X be the family of measures along the quotient map $p_X\colon X \to X/G$ as in Proposition 1.3.21.

Lemma 1.6.23. *The integration map $A_X\colon \Gamma_c(X, r_X^*(\Delta)) \to \Gamma_c(X/G; \Delta_X)$ as in Proposition 1.3.21, sending $f \mapsto \int f(\gamma x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)$, is a surjection.*

Proof. Let $f \in \Gamma_c(X/G; \Delta_X)$ be given. Using Lemma 1.3.16 we get a function $g_f^X \in \Gamma_c(X, r_X^*(\Delta))$ with $\alpha_X(g_f^X) = 1$ on $\mathrm{supp}(f)$. Then $f = \alpha_X(f \circ p_X \cdot g_f^X)$. □

Note that $A_G = A$ where $A_G\colon \Gamma_c(G, r_G^*(\Delta)) \to \Gamma_c(G^{(0)}; \Delta)$ is the integration map as in the statement of the lemma and $A\colon \Gamma_c(G, r_G^*(\Delta)) \to \Gamma_c(G^{(0)}; \Delta)$ is the map induced by the integration map $C_c(G) \to C_c(G^{(0)})$ that occurs in the definition of the Haar system α.

Now we define the Δ_X-transverse measure $\mathsf{T}(\lambda)$. Let $f \in \Gamma_c(X/G; \Delta_X)$ then

$$\mathsf{T}(\lambda)(f) := \int_{G^{(0)}} \int_{X^u} f([x])\, g_f^X(x)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\nu(u) \tag{1.6.24}$$

with $g_f^X \in \Gamma_c(X; r_X^*(\Delta))$ as in the proof of Lemma 1.6.23. It is a slightly complicated but straightforward computation to check that T does not depend on the choice of g_f^X. The reader may refer to a similar computation that we do in (iii) of Proposition 2.4.3. The proof uses a commutative diagram as in Figure 1.2.

For fixed $f \in \Gamma(X/G, \Delta_X)$ define $\mathsf{T}(\lambda)_f \colon \Gamma_c(X/G) \to [0, \infty)$ by the formula

$$\mathsf{T}(\lambda)_f(k) = \mathsf{T}(\lambda)(f \cdot k) = \iint k([x]) f([x])\, g^X(x)\, \mathrm{d}\lambda^u(x)\, \mathrm{d}\nu(u), \tag{1.6.25}$$

where $k \in \Gamma_c(X/G)$. Then $\mathsf{T}(\lambda)_f$ is continuous in the inductive limit topology (in fact, T is continuous when the convergence in $\Gamma_c(X/G)$ is Lebesgue's dominated convergence). Due to the Riesz representation theorem, $\mathsf{T}(\lambda)_f$ induces a positive Radon measure on X/G, which we denote by $\mathsf{T}(\lambda)_f$. If $g \in \mathcal{B}_+(X/G)$, then $\mathsf{T}(\lambda)_f(g) = \mathsf{T}(\lambda)(gf)$. To announce that$\mathsf{T}(\lambda)$ is the required Δ_X-transverse measure on X/G, we need to check that the assignment T is coherent.

$$\begin{array}{ccccc} Y & \xrightarrow[\beta]{\pi} & X & \xrightarrow[\lambda]{r_X} & G^{(0)} \\ {\scriptstyle p_Y}\downarrow{\scriptstyle \alpha_Y} & & {\scriptstyle p_X}\downarrow{\scriptstyle \alpha_X} & & \\ Y/H & \xrightarrow[{[\beta]}]{[\pi]} & X/H. & & \end{array}$$

Figure 1.3

To check the coherence we redraw Figure 1.2 filling in the present data to get Figure 1.3. The coherence follows easily from the commutativity of the square in Figure 1.3. Let $f \in \Gamma_c(Y/G; \Delta_Y)$, then

$$\begin{aligned} \mathsf{T}(\lambda)([B](f)) = \nu\left(\Lambda\left([B](f) \circ p_X \cdot g^X_{[B](f)}\right)\right) &= \nu\left(\Lambda\left(B\left(f \circ p_X \cdot g_f^Y\right)\, g^X_{[B](f)}\right)\right) \\ &= \nu\left(\Lambda \circ B\left(f \circ p_X \cdot g_f^Y\right)\, g^X_{[B](f)}\right) = \mathsf{T}(\lambda \circ \beta)(B(f)) \end{aligned}$$

The above computations are sketchy. Writing them with integration symbols makes things much clearer. We avoid it due to its length and complexity.

We discuss an interesting fact now. We constructed the transverse measures T using the quasi-invariant measure ν on $G^{(0)}$. What is the relation between T and ν?

For the answer, check what happens when $(X, \lambda) = (G, \alpha)$, $f = \Delta_{G,\nu}$ and $k \in \Gamma_c(G^{(0)}, \Delta)$ in Equation 1.6.24. In this case, note that f is a G-equivariant section of $\Delta_{G,\nu}$. This is the point where we use the definition of $\Delta_{G,\nu}$ and now we compute

$$\mathsf{T}(\alpha)_{\Delta_{G,\nu}}(k) = \iint k(u)\, g_k^G(\gamma)\, \mathrm{d}\alpha^{u = r_G(\gamma)}(\gamma)\, \mathrm{d}\nu(u) = \int k(u)\, \mathrm{d}\nu(u),$$

where we use that $\Delta_{G,\nu}(\gamma) f(\gamma)|_{G^{(0)}} = 1$ and $\int_G g_k^G\, \mathrm{d}\alpha^u = 1$ on $\mathrm{supp}(k)$. Thus $\mathsf{T}(\alpha)_{\Delta_{G,\nu}} = \nu$.

The above construction of a transverse measure using a quasi-invariant measure is the first half of Proposition 1.6.26 below.

Proposition 1.6.26 (Proposition A.1.15 [1]). *Let (G, α) be an analytic Borel groupoid with a Borel family of measures and let $\delta\colon G \to \mathbb{R}_+^*$ be a Borel homomorphism. Then there is a one-to-one correspondence between*

i) quasi-invariant measures μ for (G, α) with $\mathrm{d}(\mu \circ \alpha^{-1})/\mathrm{d}(\mu \circ \alpha) = \delta$ almost everywhere;

ii) transverse measures T for G of module δ.

The measure μ and the transverse measure T are related by $\mu = \mathsf{T}(\alpha)_\delta$.

We are going to prove the second half of the above proposition. We show that given a Δ-transverse measure T for (G, α) of module $\delta\colon G \to \mathbb{R}_+^*$,

a) the measure class defined by $\mathsf{T}(\alpha)$ consists of quasi-invariant measures on $G^{(0)}$;

b) there is a measure μ in the measure class defined by $\mathsf{T}(\alpha)$ with $\mathrm{d}(\mu \circ \alpha^{-1})/\mathrm{d}(\mu \circ \alpha) = \delta$ $\mu \circ \alpha$-almost everywhere.

Let $\Delta \to G^{(0)}$ be the G-equivariant Borel principal $\mathbb{R}_+^*$-bundle corresponding to the graph of δ and let T be a Δ-transverse measure. For $(X, \lambda) = (G, \alpha)$ we have $\Delta_G = \Delta$, which gives us a Δ-measure $T(\alpha)$ on $G^{(0)}$. Fix a section σ of Δ. Let $\mathsf{T}(\alpha)_\sigma$ be the corresponding measure on $G^{(0)}$. We show that this measure is quasi-invariant.

For $G \times_{r_G, r_G} G$, let π_i, $i = 1, 2$ be the projections on the first and the second factor, respectively. Define the family of measures α_i along π_i for $i = 1, 2$ by

$$\int f(\gamma, \gamma')\,\mathrm{d}\alpha_1^{\gamma}(\gamma, \gamma') = \int f(\gamma, \gamma')\,\mathrm{d}\alpha^{r_G(\gamma)}(\gamma'), \tag{1.6.27}$$

$$\int f(\gamma, \gamma')\,\mathrm{d}\alpha_2^{\gamma'}(\gamma, \gamma') = \int f(\gamma, \gamma')\,\mathrm{d}\alpha^{r_G(\gamma')}(\gamma) \tag{1.6.28}$$

for a non-negative Borel function f.

The family α_2 induces a family of measures along $[\pi_2]\colon (G \times_{r_G, r_G} G)/G \to G/G$. Identify $(G \times_{r_G, r_G} G)/G$ with G via the map $(\gamma, \gamma') \mapsto \gamma^{-1}\gamma'$. And G/G is identified with $G^{(0)}$ via the map $\gamma \mapsto r_G(\gamma)$. Then $[\pi_2]([\gamma, \gamma']) = [\pi_2]([r_G(\gamma'), \gamma']) = \pi_2([\gamma']) = r_G(\gamma')$. Thus $[\pi_2] = r_G\colon G \to G^{(0)}$.

We compute $[\alpha_2]$ now. Take a non-negative Borel function f on G. Then

$$\int f([r_G(\gamma'), \gamma'])\,\mathrm{d}[\alpha_2]^{r_G(\gamma')}([r_G(\gamma'), \gamma']) = \int f(\gamma')\,\mathrm{d}\alpha^{[\pi_2]([r_G(\gamma'), \gamma']) = r_G(\gamma')}(\gamma') = \int_{G^{r_G(\gamma')}} f\,\mathrm{d}\alpha^{r_G(\gamma')}. \tag{1.6.29}$$

Thus $[\alpha_2] = \alpha$ along the range map r_G.

A similar computation shows that $[\pi_1] = s_G\colon G \to G^{(0)}$ and $[\alpha_1] = \alpha^{-1}$ along s_G. Replace non-negative Borel functions by functions in $C_c(G)$. A simple computation shows $\alpha \circ \alpha_1 = \alpha \circ \alpha_2 = \alpha \times \alpha$. Hence $\mathsf{T}(\alpha \circ \alpha_1) = \mathsf{T}(\alpha \circ \alpha_2)$. But the coherence of T gives $\mathsf{T}(\alpha) \circ [\alpha_1] = \mathsf{T}(\alpha) \circ [\alpha_2]$, that is, $\mathsf{T}(\alpha) \circ \alpha = \mathsf{T}(\alpha) \circ \alpha^{-1}$. Thus $\mathsf{T}(\alpha)_\sigma \circ \alpha, \mathsf{T}(\alpha)_\sigma \circ \alpha^{-1} \in \mathsf{T}(\alpha) \circ \alpha_1 = \mathsf{T}(\alpha \circ \alpha_1)$. But all the measures defined by $\mathsf{T}(\alpha \circ \alpha_1)$ are equivalent, hence $\mathsf{T}(\alpha)_\sigma \circ \alpha \sim \mathsf{T}(\alpha)_\sigma \circ \alpha^{-1}$.

The discussion in the last paragraph proves that for every section $\sigma \in \Gamma(G^{(0)}, \Delta)$ the measure $\mathsf{T}(\alpha)_\sigma$ is quasi-invariant. Denote the measure $\mathsf{T}(\alpha)_\delta$ by μ.

Recall that $\Delta \to G^{(0)}$ is G-equivariant and the action of G is given by $\gamma(s_G(\gamma), t) = (r_G(\gamma), \delta(\gamma)t)$. Now we prove that $\mathsf{T}(\alpha)_\delta = \Delta_{G,\mu}$ almost everywhere. We need to keep the action in mind. We use Lemma 1.6.23; Let $f \in \Gamma_c(G, r_G^*(\Delta))$, then $A(f) \in \Gamma_c(G^{(0)}, \Delta)$ and we may write

$$\begin{aligned} \mathsf{T}(\alpha)_\delta(A(f)) &= \int_{G^{(0)}} \left(\int_u f(\gamma) \, \mathrm{d}\alpha^u(\gamma) \right) \mathrm{d}\mu(u) \\ &= \int_{G^{(0)}} \left(\int_u f(\gamma^{-1}) \, \delta(\gamma^{-1}) \, \mathrm{d}\alpha^u(\gamma) \right) \mathrm{d}\mu(u). \end{aligned} \tag{1.6.30}$$

But μ is quasi-invariant, hence $\delta(\gamma) = \Delta_{G,\nu}(\gamma^{-1}) = \frac{\mathrm{d}(\nu\circ\alpha^{-1})}{\mathrm{d}(\nu\circ\alpha)}(\gamma)$ $\mu \circ \alpha$-almost everywhere.

From Equation 1.6.30 it is clear that $\mathrm{graph}(\delta) \simeq \mathrm{graph}(\Delta_{G,\mu})$ $\mu \circ \alpha$-almost everywhere. Hence the corresponding G-equivariant $\mathbb{R}_+^*$-principal Borel bundles are Borel isomorphic. We do not distinguish between both bundles and denote them by $\Delta \to G^{(0)}$ itself.

Denote the transverse measure on $\Delta \to X$ obtained from the graph of $\Delta_{G,\mu}$ by T' (see Equation (1.6.25) for the construction). To prove the claim of Proposition 1.6.26 we need to show that T and T' are the same function. From the computation involving Equation 1.6.30 we can see that $\mathsf{T}(\alpha) = \mathsf{T}'(\alpha)$. We sketch the proof that $\mathsf{T}(\lambda) = \mathsf{T}'(\lambda)$ for all pairs (X, λ) consisting of a proper G-space and an invariant family of measures below.

Let X be a proper G-space and let λ be a G-invariant family of measures on X. Let $\pi_1 \colon G \times_{r_G, r_X} X \to G$ and $\pi_2 \colon G \times_{r_G, r_X} X \to X$ be the projection maps and let λ_1 and α_2 be the following families of measures along π_1 and π_2, respectively:

$$\begin{aligned} \int f \, \mathrm{d}\lambda_1^\gamma &= \int_{X^{r_G(\gamma)}} f(\gamma, x) \, \mathrm{d}\lambda^{r_G(\gamma)}(x), \\ \int f \, \mathrm{d}\alpha_2^x &= \int_{G^{r_X(x)}} f(\gamma, x) \, \mathrm{d}\alpha^{r_X(x)}(\gamma), \qquad \text{for } f \in C_c(G \times_{r_G, r_X} X). \end{aligned}$$

i) A small computation involving Fubini's Theorem with functions in $C_c(G \times_{r_G, r_X} X)$ shows that $\alpha \circ \lambda_1 = \lambda \circ \alpha_2$.

ii) The family of measures along $[\pi_1]$ is $[\lambda_1] = \lambda$ and the one along $[\pi_2]$ is $[\alpha_2] = \alpha_X$. Here for $f \in \mathcal{B}_+(X)$, $\int f \, \mathrm{d}\alpha_X^{[x]} = \int f(\gamma^{-1}x) \, \mathrm{d}\alpha^{r_X(x)}(\gamma)$.

The above two facts, the coherence of T and T' and the fact that $\mathsf{T}(\alpha) = \mathsf{T}'(\alpha)$ show that

$$\begin{aligned} \mathsf{T}(\lambda) \circ \alpha_X = \mathsf{T}(\lambda \circ \alpha_2) = \mathsf{T}(\alpha \circ \lambda_1) = \mathsf{T}(\alpha) \circ \lambda \\ = \mathsf{T}'(\alpha) \circ \lambda = \mathsf{T}'(\alpha \circ \lambda_1) = \mathsf{T}'(\lambda \circ \alpha_2) = \mathsf{T}'(\lambda) \circ \alpha_X \end{aligned} \tag{1.6.31}$$

The equality of measure classes in Equation 1.6.31 says that for measures $\nu \in \mathsf{T}(\Lambda)$ and $\nu' \in \mathsf{T}'(\Lambda)$ we have $\nu \circ \alpha_X \sim \nu' \circ \alpha_X$. For $f \in \Gamma_c(X/G, \Delta_X)$ we choose g_f^X as in Lemma 1.6.23 to see that

$$\begin{aligned} \nu(f) = \nu \circ \alpha_X(f \circ p_X \cdot g_f^X) &= \nu' \circ \alpha_X \left(\frac{\mathrm{d}\nu \circ \alpha_X}{\mathrm{d}\nu' \circ \alpha_X} f \circ p_X \cdot g_f^X \right) \\ &= \nu' \left(f \circ p_X \cdot \frac{\mathrm{d}\nu \circ \alpha_X}{\mathrm{d}\nu' \circ \alpha_X} \cdot \alpha_X\left(g_f^X\right) \right) = \nu' \left(f \circ p_X \cdot \frac{\mathrm{d}\nu \circ \alpha_X}{\mathrm{d}\nu' \circ \alpha_X} \right). \end{aligned}$$

Hence $\nu \sim \nu'$ with $\frac{d\nu}{d\nu'} = \frac{d\nu \circ \alpha_X}{d\nu' \circ \alpha_X}$. But then Lemma 1.6.21 gives $\mathsf{T}(\alpha) = \mathsf{T}'(\alpha)$.

Definition 1.6.32 (Transverse measure class, Definition A.1.19 [1]). For a Borel groupoid G a *transverse measure class* m on G is a coherent assignment $(X, \lambda) \mapsto m(\lambda)$, where

i) X is a proper G-space and λ is a G-invariant family of measures on X;

ii) $m(\lambda)$ is a measure class on X/G;

iii) coherence of the assignment means: for every Borel G-map $\pi\colon Y \to X$ between proper Borel G-spaces X and Y, every G-invariant Borel family of measures λ' for π and every G-invariant Borel family of measures λ for $r_X\colon X \to G^{(0)}$, one has $m(\lambda \circ \lambda') = m(\lambda) \circ [\lambda']$.

Note that a transverse measure T defines a transverse measure class m such that for a proper G-space X and an invariant family of measures λ on X, $m(\lambda)$ is the measure class defined by $\mathsf{T}(\lambda)$.

Proposition 1.6.33 (Proposition A.1.20 [1]). *Let G be a Borel groupoid.*

i) For every transverse measure class m and every G-space X with an invariant family of measures, the measure class $m(\lambda) \circ [\alpha_1]$ on $(X * X)/G$ is invariant under the symmetry $(x, y) \mapsto (y, x)$.

ii) Conversely, given a proper G-space (X, λ) with an invariant family of measures λ and a measure class $[\mu]$ on X/G such that the measure class $[\mu \circ [\alpha_1]]$ on $(X * X)/G$ is symmetric, there is a unique transverse measure class m with $[y] = m(\alpha)$.

Corollary 1.6.34. *Let G be a Borel groupoid and α a Borel family of measures for G.*

i) Every quasi-invariant measure μ on $G^{(0)}$ induces a transverse measure class m.

ii) Conversely, given a transverse measure class m, there is a quasi-invariant measure $\mu \in m(\alpha)$ such that the transverse measure class induced by μ is m.

Proof. (i): Given a quasi-invariant measure μ, Proposition 1.6.26 gives a transverse measure T of module $d(\mu \circ \alpha^{-1})/d(\mu \circ \alpha) = \Delta_{G,\mu}^{-1}$. The transverse measure T defines a transverse measure class m such that for (X, λ), $m(\lambda)$ is the measure class defined by $\mathsf{T}(\lambda)$.

(ii): Let m be a transverse measure class for G. Let $\mu \in m(\alpha)$. Then (i) of Proposition 1.6.33 says that the measure $m(\alpha) \circ [\alpha_1]$ is invariant under the symmetry. Hence $m(\alpha) \circ [\alpha_1] = m(\alpha) \circ [\alpha_2]$. But from the discussion on Page 32 we have $[\alpha_1] = \alpha$ and $[\alpha_2] = \alpha^{-1}$. Thus if $\nu \in m(\alpha)$, then $\nu \circ \alpha \sim \nu \circ \alpha^{-1}$.

If T is the transverse measure induced by ν, then Proposition 1.6.26 shows that m is the transverse measure class defined by T. $\square$

1.6.3 Representation theory II

Corollary 1.6.34 shows that there is a bijection between the classes of quasi-invariant measures on $G^{(0)}$ and transverse measure classes for (G, α). This enabled Renault to rewrite the representation theory in a new language in [34].

We know that a representation of (G, α) is given by a (G, α)-quasi-invariant measure μ on $G^{(0)}$ and a μ-measurable G-Hilbert bundle. Let $(\mu, \mathcal{H}, \pi)$ be a representation of (G, α). If μ' is equivalent to μ then μ' is also quasi-invariant (see 1.5.7). Then $(\mu, \mathcal{H}, \pi)$ and $(\mu', \mathcal{H}, \pi)$ are equivalent representations. From Equation (1.6.4) we see that the Hilbert spaces of square-integrable sections of $\mathcal{H}$ for μ and μ' are unitarily isomorphic. Thus for a fixed measurable G-Hilbert bundle $\mathcal{H}$, a representation depends only on the equivalence class of μ. Using the transverse measure theory one can work with the equivalence class of μ at once. The trick is to replace the Hilbert space of μ-square integrable sections by a Hilbert space that represents the square-integrable sections for the *class* of μ.

We are going to rewrite the representation theory of groupoids with quasi-invariant measures replaced by transverse measure classes.

Let $(\mu, \mathcal{H}, \pi)$ be a representation of (G, α). If m is the transverse measure class induced by μ (see Corollary 1.6.34), then we replace $\mathcal{L}^2(G^{(0)}, \mu; \mathcal{H})$ by the Hilbert space of half-densities:

$$\mathcal{L}^2(G^{(0)}, m\,; \mathcal{H}) = \{\zeta\sqrt{\nu} : \nu \in m \text{ and } \zeta \in \mathcal{L}^2(G^{(0)}, \nu\,; \mathcal{H})\}.$$

The inner product of $\zeta\sqrt{\nu}, \xi\sqrt{\mu} \in \mathcal{L}^2(G^{(0)}, \mu; \mathcal{H})$ is defined by

$$\langle \zeta\sqrt{\nu}\,, \xi\sqrt{\mu}\rangle = \int \langle \zeta(u)\,, \xi(u)\rangle \sqrt{\frac{\nu}{\mu}(u)}\, \mu(u).$$

Definition 1.6.35 (Representation of a groupoid, Definition 3.4 [35]). *A representation* of a groupoid with Haar system (G, α) is a quadruple $(m, \mathcal{H}, p, \pi)$, where m is a transverse measure class for (G, α) and $(\mathcal{H}, p, \pi)$ is a μ-measurable G-Hilbert bundle on $G^{(0)}$.

Two representations are equivalent if there is a unitary that intertwines the Hilbert bundles as described in the previous section. As before, we drop the projection map from our notation and call $(m, \mathcal{H}, \pi)$ a representation of (G, α).

We define non-degenerate representations of the *-algebra $C_c(G)$ to be continuous in the inductive limit topology as in [34]. One can define what it means for two representations of $C_c(G)$ to be equivalent.

The relation between the unitary representations of (G, α) (Definition 1.6.35) and the non-degenerate representations of $C_c(G)$ is given by the following two theorems.

Proposition 1.6.36 (Integration of representation, Proposition 3.5, [34]). *Let (G, α) be a groupoid with a Haar system.*

*i) A representation $(m, \mathcal{H}, \pi)$ of (G, α) as in Definition 1.6.35 induces a non-degenerate *-representation of $C_c(G)$ on the Hilbert space of $m(\alpha)$-square integrable sections of the bundle $\mathcal{H}$. This *-representation is continuous in the inductive limit topology and bounded in the I-norm.*

ii) *If two representations of (G,α) are equivalent, then the representation of $C_c(G)$ which they induce are also equivalent.*

For a representation $(m,\mathcal{H},\pi)$ of G and $f\in C_c(G)$, the operator $\bar{\pi}(f)$ on $\mathcal{L}^2(G^{(0)},\mu\,;\mathcal{H})$ is given by

$$\langle\bar{\pi}(f)\zeta\sqrt{\nu}\,,\psi\sqrt{\mu}\rangle=\int_G f(\gamma)\,\langle\pi(\gamma)(\zeta\circ s)(\gamma)\,,\psi\circ r(\gamma)\rangle\,\sqrt{\frac{\mathrm{d}(\nu\circ\alpha^{-1})}{\mathrm{d}(\mu\circ\alpha)}(\gamma)}\,\mathrm{d}(\mu\circ\alpha)(\gamma). \tag{1.6.37}$$

From the symmetry-related claim of (i) of Proposition 1.6.33 we know that $m(\alpha)\circ\alpha=m(\alpha)\circ\alpha^{-1}$. Hence $\nu\circ\alpha^{-1}\sim\mu\circ\alpha$ and thus the Radon-Nikodym derivative $\frac{\mathrm{d}(\nu\circ\alpha^{-1})}{\mathrm{d}(\mu\circ\alpha)}$ in Equation (1.6.37) makes sense.

The representation that $(m,\mathcal{H},\pi)$ induces on $C_c(G)$ is called the *integrated form* of $(m,\mathcal{H},\pi)$. The process of inducing the representation is called *integration* of the representation $(m,\mathcal{H},\pi)$.

Theorem 1.6.38 (Disintegration theorem, Theorem 4.1, [34]). *Let (G,α) be a locally compact, Hausdorff groupoid equipped with a Haar system.*

i) *Every non-degenerate continuous representation of the convolution *-algebra $C_c(G)$ into a separable Hilbert space H is the integrated form of a representation of (G,α) of the form in Definition 1.6.35.*

ii) *The process of integration establishes an equivalence between the category of representations of (G,α) defined as in Definition 1.6.35 and the category of continuous, non-degenerate representations of $C_c(G)$ on separable Hilbert spaces.*

Now we can go through the same arguments as in the last subsection to construct $\mathrm{C}^*(G,\alpha)$.

1.7 Some more definitions

1.7.1 Amenability

In this short section we quickly review some ideas of amenability of groupoids from [1]. Anantharaman Delaroche and Renault introduced the notions of amenability for a *measured* Borel groupoid ([1, Definition 3.2.8]), *measurewise amenability* for a Borel groupoid ([1, Definition 3.3.1]) and *topological* amenability for a locally compact topological groupoid ([1, Definition 2.2.7]). In [1, Proposition 3.3.5], they prove that topological amenability implies measurewise amenability. The definition of measurewise amenability implies that if G is measurewise amenable, then for any Borel Haar system on G and a quasi-invariant measure for the Haar system, G is an amenable measured groupoid.

One of the most important results for us is [1, Theorem 6.1.4]. The above discussion along with this theorem yields the following result:

Lemma 1.7.1. *Let G be a locally compact, second countable Hausdorff groupoid and α a Haar system for G. If G is topologically amenable, then* $\mathrm{C}^*(G,\alpha)\simeq\mathrm{C}^*_{\mathrm{r}}(G,\alpha)$.

Proof. Since G is topologically amenable, [1, Proposition 2.2.5] gives that for any quasi-invariant measure μ the measured groupoid (G, α, μ) is amenable. Now (ii) of Theorem 6.1.4 [1] says that for any quasi-invariant measure μ, the trivial representation of (G, α, μ) is weakly contained in the regular representation. This proves the claim. □

Corollary 1.7.2. *Let G be a proper groupoid. Given a Haar system α on G, $\mathrm{C}^*(G, \alpha) \simeq \mathrm{C}^*_{\mathrm{r}}(G, \alpha)$.*

Proof. Follows from Lemma 1.3.29 and Lemma 1.7.1. □

1.7.2 C^*-correspondences

We shall be working with separable C^**-algebras only.* We shall use the theory of Hilbert modules and assume that the reader is familiar with it. For the theory of Hilbert modules the book of Lance [23] is a good reference.

Definition 1.7.3 (C^*-correspondence)**.** Let A and B be C^* algebras. A C^**-correspondence* from A to B is a Hilbert B-module $\mathcal{H}$ with a homomorphism $A \to \mathbb{B}_B(\mathcal{H})$.

A special type of C^*-correspondence is called an *equivalence* or an *imprimitivity bimodule*. A Hilbert B-module $\mathcal{H}$ is *full* if the linear span of the image of $\mathcal{H} \times \mathcal{H}$ under the inner product map is dense in B.

Definition 1.7.4. An *imprimitivity bimodule* from A to B is an A-B-bimodule $\mathcal{H}$ such that

i) $\mathcal{H}$ is a full left Hilbert A-module with an inner product ${}_*\langle,\rangle$;

ii) $\mathcal{H}$ is a full right Hilbert B-module with an inner product $\langle,\rangle_*$;

iii) $(\mathcal{H}, {}_*\langle,\rangle)$ is a correspondence from B to A;

iv) $(\mathcal{H}, \langle,\rangle_*)$ is a correspondence from A to B;

v) for $a, b, c \in \mathcal{H}$ $a\,\langle b\,, c\rangle_* = {}_*\langle a\,, b\rangle c$.

In [36], Rieffel shows that an A-B-imprimitivity bimodule induces an isomorphism between the representation categories of B and A.

In general, if $\mathcal{H}$ is a C^*-correspondence from A to B and $\mathcal{H}$ is a Hilbert B-module, then $\mathcal{H}$ induces a functor from $\mathrm{Rep}(B)$, the representation category of B, to the one of A.

In Section 2.5.2, we shall study bicategories. The reader may refer to Section 2.2 of [9], where the authors prove that C^*-correspondences form a bicategory.

We shall call a C^*-correspondence $\mathcal{H}$ from A to B a *proper* correspondence if A acts on $\mathcal{H}$ by compact operators, that is, the action of A is given by a homomorphism $A \to \mathbb{K}_B(\mathcal{H})$.

Definition 1.7.5. Let A and B be two C^*-algebras and let $\mathcal{H}$ and $\mathcal{H}'$ be two C^*-correspondences. A homomorphism of C^*-correspondences from $\mathcal{H}$ to $\mathcal{H}'$ is a map $\phi\colon \mathcal{H} \to \mathcal{H}'$ such that

i) ϕ is a homomorphism of Hilbert B-modules,

ii) ϕ intertwines the actions of A on the Hilbert B-modules $\mathcal{H}$ and $\mathcal{H}'$.

Definition 1.7.6. Let $\mathcal{H}$ and $\mathcal{H}'$ be two C*-correspondences from A to B. An isomorphism of C*-correspondences from $\mathcal{H}$ to $\mathcal{H}'$ is a homomorphism ϕ of C*-correspondences such that ϕ is unitary.

If there is an isomorphism of C*-correspondences from $\mathcal{H}$ to $\mathcal{H}'$, we call $\mathcal{H}$ and $\mathcal{H}'$ isomorphic.

Chapter 2

Topological correspondences

We define topological correspondences in this chapter. Let (G, α) and (H, β) be locally compact Hausdorff groupoids with Haar systems. Then a correspondence from (G, α) to (H, β) is a G-H-bispace where the H-action is proper and X carries a certain family of measures. X is locally compact, Hausdorff and second countable. This chapter deals with three topics:

constructing a C^*-correspondence from a topological correspondence: this topic has two main sections, namely, constructing a $C^*(H, \beta)$-Hilbert module using X and λ, and defining an action of (G, α) on this Hilbert module. Finally, both these results are put together to get a C^*-correspondence from $C^*(G, \alpha)$ to $C^*(H, \beta)$;

composing topological correspondences;

proving the functoriality of the assignment that sends a topological correspondence to a C^*-correspondence.

The Integration and Disintegration Theorems (Proposition 1.6.36, Theorem 1.6.12) are the main ingredients for the construction of a C^*-correspondence from a topological correspondence.

Composition of correspondences turns out to be a bit involved. Composition also needs that the correspondences are second countable. We shall use the method of *pushing a family of measures* on the quotient, which is similar to methods discussed in Section 1.6.2 on *transverse measures*.

Eventually, we show that topological correspondences form a bicategory and that the assignment that sends a topological correspondence to a C^*-correspondence is a morphism of bicategories. We give detailed calculations to construct the bicategory and the homomorphism between the bicategories.

In this chapter, all the groupoids and spaces considered are locally compact, Hausdorff. The space involved in a topological correspondences is locally compact, Hausdorff and second countable. In the latter half of the chapter, when we start with the composition of correspondences (Section 2.4.1), we assume that the groupoids are also second countable. All the groupoids carry a Haar system. In fact, most of the families of measures are continuous families of measures which are invariant for the right action. Hence we usually call an *proper invariant proper continuous family of measures* merely a *family of measures*.

2.1 Construction of topological correspondences

Recall Definition 1.7.3, which says that a C*-correspondence $\mathcal{H}$ from a C*-algebra A to a C*-algebra B is an A-B-bimodule with some extra structure and some conditions on the action. Similarly, we define a topological correspondence from a groupoid with Haar system (G, α) to a groupoid with Haar system (H, β) as a G-H-bispace X with some extra structure and certain conditions on the actions.

Let G be a groupoid equipped with a Haar system α and X a left G-space. Let $G \ltimes X$ denote the transformation groupoid. Its space of arrows is $G * X := \{(\gamma, x) \in G \times X : s_G(\gamma) = r_X(x)\}$.

Definition 2.1.1 (Topological correspondence)**.** A *topological correspondence* from a locally compact, Hausdorff groupoid G with a Haar system α to a locally compact, Hausdorff groupoid H equipped with a Haar system β is a pair (X, λ), where:

i) X is a locally compact, Hausdorff, second countable G-H-bispace,

ii) $\lambda = \{\lambda_u\}_{u \in H^{(0)}}$ is an H-invariant proper continuous family of measures along the momentum map $s_X : X \to H^{(0)}$,

iii) the action of H is proper,

iv) Δ is a continuous function $\Delta : G \ltimes X \to \mathbb{R}^+$ such that for each $u \in H^{(0)}$ and $F \in C_c(G * X)$,

$$\int_{X_u} \int_{G^{r_X(x)}} F(\gamma^{-1}, x) \, \mathrm{d}\alpha^{r_X(x)}(\gamma) \, \mathrm{d}\lambda_u(x) = \int_{X_u} \int_{G^{r_X(x)}} F(\gamma, \gamma^{-1}x) \, \Delta(\gamma, \gamma^{-1}x) \, \mathrm{d}\alpha^{r_X(x)}(\gamma) \, \mathrm{d}\lambda_u(x).$$

If Δ' is another function that satisfies condition (iv) in Definition 2.1.1, then $\Delta = \Delta'$ $\lambda_u \circ \alpha$-almost everywhere for each $u \in H^{(0)}$. As both Δ and Δ' are continuous, we get $\Delta = \Delta'$. We call the function Δ *the adjoining function* of the correspondence (X, λ).

Remark 2.1.2. Note that we do not need that the momentum maps s_X and r_X are open surjections. We also do not demand that the family of measures λ has full support. Hence the Hilbert module in the resulting C*-correspondence need not be full. This C*-correspondence need not be proper.

Remark 2.1.3. Referring to Definition 1.5.1, we can see that Condition (iv) in Definition 2.1.1 says that the measure $\alpha \times \lambda_u$ on $G * X_u$ is (G, α)-quasi-invariant for each $u \in H^{(0)}$.

In short, "*A topological correspondence from G to H is a pair (X, λ) where X is a G-H-bispace and λ is an H-invariant and G-quasi-invariant family of measures on X indexed by $H^{(0)}$.*"

Remark 2.1.4. Let $u \in H^{(0)}$ and let $x \in X$ be such that $s_X(x) = u$. As in [1] or [8] it can be shown that Δ restricted to $G * X_u$ is $\alpha^{r_X(x)} \times \lambda_u$-almost everywhere a groupoid homomorphism for all $u \in H^{(0)}$. So the function Δ in (iv) of Definition 2.1.1 is a continuous 1-cocycle on the groupoid $G \ltimes X$. We shall use this fact in many computations.

Remark 2.1.5. Example 1.4.6 gives a right action of H on $G \ltimes X$. In [8], Buneci and Stachura use an adjoining function. Their topological correspondences is a special case of our construction (Example 3.1.11). They show that the adjoining function in their case is H-invariant (see [8, Lemma 11]). In the similar fashion we may prove that Δ_H is H-invariant under the right action of H, that is,

$$\Delta(\gamma, x\eta) = \Delta(\gamma, x)$$

for all composable triples $(\gamma, x, \eta) \in G * X * H$. Thus, in fact, $\Delta\colon G \ltimes X/H \to \mathbb{R}_+^*$. Now Remark 2.1.4 can be made finer by saying that Δ is an H-invariant continuous 1-cocycle on the groupoid $G \ltimes X$.

Important conventions: As we shall see later, the adjoining function Δ plays a vital role while constructing a C*-correspondence from a topological correspondence. Hence sometimes in Chapter 2 we denote a topological correspondence by a triple (X, α, Δ). By this convention, we do *not* intend to mean that a topological correspondence is a triple, but we wish to emphasis the importance of the adjoining function.

Use of the adjoining function: In the following discussion we explain the role of the adjoining function. Let (X, λ) be a topological correspondence from (G, α) to (H, β) with Δ as the adjoining function. We make $C_c(X)$ into a $C_c(H)$-module using the same formulae as in [28] or [25]. To make $C_c(X)$ into a $\mathrm{C}^*(H, \beta)$-pre-Hilbert module, we need to define a $C_c(H)$-valued inner product on $C_c(X)$. The formula for this inner product cannot be copied directly from either [28] or [25]. This formula has to be modified, and it uses the family of measures λ.

Talking about the left action, for $\phi \in C_c(G)$ and $f \in C_c(X)$ [28] and [25] define $\phi \cdot f \in C_c(X)$ by

$$(\phi \cdot f)(x) = \int_G \phi(\gamma) f(\gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma). \tag{2.1.6}$$

For our definition of topological correspondence, the action of $C_c(G)$ on the $\mathrm{C}^*(H, \beta)$-pre-Hilbert module $C_c(X)$ defined by formula 2.1.6 is not an action by adjointable operators. For ϕ and f as above we define the left action by

$$(\phi \cdot f)(x) := \int_G \phi(\gamma) f(\gamma^{-1}x)\, \Delta^{1/2}(\gamma, \gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma). \tag{2.1.7}$$

We shall see that the adjoining function gives a *nice* scaling factor for the action of $C_c(G) \subseteq \mathrm{C}^*(G, \alpha)$ on $C_c(X)$ and makes this action a *-homomorphism to the C*-algebra of adjointable operators. This is the reason we call Δ the *adjoining* function.

Two examples of topological correspondences are: an equivalence between groupoids ([28] or see Definition 2.2.1) and the correspondence of Marta-Stadler and O'uchi ([25] or see Example 3.1.8). For equivalences and Macho Stadler-O'uchi correspondences the adjoining function is the constant function 1, and then formulas (2.1.6) and (2.1.7) match. To understand the role of Δ the reader may have a look at Lemma 2.3.1.

To support the necessity of the adjoining function, consider a toy example: when a locally compact Hausdorff group G is acting on a space X with a measure λ on X. The left multiplication action of $C_c(G)$ on $C_c(X) \subseteq \mathcal{L}^2(X, \lambda)$ defined by Equation (2.1.6) is not necessarily bounded. To make this action bounded, it is sufficient that λ is G-quasi-invariant, which brings the adjoining function into the picture. Then the left action of $C_c(G)$ given by Equation (2.1.7) becomes a *-representation. This motivated us to introduce Condition (iv) in Definition 2.1.1. Buneci and Stachura [8] also use the adjoining function.

For the left multiplication action of G on G/K, where K is a closed subgroup of G, the space G/K always carries a G-quasi-invariant measure λ. Hence there is a representation of G on $\mathcal{L}^2(G/K, \lambda)$. Quasi-invariant measures and the corresponding *adjoining functions* are studied very well in the group case, for example, see Section 2.6 of [15].

Readers may peep into Section 3.1 to see some examples of adjoining functions. In Section 3.1, we list most of the definitions of correspondences which have appeared in the literature and show how our definition of topological correspondences generalises these various notions.

2.1.1 Definitions of actions and inner product

We start with the main construction now. For $\phi \in C_c(G)$, $f \in C_c(X)$ and $\psi \in C_c(H)$ define functions $\phi \cdot f$ and $f \cdot \psi$ on X as follows:

$$\begin{aligned}(\phi \cdot f)(x) &:= \int_{G^{r_X(x)}} \phi(\gamma) f(\gamma^{-1}x)\, \Delta^{1/2}(\gamma, \gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma),\\ (f \cdot \psi)(x) &:= \int_{H^{s_X(x)}} f(x\eta)\psi(\eta^{-1})\, \mathrm{d}\beta^{s_X(x)}(\eta).\end{aligned} \tag{2.1.8}$$

For $f, g \in C_c(X)$ define the function $\langle f, g\rangle$ on H by

$$\langle f, g\rangle(\eta) := \int_{X_{r_H(\eta)}} \overline{f(x)} g(x\eta)\, \mathrm{d}\lambda_{r_H(\eta)}(x). \tag{2.1.9}$$

Most of the times we write ϕf and $f\psi$ instead of $\phi \cdot f$ and $f \cdot \psi$.

Lemma 2.1.10. *The functions ϕf, $f\psi$ and $\langle f, g\rangle$ defined above are continuous compactly supported functions on their domains.*

Proof. Showing that ϕf and $f\psi$ are continuous, compactly supported is a direct application of (i) of Lemma 1.3.20. To see that $\phi f \in C_c(X)$, rewrite the lemma for a left action with (G, α) as the groupoid and X as the space, then put

$$F(\gamma, x) = \phi(\gamma) f(\gamma^{-1}x)\Delta^{1/2}(\gamma, \gamma^{-1}x) h(x)$$

where $h \in C_c(X)$ with $h|_{\mathrm{supp}(f)} = 1$. Then F is continuous with a compact support. Apply the rewritten lemma to F now to see that $\phi f \in C_c(X)$. Using the original settings of the lemma it is easy to see that $f\psi \in C_c(X)$.

We claim that $\eta \mapsto \int F(x, \eta)\, \mathrm{d}\lambda_{r_H(\eta)}(x)$ is in $C_c(H)$ for every $F \in C_c(X * H)$. An argument using the Stone-Weierstraß Theorem as in (i) of Lemma 1.3.20 proves this claim. Hence to show that $\langle f, g\rangle \in C_c(H)$, we need to show that the function $F(x, \eta) = \overline{f(x)} g(x\eta)$ is in $C_c(X * H)$.

For $K \subseteq X$ let $\Psi_2^{-1}(K) = \{\eta \in H : K \cdot \eta \cap K \neq \emptyset\}$ as in Lemma 1.2.16. Now F is clearly continuous. If $K = \mathrm{supp}(f) \cup \mathrm{supp}(g)$, then $\mathrm{supp}(F) \subseteq K * \Psi_2^{-1}(K) \subseteq X * H$, which is compact because the action is proper. ☐

Both $C_c(G)$ and $C_c(H)$ are *-algebras. Denote the convolution product on them by $*$.

Lemma 2.1.11. *Let $\phi, \phi' \in C_c(G)$, $\psi, \psi' \in C_c(H)$ and $f, g, g' \in C_c(X)$. Then*

$$(\phi * \phi')f = \phi(\phi' f), \tag{2.1.12}$$
$$f(\psi * \psi') = (f\psi)\psi', \tag{2.1.13}$$
$$(\phi f)\psi = \phi(f\psi), \tag{2.1.14}$$
$$\langle f, g + g'\rangle = \langle f, g\rangle + \langle f, g'\rangle, \tag{2.1.15}$$
$$\langle f\,, g\rangle^* = \langle g\,, f\rangle, \tag{2.1.16}$$
$$\langle f\,, g\rangle\psi = \langle f\,, g\psi\rangle, \tag{2.1.17}$$
$$\langle \phi f\,, g\rangle = \langle f\,, \phi^* g\rangle. \tag{2.1.18}$$

Proof. Let $\gamma \in G$, $x \in X$ and $\eta \in H$. The following are the detailed computations which prove the equations in the lemma.
Equation (2.1.12):

$$\begin{aligned}((\phi * \phi')f)(x) &= \int_{G^{r_X(x)}} (\phi * \phi')(\gamma) f(\gamma^{-1}x)\Delta(\gamma, \gamma^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ &= \int_{G^{r_X(x)}} \int_{G^{r_G(\gamma)}} \phi(\zeta)\phi'(\zeta^{-1}\gamma) f(\gamma^{-1}x)\Delta(\gamma, \gamma^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_G(\gamma)}(\zeta)\,\mathrm{d}\alpha^{r_X(x)}(\gamma).\end{aligned}$$

First we apply Fubini's theorem and then change the variable $\gamma \mapsto \zeta^{-1}\gamma$ and use the invariance of α to see that the last term equals

$$\int_{G^{r_G(\gamma)}} \int_{G^{r_X(x)}} \phi(\zeta)\phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x)\Delta(\zeta\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\zeta).$$

We observe that $(\zeta\gamma, \gamma^{-1}\zeta^{-1}x) = (\zeta, \zeta^{-1}x)(\gamma, \gamma^{-1}\zeta^{-1}x)$ in the transformation groupoid $G \ltimes X$. This relation, Remark 2.1.4 and the associativity of the left action together allow us to write the previous term as

$$\begin{aligned}&\int_{G^{r_G(\gamma)}} \int_{G^{r_X(x)}} \phi(\zeta)\phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x)\Delta(\zeta, \zeta^{-1}x)^{1/2}\Delta(\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\zeta)\\ &= \int_{G^{r_G(\gamma)}} \phi(\zeta)\left(\int_{G^{r_X(x)}} \phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x)\Delta(\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\right)\Delta(\zeta, \zeta^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_G(\gamma)}(\zeta)\\ &= \int_{G^{r_G(\gamma)}} \phi(\zeta)\,(\phi' f)(\zeta^{-1}x)\,\Delta(\zeta, \zeta^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_G(\gamma)}(\zeta)\\ &= (\phi(\phi' f))(x).\end{aligned}$$

Equation (2.1.13): This computation is similar to the above computation for Equation (2.1.12) except that we do not need the adjoining function.

Equation (2.1.14): See the computation below. We apply Fubini's theorem at the third step and the H-invariance of Δ:

$$\begin{aligned}((\phi f)\psi)(x) &= \int_{H^{s_X(x)}} (\phi f)(x\eta)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta)\\ &= \int_{H^{s_X(x)}} \left(\int_{G^{r_X(x)}} \phi(\gamma) f(\gamma^{-1}x\eta)\,\Delta(\gamma,\gamma^{-1}x\eta)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\right)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta)\\ &= \int_{G^{r_X(x)}} \phi(\gamma)\left(\int_{H^{r_X(x)}} f(\gamma^{-1}x\eta)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta)\right)\Delta(\gamma,\gamma^{-1}x)^{1/2}\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\\ &= \int_{G^{r_X(x)}} \phi(\gamma)\,(f\psi)(\gamma^{-1}x)\,\Delta(\gamma,\gamma^{-1}x)^{1/2}\,\mathrm{d}\beta^{s_X(x)}(\eta)\\ &= (\phi(f\psi))(x).\end{aligned}$$

Equation (2.1.15):

$$\begin{aligned}\langle f\,,g+g'\rangle(\eta) &= \int_{X_{r_H(\eta)}} \overline{f(x)}(g+g')(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \int_{X_{r_H(\eta)}} \overline{f(x)}g(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x) + \int_{X_{r_H(\eta)}} \overline{f(x)}g'(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= (\langle f\,,g\rangle + \langle f\,,g'\rangle)(\eta).\end{aligned}$$

Equation (2.1.16):

$$\langle f\,,g\rangle^*(\eta) = \overline{\langle f\,,g\rangle(\eta^{-1})} = \int_{X_{r_H(\eta^{-1})=s_H(\eta)}} f(x)\overline{g(x\eta^{-1})}\,\mathrm{d}\lambda_{s_H(\eta)}(x)$$

We change the variable $x\eta^{-1}\mapsto x$ and then use the right invariance of the family of measures λ and compute further:

$$\text{L.H.S.} = \int_{X_{r_H(\eta^{-1})}} f(x\eta)\overline{g(x)}\,\mathrm{d}\lambda_{s_H(\eta)}(x) = \langle g\,,f\rangle(\eta).$$

Equation (2.1.17):

$$\begin{aligned}(\langle f\,,g\rangle\,\psi)(\eta) &= \int_{H^{r_H(\eta)}} \langle f\,,g\rangle(\xi)\,\psi(\xi^{-1}\eta)\,\mathrm{d}\beta^{r_H(\eta)}(\xi)\\ &= \int_{H^{r_H(\eta)}}\int_{X_{r_H(\xi)}} \overline{f(x)}\,g(x\xi)\,\psi(\xi^{-1}\eta)\,\mathrm{d}\lambda_{r_H(\xi)}(x)\,\mathrm{d}\beta^{r_H(\eta)}(\xi).\end{aligned}$$

Change the variable $\xi\mapsto\eta\xi$ and use the left invariance of β to see that the last term becomes

$$\int_{H^{s_H(\eta)}}\int_{X_{r_H(\eta)}} \overline{f(x)}\,g((x\eta)\xi)\,\psi(\xi^{-1})\,\mathrm{d}\lambda_{r_H(\eta)}(x)\,\mathrm{d}\beta^{s_H(\eta)}(\xi).$$

Now we apply Fubini's theorem to the above term and compute further:

$$\begin{aligned}&\int_{X_{r_H(\eta)}} \overline{f(x)} \int_{H^{s_H(\eta)}} g((x\eta)\xi)\,\psi(\xi^{-1})\,\mathrm{d}\beta^{s_H(\eta)}(\xi)\,\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \int_{X_{r_H(\eta)}} \overline{f(x)}(g\psi)(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \langle f\,, g\psi\rangle\,(\eta).\end{aligned}$$

Equation 2.1.18:

$$\begin{aligned}\langle \phi f, g\rangle(\eta) &= \int \overline{(\phi f)(x)} g(x\eta)\;\mathrm{d}\lambda_{r_H(\eta)}(x) &(2.1.19)\\ &= \iint \overline{\phi(\gamma)}\,\overline{f(\gamma^{-1}x)} g(x\eta)\;\Delta^{1/2}(\gamma,\gamma^{-1}x)\;\mathrm{d}\alpha^{r_X(x)}(\gamma)\;\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \iint \overline{f(\gamma^{-1}x)}\,\overline{\phi(\gamma)} g(x\eta)\;\Delta^{1/2}(\gamma,\gamma^{-1}x)\;\mathrm{d}\alpha^{r_X(x)}(\gamma)\;\mathrm{d}\lambda_{r_H(\eta)}(x).\end{aligned}$$

Make a change of variables $(\gamma,\gamma^{-1}x)\mapsto(\gamma^{-1},x)$. Then we use the fact that Δ is almost everywhere groupoid homomorphism (see Remark 2.1.4). Due to Remark 2.1.5, we know that Δ is H-invariant. Thus computing further we see

$$\begin{aligned}\langle \phi f, g\rangle(\eta) &= \iint \overline{f(x)}\,\overline{\phi(\gamma^{-1})} g(\gamma^{-1}x\eta)\;\Delta^{1/2}(\gamma,\gamma^{-1}x\eta)\;\mathrm{d}\alpha^{r(x)}(\gamma)\;\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \iint \overline{f(x)}\,\overline{\phi(\gamma^{-1})} g(\gamma^{-1}x\eta)\;\Delta^{1/2}(\gamma,\gamma^{-1}x)\;\mathrm{d}\alpha^{r_X(x)}(\gamma)\;\mathrm{d}\lambda_{r_H(\eta)}(x)\\ &= \langle f, \phi^* g\rangle(\eta). \qquad \square\end{aligned}$$

Remark 2.1.20 (Some remarks on the identities in Lemma 2.1.11)*.* Equations (2.1.12), (2.1.13) and (2.1.14) show that $C_c(X)$ is a $C_c(G)$-$C_c(H)$-bimodule. Equations (2.1.12), (2.1.13) and (2.1.14) show that the map $\langle\ ,\ \rangle : C_c(X)\times C_c(X)\to C_c(H)$ is a $C_c(H)$-conjugate bilinear map. And Equation (2.1.18) says that $C_c(G)$ acts on $C_c(X)$ by $C_c(H)$-adjointable operators.

Remark 2.1.21*.* Using Lemma 2.1.10, it can be seen that the left and the right action and the map $\langle\,,\rangle$ are continuous in the inductive limit topology.

2.2 The right action—construction of the Hilbert module

In this section, we describe how to construct a $\mathrm{C}^*(H,\beta)$-Hilbert module $\mathcal{H}(X)$, where X is a proper H-space and λ is an H-invariant family of measures. Indeed, writing $\mathcal{H}(X,\lambda)$ would be more than writing $\mathcal{H}(X)$. But we shall not come across any case which involves the same space with different families of measures. Hence we write $\mathcal{H}(X)$.

First we discuss the case when the H-action is free as well as proper. Later we shall consider the case when the action is proper but not free. Indeed, the latter case implies the first one. But when the action is free, the techniques of constructing $\mathcal{H}(X)$ are discussed in the famous theory of *groupoid equivalence* introduced in [28]. The proofs and techniques for this case are different and interesting in their own right.

When the action is not free but just proper, we appeal to the theory of representations of groupoids. The techniques for this case differ drastically from those used for the case of free and proper actions. Thanks to [34] we can waive the second-countability assumption on the groupoid topology.

2.2.1 Case of free and proper actions

Definition 2.2.1 (Equivalence of groupoids, a slight modification of Definition 2.1 [28]). Let G and H be groupoids. A locally compact Hausdorff space X is a G-H-equivalence if

i) X is a left principal G-space;

ii) X is a right principal H-space;

iii) the momentum maps r_X and s_X are open;

iv) the actions of G and H commute;

v) the left momentum map $r_X\colon X \to G^{(0)}$ induces a bijection of X/H onto $G^{(0)}$;

vi) the right momentum map $s_X\colon X \to H^{(0)}$ induces a bijection of $G\backslash X$ onto $H^{(0)}$.

For $f, g \in C_c(X)$, $\phi \in C_c(G)$ and $\psi \in C_c(H)$ define $\phi \cdot f, f \cdot \psi\colon X \to \mathbb{C}$, $\langle f\,, g\rangle_* : H \to \mathbb{C}$ and ${}_*\langle f\,, g\rangle\colon G \to \mathbb{C}$ by

$$\left.\begin{aligned} (\phi \cdot f)(x) &= \int_G \phi(\gamma)f(\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma),\\ (f \cdot \psi)(x) &= \int_{H^{s_X(x)}} f(x\eta)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta),\\ \langle f, g\rangle_*(\eta) &= \int_{G^{r_X(x)}} \overline{f(\gamma^{-1}x)}g(\gamma^{-1}x\eta)\,\mathrm{d}\alpha^{r_X(x)}(\gamma), \end{aligned}\right\} \tag{2.2.2}$$

$${}_*\langle f, g\rangle(\gamma) = \int_{H^{s_X(x)}} f(\gamma^{-1}x\eta)\overline{g(x\eta)}\,\mathrm{d}\beta^{s_X(x)}(\eta). \tag{2.2.3}$$

Theorem 2.2.4 (Theorem 2.8 [28]). *Suppose that (G, α) and (H, β) are second countable, locally compact, Hausdorff groupoids with Haar systems. Then for any G-H-equivalence X, $C_c(X)$ with the above $C_c(G)$-$C_c(H)$-bimodule structure and inner products can naturally be completed into a* $\mathrm{C}^*(G, \alpha)$*-*$\mathrm{C}^*(H, \beta)$*-imprimitivity bimodule. In particular,* $\mathrm{C}^*(G, \alpha)$ *and* $\mathrm{C}^*(H, \beta)$ *are strongly Morita equivalent.*

Lemma 2.2.5. *Let X be an equivalence from (G, α) to (H, β) and let the topological hypotheses be as in Theorem 2.2.4.*

1. *There is a canonical H-invariant family of measures λ on X such that (X, λ) is a topological correspondence and the adjoining function for (X, λ) is the constant function* 1.

2. *The action and inner product formulae in the set of equations in* (2.2.2) *match those in Equation* (2.1.9) *and* (2.1.8).

Proof. For (1) see Example 3.1.10. And (2) is a direct computation. □

Example 2.2.6. Let (G, α) be a groupoid with a Haar system. Then the left and the right actions of G on itself make G into a (G, α)-(G, α)-equivalence. It can be seen that $\mathcal{H}(G) = \mathrm{C}^*(G, \alpha)$ as a C^*-correspondence. The computations in Example 3.1.8 for this special case show that the right-invariant family of measures on G is α^{-1} (see Example 1.3.12).

Example 2.2.7. Let G be a groupoid and let α and α' be two Haar systems on G. Then G is a (G, α)-(G, α')-equivalence. Hence $\mathrm{C}^*(G, \alpha)$ and $\mathrm{C}^*(G, \alpha')$ are Morita equivalent.

Equation (2.2.3) gives a $\mathrm{C}^*(G, \alpha)$-valued inner product on $C_c(X)$ which produces the imprimitivity bimodule in Theorem 2.2.4. We do not need a $\mathrm{C}^*(G, \alpha)$-valued inner product.

Remark 2.2.8 (Techniques used to prove Theorem 2.2.4). The hardest thing to prove here is to that the bilinear map $\langle\, , \rangle$ is positive. In [28] the main ingredient used to show this fact is the existence of *good* approximate identities for the *-algebras $C_c(G)$ and $C_c(H)$. Creating these approximate identities needs that the groupoid actions are free.

The technique of approximate identities is used earlier in [16], [37] and [33]. At the end of [37], Rieffel gives the calculations where he uses this approximate identity to prove the positivity of the bilinear map. The earliest appearance of this technique that we found is an article by P. Green [16].

As mentioned earlier, the existence of *good* approximate identities needs that the groupoid actions are free. Hence when X has a *free* and proper action of H, we can prove the following statement.

Proposition 2.2.9. *Let (H, β) be a locally compact Hausdorff second countable groupoid endowed with a Haar system and X a locally compact, Hausdorff right H-space with s_X open and surjective. Let λ be an H-invariant family of measures on X. If the action of H on X is free and proper, then $C_c(X)$ can be completed to a* $\mathrm{C}^*(H)$*-Hilbert module using the operations defined in Equation* (2.2.2) *or equivalently in Equations* (2.1.9) *and* (2.1.8).

Proof. It is sufficient to produce a groupoid G and Haar measure α for it such that X is a G-H-equivalence, then we appeal to Theorem 2.2.4 to get a $\mathrm{C}^*(G, \alpha)$-$\mathrm{C}^*(H, \beta)$-imprimitivity bimodule $\mathcal{H}(X)$. Then $\mathcal{H}(X)$ is the required $\mathrm{C}^*(H, \beta)$-Hilbert module. We construct the groupoid (G, α) now and show that X is a G-H-equivalence.

Construction of (G, α)**:** Since the right action is free and proper, the space $(X * X)/H$ is a locally compact, Hausdorff groupoid (see [28, page 5]), where we write $X * X = X \times_{s_X, s_X} X$. Denote this groupoid by G. Then $G^{(0)} = X/H$.

Let $\pi_2 : X * X \to X$ be the projection onto the second factor. Then $\delta \times \lambda = \{\delta_x \times \lambda_{s_X(x)}\}_{x \in X}$ is a family of measures along π_2. The actions of H on $X * X$ and X are proper. Hence Proposition 1.3.27 shows that $[\delta \times \lambda]$ is a continuous family of measures along $[\pi_2]$. Write α for $[\delta \times \lambda]$ and $\alpha^{[x]}$ for $[\delta \times \lambda]^{[x]}$. For $[x] \in G^{(0)}$ and $f \in C_c(G)$, we have

$$\int f \,\mathrm{d}\alpha^{[x]} = \int f[x, z] \,\mathrm{d}\lambda_{s(x)}(z).$$

It is not hard to see that α is invariant under the left multiplication action of G on itself. Thus (G, α) is a locally compact, Hausdorff groupoid with a Haar system.

Proof that X is a G-H-equivalence: There is a natural left action of G on X. The momentum map r_X for the action is the quotient map $X \to X/H = G^{(0)}$. Lemma 1.2.13 shows that r_X is an open map. The action is given as: $[x, y]z = x\gamma$, where $\gamma \in H$ is the unique element with $z = y\gamma$. The action is well defined because if $[x', y'] = [x, y]$, then $x' = x\eta$ and $y' = y\eta$ for some unique $\eta \in H$. And hence $z = y\gamma = y\eta(\eta^{-1}\gamma) = y'(\eta^{-1}\gamma)$. Then $[x', y']z = x'\eta^{-1}\gamma = x\gamma = [x, y]z$. It is not hard to check that this action is free and proper.

The right source map $s_X\colon X \to H^{(0)}$ induces a homeomorphism from the quotient space $G\backslash X$ to $H^{(0)}$. And r_X clearly induces a homeomorphism $X/H \to G^{(0)} = X/H$, which is nothing but the identity map.

This proves that X is an equivalence between G and H. □

Corollary 2.2.10. *Let (H, β) be a groupoid endowed with a Haar system and X a right H-space. Let the action of H on X be free and proper. Assume the same topological hypotheses as in Proposition 2.2.9 and let (G, α) be the groupoid with Haar system in the proof of Proposition 2.2.9. Then* $\mathrm{C}^*(G, \alpha)$ *is isomorphic to the algebra of compact operators on the* $\mathrm{C}^*(H, \beta)$*-Hilbert module $\mathcal{H}(X)$.*

2.2.2 Case of proper actions

Proposition 2.2.11. *Let (H, β) be a groupoid equipped with a Haar system, X a proper left H-space and λ an invariant family of measures on X. Then the bilinear map defined by Equation* (2.1.9) *is a $C_c(H)$-valued inner product on $C_c(X)$.*

We only need to prove that the bilinear map is positive. The other required properties of $\langle , \rangle$ are clear from Lemma 2.1.11.

Our strategy is the following: for every (non-degenerate) representation $\tilde{\pi}\colon \mathrm{C}^*(H, \beta) \to \mathbb{B}(\mathcal{K})$, we show that $\tilde{\pi}(\langle f, f \rangle) \in \mathbb{B}(\mathcal{K})$ is positive. Due to the Disintegration Theorem, we work with representations of (H, β) and prove the same fact there. We shall use the flavour of representation theory that uses transverse measures, see Section 1.6.3.

$$\begin{array}{ccc} X * H & \xrightarrow[\pi_2]{\lambda_2} & H \\ {\scriptstyle \tilde{\beta}_X}\big\downarrow{\scriptstyle \pi_1} & & {\scriptstyle r_H}\big\downarrow{\scriptstyle \beta} \\ X & \xrightarrow[s_X]{\lambda} & H^{(0)}. \end{array}$$

Figure 2.1

Remark 2.2.12. In Figure 2.1, π_i for $i = 1, 2$ are the projections on the i^{th} component, λ_2 is as in Lemma 1.3.17 and $\tilde{\beta}_X$ is as in Equation (1.3.19). Clearly, $\beta \circ \lambda_2 = \lambda \circ \tilde{\beta}_X$. Let m be a transverse measure class on H. We take the quotient of each space in Figure 2.1 and the corresponding induced maps and families of measures.

i) The coherence of m gives $m(\beta) \circ [\lambda_2] = m(\lambda) \circ [\tilde{\beta}_X]$.

ii) A computation very similar to Equation 1.6.29 gives $[\lambda_2] = \lambda$.

iii) Equation 1.3.26 gives $[\tilde{\beta}_X] = \beta_X$, where β_X is as in Equation (1.3.22).

iv) (i), (ii) and (iii) together say that $m(\beta) \circ [\lambda] = m(\lambda) \circ [\beta_X]$.

Hence if $\mu \in m(\beta)$ and $\nu \in m(\lambda)$, then $\mu \circ \lambda \sim \nu \circ \beta_X$.

Proposition 2.2.11 follows from Lemma 2.2.17 and Lemma 2.2.18 below. In the following discussion, we shall write $\langle f\,, f\rangle_{C_c(H)}$ instead of $\langle f\,, f\rangle$ for $f \in C_c(X)$.

Let $(m, \mathcal{H}, \pi)$ be a representation of (H, β) where m is a transverse measure class for H, $\mathcal{H} \to H^{(0)}$ is a measurable H-Hilbert bundle which has separable fibres and π is the action of H on fibres of $\mathcal{H}$. The fibre product $X * \mathcal{H}$ carries the diagonal action of H, that is, $(x, h)\eta = (x\eta, \pi(\eta^{-1})h)$. After taking the quotient by this action, we get the measurable Hilbert bundle $\pi_X \colon (X * \mathcal{H})/H \to X/H$ where $\pi_X([x, h]) = [x]$. Denote $(X * \mathcal{H})/H$ by $\mathcal{H}^X$. For each $[x] \in H^{(0)}$, there is a unitary isomorphism $\mathcal{H}^X_{[x]} \simeq \mathcal{H}_{s_X(x)}$.

By definition, the transverse measure class m induces a measure class $m(\lambda)$ on X/H (see Definition 1.6.32). We fix $\mu \in m(\beta)$ and $\nu \in m(\lambda)$, that is, μ is a measure on H^0 and ν is a measure on X/H. Furthermore, let $\lambda^X \colon C_c(X) \to C_c(H^0)$ and $\beta_X \colon C_c(X) \to C_c(X/H)$ be the integration operators for the families of measures $(\lambda_u)_{u \in H^0}$ and $(\beta^{[x]})_{[x] \in X/H}$. Remark 2.2.12 shows that $\nu \circ \beta_X$ and $\mu \circ \lambda_X$ are equivalent measures on X. Let $\tilde{\beta}_X$ be the family of measures along the map $X * H \to X$, $(x, h) \mapsto x$, defined by

$$\int f \,\mathrm{d}\tilde{\beta}_X^x = \int f(x, h) \,\mathrm{d}\beta^{s_X(x)}(h)$$

for $f \in C_c(X * H)$.

Lemma 2.2.13. *The measure $\nu \circ \beta_X$ on X is H-invariant.*

Proof. We must prove that the measure $\nu\beta_X\tilde{\beta}_X$ on $X * H$ defined by

$$f \mapsto \int_{X/H} \int_{H^{s_X(x)}} \int_{H^{s_H(\eta)}} f(x\eta, h) \,\mathrm{d}\beta^{s_H(\eta)}(h) \,\mathrm{d}\beta^{s_X(x)}(\eta) \,\mathrm{d}\nu[x]$$

for $f \in C_c(X * H)$ is invariant under the inversion map $(x, h) \mapsto (xh, h^{-1})$. For this, we substitute $\eta^{-1}h$ for h and write

$$\nu\beta_X\tilde{\beta}_X(f) = \int_{X/H} \int_{H^{s_X(x)}} \int_{H^{s_X(x)}} f(x\eta, \eta^{-1}h) \,\mathrm{d}\beta^{s_X(x)}(h) \,\mathrm{d}\beta^{s_X(x)}(\eta) \,\mathrm{d}\nu[x];$$

now replacing f by $f \circ \mathrm{inv}$ replaces $f(x\eta, \eta^{-1}h)$ by $f(x\eta\eta^{-1}h, (\eta^{-1}h)^{-1}) = f(xh, h^{-1}\eta)$. The substitution that switches $h \leftrightarrow \eta$ shows that the integrals over f and $f \circ \mathrm{inv}$ are the same. □

Since $\mu \circ \lambda$ is equivalent to $\nu \circ \beta$, this measure on X must also be quasi-invariant. We compute its Radon-Nikodym derivative. Let $f \in C_c(X * H)$, then we get

$$\begin{aligned}\mu \circ \lambda \circ \tilde{\beta}_X(f) &= \int_{H^0} \int_{X_u} \int_{H^u} f(x,h)\,\mathrm{d}\beta^u(h)\,\mathrm{d}\lambda_u(x)\,\mathrm{d}\mu(u)\\ &= \int_{H^0} \int_{H^u} \int_{X_u} f(x,h)\,\mathrm{d}\lambda_u(x)\,\mathrm{d}\beta^u(h)\,\mathrm{d}\mu(u)\end{aligned}$$

by Fubini's Theorem. When we replace f by $f \circ \mathrm{inv}$ and use the H-invariance of λ, we get

$$\begin{aligned}\mu \circ \lambda \circ \tilde{\beta}_X(f \circ \mathrm{inv}) &= \int_{H^0} \int_{H^u} \int_{X_u} f(xh, h^{-1})\,\mathrm{d}\lambda_u(x)\,\mathrm{d}\beta^u(h)\,\mathrm{d}\mu(u)\\ &= \int_{H^0} \int_{H^u} \int_{X_{s_H(h)}} f(x, h^{-1})\,\mathrm{d}\lambda_{s_H(h)}(x)\,\mathrm{d}\beta^u(h)\,\mathrm{d}\mu(u)\\ &= \int_{H^0} \int_{H_u} \int_{X_{r_H(h)}} f(x, h)\,\mathrm{d}\lambda_{r_H(h)}(x)\,\mathrm{d}\beta_u^{-1}(h)\,\mathrm{d}\mu(u),\end{aligned}$$

where the last step uses the substitution $h \mapsto h^{-1}$. In terms of the integration operator $\lambda_2\colon C_c(X * H) \to C_c(H)$ along π_2, we may rewrite this as $\mu \circ \beta^{-1}(\lambda_2(f))$, whereas $\mu \circ \lambda \circ \tilde{\beta}_X(f) = \mu \circ \beta(\lambda_2(f))$. Thus the Radon-Nikodym derivative is

$$\frac{\mathrm{d}\,\mathrm{inv}^*(\mu \circ \lambda \circ \tilde{\beta}_X)}{\mathrm{d}(\mu \circ \lambda \circ \tilde{\beta}_X)}(x,h) = \frac{\mathrm{d}(\mu \circ \beta^{-1})}{\mathrm{d}(\mu \circ \beta)}(h).$$

Now let

$$M(x) = \frac{\mathrm{d}(\mu \circ \lambda)}{\mathrm{d}(\nu \circ \beta)}.$$

Lemma 2.2.14. *Let $x \in X$ and $h \in H^1$ satisfy $s_X(x) = r_H(h)$. Then*

$$M(xh) = M(x)\frac{\mathrm{d}(\mu \circ \beta^{-1})}{\mathrm{d}(\mu \circ \beta)}(h).$$

Proof. Let $g \in C_c(X * H)$ and let $f = \tilde{\beta}_X(g)$, that is, $f(x) = \int_{H^x} f(x,h)\,\mathrm{d}\beta^x(h)$. By definition of the Radon-Nikodym derivative, we have

$$\int_X f(x)\,\mathrm{d}(\nu \circ \beta) = \int_X f(x) M(x)^{-1}\,\mathrm{d}(\mu \circ \lambda).$$

Thus

$$\int_X g(x,h)\,\mathrm{d}(\nu \circ \beta \circ \tilde{\beta}_X) = \int_X g(x,h) M(x)^{-1}\,\mathrm{d}(\mu \circ \lambda \circ \tilde{\beta}_X).$$

Since the measure $\nu \circ \beta$ is H-invariant, the left hand side is invariant under replacing g by $g \circ \mathrm{inv}$. Hence so is the right-hand side, that is,

$$\begin{aligned}\int_X g(x,h) M(x)^{-1}\,\mathrm{d}(\mu \circ \lambda \circ \tilde{\beta}_X) &= \int_X g(xh, h^{-1}) M(x)^{-1}\,\mathrm{d}(\mu \circ \lambda \circ \tilde{\beta}_X)\\ &= \int_X g(xh, h^{-1}) M(xh)^{-1} \frac{M(xh)}{M(x)}\,\mathrm{d}(\mu \circ \lambda \circ \tilde{\beta}_X).\end{aligned}$$

Letting $g'(x,h) = g(x,h)M(x)^{-1}$, we see that $M(xh)/M(x)$ has to be the Radon-Nikodym derivative

$$\frac{M(xh)}{M(x)} = \frac{\mathrm{d}(\mathrm{inv}^*\mu\circ\lambda\circ\tilde{\beta}_X)}{\mathrm{d}(\mu\circ\lambda\circ\tilde{\beta}_X)}(x,h) = \frac{\mathrm{d}(\mu\circ\beta^{-1})}{\mathrm{d}(\mu\circ\beta)}(h). \qquad \square$$

Let $\mathcal{H} = (\mathcal{H}_x)_{x\in X}$ be a μ-measurable field of Hilbert spaces over H^0 equipped with a representation π of H. The Hilbert space $L^2(H^0,\mu,\mathcal{H})$ consists of all μ-measurable sections $\xi\colon H^0\to\mathcal{H}$ such that

$$\int_{H^0} \|\xi(x)\|^2_{\mathcal{H}_x}\,\mathrm{d}\mu(x) < \infty.$$

This norm comes from an inner product on $L^2(H^0,\mu,\mathcal{H})$, of course.

We pull back $\mathcal{H}$ to a field $s_X^*\mathcal{H}$ of Hilbert spaces over X along $s_X\colon X\to H^0$. Then we take the induced field of Hilbert spaces $\mathcal{H}^X$ over X/H whose $\mu\circ\lambda$-measurable sections are those sections ζ of $s^*\mathcal{H}$ that satisfy $\pi_h(\zeta(xh)) = \zeta(x)$ for all $x\in X$, $h\in H$ with $s_X(x) = r_H(h)$. For ν as above, we define the Hilbert space $L^2(X/H,\nu,\mathcal{H}^X)$ to consist of those sections ζ of $\mathcal{H}^X$ with

$$\int_{X/H} \|\zeta(x)\|^2_{\mathcal{H}_{s_X(x)}}\,\mathrm{d}\nu[x] < \infty.$$

The function $\|\zeta(x)\|^2_{\mathcal{H}_{s_X(x)}}$ is constant on H-orbits and thus descends to X/H because $\pi_h(\zeta(xh)) = \zeta(x)$ and the operators π_h are unitary. The norm defining $L^2(X/H,\nu,\mathcal{H}^X)$ comes from an obvious inner product. Notice that an element of $L^2(X/H,\nu,\mathcal{H}^X)$ is *not* a function on X/H.

Now we define the operator $|f\rangle\!\rangle$ from $L^2(H^0,\mu,\mathcal{H})$ to $L^2(X/H,\nu,\mathcal{H}^X)$ and its adjoint $\langle\!\langle f|$. Let $\xi\in L^2(H^0,\mu,\mathcal{H})$ and $\zeta\in L^2(X/H,\nu,\mathcal{H}^X)$. Computations by Renault which are discussed in Section 3.3.1 lead to the following formulas for $\langle\!\langle f|$ and $|f\rangle\!\rangle$:

$$(|f\rangle\!\rangle\xi)(x) = \int_{H^{s_X(x)}} f(x\eta)\pi_\eta(\xi(s_H(\eta)))\sqrt{M(x\eta)}\,\mathrm{d}\beta^{s_X(x)}(\eta), \tag{2.2.15}$$

$$(\langle\!\langle f|\zeta)(u) = \int_{X_u} \overline{f(x)}\zeta(x)\frac{1}{\sqrt{M(x)}}\,\mathrm{d}\lambda_u(x). \tag{2.2.16}$$

Notice that

$$\pi_h(|f\rangle\!\rangle\xi(xh)) = \int_{H^{s_X(x)}} f(xh\eta)\pi_{h\eta}(\xi(s_H(\eta)))\sqrt{M(xh\eta)}\,\mathrm{d}\beta^{s_X(x)}(\eta) = |f\rangle\!\rangle\xi(x)$$

by the substitution $h\eta\mapsto\eta$ because β is left-invariant. Thus $|f\rangle\!\rangle\xi$ is a section of $\mathcal{H}^X$. If we pick ξ and ζ of compact support, then $|f\rangle\!\rangle\xi$ and $\langle\!\langle f|\zeta$ also have compact support in X/H and H^0, respectively. Hence our operators $|f\rangle\!\rangle$ and $\langle\!\langle f|$ are at least well-defined on dense subspaces.

Lemma 2.2.17. *Let ξ and ζ have compact support. Then $\langle\zeta, |f\rangle\!\rangle\xi\rangle = \langle\,\langle\!\langle f|\zeta,\xi\rangle$, that is, $\langle\!\langle f|$ is formally adjoint to $|f\rangle\!\rangle$.*

Proof. On the one hand,

$$\begin{aligned}
\langle \zeta, |f\rangle\rangle\xi\rangle &= \int_{X/H} \langle \zeta(x), |f\rangle\rangle\xi(x)\rangle \,\mathrm{d}\nu(x) \\
&= \int_{X/H}\int_{H^{s_X(x)}} \langle \zeta(x), f(x\eta)\pi_\eta\xi(s_H(\eta))\rangle \sqrt{M(x\eta)}\,\mathrm{d}\beta^{s_X(x)}(\eta)\,\mathrm{d}\nu[x] \\
&= \int_{X/H}\int_{H^{s_X(x)}} \langle \zeta(x\eta), \xi(s_H(\eta))\rangle f(x\eta)\sqrt{M(x\eta)}\,\mathrm{d}\beta^{s_X(x)}(\eta)\,\mathrm{d}\nu[x] \\
&= \int_{X/H}\int_{H^{s_X(x)}} \langle \zeta(x), \xi(s_X(x))\rangle f(x)\sqrt{M(x)}\,\mathrm{d}(\nu\circ\beta)(x),
\end{aligned}$$

where we used $\pi_h\zeta(xh) = \zeta(x)$, the unitarity of π_h, and the definition of the measure $\nu\circ\beta$ on X. On the other hand,

$$\begin{aligned}
\langle \langle\langle f|\zeta, \xi\rangle &= \int_{H^0} \langle \langle\langle f|\zeta(u), \xi(u)\rangle \,\mathrm{d}\mu(u) \\
&= \int_{H^0}\int_{X_u} \langle \overline{f(x)}\zeta(x)\sqrt{M(x)}^{-1}, \xi(u)\rangle \,\mathrm{d}\lambda_u(x)\,\mathrm{d}\mu(u) \\
&= \int_X \langle \zeta(x), \xi(s_X(x))\rangle f(x)\sqrt{M(x)}^{-1}\,\mathrm{d}(\mu\circ\lambda)(x) \\
&= \int_X \langle \zeta(x), \xi(s_X(x))\rangle f(x)\sqrt{M(x)}^{-1}\frac{\mathrm{d}(\mu\circ\lambda)}{\mathrm{d}(\nu\circ\beta)}(x)\,\mathrm{d}(\nu\circ\beta)(x).
\end{aligned}$$

Now the definition of M shows that this is the same as the previous integral. □

The convolution algebra $C_c(H)$ acts on $L^2(H^0, \mu, \mathcal{H})$ by

$$L(f)\xi(u) = \int_{H^u} f(\eta)\pi_\eta\xi(s_H(\eta))\sqrt{\frac{\mathrm{d}(\mu\circ\beta^{-1})}{\mathrm{d}(\mu\circ\beta)}(\eta)}\,\mathrm{d}\beta^u(\eta).$$

This is a *-representation.

Lemma 2.2.18. *Let ξ be compactly supported. Then $\langle\langle f|\circ|f\rangle\rangle(\xi) = L(\langle f, f\rangle_{C_c(H)})(\xi)$. Hence $\langle\langle f|\circ|f\rangle\rangle$ extends to a bounded operator with norm at most $\|\langle f, f\rangle\|_{C^*(H)}$. It follows that $|f\rangle\rangle$ and $\langle\langle f|$ extend to bounded operators between the Hilbert spaces $L^2(H^0, \mu, \mathcal{H})$ and $L^2(X/H, \nu, \mathcal{H}^X)$ which are adjoints of one another.*

Proof. We compute

$$\begin{aligned}
\langle\langle f|\circ|f\rangle\rangle(\xi)(u) &= \int_{X_u} \overline{f(x)}|f\rangle\rangle(\xi)(x)\sqrt{M}^{-1}(x)\,\mathrm{d}\lambda_u(x) \\
&= \int_{X_u}\int_{H^u} \overline{f(x)}f(x\eta)\pi_\eta(\xi(s_H(\eta)))\sqrt{M}(x\eta)\sqrt{M}^{-1}(x)\,\mathrm{d}\beta^u(\eta)\,\mathrm{d}\lambda_u(x)
\end{aligned}$$

Now we use Lemma 2.2.14 to identify $M(x\eta)/M(x)$ with the function

$$\delta(\eta) = \frac{\mathrm{d}(\mu\circ\beta^{-1})}{\mathrm{d}(\mu\circ\beta)}(\eta).$$

Then we use Fubini's Theorem and continue the computation:

$$\begin{aligned}\langle\!\langle f| \circ |f\rangle\!\rangle(\xi)(u) &= \int_{H^u}\int_{X_u} \overline{f(x)} f(x\eta)\pi_\eta(\xi(s_H(\eta)))\sqrt{\delta(\eta)}\,\mathrm{d}\lambda_u(x)\,\mathrm{d}\beta^u(\eta)\\ &= \int_{H^u} \langle f, f\rangle_{C_c(H)}(\eta)\pi_\eta(\xi(s_H(\eta)))\sqrt{\delta(\eta)}\,\mathrm{d}\beta^u(\eta) = L(\langle f, f\rangle_{C_c(H)})(\xi).\end{aligned}$$

Since $L(\langle f, f\rangle_{C_c(H)})$ is bounded, it follows that $\langle\!\langle f| \circ |f\rangle\!\rangle$ extends to a bounded operator on $L^2(H^0, \mu, \mathcal{H})$. Let $C > 0$ be its norm. Then

$$\||f\rangle\!\rangle\xi\|^2 = |\langle \xi, \langle\!\langle f| \circ |f\rangle\!\rangle\xi\rangle| \leq C\|\xi\|^2$$

by Lemma 2.2.17 for all compactly supported ξ. Hence $|f\rangle\!\rangle$ extends to a bounded operator from $L^2(H^0, \mu, \mathcal{H})$ to $L^2(X/H, \nu, \mathcal{H}^X)$. A similar estimate shows that $\langle\!\langle f|$ extends to a bounded operator from $L^2(X/H, \nu, \mathcal{H}^X)$ to $L^2(H^0, \mu, \mathcal{H})$. □

Proof of Proposition 2.2.11. Follows from Lemma 2.2.18. □

The last proposition shows that $C_c(X)$ is a $\mathrm{C}^*(H, \beta)$-pre-Hilbert module. Let $\mathcal{H}(X)$ denote the $\mathrm{C}^*(H, \beta)$-Hilbert module obtained by completing $C_c(X)$. Note that we did not use the second countability of X anywhere in the construction of $\mathcal{H}(X)$.

Theorem 2.2.19. *Let (H, β) be a Hausdorff, locally compact groupoid with a Haar system and let X be a locally compact, Hausdorff proper right H-space carrying an H-invariant continuous family of measures λ. Then using Formulae* (2.1.8) *and* (2.1.9) *the right $C_c(H)$-module $C_c(X)$ can be completed to a $\mathrm{C}^*(H)$-Hilbert module $\mathcal{H}(X)$.*

In the whole discussion above we worked with all representations of (H, β). The same argument used for the left regular representation of (H, β) produces the following result for the reduced C^*-algebras:

Proposition 2.2.20. *Let (H, β) be a Hausdorff, locally compact groupoid with a Haar system and let X be a locally compact, Hausdorff proper right H-space carrying an H-invariant continuous family of measures λ. Then using Formulae* (2.1.8) *and* (2.1.9) *the right $C_c(H)$-module $C_c(X)$ can be completed to a $\mathrm{C}^*_\mathrm{r}(H)$-Hilbert module $\mathcal{H}_\mathrm{r}(X)$.*

2.3 The left action

Now we turn our attention to the left action. We wish to extend the action of $C_c(G)$ on $C_c(X)$ to an action of $\mathrm{C}^*(G)$ on $\mathcal{H}(X)$. For a groupoid equivalence the adjoining function vanishes (Lemma 2.2.5), that is, it becomes the constant function 1, and the formulae for the left actions in Definition 2.1.8 and (2.2.2) match. In this case, $C_c(G)$ acts on $C_c(X)$ by $\mathrm{C}^*(H, \beta)$-adjointable operators. Our proof for the non-free case runs along the same lines as in [28].

Lemma 2.3.1. *The action of $C_c(G)$ on $C_c(X)$ defined by Definition 2.1.8 extends to a non-degenerate *-homomorphism from $\mathrm{C}^*(G)$ to $\mathbb{B}_{\mathrm{C}^*(H)}(\mathcal{H}(X))$.*

Proof. We claim that the map

$$T\colon C_c(X*G)\to C_c(X),\qquad (Tf)(x)=\int_{G^{r_X(x)}} f(\gamma,\gamma^{-1}x)\Delta(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_G(x)}(\gamma).$$

is surjective. The range map of G is open and we appeal to Lemma 1.2.13 to see that the map $m\colon G*X\to X$ sending $m\colon(\gamma,x)\mapsto\gamma x$ is open. Now we appeal to Lemma 1.3.28 to get a function $W\colon G*X\to[0,\infty)$ such that $\mathrm{supp}(W)\cap G*K$ is compact for any $K\subseteq X$ compact. Now the function $F(x):=\int W(\gamma,\gamma^{-1}x)\Delta(\gamma,\gamma^{-1}x)\mathrm{d}\alpha^{r_G(x)}(\gamma)>0$ for each $x\in X$ and hence the function $w(\gamma,\gamma^{-1}x):=\frac{W(\gamma,\gamma^{-1}x)}{F(x)}$ satisfies

$$\int w(\gamma,\gamma^{-1}x)\Delta(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)=1$$

for all $x\in X$. This process is same as what we did in the proof of Lemma 1.3.29.

Then the operator

$$S\colon C_c(X)\to C_c(X*G),\qquad S(f)(\gamma,x)=f(\gamma x)\cdot w(\gamma,x),$$

satisfies $T\circ S=\mathrm{Id}_{C_c(X)}$. This proves the claim we made at the beginning of the proof.

Equation (2.1.18) says that the action of $C_c(G)$ on $C_c(X)$ is a *-homomorphism. Now we check that the action is also bounded.

Let ϵ be a state on $\mathrm{C}^*(H)$. Then $\epsilon(\langle\,,\rangle)$ makes $\mathcal{H}(X)$ into a Hilbert space, say $\mathcal{H}(X)_\epsilon$. Take the subspace V_ϵ of this Hilbert space generated by $\{\zeta f:\zeta\in C_c(G),\, f\in C_c(X)\}$. Define a representation L of $C_c(G)$ on V_ϵ by $L(\zeta)f=\zeta f$.

i) The representation L is a non-degenerate representation of $C_c(G)$ on V_ϵ. Non-degenerate means that the set $\{\zeta f:\zeta\in C_c(G),\, f\in C_c(X)\}$ is dense in V_ϵ. This is true because $C_c(G)\otimes C_c(X)$ is dense in $C_c(G*X)$ and the map $C_c(G*X)\to C_c(X)$, $(f,\zeta)\mapsto f\zeta$, is surjective.

ii) The continuity of the operations in Lemma 2.1.11 in the inductive limit topology implies that L is continuous: for $f,g\in C_c(X)$, $L_{f,g}(\zeta)=\langle f,L(\zeta)g\rangle$ is a continuous functional on $C_c(G)$ when $C_c(G)$ is given the inductive limit topology.

iii) L preserves the involution, that is, $\langle\zeta f,g\rangle=\langle f,\zeta^*g\rangle$. This is proved in Equation 2.1.18 in Lemma (2.1.11).

Proposition 4.2 of [34] says that L is a representation of G on V_ϵ, that is, bounded with respect to the norm on $\mathrm{C}^*(G)$. Thus $\epsilon(\langle\zeta f,\zeta f\rangle)\le\|\zeta\|_{\mathrm{C}^*(G)}\,\epsilon(\langle f,f\rangle)$ for all $f\in C_c(X)$ and $\zeta\in C_c(G)$. As the state ϵ was arbitrary, the inequality holds for all states. Hence for all $f\in C_c(X)$ and $\zeta\in C_c(G)$ we get

$$\langle\zeta f,\zeta f\rangle\le\|\zeta\|_{\mathrm{C}^*(G)}\,\langle f,f\rangle.$$

This shows that the action of $C_c(G)$ on $C_c(X)$ is bounded in the topology induced by the norm of the inner product $\langle\,,\rangle$. Hence the action can be extended to $\mathrm{C}^*(G)$. The proof also shows that $C_c(G)\mathcal{H}(X)\subseteq\mathcal{H}(X)$ is dense, so the representation of $\mathrm{C}^*(G)$ is non-degenerate. $\square$

Recall the definition of $\mathcal{H}_\mathrm{r}(X)$ from Corollary 2.2.20. One can work with the left regular representations of (G, α) and (H, β) to get the following result:

Lemma 2.3.2. *Assume that the transformation groupoid $G \ltimes X$ is amenable, that is, the action of G on X is amenable. Then the action of $C_c(G)$ on $C_c(X)$ defined by Definition 2.1.8 extends to an action of $\mathrm{C}^*_\mathrm{r}(G)$ on the $\mathrm{C}^*(H)$-Hilbert module $\mathcal{H}_\mathrm{r}(X)$ by adjointable operators.*

Proof. Take the faithful representation of $C^*_r(H)$ on the continuous field of Hilbert spaces $L^2(H^1)$ over H^0. Then the C^*-algebra of adjointable operators on $\mathcal{H}_r(X)$ is represented faithfully on the induced continuous field $\mathcal{H}(X) \otimes_{C^*(H)} L^2(H^1)$; the fibre of this field at $u \in H^0$ is $L^2(X_u, \lambda_u)$. This carries a multiplication action of $C_0(X)$, which is covariant with the action of $C^*(G)$ to give a representation of the crossed product algebra $G \ltimes C_0(X)$ or, equivalently, the groupoid C^*-algebra $C^*(G \ltimes X)$ of the transformation groupoid. We check this covariance.

As in Example 1.3.11, let $\bar{\alpha}$ be the Haar system for $G \ltimes X$ which is obtained using the Haar system α. Fix $u \in H^{(0)}$. For $f \in C_c(G \ltimes X)$ define the operator $\pi(f) \colon \mathcal{L}^2(X_u, \lambda_u) \to \mathcal{L}^2(X_u, \lambda_u)$ by

$$\begin{aligned}\pi(f)(\xi)(x) &= \int f((\gamma^{-1}, x)^{-1})\xi((\gamma^{-1}, x)x)\Delta^{1/2}(\gamma, \gamma^{-1}x)\, \mathrm{d}\bar{\alpha}^x((\gamma^{-1}, x)) \\ &= \int f(\gamma, \gamma^{-1}x)\xi(\gamma^{-1}x)\Delta^{1/2}(\gamma, \gamma^{-1}x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma),\end{aligned}$$

where $\xi \in \mathcal{L}^2(X_u, \lambda_u)$. We use the obvious action of $G \ltimes X$ on X given by $(\gamma, x)x = \gamma x$ to define the above operator. If $\zeta, \xi \in \mathcal{L}^2(X_u, \lambda_u)$, then a computation similar to that in the proof of Equation (2.1.18) of Lemma 2.1.11 gives

$$\langle \pi(f)\zeta \,, \xi\rangle_{\mathcal{L}^2(X_u, \lambda_u)} = \langle \zeta \,, \pi(f^*)(\xi)\rangle_{\mathcal{L}^2(X_u, \lambda_u)} .$$

In detail,

$$\langle \pi(f)\zeta \,, \xi\rangle = \int_{X_u}\int_G \overline{f(\gamma, \gamma^{-1}x)}\, \overline{\zeta(\gamma^{-1}x)}\xi(x)\, \Delta(\gamma, \gamma^{-1}x)^{1/2}\, \mathrm{d}\alpha^{r(x)}(\gamma)\mathrm{d}\lambda_u(x)$$

Change the variable $(\gamma, \gamma^{-1}x) \mapsto (\gamma^{-1}, x)$ and use the (G, α)-quasi-invariance of λ_u to see that the last term equals

$$\begin{aligned}&\int_{X_u}\int_G \overline{f(\gamma^{-1}, x)}\, \overline{\zeta(x)}\xi(\gamma^{-1}x)\, \Delta(\gamma, \gamma^{-1}x)^{1/2}\, \mathrm{d}\alpha^{r(x)}(\gamma)\mathrm{d}\lambda_u(x) \\ &\int_{X_u}\int_G \overline{\zeta(x)}\, \overline{f(\gamma^{-1}, x)}\xi((\gamma^{-1}, x)x)\, \Delta(\gamma\gamma^{-1}x)^{1/2}\, \mathrm{d}\alpha^{r(x)}(\gamma)\mathrm{d}\lambda_u(x) \\ &= \langle \zeta \,, \pi(f^*)(\xi)\rangle\end{aligned}$$

Similar to the proof of Equation (2.1.12) of Lemma 2.1.11, it can be proved that $\pi(f_1)\pi(f_2) = \pi(f_1 * f_2)$ for $f_1, f_2 \in C_c(G \ltimes X)$. Thus π is a *-representation of $C_c(G \ltimes X)$ on $\mathcal{L}^2(X_u, \lambda_u)$. And this proves the G-covariance of the multiplication action of $C_0(X)$ on $\mathcal{L}^2(X_u, \lambda_u)$.

Now if $G \ltimes X$ is amenable, then $C^*(G \ltimes X) = C^*_\mathrm{r}(G \ltimes X)$. Hence the morphism $C^*(G) \to C^*(G \ltimes X)$ vanishes on the kernel of $C^*(G) \to C^*_r(G)$. Since the action of $C^*(G)$ on $\mathcal{H}(X) \otimes_{C^*(H)} L^2(H^1)$ factors through $C^*(G \ltimes X)$, it descends to $C^*_\mathrm{r}(G)$. And since this is a faithful representation of the adjointable operators on the reduced version $\mathcal{H}_\mathrm{r}(X)$ of $\mathcal{H}(X)$, we get the desired left action of $C^*_\mathrm{r}(G)$ on $\mathcal{H}(X)$. □

Now we are ready to state the main theorem of the present chapter.

Theorem 2.3.3. *Let (G, α) and (H, β) be locally compact, Hausdorff groupoids with Haar systems. If (X, λ) is a correspondence from (G, α) to (H, β) then using the family of measures λ the space $C_c(X)$ can be completed to a C^*-correspondence $\mathcal{H}(X)$ from $\mathrm{C}^*(G, \alpha)$ to $\mathrm{C}^*(H, \beta)$.*

Proof. Follows by putting Proposition 2.2.19 and Lemma 2.3.1 together. □

In Theorem 2.3.3, we do not need that either X or H are second countable.

Proposition 2.3.4. *Let (G, α) and (H, β) be locally compact, Hausdorff groupoids with Haar systems. Let (X, λ) be a topological correspondence from (G, α) to (H, β). If the action of G on X is amenable, then using the family of measures λ the space $C_c(X)$ can be completed to a C^*-correspondence $\mathcal{H}_{\mathrm{r}}(X)$ from $\mathrm{C}^*_{\mathrm{r}}(G, \alpha)$ to $\mathrm{C}^*_{\mathrm{r}}(H, \beta)$.*

Proof. Follows by putting Proposition 2.2.20 and Lemma 2.3.2 together. □

In Proposition 2.3.4, we do not need that either X or H is second countable.

Corollary 2.3.5. *Assume the same hypotheses as in Theorem 2.3.3. If the action of G on X is proper, then $C_c(X)$ can be completed to a C^*-correspondence $\mathcal{H}_{\mathrm{r}}(X)$ from $\mathrm{C}^*_{\mathrm{r}}(G)$ to $\mathrm{C}^*_{\mathrm{r}}(H)$.*

Proof. Since the action of G is proper, the transformation groupoid $G \ltimes X$ is a proper groupoid. The Haar system of G also gives a Haar system for $G \ltimes X$. Now we apply Lemma 1.3.29 to $G \ltimes X$ to see that it is an amenable groupoid. □

An instance when the hypothesis of Corollary 2.3.5 holds, is when X is second countable.
We need neither r_X not s_X to be open surjection.

2.4 Composition of correspondences

2.4.1 Preparation for composition

Let Z, Ω be spaces, let $\pi : Z \to \Omega$ be a surjection and λ a family of measures along π. Let $X * X := X \times_{\pi,\pi} X$ and let π_1, π_2 be the projection maps from $Z * Z$ to the first and second copy of Z, respectively. The family of measures λ induces families of measures λ_2 and λ_1 along π_1 and π_2, respectively, as in Lemma 1.3.17. For $x \in Z$ the measure λ_{1x} on $\pi_1^{-1}(x)$ is given by $\delta_x \times \lambda_{\pi(x)}$. And λ_{1x} is defined similarly. This data gives Figure 2.2

Observation 2.4.1. The composite families of measures $\lambda \circ \lambda_2$ and $\lambda \circ \lambda_1$ on $Z * Z$ are the same family of measures along $\pi \circ \pi_1 = \pi \circ \pi_2 \colon Z * Z \to \Omega$. We denote this family of measures by $\lambda \times \lambda$, where $\{(\lambda \circ \lambda)_u = \lambda_u \times \lambda_u\}_{u \in \Omega}$, that is, for $u \in \Omega$, $f \in C_c(X * X)$,

$$\int f \, \mathrm{d}(\lambda \times \lambda)_u = \int f(x, y) \, \mathrm{d}\lambda_u(x) \, \mathrm{d}\lambda_u(y).$$

Observation 2.4.2. Let $f_1, f_2 \in \mathcal{B}_+(Z)$ and let m be a measure on Z. By an abuse of notation we write $f_1 \otimes f_2$ for the restriction of $f_1 \otimes f_2$ to $Z * Z$. Then $m \circ \lambda_1(f_1 \otimes f_2) = m \circ \lambda_2(f_1 \otimes f_2)$ means $m(\lambda(f_1)f_2) = m(f_1\lambda(f_2))$. In this situation, we say that λ_1 and λ_2 are symmetric with respect to m.

$$\begin{array}{ccc} Z * Z & \xrightarrow[\pi_2]{\lambda_2} & Z \\ {\scriptstyle \lambda_1}\big\downarrow{\scriptstyle \pi_1} & & {\scriptstyle \pi}\big\downarrow{\scriptstyle \lambda} \\ Z & \xrightarrow[\pi]{\lambda} & \Omega \end{array}$$

Figure 2.2

Notice that we are a *bit* loose with the notation in Observation 2.4.2 because $m(\lambda(f_1)f_2) = m(f_1\lambda(f_2))$ means $m((\lambda(f_1) \circ \pi)f_2) = m(f_1(\lambda(f_2) \circ \pi))$.

Proposition 2.4.3. *Let Z, Ω be spaces, $\pi : Z \to \Omega$ a surjection and λ a π-family of measures on Z. Let π_1, π_2 be the projection maps from $Z * Z$ onto the first and the second copy of Z.*

i) *Let μ be a measure on Ω. Then λ_1 and λ_2 are symmetric with respect to $m = \mu \circ \lambda$ in the sense of Observation 2.4.2.*

ii) *Let m be a measure on Z. If λ_1 and λ_2 are symmetric with respect to m and there is a non-negative Borel function e on Z with $\lambda(e) = 1$, then there is a measure μ on Ω with $\mu \circ \lambda = m$.*

iii) *The measure μ in (ii) with $\mu \circ \lambda = m$ is unique.*

Recall from the discussion that followed Definition 1.3.1 that we work with proper families of measures only. Thus we always have a function e as in (ii) above.

Proof. (i): If $\mu \circ \lambda = m$ then λ_1 and λ_2 are symmetric with respect to m because $m \circ \lambda_2 = (\mu \circ \lambda) \circ \lambda_2 = \mu \circ (\lambda \circ \lambda_2) = \mu \circ (\lambda \circ \lambda_1) = (\mu \circ \lambda) \circ \lambda_1 = m \circ \lambda_1$. The equality $\lambda \circ \lambda_2 = \lambda \circ \lambda_1$ follows from Observation 2.4.1.

(ii): For $g \in \mathcal{B}_+(\Omega)$ define $\mu(g) := m((g \circ \pi) \cdot e)$. In Observation 2.4.2 let $f_1 = f$, $\lambda(e) = 1$ and take $g = \lambda(f)$ in the definition of μ in the previous sentence. Then

$$m(f) = m(f \cdot \lambda(e)) = m((\lambda(f) \circ \pi) \cdot e) = \mu(\lambda(f)) = \mu \circ \lambda(f).$$

(iii): Let μ' be another measure on Ω which satisfies the condition $\mu' \circ \lambda = m$. Since λ is a proper family of measures, the integration map $\Lambda\colon C_c(Z) \to C_c(\Omega)$ is surjective. So $\mu \circ \lambda = \mu' \circ \lambda$ implies $\mu = \mu'$.

□

For $\pi\colon Z \to \Omega$ the fibre product $Z * Z$ is the groupoid of the equivalence relation defined by $x \sim y$ if and only if $\pi(x) = \pi(y)$. For an equivalence groupoid relation $(x, y)^{-1} = (y, x)$, $s_{X*X} = \pi_2$ and $r_{X*X} = \pi_1$. Now we study the case when the measures λ_1 and λ_2 are not symmetric, but *weakly* symmetric. The measures λ_1 and λ_2 are called *weakly symmetric* if there is a continuous homomorphism $\Delta : Z * Z \to \mathbb{R}_+^*$ with $m \circ \lambda_2 = \Delta \cdot (m \circ \lambda_1)$. In Section 1.4, we saw that a homomorphism from a groupoid G to an abelian group R is also called an R-valued 1-cocycle.

It is a well-known fact that $Z * Z$ is a proper groupoid (see Lemma 2.4.9 for the proof). Assume that the measures λ_1 and λ_2 are weakly symmetric. Let $\Delta : Z * Z \to \mathbb{R}_+^*$ be the $\mathbb{R}_+^*$-valued 1-cocycle that implements the weak equivalence. Then $\log \circ \Delta : Z * Z \to \mathbb{R}$ is an $\mathbb{R}$-valued 1-cocycle. Proposition 1.4.10 says that $\log \circ \Delta = \underline{b} \circ s - \underline{b} \circ r$ for some continuous function $\underline{b} : Z \to \mathbb{R}$. Thus

$$\Delta = \frac{e^{\underline{b} \circ s}}{e^{\underline{b} \circ r}}.$$

Write $b = e^{\underline{b}}$, then $b > 0$ and

$$\Delta = \frac{e^{\underline{b} \circ s}}{e^{\underline{b} \circ r}} = \frac{e^{\underline{b}} \circ s}{e^{\underline{b}} \circ r} = \frac{b \circ s}{b \circ r} = \frac{b \circ \pi_2}{b \circ \pi_1}.$$

Now we have $m \circ \lambda_2 = \left(\frac{b \circ \pi_2}{b \circ \pi_1}\right) m \circ \lambda_1$, which is equivalent to $(b \circ \pi_1)(m \circ \lambda_2) = (b \circ \pi_2)(m \circ \lambda_1)$. An easy calculation shows that $(b \circ \pi_1)(m \circ \lambda_2) = (bm) \circ \lambda_2$ and $(b \circ \pi_2)(m \circ \lambda_1) = (bm) \circ \lambda_1$. Thus we get

Proposition 2.4.4. *Let Z, Ω, π and λ be as in Proposition 2.4.3 and let m be a measure on Z with respect to which λ_1 and λ_2 are weakly symmetric. Let Δ be the $\mathbb{R}_+^*$-valued 1-cocycle that implements the weak equivalence. Then there is a function $b : Z \to \mathbb{R}^*$ with*

i) $\frac{b(y)}{b(x)} = \Delta(x, y)$ *for all* $(x, y) \in Z * Z$*;*

ii) λ_1 *and* λ_2 *are symmetric with respect to the measure* bm*, that is,* $bm \circ \lambda_1 = bm \circ \lambda_2$*.*

2.4.2 Composition of topological correspondences

Let (X, α, Δ_1) and (Y, β, Δ_2) be correspondences[1] from (G_1, λ_1) to (G_2, λ_2) and from (G_2, λ_2) to (G_3, λ_3), respectively. This is pictured in Figure 2.3

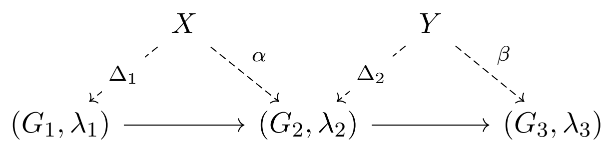


Figure 2.3

We need to create a G_1-G_3-bispace Ω equipped with a G_3-invariant and G_1-quasi-invariant family of measures $\mu = \{\mu_u\}_{u \in H_3^{(0)}}$. The $C^*(G_1)$-$C^*(G_3)$-Hilbert module $\mathcal{H}(\Omega)$ should be isomorphic to the Hilbert module $\mathcal{H}(X) \otimes_{C^*(G_2)} \mathcal{H}(Y)$.

Let $Z := X * Y$ be the fibre product over $G_2^{(0)}$ for the maps s_X and r_Y. Then Z carries the diagonal action of G_2. Since the action of G_2 on X is proper, its action on Z is proper. Define the space $\Omega = Z/G_2$.

[1]See the paragraph 'Important Conventions 2.1' on page 41.

Observation 2.4.5. The space Z is a G_1-G_3-bispace. The momentum maps are $r_Z(x,y) = r_X(y)$ and $s_Z(x,y) = s_Y(z)$. For $(\gamma_1,(x,y)) \in G_1 * Z$ and $((x,y),\gamma_3) \in Z * G_3$, the actions are $\gamma_1 \cdot (x,y) = (\gamma_1 x, y)$ and $(x,y)\cdot\gamma_3 = (x, y\gamma_3)$, respectively. These actions make Ω into a G_1-G_3-bispace.

Lemma 2.4.6. *The obvious right action of G_3 on Ω is proper.*

Proof. See [42, Proposition 7.6]. □

The quotient map $\pi : Z \to \Omega$ carries the family of measures λ_{2Z} as in Proposition 1.3.21. We write $\lambda = \{\lambda^\omega\}_{\omega\in\Omega}$ instead of $\lambda_{2Z} = \{\lambda_{2Z}^{\omega}\}_{\omega\in\Omega}$. Recall that for $f \in C_c(Z)$,

$$\int f \,\mathrm{d}\lambda^{\omega=[x,y]} := \int_{G_2^{r_Y(y)}} f(x\gamma, \gamma^{-1}y)\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma).$$

Proposition 1.3.21 shows that λ is a continuous family of measures with full support.

For a fixed $u \in G_3^{(0)}$ we define a measure m_u on the space Z as follows: for $f \in C_c(Z)$,

$$\int_Z f \,\mathrm{d}m_u = \int_Y \int_X f(x,y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y). \tag{2.4.7}$$

Lemma 2.4.8. *The family of measures $\{m_u\}_{u\in G_3^{(0)}}$ is a G_3-invariant continuous family of measures on Z.*

Proof. The G_3-invariance of the family of measures β makes $\{m_u\}_{u\in G_3^{(0)}}$ G_3-invariant.

Let $f \in C_c(X)$ and $g \in C_c(X)$, then

$$\int f \otimes g \,\mathrm{d}m_u = B((A(f)\circ r_Y)\,g)(u).$$

Using a density argument as in the proof of Lemma 1.3.20 we conclude that $\{m_u\}_{u\in G_3^{(0)}}$ is a continuous family of measures. □

We wish to prove that up to equivalence $\{m_u\}_{u\in G^{(0)}}$ can be pushed down from Z to Ω to a G_3-invariant family of measures $\{\mu^u\}$. Before we proceed we prove a small lemma. Denote $X * G_2 * Y := \{(x,\gamma_2,y) \in X \times G_2 \times Y : s_X(x) = r_G(\gamma_2) = r_Y(y)\}$. And let $X * G_2 = X \times_{s_X, r_{G_2}} G_2$ and $G_2 * Y = G_2 \times_{r_{G_2}, r_Y} Y$.

Lemma 2.4.9. *Let (X,α,Δ_1) and (Y,β,Δ_2) be correspondences from (G_1,λ_1) to (G_2,λ_2) and from (G_2,λ_2) to (G_3,λ_3), respectively. Let Z, Ω, λ, m_u, λ_i for $i = 1,2$ be as discussed above. For each $u \in G_3^{(0)}$ there is a function b_u on Z such that λ_1 and λ_2 are symmetric with respect to $b_u \cdot m_u$. Furthermore, b satisfies $b(x,y)\,b(x\gamma,\gamma^{-1}y)^{-1} = \Delta((x,y),(x\gamma,\gamma^{-1}y)) = \Delta_2(\gamma,\gamma^{-1}y)$.*

We shall write b instead of b_u. We work with a single μ_u at a time, so we prefer to drop the suffix u.

Proof. The proof follows in the steps below:

i) λ_1 and λ_2 are weakly symmetric with respect to m_u for each $u \in G_3^{(0)}$.

ii) $Z * Z$ is a proper groupoid.

iii) Appeal to Proposition 2.4.4 and get the result.

(i): Now we show that λ_1 and λ_2 are weakly symmetric families of measures. Figure 2.4 shows all maps and the families of measures along the maps:

$$\begin{array}{ccc} Z * Z & \xrightarrow[\pi_1]{\lambda_2} & Z \\ \lambda_1 \big\downarrow \pi_2 & & \pi \big\downarrow \lambda \\ Z & \xrightarrow[\pi]{\lambda} & \Omega \end{array}$$

Figure 2.4

Let $f \in C_c(Z * Z)$, then

$$\begin{aligned}(m_u \circ \lambda_2)(f) &= \iint f((x,y),(x\gamma,\gamma^{-1}y))\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\,\mathrm{d}m_u(x,y)\\ &= \iiint f((x,y),(x\gamma,\gamma^{-1}y))\,\mathrm{d}\lambda_2^{r(y)}(\gamma)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y).\end{aligned}$$

Change variables $(x\gamma, \gamma^{-1}y) \mapsto (x, y)$. Recall that the family α is G_2-invariant and β is G_2-quasi-invariant. Now calculating further:

$$\begin{aligned}\text{R. H. S.} &= \iiint f((x\gamma^{-1},\gamma y),(x,y))\,\Delta_2(\gamma^{-1},\gamma y)\ \mathrm{d}\lambda_2^{r(y)}(\gamma)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y)\\ &= (m_u \circ \lambda_1)(f \cdot \Delta_2 \circ \mathrm{inv}_{G_2 \ltimes Y}),\end{aligned}$$

where $\mathrm{inv}_{G_2 \ltimes Y}$ is the inverse function on the groupoid $G_2 \ltimes Y$.

(ii): Observe that $r_{Z*Z} \times s_{Z*Z} \colon Z * Z \to Z \times Z$ is the inclusion map. Hence to show that $Z * Z$ is proper, it suffices to prove that $Z * Z \subseteq Z \times Z$ is closed. To see this, we observe that Ω is Hausdorff, hence $\mathrm{dia}(\Omega) := \{(\omega,\omega) : \omega \in \Omega\}$ is closed in $\Omega \times \Omega$. Since π is continuous, $(\pi \times \pi)^{-1}(\mathrm{dia}(\Omega)) \subset Z \times Z$ is closed where $\pi \times \pi \colon Z \times Z \to \Omega \times \Omega$ is the canonical map.

(iii): Due to (i) and (ii), we may apply Proposition 2.4.4 which gives a function $b \colon Z \to \mathbb{R}^*$ such that λ_1 and λ_2 are symmetric with respect to bm_u. □

Remark 2.4.10. The cocycle $\Delta \colon Z * Z \to \mathbb{R}^*$, $\Delta((x,y),(x\gamma,\gamma^{-1}y)) = \Delta_2(\gamma,\gamma^{-1}y)$, implements the weak symmetry between λ_1 and λ_2. We observe:

i) since Δ does not depend on x, Δ is G_1-invariant;

ii) Δ_2 is defined on $G_2 * (Y/G_3)$ (see Remark 2.1.5). Hence $\Delta((z,z')\gamma_3) = \Delta(z\gamma_3, z'\gamma_3) = \Delta(z,z')$ with $s_Z(z) = s_Z(z') = r_{G_3}(\gamma_3)$, $\gamma_3 \in G_3$. Thus Δ depends only on γ and $[y]$.

The function b appearing in Lemma 2.4.9 can be computed explicitly. Let $p = \{p^z\}_{z\in Z}$ be a family of probability measures on $Z * Z$ as in Lemma 1.3.29. Then Corollary 2.4.4 gives

$$b(z') = \exp(\underline{b})(z') = \exp\left(\int \log \circ \Delta((z,z'))\, \mathrm{d}p^{z'}(z)\right). \tag{2.4.11}$$

This implies that b is continuous on Z.

Remark 2.4.12. i) The G_1-invariance of Δ from Remark 2.4.10 and Equation 2.4.11 clearly implies that b is G_1-invariant.

ii) The G_3-invariance of Δ (Remark 2.4.10 and Equation 2.4.11) implies b is G_3-invariant. Indeed, for $\gamma_3 \in G_3$

$$b(z'\gamma_3) = \exp\left(\int \log \circ \Delta(z\gamma_3, z'\gamma_3)\, \mathrm{d}p^{z'}(\gamma)\right) = \exp\left(\int \log \circ \Delta(z, z')\, \mathrm{d}p^{z'}(\gamma)\right) = b(z').$$

Remark 2.4.13. Once we have $bm_u \circ \lambda_1 = bm_u \circ \lambda_2$, (ii) in Proposition 2.4.3 gives a measure μ_u on Ω which lifts to bm_u on Z via λ. $\{\mu_u\}_{u\in\Omega}$ is the required family of measures. For $f \in C_c(\Omega)$

$$\int f\, \mathrm{d}\mu_u = \iint f \circ \pi(x,y) e(x,y)\, b(x,y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y).$$

Due to Proposition 2.4.3 the measure μ_u is independent of the choice of the function e. Sometimes we abuse notation and write f instead of $f \circ \pi$. We think of f as a function on Z itself.

Recall that Ω is a G_1-G_3-bispace (see Observation 2.4.5).

Proposition 2.4.14. *The family of measures $\{\mu_u\}_{u\in G_3^{(0)}}$ is a G_3-invariant continuous family of measures on Ω along the momentum map s_Ω.*

Proof. We check the invariance first and then check the continuity. Let $f \in C_c(\Omega)$ and $\gamma \in G_3$, and let e be a non-negative Borel function on $Z = X * Y$ with $\lambda(e) = 1$. Then

$$\begin{aligned}
&\int f[x, y\gamma']\, \mathrm{d}\mu_{r_{G_3}(\gamma')}[x,y] \\
&= \int \left((f\circ\pi)\cdot e\, \mathrm{d}\lambda^{[x,y]}\right)\, b\, \mathrm{d}m_{r_{G_3}(\gamma')} \\
&= \iiint f([x, y\gamma'])e(x\gamma, \gamma^{-1}y\gamma')b(x,y)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{r_{G_3}(\gamma'))}(y).
\end{aligned}$$

Change $y\gamma' \to y$ and use the G_3-invariance of the family β and that of the function b to get

$$\begin{aligned}
\text{R.H.S.} &= \iiint f([x,y])e(x\gamma, \gamma^{-1}y)b(x,y)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{s_{G_3}(\gamma')}(y) \\
&= \iint \left(f \cdot e\, \mathrm{d}\lambda^{[x,y]}\right)\, b\, \mathrm{d}m_{s_{G_3}(\gamma')} \\
&= \int f[x,y]\, \mathrm{d}\mu_{s_{G_3}(\gamma')}[x,y].
\end{aligned}$$

Thus $\{\mu_u\}_{u \in G_3{}^{(0)}}$ is G_3-invariant.

Now we check that μ is a continuous family of measures. Let M, μ and Λ denote the integration maps which the families of measures m, μ and λ induce between the corresponding spaces of continuous compactly supported functions. By the construction itself, $M : C_c(Z) \to C_c(G_3^{(0)})$ is the composite of $C_c(Z) \xrightarrow{\Lambda} C_c(\Omega) \xrightarrow{\mu} C_c(G_3^{(0)})$. Due the definition of μ the following diagram commutes:

$$\begin{array}{ccc} & & C_c(Z) \\ & \overset{\Lambda}{\swarrow} & \downarrow M \\ C_c(\Omega) & \underset{\mu}{\searrow} & \\ & & C_c(G_3^{(0)}). \end{array}$$

Lemma 2.4.8 shows that M is continuous, Proposition 1.3.21 shows that Λ is continuous and surjective. Hence μ is continuous. □

The family of measures μ on Ω is the required family of measures for the composite correspondence. We still need to show that it is G_1-quasi-invariant. Let $f \in C_c(G_1 \ltimes \Omega)$ and $u \in G_3^{(0)}$, then

$$\begin{aligned} &\int f(\eta^{-1}, [x,y])\, \mathrm{d}\lambda_1^{r_\Omega([x,y])}(\eta)\, \mathrm{d}\mu_u[x,y] \\ &= \iiint f(\eta^{-1}, [x,y])\, e(x\gamma, \gamma^{-1}y)\, b(x,y)\, \mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y). \end{aligned}$$

We apply Fubini's Theorem to the last step to get

$$\mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x) \mapsto \mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x) \mathrm{d}\lambda_2^{s_X(x)}(\gamma).$$

Now we change $(\eta^{-1}, [x,y]) \mapsto (\eta, [\eta^{-1}x, y])$. Then

$$\mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x) \mapsto \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x).$$

We incorporate this change and apply Fubini's theorem again to get the same sequence of the integrals and compute further:

$$\begin{aligned} &\iiint f(\eta, [\eta^{-1}x, y])\, e(\eta^{-1}x\gamma, \gamma^{-1}y)\, b(\eta^{-1}x, y)\, \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y) \\ &= \iiint f(\eta, [\eta^{-1}x, y])\, \frac{b(\eta^{-1}x, y)}{b(x,y)}\, \Delta_1(\eta, \eta^{-1}x)\, e(\eta^{-1}x, y) b(x,y)\, \mathrm{d}\lambda_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \lambda_2^{s_X(x)}(\gamma)\, \mathrm{d}\beta_u(y). \end{aligned}$$

But $e(\eta^{-1}x\gamma, \gamma^{-1}y)\, \mathrm{d}\lambda_1^{r_X(x)}(\eta) = 1$ and $b(x,y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y) = \mathrm{d}\mu_u[x,y]$. the last term equals

$$\iint f(\eta, [\eta^{-1}x, y])\, \frac{b(\eta^{-1}x, y)}{b(x,y)}\, \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\lambda_1^{r_\Omega([x,y]}(\eta)\, \mathrm{d}\mu_u[x,y].$$

Thus if $\Delta_{1,2} : G_1 \ltimes \Omega \to \mathbb{R}_+^*$ is defined as

$$\Delta_{1,2}(\eta^{-1}, [x,y]) = b(\eta^{-1}x, y)^{-1} \Delta_1(\eta^{-1}, x) b(x,y), \tag{2.4.15}$$

then the above computation gives

$$\int f(\eta^{-1}, [x,y])\, \mathrm{d}\lambda_1(\eta)\, \mathrm{d}\mu_u[x,y] = \iint f(\eta, [\eta^{-1}x, y])\, \Delta_{1,2}(\eta, \eta^{-1}[x,y])\, \mathrm{d}\lambda_1(\eta)\, \mathrm{d}\mu_u[x,y],$$

for all $u \in G_3^{(0)}$. One must check that the function $\Delta_{1,2}$ makes sense. We prove the following lemma for this purpose.

Lemma 2.4.16. *The function $\Delta_{1,2}$ defined in Equation* (2.4.15) *is a well-defined $\mathbb{R}_+^*$-valued continuous 1-cocycle on the groupoid $G_1 \ltimes \Omega$.*

Proof. Let $(x\gamma, \gamma^{-1}y) \in [x,y]$, then

$$\begin{aligned}
\Delta_{1,2}(\eta^{-1}, [x\gamma, \gamma^{-1}y]) &= b(\eta^{-1}x\gamma, \gamma^{-1}y)^{-1}\Delta_1(\eta^{-1}, x\gamma)b(x\gamma, \gamma^{-1}y) \\
&= b(\eta^{-1}x, y)^{-1}\Delta_1(\eta^{-1}, x)b(x,y)\left(\frac{b(\eta^{-1}x, y)}{b(\eta^{-1}x\gamma, \gamma^{-1}y)}\,\frac{b(x\gamma, \gamma^{-1}y)}{b(x,y)}\right) \\
&= \Delta_{1,2}(\eta^{-1}, [x,y])\left(\Delta_2(\eta^{-1}, \gamma^{-1}y)\Delta_2(\gamma^{-1}, \gamma^{-1}y)^{-1}\right) \\
&= \Delta_{1,2}(\eta^{-1}, [x,y]).
\end{aligned}$$

In the above computations, to get the third equality, we used the last claim in Lemma 2.4.9. Due to the continuity of b and Δ_1, $\Delta_{1,2}$ is continuous. Checking that $\Delta_{1,2}$ is a groupoid homomorphism is a routine computation. □

Proposition 2.4.17. *The family of measures $\{\mu_u\}_{u\in G_3^{(0)}}$ is G_1-quasi-invariant. The adjoining function for the quasi-invariance is given by Equation* (2.4.15)*.*

Proof. Clear from the discussion above. □

Definition 2.4.18 (Composition)**.** For correspondences

$$\begin{aligned}
(X, \alpha, \Delta_1)\colon (G_1, \lambda_1) \to (G_2, \lambda_2) \text{ and}\\
(Y, \beta, \Delta_2)\colon (G_2, \lambda_2) \to (G_3, \lambda_3),
\end{aligned}$$

their composite correspondence $(\Omega, \mu, \Delta_{1,2}) : (G_1, \lambda_1) \to (G_3, \lambda_3)$ is defined by:

i) a space $\Omega := (X * Y)/G_2$,

ii) a family of measures $\mu = \{\mu_u\}_{u\in G_3^{(0)}}$ that lifts to $\{b\alpha \times \beta_u\}_{u\in G^{(0)}}$ on Z for a cochain $b \in \mathrm{C}^0_{G_3}(Z * Z, \mathbb{R}_+^*)$ satisfying $d^0(b) = \Delta$.

The Δ above is the one in Remark 2.4.10. $\mathrm{C}^0_{G_3}$ is the zeroth cochain group of the G_3-invariant $\mathbb{R}_+^*$-valued continuous cochain complex of $X * X$ (see Definition 1.4.7). For a composite correspondence the adjoining function $\Delta_{1,2}$ is the one given by Equation (2.4.15).

Theorem 2.4.19. *Let $(X, \alpha)\colon (G_1, \lambda_1) \to (G_2, \lambda_2)$ and $(Y, \beta) : (G_2, \lambda_2) \to (G_3, \lambda_3)$ be topological correspondences. Assume that the topologies are locally compact, Hausdorff and second countable. Let $(\Omega, \mu)\colon (G_1, \lambda_1) \to (G_3, \lambda_3)$ be a composite of the correspondences. Then $\mathcal{H}(\Omega)$ and $\mathcal{H}(X)\hat{\otimes}_{\mathrm{C}^*(G_2)}\mathcal{H}(Y)$ are isomorphic correspondences from $\mathrm{C}^*(G_1, \lambda_1)$ to $\mathrm{C}^*(G_3, \lambda_3)$.*

We make a remark before commencing the proof of the theorem.

Remark 2.4.20. The function Δ in Remark 2.4.10, is a cocycle in $\mathrm{C}^1_{G_3}(Z*Z;\mathbb{R}^+_*)$, and $b \in \mathrm{C}^0(Z*Z;\mathbb{R}^+_*)$ is a cochain. Remark 2.4.12 says that $b \in \mathrm{C}^0_{G_3}(Z*Z;\mathbb{R}^+_*)$, and Corollary 2.4.4 gives that $\Delta = d^0(b)$. Let $\mathcal{H}(\Omega, b)$ denote the $\mathrm{C}^*(G_3, \lambda_3)$-Hilbert module obtained using $\{\mu_u = e \cdot b \cdot \alpha \times \beta_u\}_{u \in G_3^{(0)}}$. Let b' be another G_3-equivariant 0-cochain with $\Delta = d^0(b')$ and let $\mathcal{H}(\Omega, b')$ be the $\mathrm{C}^*(G_3, \lambda_3)$-Hilbert module obtained by using $\{\mu'_u = e \cdot b' \cdot \alpha \times \beta_u\}_{u \in G_3^{(0)}}$ as family of measures. Corollary 2.5.18 gives an isomorphism from the C^*-correspondence $\mathcal{H}(\Omega, b)$ to $\mathcal{H}(\Omega, b')$. Hence in the statement of Theorem 2.4.19 we need not refer to a certain fixed 0-cochain. In the proof of the theorem, we work with a fixed cochain $b \in \mathrm{C}^0_{G_3}(Z*Z;\mathbb{R}^+_*)$.

Proof of Theorem 2.4.19. We need to prove that $\mathcal{H}(\Omega)$ and $\mathcal{H}(X) \hat{\otimes}_{\mathrm{C}^*(G_2)} \mathcal{H}(Y)$ are isomorphic $\mathrm{C}^*(G_3, \lambda_3)$-Hilbert modules and the representations of $\mathrm{C}^*(G_1, \lambda_1)$ on $\mathcal{H}(\Omega)$ and $\mathcal{H}(X) \hat{\otimes}_{\mathrm{C}^*(G_2)} \mathcal{H}(Y)$ are isomorphic. We divide the proof into two parts: the first dealing with the isomorphism of Hilbert modules and the other dealing with the isomorphism of representations.

Due to the Stone-Weierstraß Theorem, the set $A := \{f \otimes g : f \in C_c(X) \text{ and } g \in C_c(Y)\}$ is linearly dense in $C_c(Z)$ in the inductive limit topology, where $(f \otimes g)(x, y) = f(x)g(y)$. We observe the following two facts:

i) The Hilbert module $\mathcal{H}(X) \hat{\otimes}_{\mathrm{C}^*(G_2)} \mathcal{H}(Y)$ is the completion of $A \subseteq C_c(Z)$ with respect to the norm given by the inner product $\langle f \otimes g, f \times g \rangle_{C_c(G_3)} := \langle g, \langle f, f \rangle_{\mathcal{H}(X)}\, g \rangle_{\mathcal{H}(Y)}$.

ii) As λ is a (proper) continuous family of measure along $\pi : Z \to \Omega$, we have a surjection $\Lambda' : C_c(Z) \to C_c(\Omega)$ given by

$$\Lambda'(F)[x, y] = \Lambda(F b^{-1/2})[x, y] = \int_{G_2} F(x\gamma, \gamma^{-1} y)\, b^{-1/2}(x\gamma, \gamma^{-1} y)\, \mathrm{d}\lambda_2^{s_X(x)}(\gamma)$$

for $F \in C_c(Z)$.

For b as in Proposition 2.4.4, the multiplication by $b^{-1/2}$ is an isomorphism from $C_c(Z)$ to itself. Then Λ is a surjection from $C_c(Z)$ to $C_c(\Omega)$, since $\{\lambda_u\}_{u \in G_3^{(0)}}$ is a continuous family of measures. Thus the composite $\Lambda' \colon C_c(Z) \xrightarrow{b^{-1/2}} C_c(Z) \xrightarrow{\Lambda} C_c(\Omega)$ is a continuous surjection.

Let $f, f' \in C_c(X)$, $g, g' \in C_c(Y)$ and $\psi \in C_c(G_3)$. Then $\Lambda'(f \otimes g + f' \otimes g') = \Lambda'(f \otimes g) + \Lambda'(f' \otimes g')$. Furthermore,

We show that Λ' is an isomorphism of pre-Hilbert modules, hence it extends to an isomorphism of Hilbert modules. Later we show that Λ' also intertwines the representations.

The isomorphism of Hilbert modules: Now we compute the norm of $f \otimes g \in \mathcal{H}(X) \hat{\otimes}_{\mathrm{C}^*(G_2)} \mathcal{H}(Y)$.

In the calculation below, the inner product on the left is taken in $\mathcal{H}(X) \hat{\otimes}_{\mathrm{C}^*(G_2)} \mathcal{H}(Y)$, and subscripts to other inner products tell in what space the inner product is defined. For $\underline{\gamma} \in G_3$,

$$\begin{aligned}
&\langle f \otimes g, f \otimes g\rangle_{C_c(G_3)}(\underline{\gamma}) \\
&:= \langle g, \langle f, f\rangle_{\mathcal{H}(X)}\, g\rangle_{\mathcal{H}(Y)}(\underline{\gamma}) \\
&= \int \overline{g(y)} \left(\langle f, f\rangle_{\mathcal{H}(X)}\, g\right)(y\underline{\gamma})\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \int \overline{g(y)} \left(\int \langle f, f\rangle_{\mathcal{H}(X)}(\gamma)\, g(\gamma^{-1}y\underline{\gamma})\Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right) \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \iint \overline{g(y)} \left(\int \overline{f(x)} f(x\gamma)\, \mathrm{d}\alpha_{r(y)}(x)\right) g(\gamma^{-1}y\underline{\gamma})\, \Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \iiint \overline{f(x)g(y)}\, f(x\gamma)g(\gamma^{-1}y\underline{\gamma})\, \Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

Now we calculate the norm of $\Lambda'(f \otimes g \cdot b^{-1/2})$ in $C_c(\Omega)$:

$$\begin{aligned}
&\langle \Lambda'(f \otimes g \cdot b^{-1/2}), \Lambda'(f \times g \cdot b^{-1/2})\rangle(\underline{\gamma}) \\
&\qquad := \int \overline{\Lambda'(f \otimes g \cdot b^{-1/2})[x, y]}\, \Lambda'(f \otimes g \cdot b^{-1/2})[x, y\underline{\gamma}]\, \mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x, y].
\end{aligned}$$

After plugging in the definitions, the last term of the above equation becomes

$$\begin{aligned}
&\int \left(\int \overline{f(x\gamma_*)}\,\overline{g(\gamma_*^{-1}y)} b^{-1/2}(x\gamma_*, \gamma_*^{-1}y)\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma_*)\right) \\
&\qquad \left(\int f(x\gamma)g(\gamma^{-1}y\underline{\gamma})b^{-1/2}(x\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right) \mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x, y] \\
&= \iint \overline{f(x\gamma_*)}\,\overline{g(\gamma_*^{-1}y)} b^{-1/2}(x\gamma_*, \gamma_*^{-1}y) \\
&\qquad \left(\int f(x\gamma)g(\gamma^{-1}y\underline{\gamma})b^{-1/2}(x\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right) \mathrm{d}\lambda_2^{r_Y(y)}(\gamma_*)\, \mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x, y] \\
&= \iint \overline{f(x)}\,\overline{g(y)} b^{-1/2}(x, y) \\
&\qquad \left(\int f(x\gamma)g(\gamma^{-1}y\underline{\gamma})b^{-1/2}(x\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right) b(x, y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

The last equality is due to Lemma 2.4.9, which says that

$$\mathrm{d}\lambda_2^{r_Y(y)}(\gamma_*)\, \mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x, y] = b(x, y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).$$

Continuing further,

$$\begin{aligned}
&\text{L.H.S.} \qquad (2.4.21) \\
&= \iint \overline{f(x)}\,\overline{g(y)} \left(\int f(x\gamma)g(\gamma^{-1}y\underline{\gamma})b^{-1/2}(x\gamma, \gamma^{-1}y\underline{\gamma})\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\right) b^{1/2}(x, y)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \iiint \overline{f(x)}\,\overline{g(y)} f(x\gamma)g(\gamma^{-1}y\underline{\gamma}) \left(\frac{b(x, y)}{b(x\gamma, \gamma^{-1}y\underline{\gamma})}\right)^{1/2} \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

Using Remark 2.4.12 we add a factor of $\underline{\gamma}$ in $b(x,y)$. The previous term equals

$$\iiint \overline{f(x)g(y)} f(x\gamma)g(\gamma^{-1}y\underline{\gamma}) \left(\frac{b(x,y\underline{\gamma})}{b(x\gamma,\gamma^{-1}y\underline{\gamma})}\right)^{1/2} \,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\,\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y). \tag{2.4.22}$$

By Lemma 2.4.4 we relate the factors of b to see that last equation is equal to

$$\iiint \left(\overline{f(x)g(y)} f(x\gamma)g(\gamma^{-1}y\underline{\gamma})\right) {\Delta_2}^{1/2}(\gamma,\gamma^{-1}y\underline{\gamma})\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\,\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y). \tag{2.4.23}$$

Finally, we apply Fubini's Theorem to $\lambda_2^{r_Y(y)}$ and $\alpha_{r_Y(y)}$ to get

$$\begin{aligned}&\langle \Lambda'(f\otimes g\cdot b^{-1/2}), \Lambda'(f\times gb^{-1/2})\rangle(\underline{\gamma})\\ &\quad= \iiint \left(\overline{f(x)g(y)} f(x\gamma)g(\gamma^{-1}y\underline{\gamma})\right)\ {\Delta_2}^{1/2}(\gamma,\gamma^{-1}y\underline{\gamma})\,\mathrm{d}\lambda_2^{r_Y(y)}(\gamma)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).\end{aligned}$$

Comparing the values of both inner products, we conclude that

$$\langle f\otimes g, f\otimes g\rangle_{C_c(G_3)} = \langle \Lambda'(f\otimes g\cdot b^{-1/2}),\ \Lambda'(f\otimes gb^{-1/2})\,\rangle_{C_c(G_3)}. \tag{2.4.24}$$

The isomorphism of representations: We denote the actions of $\mathrm{C}^*(G_1,\lambda_1)$ on $\mathcal{H}(X)\hat{\otimes}_{\mathrm{C}^*(G_2)}\mathcal{H}(Y)$ and $\mathcal{H}(\Omega)$ by ρ_1 and ρ_2, respectively, that is, $\rho_1\colon \mathrm{C}^*(G_1,\lambda_1)\to\mathbb{B}(\mathcal{H}(X)\hat{\otimes}_{\mathrm{C}^*(G_2)}\mathcal{H}(Y))$ and $\rho_2\colon \mathrm{C}^*(G_1,\lambda_1)\to\mathbb{B}(\mathcal{H}(\Omega))$ are the *-homomorphisms that give the C^*-correspondences from $\mathrm{C}^*(G_1,\lambda_1)$ to $\mathrm{C}^*(G_3,\lambda_3)$. We are going to show that Λ' intertwines ρ_1 and ρ_2.

Let $\phi\in C_c(G_1)$, $f,g\in C_c(X)$, then

$$\begin{aligned}&(\rho_2(\phi)\Lambda')(f\otimes g)[x,y]\\ &= (\phi*\Lambda'(f\otimes g))[x,y]\\ &= \int_{G_1}\phi(\eta)\Lambda'(f\otimes g))[\eta^{-1}x,y]\,\Delta_{1,2}^{1/2}(\eta,[\eta^{-1}x,y])\,\mathrm{d}\lambda_1^{r_X(x)}(\eta)\\ &= \iint \phi(\eta)f(\eta^{-1}x\gamma)g(\gamma^{-1}y)\,b^{-1/2}(\eta^{-1}x\gamma,\gamma^{-1}y)\,\Delta_{1,2}^{1/2}(\eta,[\eta^{-1}x,y])\ \mathrm{d}\lambda_1^{r_X(x)}(\eta)\,\mathrm{d}\lambda_2^{s_X(x)}(\gamma).\end{aligned}$$

Equation (2.4.15) gives $\Delta_{1,2}(\eta,[\eta^{-1}x,y]) = \Delta_{1,2}(\eta,[\eta^{-1}x\gamma,\gamma^{-1}y]) = \Delta_1(\eta,\eta^{-1}x\gamma)\,\frac{b(\eta^{-1}x\gamma,\gamma^{-1}y)}{b(x\gamma,\gamma^{-1}y)}$. Thus

$$\begin{aligned}\mathrm{R.H.S.} &= \int\left(\int \phi(\eta)f(\eta^{-1}x\gamma)\,\Delta_1^{1/2}(\eta,\eta^{-1}x\gamma)\,\mathrm{d}\lambda_1^{r_X(x)}(\eta)\right) g(\gamma^{-1}y)\ b^{-1/2}(x\gamma,\gamma^{-1}y)\,\mathrm{d}\lambda_2^{s_X(x)}(\gamma)\\ &= \int(\phi*f)(x\gamma)g(\gamma^{-1}y)\,b^{-1/2}(x\gamma,\gamma^{-1}y)\,\mathrm{d}\lambda_2^{s_X(x)}(\gamma)\\ &= \Lambda'((\phi*f)\otimes g)[x,y]\\ &= \Lambda'(\rho_2(\phi)(f\otimes g))[x,y].\end{aligned}$$

□

2.5 Bicategory of correspondences

It is clear from the definition of *a* composite of topological correspondences (Definition 2.4.18) and Theorem 2.4.19 that the isomorphism classes of topological correspondences form a category. But Remark 2.4.20 gives a subtler idea, namely, topological correspondences are likely to form a bicategory.

This section explores categorical aspects of our construction. We show that groupoid correspondences form a bicategory. We follow Bénabou's notation from [3] on bicategories. We also adopt his terminology. A bicategory is biequivalent to a 2-category (for a proof see [24]). Bénabou's convention for composition is the other way round than the standard one.

2.5.1 Bicategory

Definition 2.5.1 (Bicategory). A bicategory $\mathfrak{S}$ is determined by the following data:

i) a set $\mathfrak{S}_0$ called set of objects or vertices;

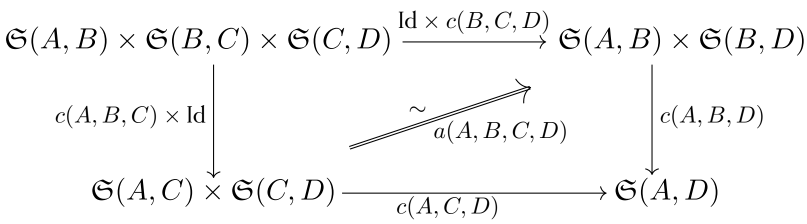
ii) for each pair (A, B) of objects, a category $\mathfrak{S}(A, B)$;

iii) for each triple (A, B, C) of objects of $\mathfrak{S}$ a *composition functor*

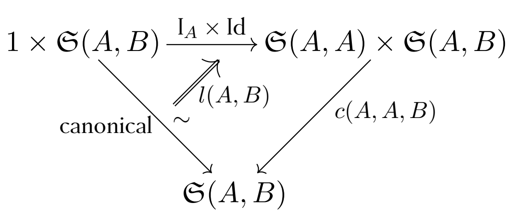
$$c(A, B, C)\colon \mathfrak{S}(A, B) \times \mathfrak{S}(B, C) \to \mathfrak{S}(A, C);$$

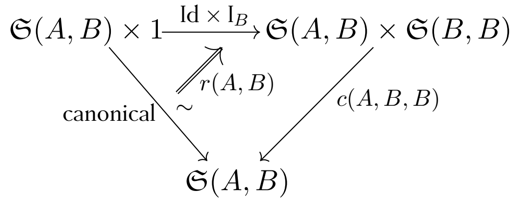
iv) for each object A of $\mathfrak{S}$ an object I_A of $\mathfrak{S}(A, A)$ called *identity arrow* of A (the identity map of I_A in $\mathfrak{S}(A, A)$ is denoted $i_A\colon I_A \implies I_A$ and is called *identity 2-cell* of A);

v) for each quadruple (A, B, C, D) of objects of $\mathfrak{S}$, a natural isomorphism $a(A, B, C, D)$ called *associativity isomorphism* between the two composite functors making the following diagram commute:



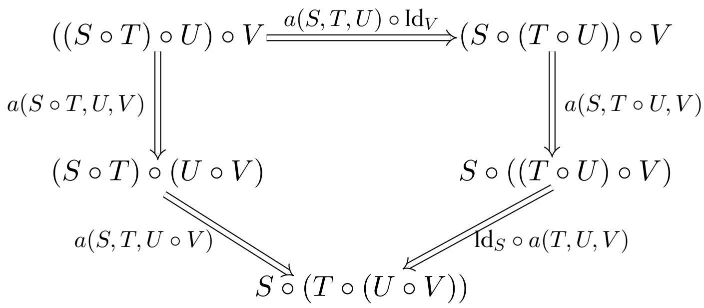
vi) for each pair (A, B) of objects of $\mathfrak{S}$, two natural isomorphisms $l(A, B)$ and $r(A, B)$, called left and right identities such that the following diagrams commute:



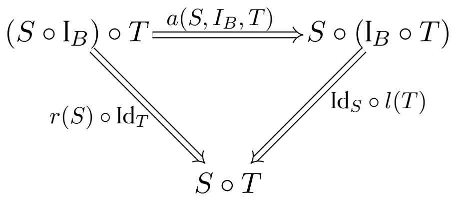


This data satisfies the following conditions:

vii) *associativity coherence*: If (S,T,U,V) is an object of $\mathfrak{S}(A,B)\times\mathfrak{S}(B,C)\times\mathfrak{S}(C,D)\times\mathfrak{S}(D,E)$, then the following diagram commutes:



viii) *identity coherence*: If (S,T) is an object of $\mathfrak{S}(A,B)\times\mathfrak{S}(B,C)$, then the following diagram commutes:



In modern literature, a vertex, an arrow (or a 1-cell) and a 2-cell are called an object, a 1-arrow and a 2-arrow, respectively. Let A and B be two objects and let t,u be two arrows in the category $\mathfrak{S}(A,B)$. Then we call the rule of composition of t and u in $\mathfrak{S}(A,B)$ the vertical composition of 1-arrows. The composite functor c in (iii) above gives the horizontal composition of 2-arrows. Let (S,T) and (S',T') be two objects in $\mathfrak{S}(A,B)\times\mathfrak{S}(B,C)$, respectively, and let $s\colon S\to S'$ and $t\colon T\to T'$ be 2-arrows. Then s and t induce a 2-arrow $s\cdot_h t\colon S\circ T\to S'\circ T'$. The 2-arrow $s\cdot_h t$ is called the vertical composite of the 2-arrows s and t.

Example 2.5.2 (C*-correspondences). In Section 2.2 of [9] Buss, Meyer and Zhu form a bicategory of C*-algebraic correspondences. In this bicategory the objects are the C*-algebras, 1-arrows are the C*-algebraic correspondences and 2-arrows are the equivariant unitary intertwiners of C*-correspondences.

Definition 2.5.3 (Morphisms of bicategories)**.** Let $\mathfrak{S}$ and $\mathfrak{S}'$ be bicategories. A morphism $\mathfrak{V} = (V, v)$ from $\mathfrak{S}$ to $\mathfrak{S}'$ consists of:

i) a map $V\colon \mathfrak{S}_0 \to \mathfrak{S}'_0$ sending an object A to $V(A)$;

ii) a family of functors $V(A,B)\colon \mathfrak{S}(A,B) \to \mathfrak{S}'(V(A),V(B))$ sending a 1-cell S to $V(A)$ and a 2-cell s to $V(s)$;

iii) for each object A of $\mathfrak{S}$, a 2-cell $v_A \in \mathfrak{S}(V(A),V(B))$

$$v_A\colon I_{V(A)} \Rightarrow V(I_A);$$

iv) a family of natural transformations

$$v(A,B,C)\colon c(V(A),V(B),V(C)) \circ (V(A,B) \times V(B,C)) \to V(A,C) \circ c(A,B,C).$$

If (S,T) is an object of $\mathfrak{S}(A,B) \times \mathfrak{S}'(B,C)$, the (S,T)-components of $v(A,B,C)$

$$v(A,B,C)(S,T)\colon V(S) \circ V(T) \Rightarrow V(S \circ T)$$

shall be abbreviated v or $v(S,T)$.

This data satisfies the following coherence conditions:

v) If (S,T,U) is an object of $\mathfrak{S}(A,B) \times \mathfrak{S}(B,C) \times \mathfrak{S}(C,D)$ the diagram in Figure 2.5 is commutative.

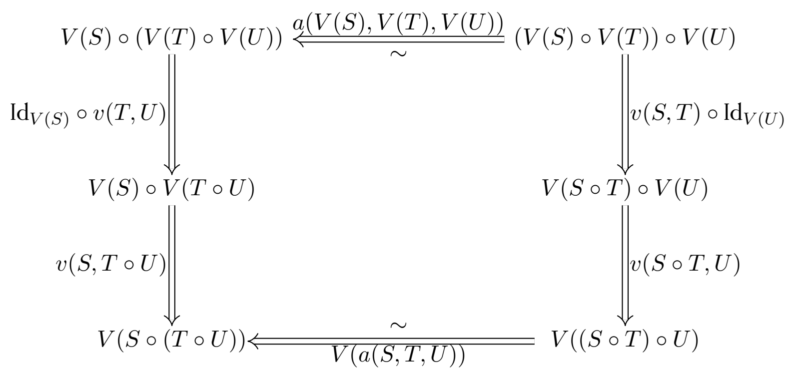


Figure 2.5: Associativity coherence for a transformation between bicategories

vi) If S is an object of $\mathfrak{S}(A,B)$ then the diagram in Figure 2.6, for the right identity commutes.

A similar diagram for the left identity commutes.

$$\begin{array}{ccc} V(S) & \overset{\sim}{\Longleftarrow} & V(S \circ I_B) \\ \Uparrow \sim & & \Uparrow \\ V(S) \circ \mathrm{I}_{V(B)} & \underset{\mathrm{Id} \circ \phi_B}{\Longrightarrow} & V(S) \circ V(\mathrm{I}_B)[swap]v(S, \mathrm{I}_B) \end{array}$$

Figure 2.6: Coherence of the right identity (and a similar diagram is drawn for the the left identity)

2.5.2 The bicategory of topological correspondences

In this subsection, we show that topological correspondences between groupoids endowed with Haar systems form a bicategory. To two groupoids equipped with Haar systems (G, α) and (H, β), we associate the category of topological correspondences. A morphism between topological correspondences is a *measure-preserving* equivariant homeomorphism. Then we show that sending a topological correspondence to a C*-correspondence is a homomorphism of bicategories. We start the discussion by explaining what it means if two *systems of measures* are equivalent.

Definition 2.5.4. Let $\pi\colon X \to Y$ be an open surjection and λ, λ' families of measures along π. We call λ and λ' equivalent if $\lambda^y \sim \lambda'^y$ for each $y \in Y$ and the Radon-Nikodym derivative $\mathrm{d}\lambda^y/\mathrm{d}\lambda'^y$ is continuous.

When λ and λ' are equivalent, we write $\lambda \sim \lambda'$. In fact, Definition 2.5.4 defines a *continuous* equivalence. Since we are not going to deal with the non-continuous case, we prefer to drop the adjective *continuous*.

Definition 2.5.5 (Isomorphism between correspondences)**.** Let (X, λ, Δ) and (X', λ', Δ') be two correspondences from (G, α) to (H, β). An isomorphism from (X, λ, Δ) to (X', λ', Δ') is the function $\phi : X \to X'$ such that:

i) ϕ is a G-H-equivariant homeomorphism;

ii) the families of measures λ' and $\lambda \circ \phi^{-1}$ on X' are equivalent, that is, $\lambda' \sim \lambda \circ \phi^{-1}$.

Remark 2.5.6. Let (X, B, μ) be a measure space. In [41, Definition 2.2], Sundar defines an automorphism of (X, B, μ) as a B-measurable function $T\colon X \to X$ such that there is another B-measurable function $T^{-1}\colon X \to X$ with $T \circ T^{-1} = T^{-1} \circ T = \mathrm{Id}_X$ μ-almost everywhere. in Definition 2.5.4, when G and H are trivial groups and $X = X'$, λ and λ' become Borel measures on X. When $\lambda = \lambda'$, a function ϕ is an automorphism of the Borel measure space X. Sundar shows that if $\mathcal{H}$ is a separable Hilbert space, then the automorphisms of (X, B, μ) form a group, and this group has a unitary representation on $\mathcal{L}^2(X, \mu; \mathcal{H})$ (see [41, Proposition 2.4]).

Lemma 2.5.7 (Chain rule)**.** *For $i = 1, 2, 3$, let $\pi_i : X_i \to Z$ be surjections and λ_i families of measures along π_i. Let $a_i : X_i \to X_{i+1}$ be two functions which make the following diagram commute:*

$$\begin{array}{ccccc} X_1 & \xrightarrow{a_1} & X_2 & \xrightarrow{a_2} & X_3 \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle \pi_2} & \swarrow {\scriptstyle \pi_3} & \\ & & Z & & \end{array}$$

If $\lambda_i \circ a_i^{-1}$ is equivalent to λ_{i+1} for $i = 1, 2$, then $\lambda_1 \circ a_1^{-1} \circ a_2^{-1}$ is equivalent to λ_3 and, for all $z \in Z$, and

$$\frac{\mathrm{d}(\lambda_1^z \circ a_1^{-1} \circ a_2^{-1})}{\mathrm{d}\lambda_3^z} = \left(\frac{\mathrm{d}(\lambda_1^z \circ a_1^{-1})}{\mathrm{d}\lambda_2^z} \circ a_2^{-1}\right) \frac{\mathrm{d}(\lambda_2^z \circ a_2^{-1})}{\mathrm{d}\lambda_3^z}.$$

Proof. This is a straightforward computation. Let f be a measurable function on X_3 and let $z \in Z$. Then

$$\begin{aligned}\int f \left(\frac{\mathrm{d}(\lambda_1^z \circ a_1^{-1})}{\mathrm{d}\lambda_2^z} \circ a_2^{-1}\right) \frac{\mathrm{d}(\lambda_2^z \circ a_2^{-1})}{\mathrm{d}\lambda_3^z} \,\mathrm{d}\lambda_3^z &= \int f \circ a_2 \left(\frac{\mathrm{d}(\lambda_1^z \circ a_1^{-1})}{\mathrm{d}\lambda_2^z}\right) \mathrm{d}\lambda_2^z \\ &= \int f \circ a_2 \circ a_1 \,\mathrm{d}\lambda_1^z \\ &= \int f \,\mathrm{d}(\lambda_1^z \circ a_1^{-1} \circ a_2^{-1}).\end{aligned}$$

□

Remark 2.5.8. Since ϕ is a homeomorphism, Condition (ii) in Definition 2.5.5 is equivalent to saying $\lambda \sim \lambda' \circ \phi$. To see this, apply the chain rule in Lemma 2.5.7 to $(X, \lambda) \xrightarrow{\phi} (X', \lambda') \xrightarrow{\phi^{-1}} (X, \lambda)$. This gives $\frac{\mathrm{d}(\lambda^u)}{\mathrm{d}(\lambda'^u \circ \phi)} = \frac{\mathrm{d}(\lambda'^u)}{\mathrm{d}(\lambda^u \circ \phi^{-1})} \circ \phi$ for all $u \in G^{(0)}$. Thus $\phi^{-1} \colon Y \to X$ is an isomorphism of correspondences.

When we composed correspondences we observed that the composite is defined up to a positive function b on Z. We show that given two composites, the corresponding correspondences are isomorphic.

To state and prove the following proposition, recall the terminology introduced in Subsection 2.4.2 on composition.

Proposition 2.5.9. *Let*

$$\begin{aligned}(X, \alpha) &\colon (G_1, \lambda_1) \to (G_2, \lambda_2) \\ (Y, \beta) &\colon (G_2, \lambda_2) \to (G_3, \lambda_3)\end{aligned}$$

be correspondences. Let

$$(\Omega, \mu), (\Omega, \mu') \colon (G_1, \lambda_1) \to (G_3, \lambda_3)$$

be two composites of these correspondences lifting bm and $b'm$ for 0-cochains b and b', respectively. Then (Ω, μ) and (Ω, μ') are isomorphic correspondences.

Proof. We use the same notation as for the composition of correspondences and let $Z := X *_{G_2^{(0)}} Y$ and let $\pi \colon Z \to \Omega$ be the projection map. Since $b, b' \in \mathrm{C}^0_{G_3}(Z, \mathbb{R}_+^*)$ with $d^0(b) = d^0(b') = \Delta$, Remark 1.4.9 gives a positive function $c \colon \Omega \to \mathbb{R}_+^*$ with $b' = (c \circ \pi) b$. Since π is open, the continuity of b, b' implies that c is continuous. Let $f \in C_c(\Omega)$, then

$$\begin{aligned}\int f[x, y] \,\mathrm{d}\mu'_u([x, y]) &= \int f \circ \pi(x, y) e(x, y)\, b'(x, y) \,\mathrm{d}\alpha_{r_Y(y)}(x) \,\mathrm{d}\beta_u(y) \\ &= \int f \circ \pi(x, y) e(x, y)\, c \circ \pi(x, y) b(x, y) \,\mathrm{d}\alpha_{r_Y(y)}(x) \,\mathrm{d}\beta_u(y) \\ &= \int f[x, y] c[x, y] \,\mathrm{d}\mu_u([x, y])\end{aligned}$$

Thus $\mu'_u \sim \mu_u$ with $\frac{d\mu'_u}{d\mu_u} = c$, where $c\colon \Omega \to \mathbb{R}^*_+$ is such that $b' = (c \circ \pi) \cdot b$. □

Given three correspondences (X_i, λ_i) for $i = 1, 2, 3$ from (G, α) to (H, β) and isomorphisms $\phi_i\colon X_i \to X_{i+1}$ for $i = 1, 2$, the composite $\phi_2 \circ \phi_1\colon X_1 \to X_3$ gives an isomorphism from (X_1, λ_1) to (X_3, λ_3). To see this, we need to check that $\lambda_1 \circ \phi_1^{-1} \circ \phi_2^{-1} \sim \lambda_3$. We first prove the following simple lemma and then show that $\lambda_1 \sim \lambda_3 \circ \phi_1^{-1} \circ \phi_2^{-1}$.

Lemma 2.5.10. *Let $f\colon X \to Y$ be an open surjection and let λ and λ' be measures on X. If $\lambda \sim \lambda'$ then $\lambda \circ f^{-1} \sim \lambda' \circ f^{-1}$.*

Proof. Let $U \subseteq Y$. Then

$$\begin{aligned} \lambda \circ f^{-1}(U) = 0 &\iff \lambda(f^{-1}(U)) = 0 \\ &\iff \lambda'(f^{-1}(U)) = 0 \text{ (because } \lambda \sim \lambda') \\ &\iff \lambda' \circ f^{-1}(U) = 0 \end{aligned}$$

□

We continue the discussion we started before Lemma 2.5.10. Since ϕ_1 is an isomorphism of correspondences, $\lambda_1 \circ \phi_1^{-1} \sim \lambda_2$. Fix $u \in H^{(0)}$ and use Lemma 2.5.10 fibrewise to see that $\lambda_{1u} \circ \phi_1^{-1} \circ \phi_2^{-1} \sim \lambda_{2u} \circ \phi_2^{-1}$. Since ϕ_2 is an isomorphism of correspondences, $\lambda_{2u} \circ \phi_2^{-1} \sim \lambda_{3u}$. The transitivity of equivalence of measures gives $\lambda_{1u} \circ \phi_1^{-1} \circ \phi_2^{-1} \sim \lambda_{3u}$.

Remark 2.5.11. Let (X, λ) and (X', λ') be correspondences from (G, α) to (H, β) and (Y, κ) and (Y, κ) be correspondence from (H, β) to (K, μ). If $\phi\colon X \to X'$ and $\phi'\colon Y \to Y'$ are isomorphisms of correspondences, then it can be checked that $\phi \otimes \phi'\colon X * Y/H \to X' * Y'/H$ is an isomorphism of correspondences, where $\phi \otimes \phi'([x, y]) = [\phi(x), \phi'(y')]$.

Remark 2.5.12. Let (G, α) and (H, β) be groupoids with Haar systems. It is easy to see that isomorphism of correspondences is an equivalence relation on the set of correspondences from (G, α) to (H, β). Let (X, λ_1), (Y, λ_2) and (Z, λ_3) be correspondences from G to H.

Reflexivity: the identity function from X to X gives reflexivity.

Symmetry: if ϕ is an isomorphism from (X, λ_1) to (Y, λ_2), then ϕ^{-1} is an isomorphism from (Y, λ_2) to (X, λ_1), see Remark 2.5.8.

Transitivity: Follows from the discussion just before this remark.

We form a bicategory of topological correspondences:

Objects or vertices: second countable, locally compact, Hausdorff groupoids with Haar system.

1-*arrows or edges:* topological correspondences with locally compact, Hausdorff, second countable topologies.

2-*arrows or* 2-*cells:* isomorphisms of topological correspondences (Definition 2.5.5).

Vertical composition of 2-arrows: vertical arrows are merely functions between spaces. Their composition is the usual composition of functions.

1-*identity arrow:* the identity 1-arrow on (G, α) is (G, α).

2*-identity arrow:* the identity 2-arrow on a correspondence (X, μ, Δ) is the identity map

$$\mathrm{Id}_X : X \to X.$$

Composition of 1-arrows: composition of correspondences as in Definition 2.4.18.

Horizontal composition of 2-arrows: with the data in Remark 2.5.11, we call $\phi \otimes \phi'$ the horizontal product of ϕ and ϕ'.

The associativity isomorphism: Proposition 2.5.13 below.

The identity isomorphism: described in Proposition 2.5.13 below.

We need to describe associators and left and right identities. We also need to verify that this data satisfies the coherence conditions.

Proposition 2.5.13. *There are (obvious) associativity and identity isomorphisms, which along with the above data form the bicategory $\mathfrak{T}$ of topological correspondences.*

Proof. We have the data required in i–iv in Definition 2.5.1. We define the associativity isomorphism and the identity isomorphism. Then we check the coherence conditions. We explain the notation used in the proof first.

In the proof we denote a groupoid with Haar system by (G_i, α_i) for $i = 1, 2, \dots, 5$. We assume that (X_i, λ_i) is a correspondences from (G_i, α_i) to (G_{i+1}, α_{i+1}) for $i = 1, 2, 3, 4$. We denote the composite $\left(X_i *_{G_{i+1}^{(0)}} X_{i+1}\right)/G_{i+1}$ by $X_i \circ X_{i+1}$. The 0-cochain on $X_i * X_{i+1}$ that appears in Proposition 2.4.4 will be denoted by $b_{i\,i+1}$. When there are too many X's, G's or b's we adopt the following notations, we write $X_{1((23)4)}$ for $X_1 \circ (X_2 \circ X_3) \circ X_4)$ and similarly for groupoids and 0-cochains. For example, $b_{1(23)}$ means the b-function in Corollary 2.4.4 for the space $X_{1(23)} = X_1 \circ (X_2 \circ X_3)$, and so on. Note that $b_{1(23)}$ is the product of b and b_{23} for $X_1 \circ X_{23}$. Since $i = 1, 2, 3, 4$, X_{12} means the composite of X_1 and X_2 and not the twelfth space. Since we do not have a two digit index, this notation does not cause any confusion.

v) *Associativity isomorphism*: Let (G_i, α_i) be four objects for $i = 1, 2, 3, 4$ and let $(X_i, \lambda_i, \Delta_{i,i+1})$ for $i = 1, 2, 3$ be correspondences from G_i to G_{i+1}. The spaces X_i for $i = 1, 2, 3$ are locally compact, Hausdorff and the action of the groupoid G_{i+1} on the space X_i is proper for $i = 1, 2$. Hence the induced action of G_{i+1} on $X_i * X_{i+1}$ is proper. Similarly, the obvious action of $G_i \times G_{i+1}$ on $X_{i-1} * X_i * X_{i+1}$ is proper. Define

$$a'(X_1, X_2, X_3) \colon (X_1 * X_2 * X_3)/(G_2 \times G_3) \to (X_1 \circ X_2) \circ X_3,$$

sending

$$[x_1, x_2, x_2] \mapsto [[x_1, x_2], x_3]$$

and

$$a''(X_1, X_2, X_3) \colon (X_1 * X_2 * X_3)/(G_2 \times G_3) \to X_1 \circ (X_2 \circ X_3),$$

sending

$$[x_1, x_2, x_2] \mapsto [x_1, [x_2, x_3]].$$

We claim that both a' and a'' are homeomorphisms. We prove that a' is a homeomorphism, and the claim for a'' can be proved similarly. First we check that a' is well-defined. Let $p\colon X_1 * X_2 * X_3 \to (X_1 \circ X_2) * X_3$ and $p'\colon (X_1 \circ X_2) * X_3 \to (X_1 \circ X_2) \circ X_3$ be the quotient maps. Then $p \circ p'$ is a well-defined continuous surjection. For $(x_1, x_2, x_3) \in X_1 * X_2 * X_3$ and appropriate $(\gamma_1, \gamma_2) \in G_1 \times G_2$,

$$\begin{aligned} p'(p(x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2, \gamma_2^{-1}x_3)) &= [[x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2], \gamma_2^{-1}x_3] \\ &= [[x_1\gamma_1, \gamma_1^{-1}x_2]\gamma_2, \gamma_2^{-1}x_3] = [[x_1, x_2]\gamma_2, \gamma_2^{-1}x_3] = [[x_1, x_2], x_3] = p'(p(x_1, x_2, x_3)). \end{aligned}$$

Hence by the universal property of the quotient, $p' \circ p$ induces a continuous map

$$(X_1 * X_2 * X_3)/(G_2 \times G_3) \to (X_1 \circ X_2) \circ X_3,$$

which is nothing but a'. Let $a'[x_1, x_2, x_3] = a'[y_1, y_2, y_3]$, that is, $[[x_1, x_2], x_3] = [[y_1, y_2], y_3]$. Then there is $\gamma_2 \in G_2$ with $([x_1, x_2\gamma_2], \gamma_2^{-1}x_3) = ([x_1, x_2]\gamma_2, \gamma_2^{-1}x_3) = ([y_1, y_2], y_3)$. This in turn gives $\gamma_1 \in G_1$ with $(x_1\gamma_1, \gamma_1^{-1}x_2\gamma_2, \gamma_2^{-1}x_3) = (y_1, y_2, y_3)$. Thus $[x_1, x_2, x_3] = [y_1, y_2, y_3] \in (X_1 * X_2 * X_3)/(G_2 \times G_3)$. Hence a' is a bijection.

Let $\pi\colon X_1 * X_2 * X_3 \to (X_1 * X_2 * X_3)/(G_2 \times G_3)$ be the projection map and let $U \subseteq (X_1 * X_2 * X_3)/(G_2 \times G_3)$ be open. Then $\pi^{-1}(U)$ is open. From Lemma 1.2.13 and Remark 1.2.15, we infer that p and p' are open maps. The universal property of the quotient implies $\pi(U) = p'(p(\pi^{-1}(U)))$, where the latter is an open set. Hence a' is an open map. Hence a' is a homeomorphism.

It is not hard to see that a' and a'' are G_1-G_4-invariant. Define

$$a(X_1, X_2, X_3) = a''(X_1, X_2, X_3) \circ a'^{-1}(X_1, X_2, X_3).$$

Then $a(X_1, X_2, X_3)$ sends $[[x_1, x_2], x_3]$ to $[x_1, [x_2, x_3]]$. Whenever the X_i are clear, we write a instead of $a(X_1, X_2, X_3)$. This a is the required associativity isomorphism. We need to show that $a(X_1, X_2, X_3)$ satisfies (ii) of Definition 2.5.5 to conclude that it is an isomorphism of correspondences. The proof is below.

This is a pretty long computation and we recall what we need. For $i = 1, 2, 3$,

i) $(X_i, \lambda_i.\Delta_i)$ is a correspondence from (G_i, α_i) to (G_{i+1}, α_{i+1});
ii) $X_{i(i+1)}$ denotes the quotient $X_i * X_{(i+1)}/G_{i+1}$ for $i = 1, 2$;
iii) $(X_{12} \circ X_3, \mu_{(12)3})$ and $(X_1 \circ X_{23}, \mu_{1(23)})$ are given composites;
    a) $b_{(12)3}$ and $b_{1(23)}$ are cochains in $C^0(X_{12} * X_3, \mathbb{R}_+^*)$ and $C^0(X_1 * X_{23}, \mathbb{R}_+^*)$, respectively, which give $\mu_{(12)3}$ and $\mu_{(12)3}$, respectively (then we have $d^0(b_{(12)3}) = \Delta_3$ in a suitable sense, and similarly for $b_{1(23)}$);
iv) $(X_{i(i+1)}, \mu_{i(i+1)})$ denotes a composite of (X_i, λ_i) and (X_{i+1}, λ_{i+1}) used to get the given composites for $i = 1, 2$.
    a) $b_{i(i+1)}$ is the cochain in $C^0_{G_{i+1}}(X_i * X_{i+1}, \mathbb{R}_+^*)$ that gives $\mu_{i(i+1)}$ (hence $d^0(b_{i(i+1)}) = \Delta_{i+1}$ in a suitable sense);

b) $e_{i(i+1)}$ is an e-function as in Proposition 2.4.3 for the quotient map $X_i{*}X_{i+1} \to X_{i(i+1)}$ for $i = 1, 2$.

Note that (i), (iii), (iii)a, (iv) and (iv)a is given data, (iv)b is derived information from (iv), and (ii) is a notation. Proposition 2.4.3 says that a composite does not depend on the choice of the e-function.

Our first observation is that

$$e_{(12)3}([x,y],z) := \int_{G_2} e_{12}(x\gamma, \gamma^{-1}y)e_{23}(\gamma^{-1}y,z)\,\mathrm{d}\alpha_2^{r_{X_2}(y)}(\gamma),$$
$$e_{1(23)}(x,[y,z]) := \int_{G_3} e_{12}(x, y\eta)\, e_{23}(y\eta, \eta^{-1}z)\,\mathrm{d}\alpha_3^{s_{X_2}(y)}(\eta)$$

are e-functions for the quotient maps $X_{12} * X_3 \to X_{12} \circ X_3$ and $X_1 * X_{23} \to X_1 \circ X_{23}$, respectively.

Remark 2.4.12 shows that b_{23} is G_2-invariant, that is, $b_{23}(\gamma^{-1}y, z) = b_{23}(y, z)$. Thus $([x, y], z) \mapsto b_{23}(y, z)$ is a well-defined function $X_{12} * X_3 \to \mathbb{R}_+^*$. By abuse of notation, we say that b_{23} is a cochain in $\mathrm{C}^0_{G_4}(X_{12} * X_3, \mathbb{R}_+^*)$. Clearly, $d^0(b_{23}) = \Delta_3$. Let $\mu'_{(12)3}$ be the family of measures induced by b_{23} using the function $e_{(12)3}$ above on $X_{12} \circ X_3$. Then $(X_{12} \circ X_3, \mu'_{(12)3})$ is a composite of (X_{12}, μ_{12}) and (X_3, λ_3).

Since $d^0(b_{23}) = d^0(b_{(12)3}) = \Delta_3$, Remark 1.4.9 gives $c\colon X_{12} \circ X_3 \to \mathbb{R}_+^*$ with $b_{(12)3} = (c \circ \pi) \cdot b_{23}$. Here $\pi\colon X_{12} * X_3 \to X_{12} \circ X_3$ is the projection map. Now Proposition 2.5.9 says that $(X_{12} \circ X_3, \mu_{(12)3})$ and $(X_{12} \circ X_3, \mu'_{(12)3})$ are isomorphic correspondences with $\frac{\mu^u_{(12)3}}{\mu'^u_{(12)3}} = c$ for $u \in G_4^{(0)}$.

Use a similar notation and argument to get the composite $(X_1 \circ X_{23}, \mu'_{1(23)})$, where the family of measures $\mu'_{1(23)}$ on $X_1 \circ X_{23}$ is induced by the cochain b_{12} using $e_{1(23)}$. Then $(X_1 \circ X_{23}, \mu'_{1(23)})$ is isomorphic to $(X_1 \circ X_{23}, \mu_{1(23)})$. And $\frac{\mu'^u_{1(23)}}{\mu^u_{1(23)}} = c'$ for $c'\colon X_1 \circ X_{23} \to \mathbb{R}_+^*$ with $b_{12} = (c' \circ \pi') \cdot b_{1(23)}$ and $u \in G_4^{(0)}$. Here $\pi'\colon X_{12} * X_3 \to X_{12} \circ X_3$ is the projection map.

Before we go to the main computations, we introduce some more notation. Without this notation the computation would be very complicated and long.

i) Let $f \in C_c(X_2 \circ X_3)$ and $u \in G_4^{(0)}$, then $\lambda_2 \times \lambda_3(f)(u) := \iint f(y,z)\,\mathrm{d}\lambda_2^{r_{X_3}(z)}(y)\mathrm{d}\lambda_3^u(z)$. Define $\lambda_1 \times \lambda_2$, $\lambda_1 \times \mu_{23}$ and other possible combinations and triple integrals similarly.
ii) For $i = 1, 2$, along the projection map $X_i * X_{i+1} \to X_i \circ X_{i+1}$ there is a family of measures $\alpha_{i+1_{X_i * X_{i+1}}}$ as in Equation 2.4.2. We write $\dot{\alpha}_{i+1}$ for $\alpha_{i+1_{X_i * X_{i+1}}}$. For $f \in C_c(X_i * X_{i+1})$ and $[a,b] \in X_i \circ X_{i+1}$ define $\bar{\alpha}_{i+1}(f)[a,b] = \int f\,\mathrm{d}\dot{\alpha}_{i+1}^{[a,b]}$.

Indeed, we keep in mind that all λ_i and α_k are families of measures and not a single measure. We have to keep track of the fibres in the computations, which is not obvious in the above notation. This notation reduces the complexity and size of the actual computations, but also introduces a bit of naiveness.

From Lemma 2.4.9 we know that $b_{i(i+1)}$ implements the symmetry between $(\lambda_i \times \lambda_{i+1}) \circ (\dot{\alpha}_{i+1})_1$ and $(\lambda_i \times \lambda_{i+1}) \circ (\dot{\alpha}_{i+1})_2$ for $i = 1, 2$. Observation 2.4.2 gives $\lambda_i \times \lambda_{i+1}(b_{i(i+1)} f \bar{\alpha}_{i+1}(g)) = \lambda_i \times \lambda_{i+1}(b_{i(i+1)} \bar{\alpha}_{i+1}(f) g)$ for $f \in C_c(X_i)$ and $g \in C_c(X_{i+1})$.

$$\begin{aligned}\mu_{1(23)}(f)(u) &= (\lambda_1 \times \mu_{23})(f\, b_{1(23)}\, e_{1(23)})(u)\\ &= \lambda_1 \times (\lambda_2 \times \lambda_3)(f\, b_{23} b_{1(23)}\; e_{23} e_{1(23)})(u)\\ &= \lambda_1 \times \Big((\lambda_2 \times \lambda_3)(f\, b_{23} b_{1(23)}\; e_{23}\, \bar{\alpha}_3(e_{12} e_{23}))\Big)\,(u)\\ &= \lambda_1 \times \Big((\lambda_2 \times \lambda_3)(b_{23} \bar{\alpha}_3(f\, b_{1(23)}\; e_{23})\, e_{12} e_{23})\Big)\,(u).\end{aligned}$$

The previous line is due to the symmetry of the measures $b_{23}\lambda_2 \times \lambda_3$ with respect to α_{23}. Observe that f and $b_{1(23)}$ are G_3-invariant. Hence

$$\begin{aligned}\text{R.H.S.} &= \lambda_1 \times \Big((\lambda_2 \times \lambda_3)(b_{23} \bar{\alpha}_3(f\, b_{1(23)}\; e_{23})\, e_{12} e_{23})\Big)\,(u)\\ &= \lambda_1 \times \Big((\lambda_2 \times \lambda_3)(f\, b_{1(23)}\, b_{23}\, \bar{\alpha}_3(e_{23})\, e_{12} e_{23}))\Big)\,(u).\end{aligned}$$

Now we use $\bar{\alpha}_3(e_{23}) = 1(u)$ and also introduce the identity homeomorphism a in the further computations. Hence the previous term equals

$$\begin{aligned}&\lambda_1 \times \Big((\lambda_2 \times \lambda_3)(f\, b_{1(23)}\, b_{23}\, e_{12} e_{23})\Big)\\ &= (\lambda_1 \times \lambda_2) \times \lambda_3 \left(\frac{f \circ a^{-1} \cdot (c' \circ a^{-1}) \circ \pi}{c \circ \pi}\; b_{12}\, b_{(12)3}\, e_{12} e_{23}\right)(u).\end{aligned}$$

Now we introduce e_{12} using the relation $\bar{\alpha}_2(e_{12}) = 1$. Then in later steps we use the symmetry of $b_{12}\lambda_1 \times \lambda_2$ with respect to α_2. Hence

$$\begin{aligned}\text{R.H.S.} &= (\lambda_1 \times \lambda_2) \times \lambda_3 \left(\frac{(f \circ a^{-1}) \cdot (c' \circ a^{-1}) \circ \pi}{c \circ \pi}\; b_{12}\, \bar{\alpha}_2(e_{12})\, b_{(12)3}\, b_{23}\, e_{12} e_{23}\right)(u)\\ &= (\lambda_1 \times \lambda_2) \times \lambda_3 \left(\frac{(f \circ a^{-1})\, c' \circ a^{-1} \circ \pi}{c \circ \pi}\; b_{12}\, e_{12}\, \bar{\alpha}_2(b_{(12)3}\, e_{12}\, e_{23})\right)(u)\\ &= (\lambda_1 \times \lambda_2) \times \lambda_3 \left(\frac{f \circ a^{-1} \cdot (c' \circ a^{-1}) \circ \pi}{c \circ \pi}\; b_{12}\, e_{12}\, b_{(12)3}\, \bar{\alpha}_2(e_{12}\, e_{23})\right)(u).\end{aligned}$$

The last step is due to the G_2-invariance of $b_{(12)3}$. Now apply Fubini's Theorem and compute further:

$$\begin{aligned}
&= \lambda_3 \times \left((\lambda_1 \times \lambda_2) \left(\left(\frac{f \circ a^{-1} \cdot (c' \circ a^{-1}) \circ \pi}{c \circ \pi} b_{(12)3}\, e_{(12)3} \right) b_{12}\, e_{12} \right) \right) (u) \\
&= \lambda_3 \times \mu_{12} \left(\frac{f \circ a^{-1} \cdot (c' \circ a^{-1}) \circ \pi}{c \circ \pi} b_{(12)3}\, e_{(12)3} \right) (u) \\
&= \mu_{(12)3} \left(f \circ a^{-1} \cdot \frac{(c' \circ a^{-1}) \circ \pi}{c \circ \pi} \right) (u).
\end{aligned}$$

Thus $\mu^u_{1(23)} \sim \mu^u_{(12)3}$ for all $u \in G_4^{(0)}$ with the Radon-Nikodym derivative $\frac{\mathrm{d}\mu^u_{1(23)}}{\mathrm{d}\mu^u_{(12)3} \circ a^{-1}} = \frac{(c' \circ a^{-1})}{c} \circ \pi$.

vi) *Identity isomorphisms*: Let (G_i, α_i) be groupoids for $i = 1, 2$ and let (X, λ) be a correspondence from G_1 to G_2. There is a homeomorphism

$$l(G_1, G_2) \colon (G_1 * X)/G_1 \to X, \quad [\gamma, x] \mapsto \gamma^{-1}x.$$

The inverse of this homeomorphism $l(G_1, G_2)$ sends $x \mapsto [r_X(x), x]$. Then $l(G_1, G_2)$ is the left identity coherence arrow. We need to check that $(G_1 * X)/G_1$ is equipped with the family of measure Λ.

The fibre product $G_1 * X$ carries the family of measures $\{\alpha_1 \circ \lambda_u\}_{u \in G_2{}^{(0)}}$ which we denoted by $\{\mu_u\}_{u \in G_2{}^{(0)}}$ in Proposition 2.4.14. The map $G_1 * X \to (G_1 * X)/G_1 \approx X$ carries a family of measures $\alpha_1^{-1 G_1 * X}$ induced by α_1^{-1} which is defined by $\alpha_1^{-1 G_1 * X}(f)[r_X(x), x] = \int f(\gamma\eta, \eta^{-1}x)\, \mathrm{d}(\alpha_1^{-1})_{r_X(x)}(\eta)$ for $f \in C_c(G_1 * X)$. In Proposition 2.4.14, we denoted $\alpha_1^{-1 G_1 * X}$ by λ. Using the right invariance of α_1^{-1} it can be checked that

$$\mu(f) = \lambda(\alpha_1^{-1 G_1 * X}(f)) \tag{2.5.14}$$

for $f \in C_c(G_1 * X)$. Now we may draw a diagram similar to the one in Figure 2.2 and use Equation 2.5.14 to see that the families of measures $(\alpha_1^{-1 G_1 * X})_1$ and $(\alpha_1^{-1 G_1 * X})_2$ in this new diagram are weakly symmetric with respect to the measure μ_u for each $u \in G_2$ and then Proposition 2.4.3 along with the definition of composite of topological correspondences (Definition 2.4.18) gives that λ is the family of measures on the composite $(G_1 * X)/G_1$.

Similarly, the map $r(G_1, G_2) \colon (X * G_2)/G_2 \to X$ sending $[x, \gamma] \mapsto x\gamma$ is the right identity coherence.

vii) *Horizontal composition of 2-arrows*: Let $(X_i, \lambda_i), (X_i', \lambda_i')$ be correspondences from (G_i, α_i) to (G_{i+1}, α_{i+1}) for $i = 1, 2$ and let $\phi_i \colon X_i \to X_i'$ be isomorphisms of correspondences. Let $(X_1 \circ X_2, \mu)$ and $(X_1' \circ X_2', \mu')$ be the composites. And assume that b and b' are the cochains which produce μ and μ', respectively.

Since ϕ_i is a G_i-G_{i+1}-equivariant homeomorphism for $i = 1, 2$, ϕ_1 and ϕ_2 induce a G_1-G_3-equivariant homeomorphism $\phi_1 \cdot_h \phi_2 \colon X_1 \circ X_2 \to X_1' \circ X_2'$. We claim that $\phi_1 \cdot_h \phi_2$ is the horizontal product of ϕ_1 and ϕ_2. To prove the claim, we need to check $\mu_u \circ (\phi_1 \cdot_h \phi_2)^{-1} \sim \mu'_u$ for each $u \in G_3^{(0)}$. Before we proceed, note that $\phi_1 \cdot_h \phi_2$ is induced by $\phi_1 \times \phi_2 \colon X_1 * X_2 \to X'_1 * X'_2$. As ϕ_1 and ϕ_2 are isomorphisms of correspondences, $(\lambda_1 \times \lambda_2)_u \circ (\phi_1 \times \phi_2)^{-1} \sim (\lambda'_1 \times \lambda'_2)_u$ on $X'_1 * X'_2$, for all $u \in G_3^{(0)}$. But then $(b \circ (\phi_1 \times \phi_2)^{-1}) \cdot ((\lambda_1 \times \lambda_2)_u \circ (\phi_1 \times \phi_2)^{-1}) = (b\lambda_1 \times \lambda_2)_u \circ (\phi_1 \times \phi_2)^{-1} \sim b'(\lambda'_1 \times \lambda'_2)_u$ on $X'_1 * X'_2$. But $b \circ (\phi_1 \times \phi_2)^{-1} \in \mathrm{C}^0_{G_3}(X'_1 * X'_2, \mathbb{R}^*_+)$. As in the proof of Proposition 2.5.9, we get $b \circ (\phi_1 \times \phi_2)^{-1} \cdot (\lambda_1 \times \lambda_2)_u \circ (\phi_1 \times \phi_2)^{-1} \sim b'(\lambda'_1 \times \lambda'_2)_u$ on $X'_1 * X'_2$. Now use Proposition 2.5.9 to see that $\mu_u \circ (\phi_1 \cdot_h \phi_2)^{-1} \sim \mu'_u$ on $X'_1 \circ X'_2$ for each $u \in G_3^{(0)}$.

viii) *Associativity coherence*: Let (G_i, α_i) be groupoids equipped with Haar systems for $i = 1, \dots, 5$ and let (X_i, λ_i) be a correspondence from G_i to G_{i+1} for $i = 1, \dots, 4$.

The associativity coherence says that the pentagon in Figure 2.7 commutes:

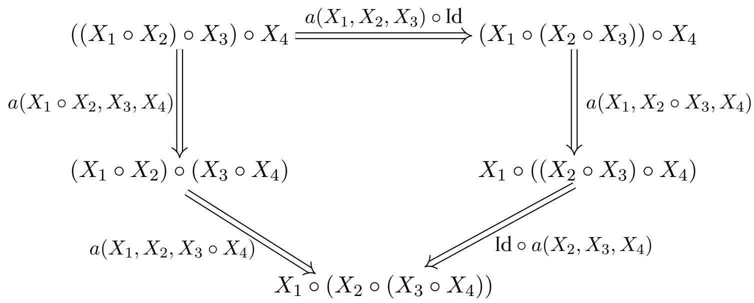
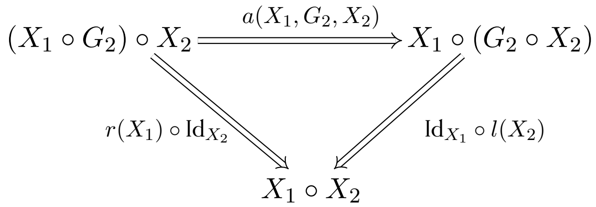


Figure 2.7: Associativity coherence

Let $x_{((12)3)4}$ be a point in $X_{((12)3)4} := (X_1 \circ X_2) \circ X_3) \circ X_4$. Following the left top vertex of the pentagon along the right top sides till the vertex at the bottom, an element $x_{((12)3)4} := [[[x_1, x_2], x_3]x_4]$ goes to $x_{1(2(34))} := [x_1, [x_2, [x_3, x_4]]]$.

The lower left path between the same vertices gives the same map.

ix) *Identity coherence*: Let (X_i, λ_i) be topological correspondences from (G_i, α_i) to (G_{i+1}, α_{i+1}) for $i = 1, 2$. We need to show that the following diagram is commutative:



For $[[x_1, \gamma], x_2] \in (X_1 \circ G_2) \circ X_2$

$\mathrm{Id}_{X_1} \circ l(X_2)(a(X_1, G_2, X_2)([[x_1, \gamma], x_2])) = [x_1, \gamma x_2] = [x_1\gamma, s_{G_2}(\gamma x_2)] = r(X_1) \circ \mathrm{Id}_{X_2}([[x_1, \gamma], x_2])$.

This proves all the axioms. $\square$

Convention: Let $(X, \lambda, \Delta_X), (Y, \tau, \Delta_Y)$ be correspondences from (G, α) to (H, β) and let $\mathrm{t}\colon X \to Y$ be an isomorphism between the correspondences. Let $\frac{\mathrm{d}(\lambda_u \circ \mathrm{t}^{-1})}{\mathrm{d}\tau_u} = M_u$, $u \in H^{(0)}$. Write $M(y) = M_{s_Y(y)}(y)$ for $s_Y(y) \in H^{(0)}$.

Lemma 2.5.15. *Let $(X, \lambda, \Delta_X), (Y, \tau, \Delta_Y)$ be correspondences from (G, α) to (H, β) and let $\mathrm{t}\colon X \to Y$ be an isomorphism between the correspondences.*

i) M is H-invariant, that is, $M(y\eta) = M(y)$ for all $(y, \eta) \in Y * H$.

ii) $\Delta_X(\gamma, x) = (M \circ \mathrm{t})(\gamma x)\, \Delta_Y \circ (\mathrm{Id} \circ \mathrm{t})(\gamma, x)\, (M \circ \mathrm{t})(x)^{-1}$.

Proof. (i): Use the invariance of the families of measures λ and τ:

$$M(y\eta) = \frac{\mathrm{d}(\lambda_{s_H(\eta)} \circ \mathrm{t}^{-1})}{\mathrm{d}\tau_{s_H(\eta)}}(y\eta) = \frac{\mathrm{d}(\lambda_{r_H(\eta)} \circ \mathrm{t}^{-1})}{\mathrm{d}\tau_{r_H(\eta)}}(y) = M(y).$$

(ii): t induces an obvious homeomorphism $\mathrm{Id} * \mathrm{t}\colon G * X \to G * Y$. For $f \in C_c(G \ltimes X)$ and $u \in H^{(0)}$ we have

$$\begin{aligned}
&\int f(\gamma^{-1}, x)\, \mathrm{d}\alpha^{r_X(x)}(\gamma)\, \mathrm{d}\lambda_u(x)\\
&= \int f(\gamma^{-1}, t^{-1}(y))\, \mathrm{d}\alpha^{r_Y(y)=r_X(x)}(\gamma)\, \mathrm{d}\lambda_u(\mathrm{t}^{-1}(y))\\
&= \int (f \circ (\mathrm{Id} \circ t^{-1}))(\gamma^{-1}, y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}(\lambda_u \circ \mathrm{t}^{-1})(y))\\
&= \int (f \circ (\mathrm{Id} \circ t^{-1}))(\gamma^{-1}, y)\, M(y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\tau_u(y)\\
&= \int (f \circ (\mathrm{Id} \circ t^{-1}))(\gamma, \gamma^{-1}y)\, \Delta_Y(\gamma, \gamma^{-1}y)\, M(y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \mathrm{d}\tau_u(y)\\
&= \int (f \circ (\mathrm{Id} \circ t^{-1}))(\gamma, \gamma^{-1}y)\, \Delta_Y(\gamma, \gamma^{-1}y)\, M(y) M(\gamma^{-1}y)^{-1}\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\, \lambda_u \circ \mathrm{t}^{-1}(y)\\
&= \int f(\gamma, \gamma^{-1}x)\, \Delta_Y(\gamma, \gamma^{-1}t(x))\, M(t(x)) M(t(\gamma^{-1}x))^{-1}\, \mathrm{d}\alpha^{r_X(x)}(\gamma)\, \lambda_u(x).
\end{aligned}$$

Thus $\Delta_X(\gamma, x) = (M \circ \mathrm{t})(\gamma^{-1}x)\, \Delta_Y \circ (\mathrm{Id} \times \mathrm{t})(\gamma, x)\, (M \circ \mathrm{t})(x)^{-1}$ $\lambda_u \circ \alpha$-almost everywhere on $G * X_u$. But Δ_X, Δ_Y and M are continuous functions, and λ_u as well as all measures α^v for $v \in G^{(0)}$ are regular. Hence $\Delta_X(\gamma, x) = (M \circ \mathrm{t})(\gamma^{-1}x)\, \Delta_Y \circ (\mathrm{Id} \times \mathrm{t})(\gamma, x)\, (M \circ \mathrm{t})(x)^{-1}$. □

Proposition 2.5.16. *Let $(X, \lambda, \Delta_X), (Y, \tau, \Delta_Y)$ be correspondences of groupoids from (G, α) to (H, β) and let $\mathrm{t}\colon X \to Y$ be an isomorphism between the correspondences. Then t induces an isomorphism from $\mathcal{H}(X)$ to $\mathcal{H}(Y)$.*

Proof. Denote $\mathrm{d}(\lambda_u \circ \mathrm{t}^{-1})/\mathrm{d}\tau_u = M_u$ and let M be as above. Define $\mathrm{T}\colon C_c(X) \to C_c(Y)$ by $\mathrm{T}(f) = (f \circ \mathrm{t}^{-1}) \cdot M^{1/2}$ for $f \in C_c(X)$.

Right side: We first prove that T extends to a unitary operator $\mathcal{H}(X) \to \mathcal{H}(Y)$. Let $\psi \in C_c(H)$ and $f, g \in C_c(X)$. Clearly $\mathrm{T}(f+g) = \mathrm{T}(f) + \mathrm{T}(g)$. Furthermore,

$$\begin{aligned}\mathrm{T}(f\psi)(y) &= (f\psi)(\mathrm{t}^{-1}(y))\, M^{1/2}(y)\\ &= \int f(\mathrm{t}^{-1}(y)\eta)\psi(\eta^{-1})\, M^{1/2}(y)\, \mathrm{d}\beta^{s_X(x)}(\eta)\\ &= \int f(\mathrm{t}^{-1}(y)\eta)\psi(\eta^{-1})\, M^{1/2}(y\eta)\, \mathrm{d}\beta^{s_X(x)}(\eta)\\ &= \int \mathrm{T}(f)(y\eta)\psi(\eta^{-1})\, \beta^{s_Y(y)}(\eta).\end{aligned}$$

In the third equality above, we used the H-invariance of M, which is proved in Lemma 2.5.15. Thus T is $C_c(H)$-linear.

Define $\mathrm{T}^*\colon C_c(Y) \to C_c(X)$ by $\mathrm{T}^*(g) = (g \circ \mathrm{t})\bar{M}^{1/2}$ for $g \in C_c(Y)$ where $\bar{M} = \mathrm{d}(\tau_u \circ \mathrm{t})/\mathrm{d}\lambda_u$. Then a routine computation shows that T^* is the adjoint of T; half of the computations are written below, the other half are similar. An argument similar to the one in Lemma 2.3.1 proves the continuity of T and T^*.

Let $f \in C_c(X)$, then

$$\begin{aligned}\mathrm{T}^*(\mathrm{T}(f))(x) &= \mathrm{T}(f) \circ t(x) \cdot \bar{M}^{1/2}(\mathrm{t}(x))\\ &= f(x) \cdot \bar{M}^{1/2}(t(x)) \cdot M^{1/2}(x) \quad \text{(now we use the chain-rule for } \mathrm{t}^{-1} \circ t = \mathrm{Id}_X)\\ &= f(x).\end{aligned}$$

Similarly, $\mathrm{T} \circ \mathrm{T}^* = \mathrm{Id}_{C_c(Y)}$.

Left side: Let $\pi_1\colon \mathrm{C}^*(G,\alpha) \to \mathbb{B}(\mathcal{H}(X))_{\mathrm{C}^*(H,\beta)}$ and $\pi_2\colon \mathrm{C}^*(G,\alpha) \to \mathbb{B}(\mathcal{H}(Y))_{\mathrm{C}^*(H,\beta)}$ denote the representations that gives the correspondences $\mathcal{H}(X)$ and $\mathcal{H}(Y)$. Now we show that T intertwines π_1 and π_2. It suffices to show that $\mathrm{T} \circ \pi_1(\psi)(f) = \pi_2 \circ \mathrm{T}(f)$ for $\psi \in C_c(G)$ and $f \in C_c(X)$.

$$\begin{aligned}&\pi_2(\psi)(\mathrm{T}(f))(y)\\ &= \int \psi(\gamma)\mathrm{T}(f)(\gamma^{-1}y)\, \Delta_Y^{1/2}(\gamma, \gamma^{-1}y)\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\\ &= \int \psi(\gamma) f \circ \mathrm{t}(\gamma^{-1}x)\, M^{1/2}(\gamma^{-1}\mathrm{t}(x))\, \Delta_Y(\gamma, \gamma^{-1}y)^{1/2}\, \mathrm{d}\alpha^{r_Y(y)}(\gamma)\\ &= \int \psi(\gamma) f \circ \mathrm{t}(\gamma^{-1}x)\, \Big(M^{1/2}(\gamma^{-1}\mathrm{t}(x))\, \Delta_Y(\gamma, \gamma^{-1}y)^{1/2} M^{-1/2}(\mathrm{t}(x))\Big)\, M^{1/2}(\mathrm{t}(x))\mathrm{d}\alpha^{r_Y(y)}(\gamma)\\ &= \int \psi(\gamma) f \circ \mathrm{t}(\gamma^{-1}x)\, \Delta_X(\gamma, \gamma^{-1}y)^{1/2}\mathrm{d}\alpha^{r_Y(y)}(\gamma) M^{1/2}(\mathrm{t}(x)) \quad \text{(using Lemma 2.5.15)}\\ &= (\pi_1(\psi) f \circ \mathrm{t}(x)) M^{1/2}(\mathrm{t}(x))\\ &= \mathrm{T}(\pi_1(\psi) f)(y).\end{aligned}$$

□

Example 2.5.17. We can explain Example 2.2.6 using isomorphisms of correspondences. Let (H, β) be a groupoid endowed with a Haar system. Then $\mathrm{C}^*(H, \beta)$ is the identity correspondence from $\mathrm{C}^*(H, \beta)$ to itself. But remember, as a correspondence $\mathrm{C}^*(H, \beta)$ is a Hilbert module and not a

C*-algebra. Let $C_c(H,\beta)$ denote the continuous compactly supported functions on H. This is a *-algebra as in Proposition 1.6.9. We get the identity correspondence on the C*-algebra $\mathrm{C}^*(H,\beta)$ from this *-algebra. The involution is used to get the $\mathrm{C}^*(H,\beta)$-valued inner products. Let H be *the* identity equivalence from the groupoid with the Haar system (H,β) to itself and let $C_c(H)$ denote the space of continuous compactly supported functions on H. Equation (2.2.2) and Equation (2.2.3) define operations on $C_c(H)$. The main difference in $C_c(H,\beta)$ and $C_c(H)$ is that the operations on $C_c(H,\beta)$ use the *left* invariant Haar system β, whereas $C_c(H)$ uses the right invariant Haar system β^{-1}. Then the identity map $\mathrm{Id}_H\colon H\to H$ gives an isomorphism of correspondences. It is not hard to see that inv_H is an isomorphism of correspondences. Hence $\mathrm{C}^*(H,\beta)$ and $\mathcal{H}(H)$ are isomorphic correspondences from $\mathrm{C}^*(H,\beta)$ to $\mathrm{C}^*(H,\beta)$.

Corollary 2.5.18. *Let*

$$(X,\alpha)\colon(G_1,\lambda_1)\to(G_2,\lambda_2),$$
$$(Y,\beta)\colon(G_2,\lambda_2)\to(G_3,\lambda_3)$$

be correspondences and let $(\Omega,\mu),(\Omega,\mu')\colon(G_1,\lambda_1)\to(G_3,\lambda_3)$ *be two composites of them. Then* $\mathcal{H}(\Omega,\mu)$ *and* $\mathcal{H}(\Omega,\mu')$ *are isomorphic* C*-*correspondences.*

Proof. Follows directly from Proposition 2.5.9 and Proposition 2.5.16. □

Denote the bicategory of topological correspondences by $\mathfrak{T}$. The bicategory of C*-correspondences is denoted by $\mathfrak{C}$.

Now we prove that the assignment $X\mapsto\mathcal{H}(X)$ is functorial.

Theorem 2.5.19. *The assignment* $X\mapsto\mathcal{H}(X)$ *is a bifunctor from* $\mathfrak{T}$ *to* $\mathfrak{C}$.

Proof. Recall Definition 2.5.3. We define the bifunctor $\mathfrak{F}=(F,\phi)$ as the following assignment from $\mathfrak{T}$ to $\mathfrak{C}$:

Object: $F((G,\alpha))=\mathrm{C}^*(G,\alpha)$ (Data (i) in Definition 2.5.3)

1-arrow: map a 1-arrow (X,λ) from (G,α) to (H,β) to the arrow $F((X,\lambda))=\mathcal{H}(X,\lambda)$ in $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta))$. (Data (ii) in Definition 2.5.3)

2-arrow: map a 2-arrow t in $\mathfrak{T}((G,\alpha),(H,\beta))$ to the isomorphism of C*-correspondences $F(t)$ in $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta)$ as in Proposition 2.5.16. Note that F is a functor from $\mathfrak{T}((G,\alpha),(H,\beta))$ to $\mathfrak{C}(\mathrm{C}^*(G,\alpha),\mathrm{C}^*(H,\beta))$. (Data (ii) in Definition 2.5.3)

Identity 2-morphism: The isomorphism of C*-correspondences in Example 2.2.6 , $\phi_{\mathcal{H}(G)}\colon\mathrm{C}^*(G,\alpha)\to\mathcal{H}(G)$. More precisely, $\phi_{\mathcal{H}(G)}$ is the isomorphism induced by the identity map Id_H as in Example 2.5.17. (Data (iii) in Definition 2.5.3)

Natural transformation between composites: Let (X,λ) be a 1-arrow from (G,α) to (H,β) and let (Y,μ) be a 1-arrow from (H,β) to (K,ν). Then Λ' is the natural transformation

$$\phi((G,\alpha),(H,\beta),(K,\nu))\colon\mathcal{H}(X)\otimes_{\mathrm{C}^*(H,\beta)}\mathcal{H}(Y)\to\mathcal{H}(X\circ Y)$$

defined in Theorem 2.4.19.

Claim: the pair $(F, f) = \mathfrak{F}$ is a morphism from the bicategory $\mathfrak{T}$ to the bicategory $\mathfrak{C}$.

Checking that Figure 2.5 is commutative is a complicated but straightforward computation. In this diagram, the maps denoted by $v(S, T)$ are the maps Λ' defined in Theorem 2.4.19, which integrates with respect to the middle action.

We check coherence for the right identity. The coherence for the left identity can be checked similarly. Let (X, λ) be a correspondence from (G, α) to (H, β) and let (H, β^{-1}) be the identity correspondence from (H, β) to itself. Checking the coherence translates to checking the commutativity of Figure 2.8.

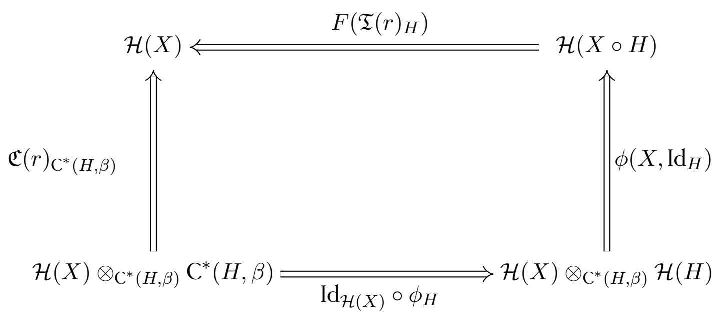


Figure 2.8: Coherence of the right identity

We first explain the maps in Figure 2.8.

Bottom: The map of C*-correspondences $\phi_{\mathcal{H}} \colon \mathrm{C}^*(H, \beta) \to \mathcal{H}(H)$ is induced by the identity map $\mathrm{Id}_H \colon H \to H$ (see Example 2.5.17).

Right: $\phi(X, \mathrm{Id}_H)$ is the map Λ' in Theorem 2.4.19. Λ' integrates over the middle action to go from the fibered product to the quotient.

Top: The map $F(\mathfrak{T}(r)_H)$ is induced by the quotient homeomorphism $[x, \eta] \mapsto x\eta$ inverse to $x \mapsto [x, s_X(x)]$.

Left: The algebraic map $\mathfrak{C}(r)_{\mathrm{C}^*(H,\beta)}$ comes from the tensor product of C*-correspondences. In this case, $\mathfrak{C}(r)_{\mathrm{C}^*(H,\beta)}(f, g)$ takes the convolution product of the element $f \in \mathcal{H}(X)$ and $g \in \mathrm{C}^*(H, \beta)$.

We show that the diagram commutes at the level of continuous compactly supported functions. Let $f \otimes g \in C_c(X) \otimes C_c(H, \beta)$. Then starting from the bottom of the diagram to the right top, the element travels as

$$f \otimes g \mapsto f \otimes (g \circ \mathrm{inv}_H) \mapsto \Lambda'(f \otimes (g \circ \mathrm{inv}_H)).$$

Denote $M = \Lambda(f \otimes (g \circ \mathrm{inv}_H)) \in C_c(X \circ H)$. Then

$$M[x, \eta] = \int f(x\gamma) g(\gamma^{-1}\eta) \, \mathrm{d}\beta^u(\gamma) = M[x\eta, s_H(\eta)].$$

Hence

$$F(\mathfrak{T}(r)_H)(M)(x) = M[x, s_X(x)] = \int f(x\gamma) g(\gamma^{-1}) \, \mathrm{d}\beta^{s_X(x)}(\gamma) = f * g(x) = \mathfrak{C}(r)_{\mathrm{C}^*(H,\beta)}(f \otimes g)(x).$$

□

Chapter 3

Applications of the theory

This chapter discusses examples and applications of the theory we have developed so far. First of all, we give a few examples of topological correspondences and also relate our work with previous definitions of correspondences between groupoids.

A C^*-algebraic correspondence from A to B induces a functor from the representation category of B to that of A. Analogously, a topological correspondence from G to H induces a functor between the representation categories of the groupoids.

While working with groupoid actions, we came across examples and questions which lead to the notion of a *spatial hypergroupoid*. We discuss it briefly here.

In [21] the authors define the Brauer group for locally compact groupoids and prove two isomorphism theorems concerned with it. Given two groupoids, we construct correspondences between groupoids which induce one-way homomorphisms between the Brauer groups of the groupoids.

As the last application, we establish a *tiny* link between our theory of correspondences and KK-theory. Given a groupoid G and some more data, we produce a KK-cycle between certain subgroupoids of G.

3.1 General examples

Example 3.1.1. Let X and Y be spaces, and let $f\colon X \to Y$ be a continuous function. We view X and Y as groupoids with Haar systems consisting of Dirac measures on X and Y $\delta_X = \{\delta_x\}_{x\in X}$ and $\delta_Y = \{\delta_y\}_{y\in Y}$, respectively, as in Example 1.3.7. We write X' for the *space* X. We use this notation to avoid confusing the space and the groupoid structures.

The function f is the momentum map for the trivial left action of Y on X', that is, $Y * X = Y \times_{\mathrm{Id}_Y, f} X$ and $f(x)\cdot x = x$ for all $x \in X$ (in fact, this is the only possible action). There is an obvious *proper* right action of X on X', namely, the trivial action. The momentum map for this action is $s_{X'} = \mathrm{Id}_X$, the fibre product $X' * X = \{(x, x) : x \in X\}$ and the action is $x \cdot x = x$ for all $x \in X$. The family of Dirac measures δ_X mentioned above is an X-invariant family of measures on

X'. If $h \in C_c(Y * X')$, then

$$\iint h(y,x)\,\mathrm{d}(\delta_Y)^b\,\mathrm{d}(\delta_X)^a = h(f(a),a) = \iint h(y^{-1},yx)\,\mathrm{d}(\delta_Y)^b\,\mathrm{d}(\delta_X)^a.$$

Therefore δ_X is Y-invariant. Thus (X',δ_X) is a topological correspondence from Y to X with the constant function 1 as the adjoining function. The action of $C_c(X)$ on $C_c(X')$ as well as the $C_c(X)$-valued inner product on $C_c(X')$ are the pointwise multiplication of two functions. For $h \in C_c(Y)$, $k \in C_c(X')$, $(h \cdot k)(x) = h(f(x))k(x)$.

Let $g\colon Y \to Z$ be another map. Then it is not hard to see that the composite $(Y',\delta_Y)\circ(X',\delta_X)$ from X to Z is isomorphic to the correspondence obtained from the map $g \circ f\colon X \to Z$.

Example 3.1.2. Let X, Y, X' and f be as in Example 3.1.1. Let $\lambda = \{\lambda_y\}_{y\in Y}$ be a continuous family of measures along f. We make X' into a proper X-Y-bispace as follows: the momentum maps are $r_{X'} = \mathrm{Id}_X$, $s_{X'} = f$ and both actions are the trivial actions as in Example 3.1.1 above. For $h \in C_c(X * X')$,

$$\iint h(x^{-1},xz)\,\mathrm{d}(\delta_X)^x(z)\,\mathrm{d}\lambda_y(x) = \iint h(x,z)\,\mathrm{d}(\delta_X)^x(z)\,\mathrm{d}\lambda_y(x) = \int h(f(x),x)\,\mathrm{d}\lambda_y(x).$$

The first equality above is due to the triviality of the action. Hence λ is X-quasi-invariant and the modular function is the constant function 1. Thus (X',λ) is a correspondence from X to Y.

Example 3.1.3. Let X, Y, X' be as in Example 3.1.1. Let $f, g\colon X \to Y$ be continuous maps and let λ be a continuous family of measures along f. For $g\colon X \to Y$ in Example 3.1.1 define a left action of $C_c(Y)$ on $C_c(X)$. For $f\colon X \to Y$ use the family of measures λ and the formulae in Example 3.1.2 to define a right action of $C_c(Y)$ on $C_c(X)$. It is straightforward to check that (X',λ) is a topological correspondence from Y to itself. When the spaces are second countable, the quintuple (Y, X, s, r, λ) is called a *topological quiver* [29].

The reader may check that (X',λ) is the product of $(X',\delta_X)\colon Y \to X$ and $(X',\lambda)\colon X \to Y$; these are the correspondences in the previous two examples.

Remark 3.1.4. In [29], Muhly and Tomforde discuss topological quivers. We have talked about this paper in the introduction. Topological quivers justify our use of families of measures in our construction. At a first glance, the families of measures and their quasi-invariance for the left action might look artificial. However, as discussed on page 41, the quasi-invariance of families of measures is natural to ask for. And topological quivers justify the use of families of measures.

Example 3.1.5. Let G and H be locally compact groups, $\phi\colon H \to G$ a (continuous) group homomorphism and α and β the Haar measures on G and H, respectively. The right multiplication action is a proper action of G on itself. The measure α^{-1} is invariant under this action. Using ϕ define an action of H on G as $\eta\gamma = \phi(\eta)\gamma$ for $(\eta,\gamma) \in H \times G$. We claim that α^{-1} is H-quasi-invariant for this H-action. Let δ_G and δ_H be the modular functions of G and H, respectively. The modular functions allow to switch between the left invariant Haar measures α and β and the right invariant ones α^{-1} and β^{-1}, respectively. The relations are $\alpha^{-1} = \delta_G^{-1}\alpha$ and $\beta^{-1} = \delta_H^{-1}\beta$. If $R_\gamma\colon G \to G$ is the right multiplication operator, then

$$\int_G R_\gamma f\,\mathrm{d}\alpha = \delta_G(\gamma)^{-1}\int_G f\,\mathrm{d}\alpha$$

for $f \in C_c(G)$ and a similar reasoning holds for $g \in C_c(H)$. Now let $f \in C_c(G \times H)$,

$$\begin{aligned}
&\iint f(\eta, \phi(\eta)^{-1}\gamma)\, \frac{\delta_H(\eta)}{\delta_G(\phi(\eta))}\, \mathrm{d}\beta(\eta)\, \mathrm{d}\alpha^{-1}(\gamma) \\
&= \iint f(\eta^{-1}, \phi(\eta)\gamma)\, \frac{1}{\delta_G(\phi(\eta))}\, \mathrm{d}\beta(\eta)\, \mathrm{d}\alpha^{-1}(\gamma) \quad \text{(by sending } \eta \text{ to } \eta^{-1} \text{ in } H) \\
&= \iint f(\eta^{-1}, \gamma)\, \mathrm{d}\beta(\eta)\, \mathrm{d}\alpha^{-1}(\gamma) \quad \text{(by removing } \phi(\eta)^{-1} \text{ in } G).
\end{aligned}$$

If one compares the first term of the above computation with the equation in (iv) Definition 2.1.1, and uses that the adjoining function is a groupoid homomorphism, then one can see that $\Delta(\eta, \eta^{-1}\gamma) = \frac{\delta_H(\eta)}{\delta_G \circ \phi(\eta)}$. Hence $\Delta(\eta^{-1}, \gamma) = \Delta(\eta, \eta^{-1}\gamma)^{-1} = \frac{\delta_G \circ \phi(\eta)}{\delta_H(\eta)}$. Thus a group homomorphism $\phi\colon H \to G$ gives a topological correspondence (G, α^{-1}) from (H, β) to (G, α) and $\frac{\delta_H \circ \phi}{\delta_G}$ is the adjoining function.

Let $\psi\colon G \to K$ be another homomorphism and let τ be the Haar measure on K. Then the composite $(K, \tau^{-1}) \circ (G, \alpha^{-1})$ from (G, α) to (K, τ) is isomorphic to the correspondence obtained from the homomorphism $\psi \circ \phi\colon G \to K$.

Example 3.1.6. Let G, H, α, β, δ_H and ϕ be as in Example 3.1.5. Additionally, assume that $\phi\colon H \to G$ is a proper function. For the time being, *assume* that the action of H on G given by $\gamma\eta := \gamma\phi(\eta)$ for $(\gamma, \eta) \in G \times H$ is proper, which is a fact and we prove it towards the end of this example. With this action of H and the left multiplication action of G on itself, G is a proper G-H-bispace. α^{-1} is an H-invariant measure. The adjoining function of this action is the constant function 1. To see this, let $f \in C_c(G \times G)$,correspondences then

$$\begin{aligned}
\iint f(\gamma^{-1}, \eta)\, \mathrm{d}\alpha(\gamma)\mathrm{d}\alpha^{-1}(\eta) &= \iint f(\gamma, \eta)\, \delta_G(\gamma)^{-1}\mathrm{d}\alpha(\gamma)\mathrm{d}\alpha^{-1}(\eta) \quad (\text{because } \alpha^{-1} = \delta^{-1}\alpha) \\
&= \iint f(\gamma, \gamma^{-1}\eta)\, \mathrm{d}\alpha(\gamma)\mathrm{d}\alpha^{-1}(\eta) \qquad (\text{because } L_\gamma\alpha^{-1} = \delta(\gamma)\alpha^{-1}).
\end{aligned}$$

Now we prove that the action of G on H is proper, that is, the map $\Psi\colon H \times G \to H \times H$ sending $(\gamma, \eta) \mapsto (\gamma, \gamma\phi(\eta))$ is proper. The maps

$$\begin{aligned}
&\mathrm{Id}_H \times \phi\colon H \times G \to H \times H \text{ and} \\
&\qquad m\colon (\gamma, \gamma') \mapsto (\gamma, \gamma\gamma') \text{ from } H \times H \to H \times H
\end{aligned}$$

are proper, and $\Psi = m \circ (\mathrm{Id}_H \times \phi)$. Hence Ψ is proper.

Example 3.1.7. Let $(G, \alpha), (H, \beta)$ and (K, τ) be locally compact, Hausdorff groups with Haar measures, and let $\phi\colon H \to H$ and $\psi\colon K \to G$ be continuous homomorphism with ψ proper. Using Example 3.1.5 we get the correspondences (G, α^{-1}) from (H, β) to (G, α) and using Example 3.1.6 we get the correspondence (G, α^{-1}) from (G, α) to (K, τ). In the composite correspondences from (H, β) to (K, τ), the space is G, the actions of H and K are the left and right multiplication via ϕ and ψ, the K-invariant family of measures on G is α^{-1}, and the adjoining function for this action is $\frac{\delta_G}{\delta_H \circ \phi}$ which can be checked as in Example 3.1.5.

An interesting situation is when $H, K \subseteq G$ are subgroups, K is closed, and ϕ and ψ are the inclusion maps. Then (G, α^{-1}) is a correspondence from (H, β) to (K, τ) where G is made into an

H-K bispace using the left and right multiplications, respectively. The adjoining function in this case is $\frac{\delta_G}{\delta_H}$.

Example 3.1.8 (Macho Stadler and O'uchi's correspondences). In [25], Macho Stadler and O'uchi present a notion of groupoid correspondences. We change the direction of correspondence in their definition to fit our construction and reproduce the definition here:

Definition 3.1.9. A correspondence from a groupoid with Haar system (G, α) to a groupoid with Haar system (H, β) is a G-H-bispace X such that:

i) the action of H is proper and the momentum map for the right action s_X is open,

ii) the action of G is proper,

iii) the actions of G and H commute,

iv) the right momentum map induces a bijection from $G\backslash X$ to $H^{(0)}$.

Like us, Macho Stadler and O'uchi do not assume that the left momentum map is open.

Macho Stadler and O'uchi do not require a family of measures on the G-H-bispace X. We show that a correspondence of Macho Stadler and O'uchi carries a canonical H-invariant family of measures λ.

$$\int_{X_u} f \,\mathrm{d}\lambda^u := \int_G f(\gamma^{-1}x) \,\mathrm{d}\alpha^{r_X(x)}(\gamma) \quad \text{for } f \in C_c(X),$$

where $u = s_X(x)$. An application of Proposition 1.3.21, for a left action gives that this is a continuous G-invariant family of measures along the quotient map $X \to G\backslash X$ and $G\backslash X$ is in bijection with $H^{(0)}$. This also gives us that $\Delta = 1$.

Thus (X, λ) is a *topological correspondence* from (G, α) to (H, β) in our sense.

To show that λ is a continuous family of measures, we need that s_X induces a homeomorphism from $G\backslash X \to H^{(0)}$. But from Proposition 8 in [5, Chapter I, §3, no. 5], for this map to be a homeomorphism, it is sufficient (an necessary) that $X \to G\backslash X$ is open. But as G has a Haar system, this condition is satisfied by Lemma 1.2.13. Hence, we need not ask that s_X is open.

Macho Stadler and O'uchi prove that such a correspondence from (G, α) to (H, β) induces a C^*-correspondence from $\mathrm{C}^*_\mathrm{r}(G, \alpha)$ to $\mathrm{C}^*_\mathrm{r}(H, \beta)$, which is clear to us from Corollary 2.3.5. Since the left action is proper in this case, the groupoid $G \times X$ is amenable and to Corollary 2.3.5 applies.

Example 3.1.10 (Equivalence of groupoids). Equivalences of groupoids as defined by Muhly-Renault-Williams in [28] are a special case of Macho Stadler-O'uchi correspondences. Hence equivalences of groupoids are topological correspondences as well. Furthermore, an equivalence of groupoids is an invertible correspondence.

Example 3.1.11 (Generalised morphisms of Buneci and Stachura). Buneci and Stachura define *generalised morphisms* in [8]. We modify this definition to fit our conventions and repeat it here:

Definition 3.1.12. A generalised morphism from (G,α) to (H,β) is a left action Θ of G on the *space*[1] H with r_{GH} as the anchor map, the action commutes with the right multiplication action of H on itself and there is a continuous positive function Δ_Θ on $G\times_{s_G,r_{GH}} H$ such that

$$\iint f(\gamma,\gamma^{-1}\eta)\,\Delta_\Theta(\gamma,\gamma^{-1}\eta)\,\mathrm{d}\alpha^{r_{GH}(\eta)}(\gamma)\,\mathrm{d}\beta_u^{-1}(\eta)=\iint f(\gamma^{-1},\eta)\,\mathrm{d}\alpha^{r_{GH}(\eta)}(\gamma)\mathrm{d}\beta_u^{-1}(\eta)$$

for all $f\in C_c(G\times_{s_G,r_{GH}} H)$ and $u\in H^{(0)}$.

If Θ is a generalised morphism from (G,α) to (H,β) then (H,β^{-1}) is a topological correspondence from (G,α) to (H,β), where β^{-1} is the family of measures

$$\int_G f\,\mathrm{d}(\beta^{-1})_u=\int f(\eta^{-1})\,\mathrm{d}\beta^u(\eta)$$

for $f\in C_c(G)$ and $u\in H^{(0)}$. It is obvious from the definition itself that Δ_Θ is the adjoining function for this correspondence.

In [8], Buneci and Stachura prove that a generalised morphism induces a *-homomorphism from $\mathrm{C}^*(G,\alpha)$ to $\mathcal{M}(\mathrm{C}^*(H,\beta))$. This is a C^*-correspondence from $\mathrm{C}^*(G,\alpha)$ to $\mathcal{M}(\mathrm{C}^*(H,\beta))$ with the underlying Hilbert module $\mathrm{C}^*(H,\beta)$.

Example 3.1.13. Let X be a right G-space for a locally compact Hausdorff group G and let λ be the Haar measure on G. Let H and K be subgroups of G. Assume that K is closed and let α and β be the Haar measures on H and K, respectively. Then $X\rtimes H$ and $X\rtimes K$ are subgroupoids of $X\rtimes G$. Denote these three transformation groupoids by $\mathbf{H}$, $\mathbf{K}$ and $\mathbf{G}$, respectively. Then $\mathbf{G}$ is an $\mathbf{H}$-$\mathbf{K}$-bispace for the left and the right multiplication actions, respectively. We bestow $\mathbf{H}$ and $\mathbf{K}$ with the families of measures $\{\alpha^y\}_{y\in X}$ and $\{\beta^z\}_{z\in X}$, respectively, where $\alpha^y=\alpha$ and $\beta^z=\beta$ for each $y,z\in X$ (see Example 1.3.10). If $\lambda_x^{-1}=\lambda^{-1}$ for all $x\in X$, then the family of measures $\{\lambda_x^{-1}\}_{x\in X}$ is $\mathbf{K}$-invariant. We show, that this family is $\mathbf{H}$-quasi-invariant with the adjoining function δ_G/δ_H. To see this, we do this example.

For $((x,\gamma),(y,\kappa))\in\mathbf{G}\rtimes\mathbf{K}$ we have $y=x\gamma$, hence the map $(x,\gamma,y,\kappa)\mapsto(x,\gamma,\kappa)$ gives an isomorphism between the groupoids $\mathbf{G}\rtimes\mathbf{K}$ and $X\rtimes(G\times K)$. Using this identification, it can be checked that the right action of $\mathbf{K}$ on $\mathbf{G}$ is proper, which is implied by the fact that $K\subset G$ is closed. Another quicker way to see this is to observe that $\mathbf{K}\subseteq\mathbf{G}$ is a closed subgroupoid.

Let $f\in C_c(\mathbf{H}*\mathbf{G})$, $u\in\mathbf{K}^{(0)}=\mathbf{G}^{(0)}\approx X$. Let $(u\gamma^{-1},\gamma)\in s_{\mathbf{G}}^{-1}(u)\subseteq\mathbf{G}$. If $(u\gamma^{-1},\eta)\in\mathbf{H}$, then $(u\gamma^{-1},\eta)^{-1}=(u\gamma^{-1}\eta,\eta^{-1})$ is composable with $(u\gamma^{-1},\gamma)$ and $(u\gamma^{-1},\eta)^{-1}(u\gamma^{-1},\gamma)=(u\gamma^{-1}\eta,\eta^{-1}\gamma)$. Now a computation similar to that in Example 3.1.5 shows that

$$\begin{aligned}
&\iint f((u\gamma^{-1},\eta),(u\gamma^{-1}\eta,\eta^{-1}\gamma))\,\frac{\delta_H(\eta)}{\delta_G(\eta)}\,\mathrm{d}\alpha^{r_{\mathbf{G}}(u\gamma^{-1},\gamma)}(u\gamma^{-1},\eta)\,\mathrm{d}\lambda_u^{-1}(u\gamma^{-1},\gamma)\\
&=\iint f((u\gamma^{-1},\eta),(u\gamma^{-1}\eta,\eta^{-1}\gamma))\,\frac{\delta_H(\eta)}{\delta_G(\eta)}\,\mathrm{d}\alpha(\eta)\,\mathrm{d}\lambda^{-1}(\gamma)\\
&=\iint f((u\gamma^{-1},\eta^{-1}),(u\gamma^{-1}\eta^{-1},\eta\gamma))\frac{1}{\delta_G(\eta)}\,\mathrm{d}\alpha(\eta)\,\mathrm{d}\lambda^{-1}(\gamma) \qquad \text{(by changing } \eta\mapsto\eta^{-1}\text{)}.
\end{aligned}$$

[1]In the section on Cohomology for groupoids in Chapter 1, we defined an action of a groupoid on another groupoid, which is different.

Now we change $\gamma \mapsto \eta^{-1}\gamma$. Then we use the relation $\mathrm{d}\lambda^{-1}(\eta^{-1}\gamma) = \frac{\mathrm{d}\lambda^{-1}(\gamma)}{\delta_G(\eta)}$. Thus the previous term equals

$$\begin{aligned}&\iint f((u\gamma^{-1}\eta, \eta^{-1}), (u\gamma^{-1}, \gamma))\,\mathrm{d}\alpha(\eta)\,\mathrm{d}\lambda^{-1}(\gamma)\\ &= \iint f((u\gamma^{-1}, \eta)^{-1}, (u\gamma^{-1}, \gamma))\,\mathrm{d}\alpha(u\gamma^{-1})\,\mathrm{d}\alpha^{r_{\mathbf{G}}(u\gamma^{-1},\gamma)}(u\gamma^{-1}, \eta)\,\mathrm{d}\lambda_u^{-1}(u\gamma^{-1}, \gamma).\end{aligned}$$

Example 3.1.14. Let G be a locally compact Hausdorff group, α the Haar measure on G and let X be a proper left G-space. Let λ be a *strongly* G-quasi-invariant measure on X, that is, there is a continuous function $\Delta : G \times X \to \mathbb{R}^+$ such that $\mathrm{d}(g\lambda)(x) = \Delta(g, x)\mathrm{d}\lambda(x)$ for every g in G. In this setting, (X, λ) is a correspondence from (G, α) to $(\mathrm{Pt}, \delta_{\mathrm{Pt}})$, with Δ as the adjoining function. The C^*-algebra for Pt is $\mathbb{C}$, the Hilbert module $\mathrm{C}^*(X, \lambda)$ is the Hilbert space $L^2(X, \lambda)$ and the action of $\mathrm{C}^*(G)$ on this Hilbert module is the representation of $\mathrm{C}^*(G)$ obtained from the representation of G on $C_c(X)$.

An example of this situation is: when X is a homogenous space for G, X carries a G-strongly quasi-invariant measure. For details, see Section 2.6 in [15].

Example 3.1.15 (An example of composition). Let G be a locally compact group, let H and K be closed subgroups of G and let λ, μ and β be the Haar measures on G, H and K, respectively. Let (X, α) be a left K-space carrying a strongly K-quasi-invariant measure α. Let δ_G, δ_H and Δ be the modular functions of G and H and the Radon-Nikodym derivative of $g\alpha$ with respect to α, respectively. Then (G, λ^{-1}) is a correspondence from H to K with $\frac{\delta_G}{\delta_H}$ as adjoining function, as in Example 3.1.7. And (X, α) is a correspondence from K to Pt with Δ as adjoining function, as in Example 3.1.14. The topological space in the product of (G, λ^{-1}) and (X, α) is $(G \times X)/K$, which we denote by Z. In this example, writing the measure ν on Z concretely is not always possible. However, when $(X, \alpha) = (K, \beta)$, we get $Z \approx G$ and $\nu = \lambda^{-1}$.

The correspondence (X, α) gives a representation of K on $\mathcal{L}^2(X, \alpha)$ and the product correspondence is the representation of H *induced* by this representation of K.

Example 3.1.16 (The induction correspondence). Let (G, α) be a groupoid with Haar system, H a closed subgroupoid of G and β a Haar system for H. Let $X = G_{H^{(0)}} := s_G^{-1}(H^{(0)})$. Then X is a G-H-bispace, where the left and right actions are multiplication from the left and right, respectively. Both actions are free. We claim that the actions of G and H are proper.

First of all, we observe that $X \subseteq G$ is closed, hence $X \times X \subseteq G \times G$ is closed. Let $\Psi \colon X * H \to X \times X$ be the map $\Psi(x, \eta) = (x, x\eta)$. Since $H \subseteq G$ is closed, $X * H \subseteq= G \times_{s_G, r_G} G$ is closed. Now we notice that Ψ is the restriction of the map corresponding to the right multiplication action of G on itself to closed subspaces. Since the right multiplication is a proper action, Ψ is proper. Similarly, the left action of G on X is proper.

It is not hard to see that $G\backslash X \approx H^{(0)}$. By Example 3.1.8, X produces a topological correspondence from (G, α) to (H, β).

Remark 3.1.17. In Example 3.1.16 , both actions are free and proper. However, the correspondence is not a groupoid equivalence, since it might fail to satisfy Condition (v) of Definition 2.2.1.

3.2 Induced representations

A convention: In this section, we work with right H-spaces carrying right invariant families of measures. However, in the original works, namely, [17] and [35] the spaces are left spaces, carrying left invariant families of measures.

In the theory of representations of groups, the induction process introduced by Mackey is an important technique. In one of his famous works [36], Rieffel introduces an induction process for representations of C*-algebras. Let B be a C*-algebra and let $\mathcal{H}$ be a Hilbert B-module. If a C*-algebra A acts on $\mathcal{H}$ from the left by adjointable operators in a non-degenerate fashion, then $\mathcal{H}$ induces a functor from $\text{Rep}(B)$ to $\text{Rep}(A)$. If $\mathcal{H}$ is an imprimitivity bimodule, then it induces an equivalence from $\text{Rep}(B)$ to $\text{Rep}(A)$.

If G and H are groupoids, then in [28] Muhly, Renault and Williams define when an H-G-bispace X is an *equivalence*. If the groupoids are equipped with Haar systems then an equivalence between H and G gives a $\mathrm{C}^*(H,\beta)$-$\mathrm{C}^*(G,\alpha)$-imprimitivity bimodule, where β and α are Haar systems on H and G, respectively. Though the proof is not written, using groupoid representation theory [34], it can be seen that a groupoid equivalence induces an isomorphism from $\text{Rep}(G)$ to $\text{Rep}(H)$.

If X is a topological correspondence from H to G, then X induces a C*-correspondence from $\mathrm{C}^*(H)$ to $\mathrm{C}^*(G)$. At the C*-algebraic level, this gives a functor from $\text{Rep}(\mathrm{C}^*(G))$ to $\text{Rep}(\mathrm{C}^*(H))$. Due the the construction itself $\text{Rep}(G) \cong \text{Rep}(\mathrm{C}^*(G))$. Hence X induces a functor from $\text{Rep}(G)$ to $\text{Rep}(H)$. In his very recent work [35], Renault discusses this functor in detail. He describes the induction functor topologically, and his techniques form a topological analogue of Rieffel's induction process. This shows that topological correspondences form an analogue of C*-correspondence in the theory of topological groupoids.

We do not give details of this construction of Renault's here. Interested readers can refer to [35]. One of the main theorems in the paper says:

Theorem 3.2.1 ([35]). *If (X,λ) is a topological correspondence from a groupoid with Haar system (H,β) to a groupoid with Haar system (G,α) then a representation $(m,\mathcal{H})$ of (G,β) induces a representation $(m',\mathcal{H}')$ of (H,β), and this construction generalises the classical construction of induced representations of groups.*

The following result of ours, namely, Proposition 3.2.2, is a nice example of induction techniques in [35].

Proposition 3.2.2. *Let (G,α) be a locally compact, Hausdorff groupoid with a Haar system and $H \subseteq G$ a closed subgroupoid. Let β be a Haar system for H. The G-H-bispace $G_{H^{(0)}}$ gives a correspondence from (G,α) to (H,β).*

Proof. See Example 3.1.16. □

The following Corollary is copied from [15], and we do not explain the terminology used in it, since it is the standard terminology used in the representation theory of locally compact groups. Interested readers can refer to the original book for details. Unlike us, for a locally compact Hausdorff space X, Folland calls a continuous linear functional on $C_c(X)$ a pseudomeasure and not a measure.

Corollary 3.2.3 (Theorem 6.13, in [15]). *Let G be a locally compact group and H a closed subgroup, with modular functions Δ_G and Δ_H. Let μ be a pseudomeasure of positive type on H, let σ_μ be the associated unitary representation of H, and let ν be the injection of $\sqrt{\Delta_G/\Delta_H}\,\mu$ into G, that is, the pseudomeasure on G defined by*

$$\nu(f) = \int_H \sqrt{\frac{\Delta_G(\xi)}{\Delta_H(\xi)}}\, f(\xi)\,\mathrm{d}\mu(\xi).$$

Then ν is of positive type, and the associated unitary representation π_ν of G is unitarily equivalent to the induced representation $\Pi = \mathrm{ind}_H^G(\sigma_\mu)$.

Since Corollary 3.2.3 is a classical result discussing induction and is implied by Proposition 3.2.2 and 3.2.1, the induction process or Renault in [35]] is an appropriate generalisation of the classical induction process.

3.3 Spatial hypergroupoids

Let X be a left free and proper H-space. Let β be a Haar system for H and λ an invariant family of measures on X. The proof of Proposition 2.2.9 shows that $(X * X)/H$ is a groupoid with a Haar system. Denote this groupoid by G. The Haar system α on G is constructed using λ. Furthermore, X is an equivalence between (G, α) and (H, β). Corollary 2.2.10 is a consequence of Proposition 2.2.9, which says that $\mathrm{C}^*(G, \alpha) \simeq \mathbb{K}(\mathcal{H}(X))$.

The key ingredient here is the groupoid structure of $X * X/H$. What happens if freeness is dropped?

In that case, $X * X/H$ need not be a groupoid. However, Theorem 2.2.19 says that if X is a proper H-space with an invariant family of measures λ, then we still have a $\mathrm{C}^*(H, \beta)$-Hilbert module $\mathcal{H}(X)$. However, we could not find an analogue of Corollary 2.2.10 in the literature.

We describe how to generate a C^*-algebra for the object $X * X/H$. An observation of Renault is that $X * X/H$ need not be a groupoid but is a *spatial hypergroupoid*. Spatial hypergroupoids gave rise to the theory of representations of locally compact hypergroupoids with Haar system and their C^*-algebras. This theory generalises the representation theory of locally compact groupoids. The representation theory of locally compact hypergroupoids with Haar system is discussed in [17]. In this section, we review a special and first case of hypergroupoids, namely, spatial hypergroupoids. The content of the present section is from our work in [17].

3.3.1 A C^*-category of groupoids

For a groupoid H equipped with a Haar system β define a *-category (see [43]) as follows:

Objects: Objects are pairs (X, λ) where X is a proper right H-space and λ is an H-invariant family of measures.

Arrows: An arrow from (X, λ) to (Y, μ) is a triple (λ, f, μ) where $f \in C_c((X * Y)/H)$. Note that the set of arrows between two objects is a complex vector space under pointwise addition and scalar multiplication.

Composition: The product of (λ, f, μ) and (μ, g, ν) is $(\lambda, f *_\mu g, \nu)$ where $f *_\mu g$ is the convolution

$$(f *_\mu g)[x, z] := \int f[x, y]g[y, z] \,\mathrm{d}\mu_{s_X(x)}(y),$$

which is in $C_c((X * Y)/H)$.

Involution: For (λ, f, μ) the adjoint is $(\lambda, f, \mu)^* = (\mu, f^*, \lambda)$ with $f^*[x, y] = \overline{f[y, x]}$.

We denote this category by $\mathfrak{C}_c(H)$.

To check that the convolution above is well-defined, it is enough to check that for fixed $x \in X$ the integral is taken over a compact set. We check this now. For fixed $x \in X$, let $\phi^x\colon Y^{s_X(x)} \to (X * Y)/H$ be the map $y \mapsto [x, y]$. This map is proper. The reason is the following: if $K \subseteq (X * Y)/H$ is compact, then choose $K' \subseteq X * Y$ compact such that $K'/H = K$. This can be done, since the action of H on $X * Y$ is proper. But $X * Y \subseteq X \times Y$ and $Y^{s_X(x)} \subseteq Y$ are closed, hence the map $\bar{\phi}^x\colon Y^{s_X(x)} \to X * Y$, $y \mapsto (x, y)$ is proper. Now it can be checked that $(\phi^x)^{-1}(K) = (\bar{\phi}^x)^{-1}(K') \subseteq Y^{s_X(x)}$ where $(\bar{\phi}^x)^{-1}(K')$ is compact. Thus the function $y \mapsto f[x, y]$ is compactly supported, which implies that the integral in the definition of the convolution is taken on a compact set.

The above operations generalise all of the formulae which appear for left and right actions and inner products in the theory of topological correspondences.

Identify $(X * H)/H$ with X via the map $[x, \gamma] \mapsto x\gamma^{-1}$. Thus we identify H with $(H * H)/H$ and $H^{(0)}$ with H/H. Fix an object (X, λ) in $\mathfrak{C}_c(H)$. Then i) the left and right actions of $C_c((X * X)/H)$ and $C_c(H)$ on $C_c(X)$, and ii) the $C_c(H)$-valued inner product on $C_c(X)$ can be seen as product in $\mathfrak{C}_c(H)$. The following tables show the correspondence between action-inner product and composition of arrows in $\mathfrak{C}_c(H)$. The computations in these table need the above identifications and sometime they resemble the computations we did on page 16 for the computation for the family of measures β_X (see Equation (1.3.26)), and the one on page 32 in Equation (1.6.29) where we compute the family of measures along the quotient map $(H * H)/H \to H/H$.

Below ${}_*\langle,\rangle$ and $\langle,\rangle_*$ denote the left and right inner products, respectively. For $h \in C_c(H) = C_c((H * H)/H)$ and $\xi, \zeta \in C_c((X * H)/H)$ we make Table 3.1.

$\xi h(x) = \int \xi(x\eta^{-1})h(\eta) \,\mathrm{d}\beta^{s(x)}(\eta)$	$\leftrightarrow$	$\xi h[x, s(x)] = \int \xi[x, \eta]h[\eta, s(\eta)] \,\mathrm{d}\beta^{-1}_{s(x)}(\eta)$
$\langle \xi\,, \zeta\rangle_*(\eta) = \int \overline{\xi(x)}\zeta(x\eta) \,\mathrm{d}\lambda_{r(\eta)}(x)$	$\leftrightarrow$	$\langle \xi\,, \zeta\rangle_*[\eta, s(\eta)] = \int \xi^*[\eta, x]\zeta[x, s(\eta)] \,\mathrm{d}\lambda_{s(\eta)}(x)$

Table 3.1: Equivalence of operations

In general, $f \in C_c((X * X)/H)$ is composable with $\zeta \in C_c(X) = C_c((X * H)/H)$ and the composite $f *_\lambda \zeta \in C_c(X) = C_c((X * H)/H)$. When the action of H is free, the composite $f *_\lambda g$ is a very well-know formula in the theory of groupoid equivalences. We discuss it ahead.

Assume that the action of H on X is free, then $(X * X)/H$ is a groupoid with a Haar system (Proposition 2.2.9) which we denote by (G, α). The Haar system α is derived from λ. If $f \in C_c(G)$ and $\zeta \in C_c(X)$, then $f *_\lambda \zeta$ is the left action of $C_c(G)$ on $C_c(X)$ as in Equation (2.2.2). The $\mathrm{C}^*(G)$-valued inner product on $C_c(X)$ is also a special case of composition of arrows in $\mathfrak{C}_c(H)$. Table 3.2 gives the correspondence between these operations. In Table 3.2, $\xi, \zeta \in C_c((X * H)/H)$ and $f \in C_c(X * X/H) = C_c(G)$.

$f\zeta(x) = \int f[x,y]\zeta(y)\,\mathrm{d}\lambda_{s(x)}(y)$	$\leftrightarrow$	$f\zeta[x,s(x)] = \int f[x,y]\zeta[y,s(y)]\,\mathrm{d}\lambda_{s(x)}(y)$
${}_*\langle \xi\,,\zeta\rangle[x,y] = \int \xi(x\eta^{-1})\overline{\zeta(y\eta^{-1})}\,\mathrm{d}\beta^{-1}_{s_X(x)}(\eta)$	$\leftrightarrow$	${}_*\langle \xi\,,\zeta\rangle[x,y] = \int \xi[x,\eta]\zeta^*[\eta,y]\,\mathrm{d}\beta^{-1}{}_{s_Y(y)=s(x)}(\eta)$

Table 3.2: Equivalence of operations

The equation ${}_*\langle \xi\,,\zeta\rangle[x,y] = \int \xi(x\eta^{-1})\overline{\zeta(y\eta^{-1})}\,\mathrm{d}\beta^{-1}_{s_X(x)}(\eta)$ in Table 3.2 needs an explanation. The inner product formula in Equation 2.2.3 is meant for the left invariant families of measures and left Haar systems. Now we are using the right invariant settings. Hence the appropriate version of Equation 2.2.3 for the right invariant families of measures is

$$ {}_*\langle \xi\,,\zeta\rangle(\gamma) = \int \xi(\gamma t\eta^{-1})\overline{\zeta(t\eta^{-1})}\,\mathrm{d}\beta^{-1}_{s_X(t)}(\eta), \tag{3.3.1} $$

where f,ζ are as in the table, $\gamma \in G$, $x \in X$ and $\eta \in H$. As usual, t can be replace by any element in the H-orbit of t. Recall the action of $[x,y] \in G$ on $z \in X$ from Proposition 2.2.9, that is, $[x,y]z = x\gamma$ where $\gamma \in H$ is the unique element with $z = y\gamma$. Substituting $\gamma = [x,y]$ in Equation (3.3.1) and then choosing y as the representative in the H-orbit of y (we can choose y, see the comment below Equation (3.3.1) which is due to discussion at end of page 11 of [28]), we get

$$ {}_*\langle \xi\,,\zeta\rangle[x,y] = \int \xi([x,y]y\eta^{-1})\overline{\zeta(y\eta^{-1})}\,\mathrm{d}\beta^{-1}_{s_Y(y)}(\eta) = \int \xi(x\eta^{-1})\overline{\zeta(y\eta^{-1})}\,\mathrm{d}\beta^{-1}_{s_Y(y)=s_X(x)}(\eta). $$

Now we come back to the case when the action of H on X is proper but not free. For $f,g \in C_c(X*X/H)$ the convolution $f*_\lambda g \in C_c(X*X/H)$ defines a convolution and $f^*[x,y] = \overline{f[y,x]}$ defines the involution. The convolution and the involution makes $C_c((X*X)/H)$ into a *-algebra. This is an important observation. For a free H-space X, these operation give the convolution *-algebra $C_c(G)$ for the groupoid $G = (X*X)/H$. To see this note that $[x,y]^{-1} = [y,x]$ in G.

Lemma 3.3.2 ([17], Lemma 2.1). *The operations on $\mathfrak{C}_c(G)$ described above are well defined and they make $\mathfrak{C}_c(G)$ into a *-category.*

Here is a small comment on the above lemma: As seen at the beginning of this section, the properness of the action makes the operation well-defined. An important observation is that for $f \in C_c((X*Y)/H)$, the arrow $f^* \in C_c((Y*X)/H)$ produces the conjugate arrow.

Plan: Our plan is to define a C*-norm on this category and complete it to a C*-category. We extend the technique used in the proof of Theorem 2.2.19. We recall what we did there. Given a proper H-space X and an invariant family of measures λ on X, we showed $\langle f\,,f\rangle$ is positive in C$^*(H,\beta)$. To do this, given a representation $(m,\mathcal{H},\pi)$ of (H,β), we define an operator $|f\rangle\!\rangle\colon \mathcal{L}^2(H^{(0)},m(\beta),\mathcal{H}) \to \mathcal{L}^2(X/H,m(\lambda),\mathcal{H}_X)$ and its adjoint $\langle\!\langle f\,|$. Here m is a transverse measure class. Finally, we showed that $\langle f\,,f\rangle = \langle\!\langle f| \circ |f\rangle\!\rangle$ is a positive operator on $\mathcal{L}^2(H^{(0)},m(\beta),\mathcal{H})$. We generalise this setup. Let (X,λ) and (Y,μ) be two objects in $\mathfrak{C}_c(H)$. If f is an arrow from (X,λ) to (Y,μ) and $(m,\mathcal{H},\pi)$ is a representation of (H,β), we define operators $|f\rangle\!\rangle\colon \mathcal{L}^2(X/H,m(\lambda),\mathcal{H}_X) \to \mathcal{L}^2(Y/H,m(\mu),\mathcal{H}_Y)$ and $\langle\!\langle f\,|\colon \mathcal{L}^2(Y/H,m(\mu),\mathcal{H}_Y) \to \mathcal{L}^2(X/H,m(\lambda),\mathcal{H}_X)$. Then we show that they are adjoints of each other. Using the positivity of $\langle\langle f|\circ|f\rangle\rangle$ for every representation we define a C*-norm.

We proceed to the concrete formulation of the above plan now. Let $(m, \mathcal{H}, \pi)$ be a representation of (H, β), where m is a transverse measure class for H, $\mathcal{H} \to H^{(0)}$ is a measurable Hilbert bundle with separable fibres and π is an action of H on $\mathcal{H}$. Let X be a proper right H-space carrying an invariant family of measures λ. We know that m induces a measure class $m(\lambda)$ on X/H. Let $\mathcal{H}_X$ denote the Hilbert bundle $s_X^*(\mathcal{H})/H \to X/H$, similar to the one in Section 2.2.2. For an arrow $(\lambda, f, \mu)\colon (X, \lambda) \to (Y, \beta)$, define the operator $L(\lambda, f, \mu)\colon \mathcal{H}(\lambda) \to \mathcal{H}(\mu)$ by

$$\left\langle \zeta\sqrt{\delta}\,,\, L(\lambda, f, \mu)\eta\sqrt{\kappa}\right\rangle = \int f[x,y]\,\langle \zeta[x]\,,\,\eta[y]\rangle \sqrt{\frac{\mathrm{d}(\delta\circ[\mu_1])}{\mathrm{d}(\kappa\circ[\lambda_2])}}\mathrm{d}(\kappa\circ[\lambda_2])([x,y]).$$

Here μ_1 and λ_2 are defined like α_1 and α_2 in Equation (1.6.27). Fubini's Theorem implies $\lambda\circ\mu_1 = \mu\circ\lambda_2$, and then the coherence of m gives that $\delta\circ[\mu_1] \sim \kappa\circ[\lambda_2]$. Hence the Radon-Nikodym derivative above makes sense.

This satisfies the Cauchy-Schwarz inequality (see page 80 of [34]):

$$\|L(\lambda, f, \mu)\| \leq \max\left(\sup_x \int |f[x,y]|\,\mathrm{d}\mu_{s_X(x)}(y),\ \sup_y \int |f[x,y]|\,\mathrm{d}\lambda_{s_Y(y)}(x)\right).$$

Note that the term on the right side of the above equality is finite, since both μ and λ are continuous families of measure and f is continuous with a compact support. Define the I-norm of f as the term $\max\left(\sup_x \int |f[x,y]|\,\mathrm{d}\mu_{s_X(x)}(y),\ \sup_y \int |f[x,y]|\,\mathrm{d}\lambda_{s_Y(y)}(x)\right)$.

Theorem 3.3.3 (Theorem 2.2, [17]). *1. Let $(\mathcal{H}, m)$ be a unitary representation of a locally compact groupoid G. Then the above formulae define a representation L of the *-category $\mathfrak{C}_c(H)$, called integrated representation, which is continuous for the inductive limit topology and bounded for the I-norm.*

*2. Let (G, α) be a second countable locally compact groupoid with Haar system. Every representation of the *-algebra $C_c(G, \alpha)$ in a separable Hilbert space that is non-degenerate and continuous for the inductive limit topology is equivalent to an integrated representation.*

Remark 3.3.4. Let (X, λ), (Y, μ) and (Z, ν) be objects in $\mathfrak{C}_c(H, \beta)$. Then the formulae above along with Theorem 3.3.3 imply that each $C^*(X * X/H)$ is a C*-algebra and that $C^*(X * Y/H)$ is a Hilbert $C^*(X * X/H)$-$C^*(Y * Y/H)$-bimodule.

We know that $X * X$ is a proper H-space. We observe that for $f, g \in C_c(X * X)$, $f *_\beta g = B_{X*X}(f \otimes g)$, where B_{X*X} is the integration function associated with the family of measures β_{X*X} along the quotient map $X * X \to X * X/H$ (see Proposition 1.3.21). Since B_{X*X} is a continuous surjection, which follows from Proposition 1.3.21, the set of function $I := \{f *_\beta g : f, g \in C_c(X)\} \subseteq C_c((X * X)/H)$ is dense. But the second entry of Table 3.2 says that $I = \{{}_*\langle f\,,\, g\rangle : f, g \in C_c(X)\}$. Hence we may conclude that $\mathcal{H}(X)$ is full as left Hilbert $C^*((X * X)/H)$-module.

Let A and B be C*-algebras. An A-B-bimodule H is called a Hilbert A-B-bimodule if H is a right Hilbert A-module, left Hilbert B-module, A acts on H by B-adjointable operators and B acts on H by A-adjointable operators and ${}_*\langle x\,,\, y\rangle z = x\,\langle y\,,\, z\rangle_*$, where $x, y, z \in H$ and ${}_*\langle,\rangle$ and $\langle,\rangle_*$ have the obvious meaning.

Remark 3.3.5. Theorem 2.2.19 is a consequence of Theorem 3.3.3 now. This can be seen by the identifications in Table 3.1.

Proposition 3.3.6. *Let X be a proper H-space and let λ be an invariant family of measures on X. Let $C^*(X * X/H)$ be the completion of the *-algebra $C_c((X * X)/H)$ as in Theorem 3.3.3. Then $C^*(X * X/H) \simeq \mathbb{K}(\mathcal{H}(X, \lambda))$.*

Proof. Remark 3.3.4 gives that $C^*((X * X)/H)$ is a C^*-algebra, and Remark 3.3.5 gives that $C^*((X * H)/H) \simeq \mathcal{H}(X)$. Putting this together and looking at the *-algebras $C_c((X * X)/H)$ and $C_c(H * H)/H) = C_c(H)$ one can conclude that $C^*(X * X/H) \simeq \mathbb{K}(\mathcal{H}(X, \lambda))$. □

Proposition 3.3.6 answers the question we raised at the beginning of this section.

3.3.2 Hypergroupoids

Motivation: *Hypergroups* are structures which resemble *groups*, except that the product of two elements is not an element, but a probability measure on the set ([18]). Equivalently, a hypergroup is a convolution algebra of measures on a space with certain properties, see [19]. We adopt the latter notion that a hypergroup is a convolution algebra of measures on a space. The representation theory of hypergroups is studied thoroughly, for example, in [13].

For a hypergroup L with a Haar system, the space of compactly supported functions $C_c(L)$ is a convolution algebra. A similar convolution can be defined on $C_c(X * X/H)$ for a proper H-space X. The notion of hypergroupoid is conceptually important, since it offers the explanation for the *-algera structure of $C_c(X * X/H)$ and the C^*-algebra it gives as in Theorem 3.3.3. Remark 3.3.4 shows that the space $C_c(X * X/H)$ carries a convolution and an involution structure. The category $\mathfrak{C}_c(H, \beta)$ gives $C_c(X * X/H)$ these structure. The category $\mathfrak{C}_c(H, \beta)$ also gives $C_c(X)$ a pre-Hilbert $C_c(X * X/H)$-$C_c(H)$-bimodule structure. The completion of $\mathfrak{C}_c(H, \beta)$ completes $C_c(X * X/H)$ into C^*-algebra. While this all is happening in the algebraic settings, it is a good question to ask, if there is any geometric object whose C^*-algebra is $C^*(X * X/H)$. The answer looks affirmative, because if X is a free H-space, then we know that $X * X/H$ is a groupoid with Haar system and $C^*(X * X/H)$ is a groupoids C^*-algebra.

The answer to the question above is that yes, there is a geometrical object called hypergroupoid with Haar system which gives rise to the C^*-algebra $C^*(X * X/H)$. We introduce this structure briefly. The remaining part of the section is based on [35].

Following the approach in [19], this convolution structure can be abstractly interpreted as a *hypergroupoid* structure on $X * X/H$.

Definition 3.3.7 (Hypergroupoid; Definition 4.1 in [35]). A locally compact *hypergroupoid* is a pair $(H, H^{(0)})$ of locally compact spaces with continuous open surjective range and source maps $r, s\colon H \to H^{(0)}$, a continuous injection $i\colon H^{(0)} \to H$ such that $r \circ i$ and $s \circ i$ are the identity map, a continuous involution $\mathrm{inv}\colon h \to h^*$ of H such that $r \circ \mathrm{inv} = s$, and a product map $m\colon H^{(2)} \to P(H)$, where $H^{(2)}$ is the set of composable pairs, such that

i) the support of $m(x, y)$ is a compact subset of $H^{r(x)}_{s(x)}$;

ii) for all $(x, y, z) \in H^{(3)}$ $\int m(x, \cdot)\, \mathrm{d}m(y, z) = \int m(\cdot, z)\, \mathrm{d}m(x, y)$;

iii) for all $x \in H$, $m(r(x), x) = m(x, s(x)) = \delta_x$;

iv) for all $(x, y) \in H^{(2)}$, $m(x, y)^* = m(y^*, x^*)$ where $m(x, y)^*$ is the image of the measure $m(x, y)$ by the involution;

v) $x = y^*$ if and only if the support of $m(x, y)$ meets $i(H^{(0)})$;

vi) for all $f \in C_c(H)$ and $\epsilon > 0$ there exists a neighbourhood U of $i(H^{(0)})$ in H such that $|f(x) - f(y^*)| \leq \epsilon$ if the support of $m(x, y)$ meets U;

vii) for all $x \in H$ the left translation operator $L(x)$ is defined by

$$(L(x)f)(y) = f(x^* * y) = \int f \, \mathrm{d}m(x^*, y).$$

sends $C_c(H^{s_X(x)})$ to $C_c(H^{r_X(x)})$.

Definition 3.3.8 (Haar system for a hypergroupoid; Definition 4.3 [35]). A Haar system on a locally compact hypergroupoid H is a system of Radon measures $\lambda = \{\lambda^u\}_{u \in H^{(0)}}$ for the range map such that

i) for all $f \in C_c(H)$, $u \in H^{(0)}$ the map $u \mapsto \int f \, \mathrm{d}\lambda^u$ is continuous;

ii) for all $f, g \in C_c(H)$ and all $x \in H$,

$$\int f(x * y) g(y) \, \mathrm{d}\lambda^{s(x)}(y) = \int f(y) g(x^* * y) \, \mathrm{d}\lambda^{r(x)}(y);$$

iii) for all $f, g \in C_c(H)$, $x \in H$, the map $x \mapsto \int f(x * y) g(y) \, \mathrm{d}\lambda^{s(x)}(y)$ is continuous with compact support.

Here

$$f(x * y) := \int f \, \mathrm{d}m(x, y).$$

In [35], for a hypergroupoid with a Haar system, Renault introduces the integration-disintegration techniques, and studies the representation theory of hypergroupoids with a Haar system. Eventually, he uses this machinery to construct C*-algebras for hypergroupoids with Haar systems. We state the result which was mentioned in the motivating discussion at the beginning of this subsection.

Theorem 3.3.9 (Theorem 4.5 [35]). *Let (H, β) be a locally compact groupoid endowed with a Haar system and let (X, λ) be a proper right H-space with an H-invariant family of measures. Then $X * X/H$ is a locally compact hypergroupoid with a Haar system.*

Main ideas involved in the theorem are as follows: we call $[x, y], [w, z] \in X * X/H$ composable if and only if $y = w$, and then the *product* is the probability measure defined as follows: Two pairs $([x, y], [y, z]), ([x', y'], [y', z']) \in X * X/H$ are equal if and only if there is a pair $(\eta, \gamma) \in H(y) \times_{s_H, r_H} H$ with $x' = x\eta, y' = y\eta$ and $z' = z\gamma\eta$. Here $H(y) := \{\gamma \in H : y\gamma = y\}$. Clearly, if there is such a pair, then $[x, y] - [x', y']]$ and $y' = y\eta = \gamma\eta$. Hence $[y, z] = [y'z']$. Conversely, $[x', y'] = [x, y]$ implies that there is $\eta \in H$ with $x' = x\eta$ and $y' = y\eta$, and $[y,z] = [y',z']$ implies that there is τ with $y' = y\tau$ and $z = z\tau$. Then $\gamma = \tau\eta^{-1} \in H(y)$ and $z' = z\tau = z\gamma\eta$. The isotropy group $H(y)$ is compact, because if $\Psi\colon X * H \to X \times X$ is the proper map $(x,\eta) \mapsto (x, x\eta)$, then $H(y) = \Psi^{-1}(\{y\} \times \{y\})$. Let κ_y be the left invariant probability measure on $H(y)$. Then for $[x,y],[y,z] \in X * X/H$ and $f \in C_c(X * X/H)$ define

$$\int f \,\mathrm{d}m_{[x,y][y,z]} = \int f[x\gamma, z]\,\mathrm{d}\kappa_y(\gamma).$$

This makes $X * X/H$ into a hypergroupoid.

The family of measures λ is used to define the Haar system $\bar{\lambda}$ on $X * X/H$. For $f \in C_c(X * X/H)$

$$\int f \,\mathrm{d}\bar{\lambda}^{[x]} = \int f([x,y])\,\mathrm{d}\lambda^{s_X(x)}.$$

We conclude the description of the hypergroupoid structure of $X * X/H$ here, which is the answer to the question we asked at the beginning of this section, namely, what is the geometric object that gives rise to $\mathrm{C}^*((X * X)/H)$ which obtained by completing the category $\mathfrak{C}_c(H,\beta)$ abstractly.

3.4 The Brauer group of a groupoid

The set of isomorphism classes of *certain* equivariant continuous bundles over the space of units of a groupoid G can be made into a *group*. This group is denoted by $\mathrm{Br}(G)$ and is called the *Brauer group* of G. Elements of the Brauer group of G are equivalence classes of equivariant continuous trace C^*-algebras $C_0(G^{(0)}, \mathcal{A})$ with spectrum $G^{(0)}$.

In [21], the authors define the Brauer group for locally compact groupoids and prove two isomorphism theorems for it. The first isomorphism, which is our point of attention, says that an equivalence X from H to G produces an isomorphism from $\mathrm{Br}(G)$ to $\mathrm{Br}(H)$. We show that a Hilsum-Skandalis morphism from H to G gives a homomorphism from $\mathrm{Br}(G)$ to $\mathrm{Br}(H)$.

Important convention: In this section we shall work with locally compact, Hausdorff, second countable groupoids which *need* not have *Haar systems*. Only the topological properties of the groupoids will be used. We shall work with Hilsum-Skandalis morphism in this section and not the topological correspondences we have beed discussing so far. Since the right anchor map in a Hilsum-Skandalis morphism is not required to be open, there usually cannot be any continuous invariant family of measures with full support. So Hilsum-Skandalis morphisms are quite different from the topological correspondences introduced here. The following results show, therefore, that Brauer groups and groupoid C^*-algebras are functorial for very different kinds of morphisms of groupoids.

3.4.1 The Brauer group

Definition 3.4.1 (Hilsum-Skandalis morphism). *A Hilsum-Skandalis morphism* from a groupoid H to a groupoid G is an H-G-bispace X such that

i) the action of G is free and proper;

ii) the left momentum map induces a bijection from X/G to $H^{(0)}$.

This is a bibundle functor in the notation of Meyer and Zhu [27]. The terminology '*Hilsum-Skandalis* morphism' is originally from geometry and we continue using it.

If the action of H is proper and s_X is open, then an Hilsum-Skandalis morphism is a correspondence in the sense of Macho Stadler and O'uchi as in Example 3.1.8.

Definition 3.4.2 (Upper semicontinuous Banach bundle)**.** *An upper semicontinuous Banach bundle* over a topological space X is a topological space $\mathcal{A}$ together with a continuous open surjection $\pi_X\colon \mathcal{A} \to X$ and complex Banach space structures on each fibre $\mathcal{A}_x := p^{-1}(x)$ satisfying the following axioms:

i) if $\mathcal{A} * \mathcal{A} := \{(a, b) \in \mathcal{A} \times \mathcal{A} : p(a) = p(b)\}$, then $(a, b) \mapsto a + b$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$;

ii) for each $\lambda \in \mathbb{C}$, the map $\mathcal{A} \to \mathcal{A}$ sending $a \mapsto \lambda a$ is continuous;

iii) if $\{a_i\}$ is a net in $\mathcal{A}$, with $p(a_i) \to x$ and $\|a\| \to 0$, then $a_i \to 0 \in \mathcal{A}_x$;

iv) the map $a \mapsto \|a\|$ is upper semicontinuous from $\mathcal{A}$ to $\mathbb{R}^+$.

We abbreviate the phrase "upper semicontinuous" as u. s. c.. We call $\mathcal{A}$ the *total space* of the u. s. c. Banach bundle, X the base space and π_X the bundle projection of the bundle. In the fourth condition, if the norm function is continuous instead of being u. s. c., then the bundle is called *a continuous Banach bundle*. In this section, we shall be working with *continuous* Banach bundles *only*.

Definition 3.4.3. A C*-bundle over X is a Banach bundle $\pi_X\colon \mathcal{A} \to X$ such that each fibre is a C*-algebra satisfying, in addition to all axioms in Definition 3.4.2, the following axioms.

v) The map $(a, b) \mapsto ab$ is continuous from $\mathcal{A} * \mathcal{A} \to \mathcal{A}$.

vi) The map $a \mapsto a^*$ is continuous from $\mathcal{A} \to \mathcal{A}$.

An elementary C*-algebra is a C*-algebra which is isomorphic to the compact operators on a Hilbert space.

Definition 3.4.4 (Elementary C*-bundle)**.** A C*-bundle is called *elementary* if every fibre is an elementary C*-algebra.

Definition 3.4.5 (Fell's condition)**.** An elementary C*-bundle over a space X satisfies *Fell's condition* if each $x \in X$ has a neighbourhood U such that there is a section f for which $f(y)$ is a rank-one projection for each $y \in U$.

Proposition 10.5.8 of [12] says that an elementary C*-bundle satisfies Fell's condition *if and only if* its algebra of sections vanishing at infinity is a continuous trace C*-algebra.

Definition 3.4.6 (Right action of a groupoid on a continuous Banach bundle)**.** Let $\pi_X\colon \mathcal{E} \to X$ be a Banach bundle, let G be a groupoid acting on X on the right. A G-action on $\mathcal{E}$ is a G-action on X by isometric isomorphisms $\alpha_{x,\gamma}\colon \mathcal{E}_{x\gamma} \to \mathcal{E}_x$ for each $\gamma \in G$ such that

i) $\alpha_{x,s(x)} = \mathrm{id}_{\mathcal{E}_x}$ for each $s(x) \in G^{(0)}$;

ii) if γ and γ' are composable then $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$;

iii) α makes $\mathcal{E}$ into a continuous left G-space.

There is a similar definition for a left G-space X. When G acts on a Banach bundle $\mathcal{E}$, then we say that $\mathcal{E}$ is a G-bundle.

Definition 3.4.7 (C*-G-bundle)**.** A G-C*-bundle is a pair $(\mathcal{A}, \alpha)$ where $\pi_{G^{(0)}}\colon \mathcal{A} \to G^{(0)}$ is a G-bundle and α is an action of G on $\mathcal{A}$ by *-isomorphisms.

Definition 3.4.8. For a groupoid G, let $\mathfrak{Br}(G)$ denote the collection of continuous C*-G-bundles $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is an elementary C*-bundle with separable fibres and which satisfies Fell's conditions.

Let H and G be groupoids, let X be a Hilsum-Skandalis morphism from H to G and let $(p\colon \mathcal{A} \to G^{(0)}, \alpha)$ be a G-C*-bundle. We induce an H-bundle $\mathcal{A}^X$ as follows: Define the fibre product $s_X^*(\mathcal{A}) := \{(x, a) \in X \times \mathcal{A} : s_X(x) = p(a)\}$. Define an action of G on this fibre product by

$$(x, a)\gamma = (x\gamma,\ \alpha_{\gamma^{-1}}(a)).$$

Then $s_X^*(\mathcal{A})$ becomes a principal G-space. Let $\mathcal{A}^X$ denote the quotient of $s_X^*(\mathcal{A})$ by the G-action, and denote the class of $(x, a) \in s_X^*(\mathcal{A})$ in the quotient by $[x, a]$. We show that $\mathcal{A}^X$ is an H-C*-bundle.

The C*-bundle: The assignment $[x, a] \mapsto r_X(x)$ defines a surjection from $\mathcal{A}^X$ to $H^{(0)}$. If r_X is an open map, then this surjection is also open [21, the discussion after Definition 2.14]. The map $a \mapsto [x, a]$ defines an isomorphism from $\mathcal{A}_{s_X(x)}$ to $\mathcal{A}^X_x$ showing that each fibre of the surjection $p_X\colon \mathcal{A}^X \to H^{(0)}$ is a C*-algebra.

H**-action:** For $\eta \in H$ define $\alpha^X_\eta : \mathcal{A}^X_x \to \mathcal{A}^X_{\eta x)}$ by

$$\alpha^X_\eta[x, a] := [\eta x, a].$$

Proposition 3.4.9. *Let H and G be groupoids and let X be a Hilsum-Skandalis morphism from H to G. If $(A, \alpha) \in \mathfrak{Br}(G)$, then $(A^X, \alpha^X) \in \mathfrak{Br}(H)$.*

Proof. The proof is the same as the proof of Proposition 2.15 in [21]. To prove continuity of the addition on $\mathcal{A}^X$ the freeness of the G-action and condition *iii)* in the definition of a Hilsum-Skandalis morphism in Definition 3.4.1 are used. ☐

The following is Definition 3.1 in [21]

Definition 3.4.10 (Morita equivalence of G-C*-bundles)**.** Two G-C*-bundles $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are *Morita equivalent* if there is an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bundle $\pi_X \colon \mathcal{X} \to X$ with an action V of G by isomorphisms such that

$$
\begin{aligned}
{}_{\mathcal{A}}\langle V_\gamma(x)\,,\, V_\gamma(y)\rangle &= \alpha_\gamma({}_{\mathcal{A}}\langle x\,,\, y\rangle),\\
\langle V_\gamma(x)\,,\, V_\gamma(y)\rangle_{\mathcal{B}} &= \beta_\gamma(\langle x\,,\, y\rangle_{\mathcal{B}}).
\end{aligned}
$$

In this case, we say that $(\mathcal{X}, V)$ implements a Morita equivalence between $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ and write $(\mathcal{A}, \alpha) \sim_{(\mathcal{X},V)} (\mathcal{B}, \beta)$. Morita equivalence is an equivalence relation of G-C*-bundles [21, Lemma 3.2].

Definition 3.4.11 (The Brauer group)**.** The set $\mathrm{Br}(G)$ of Morita equivalence classes of bundles in $\mathfrak{Br}(G)$ is called the *Brauer group* of G.

Let $\mathcal{A}$ and $\mathcal{B}$ be elementary C*-bundles over $G^{(0)}$. For every $(u, v) \in G^{(0)} \times G^{(0)}$, let $\mathcal{T}'_{(u,v)} := \mathcal{A}_u \otimes \mathcal{B}_v$, where $\mathcal{A}_u \otimes \mathcal{B}_v$ is the minimal tensor product. For $\phi \in \Gamma_0(G^{(0)}; \mathcal{A})$ and $\psi \in \Gamma_0(G^{(0)}; \mathcal{A})$ the map $(u, v) \mapsto \|\phi(u) \otimes \psi(v)\|$ is continuous. The set $\{\phi(u) \otimes \psi(v) : \phi \in \Gamma_0(G^{(0)}; \mathcal{A}),\ \ \psi \in \Gamma_0(G^{(0)}; \mathcal{B})\}$ is dense in $T_{(u,v)}$ for each $(v, u) \in G^{(0)} \times G^{(0)}$. We appeal to Theorem II.13.18 [14], which ensures that there is a Banach bundle $\mathcal{T}'$ over $G^{(0)} \times G^{(0)}$ that has fibres $\mathcal{T}'_{(u,v)}$ so that the functions $(u, v) \mapsto \phi(u) \otimes \psi(v)$ generate $\Gamma_0(G^{(0)} \times G^{(0)}; \mathcal{T}')$. We identify $G^{(0)}$ inside $G^{(0)} \times G^{(0)}$ via the diagonal embedding $u \mapsto (u, u)$ and denote the restriction of the bundle $\mathcal{T}'$ to $G^{(0)}$ by $\mathcal{T}$. To get a better idea about the fibres of the bundles, we denote the bundle $\mathcal{T}'$ by $\mathcal{A} \otimes \mathcal{B}$ and the bundle $\mathcal{T}$ by $\mathcal{A} \otimes_{G^{(0)}} \mathcal{B}$.

The bundle $\mathcal{A} \otimes \mathcal{B}$ and its restriction $\mathcal{A} \otimes_{G^{(0)}} \mathcal{B}$ are both elementary C*-bundles. If $\mathcal{A}$ and $\mathcal{B}$ satisfy Fell's condition, then so do $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{A} \otimes_{G^{(0)}} \mathcal{B}$.

Restriction of the action $\alpha \otimes \beta = \{\alpha_u \otimes \beta_v\}_{u,v \in G^{(0)}}$ to $G^{(0)}$ gives an action of G on $\mathcal{A} \otimes_{G^{(0)}} \mathcal{B}$. The continuity of the action is shown in [21, page 18].

Definition 3.4.12 (Conjugate Banach bundle)**.** Let $(p\colon \mathcal{A} \to G^{(0)}, \alpha)$ be a G-C*-bundle. The *conjugate G-C*-bundle* of $(\mathcal{A}, \alpha)$ is given by $(\bar{p}\colon \bar{\mathcal{A}} \to G^{(0)}, \bar{\alpha})$ where

i) $\bar{\mathcal{A}} = \mathcal{A}$ as a topological space;

ii) $\mathrm{id}\colon \mathcal{A} \to \bar{\mathcal{A}}$ is the identity map, $\bar{p}\colon \bar{\mathcal{A}} \to G^{(0)}$ is defined by $\bar{p}(\mathrm{id}(a)) = \mathrm{id}(p(a))$, the fibre $\bar{\mathcal{A}}_{\mathrm{id}(p(a))}$ is identified with the conjugate of $\mathcal{A}_{p(a)}$;

iii) $\bar{\alpha}_\gamma(\mathrm{id}(a)) := \mathrm{id}(\alpha_\gamma(a))$;

If $(\mathcal{A}, \alpha) \in \mathfrak{Br}(G)$ then $(\bar{\mathcal{A}}, \bar{\alpha}) \in \mathfrak{Br}(G)$.

Let $\mathcal{I}$ denote the trivial line bundle $G^{(0)} \times \mathbb{C}$ with the G-action I given by $(s(\gamma), z)\gamma = (r(\gamma), z)$.

Theorem 3.4.13. *The binary operation*

$$
[\mathcal{A}, \alpha][\mathcal{B}, \beta] = [\mathcal{A} \otimes_{G^{(0)}} \mathcal{B}, \alpha \otimes_{G^{(0)}} \beta], \tag{3.4.14}
$$

is well defined on $\mathrm{Br}(G)$*. Furthermore,* $\mathrm{Br}(G)$ *can be made into an abelian group where*

a. *the addition is defined by* (3.4.14)*;*

b. *the identity element is the class* $[\mathcal{I}, I]$*;*

c. *the inverse of* $[A, \alpha]$ *is given by* $[\bar{\mathcal{A}}, \bar{\alpha}]$*.*

Proof. Use Proposition 3.6 and Theorem 3.7 of [21]. □

Theorem 3.4.15. *If X is a Hilsum-Skandalis morphism from H to G then $[A, \alpha] \mapsto [\mathcal{A}^X, \alpha^X]$ is a homomorphism from* $\mathrm{Br}(G)$ *to* $\mathrm{Br}(H)$*.*

Proof. The proof of Theorem 4.1 in [21] goes through. □

3.5 Correspondences and KK-theory

A result of Renault from [33] says that a real 1-cocycle c, that is, an element of $\mathrm{Z}^1(G; \mathbb{R})$ gives an automorphism of $\mathrm{C}^*(G)$. In [26], Mesland shows that this automorphism along with the C^*-correspondence $\mathcal{H}(G)$ from $\mathrm{C}^*(G)$ to $\mathrm{C}^*(\ker(c))$ obtained by the obvious actions, gives an element of $\mathrm{KK}^{\mathbb{R}}(\mathrm{C}^*(G), \mathrm{C}^*(\ker(c))$.

In this section, we extend this result of Mesland using our definition of topological correspondences.

Mesland uses the theme that (G, Id_G) is a topological correspondence from G to $\ker(c)$. We shall assume that an open measured subgroupoid $H \subseteq G$ is given, such that the Haar system of G is H-quasi-invariant and there is a real continuous 1-cocycle c on G. Out of this data, we shall construct an $\mathbb{R}$-equivariant unbounded KK-cycle going from $\mathrm{C}^*(H)$ to $\mathrm{C}^*(\ker(c))$. In this section, *cocycle* means a continuous 1-cocycle, unless stated otherwise.

3.5.1 Unbounded KK-theory and construction of odd KK-cycles

Let R be a group, let A and B be C^*-algebras.

Definition 3.5.1 (Equivariant Hilbert module)**.** If B is an R-C^*-algebra, then a Hilbert B-module $\mathcal{H}$ is called R-equivariant if $\mathcal{H}$ is equipped with a strictly continuous R-action satisfying

i) $t(eb) = (te)tb$,

ii) $\langle te_1 , te_2 \rangle = t \langle e_1 , e_2 \rangle$

for all $t \in R$, $e, e_2, e_2 \in E$ and $b \in B$.

In this case, we call $\mathcal{H}$ a Hilbert R-B-module.

Let A and B be C^*-algebras and let $\mathcal{H}$ be a C^*-correspondence from A to B. Assume that A and B are R-C^*-algebras and that $\mathcal{H}$ is a Hilbert R-B-module. Let $\theta^{\mathcal{H}}_A \colon A \to \mathbb{B}_B(\mathcal{H})$ be the *-homomorphism which makes $\mathcal{H}$ into a C^*-correspondence from A to B. Let $\mathrm{a}^A_R \colon R \to \mathrm{Aut}(A)$ be the homomorphism that gives the action of R on A and let $\mathrm{U}^{\mathcal{H}}_R \colon R \to \mathcal{U}_H(\mathcal{H})$ be the homomorphism that gives the action of R on $\mathcal{H}$.

Definition 3.5.2. The action of A on $\mathcal{H}$ is R-*equivariant* if for every $t \in R$ and every $a \in A$ we have
$$\theta_A^{\mathcal{H}}(\mathrm{a}_R^A(t)(a)) = \mathrm{U}_R^{\mathcal{H}}(t)\,\theta_A^{\mathcal{H}}(a)\,\mathrm{U}_R^{\mathcal{H}}(t)^{-1}.$$

Definition 3.5.3. Let A and B be R-C*-algebras. An R-equivariant C*-correspondence from A to B is a C*-correspondence $\mathcal{H}$ from A to B such that

i) $\mathcal{H}$ is a Hilbert R-B-module,

ii) the action of A on $\mathcal{H}$ is R-equivariant.

In this case, we call $\mathcal{H}$ an R-C*-correspondence from A to B or simply an equivariant C*-correspondence from A to B, when the group is obvious.

Definition 3.5.4 (Regular operator [2])**.** Let $\mathcal{H}$ be a Hilbert B-module. A densely defined closed operator $D\colon \mathrm{Dom}(D) \to E$ is called regular if

i) D^* is densely defined in $\mathcal{H}$,

ii) $1 + DD^*$ has dense range.

Such an operator is automatically B-linear, and $\mathrm{Dom}(D)$ is a B-submodule of $\mathcal{H}$. There are two bounded operators related to D, which are called the *resolvent*[2] of D and the *bounded transform*. They are given as follows:
$$\begin{aligned}\text{the resolvent:}\quad & r(D) := (1 + D^*D)^{-1/2}.\\ \text{the bounded transform:}\quad & b(D) := D(1 + D^*D)^{-1/2}.\end{aligned}$$

Definition 3.5.5. An R-equivariant odd unbounded bimodule (or an odd unbounded KK-cycle) from an R-algebra A to an R-algebra B is a pair (E, D), where $\mathcal{H}$ is an R-C*-correspondence from A to B together with an unbounded regular operator D on $\mathcal{H}$ such that:

i. $[D, a] \in \mathbb{B}(E)$ for all a in a dense sub-algebra of A;

ii. $a \cdot r(D) \in \mathbb{K}_B(E)$ for all a in a dense sub-algebra of A;

iii. the map $g \mapsto D - gDg^{-1}$ is a strictly continuous map $R \to \mathbb{B}(E)$.

Let (G, λ, σ) be a measured groupoid, let $c \in \mathrm{Z}^1(G, \mathbb{R})$ and let K denote the kernel of the cocycle, that is, $K := \{\gamma \in G : c(\gamma) = 0\}$. Then K is a closed subgroupoid of G with $G^{(0)} = K^{(0)}$ which acts on G from the left as well as from the right by multiplication. The momentum maps for these actions are the range map and the source map from K to $G^{(0)} = K^{(0)}$, respectively.

We recall some of Mesland's definitions:

Definition 3.5.6. A cocycle $c \in \mathrm{Z}^1(G; \mathbb{R})$ is regular if $\ker(c) := H$ admits a Haar system, and c is exact if it is regular and the map
$$\begin{aligned} r \times c\colon G &\to G^{(0)} \times \mathbb{R},\\ \gamma &\mapsto (r(\gamma), c(\gamma))\end{aligned}$$
is a quotient map onto its image.

[2]This is not a good terminology, since the terms *resolvent* and *resolvent set* are used in the elementary theory of C*-algebras . But we adopt the terminology that Mesland created. Baaj and Julg [2] do not name the operators.

The term *exact* is not an appropriate term *here*, because it means something different in the groupoid cohomology. I am not aware if these two meanings are related to each other. But we shall stick to the terminology in [26] in this section.

An important observation regarding exact cocycles is:

Lemma 3.5.7 ([26] Lemma 3.1.3). *If c is an exact real cocycle on G, then the map $\gamma \mapsto (r(\gamma), c(\gamma))$ from G/K to $G^{(0)} \times \mathbb{R}$ is a homeomorphism onto its image.*

Assume that c is a regular cocycle and let β be a Haar system on the subgroupoid $\ker(c) = K$. Then (G, λ^{-1}) is a right K-space with the right multiplication action and λ^{-1} is a continuous K-invariant family of measures. The inner product of $f, g \in C_c(G)$ is simply $(f^* * g)|_K$. Proposition 5.1 in [33] says that for each $t \in \hat{\mathbb{R}} = \mathbb{R}$ a cocycle $c \in \mathrm{Z}^1(G, \mathbb{R})$ gives an automorphism u_t of the *-algebra $C_c(G)$ by the formula

$$u_t(f)(\gamma) = \mathrm{e}^{\mathrm{i}t\, c(\gamma)} f(\gamma).$$

Furthermore, the proposition also says that this automorphism extends to an automorphism of $\mathrm{C}^*(G)$. Proposition 5.3 in the same book says that the group of automorphisms $\{u_t\}_{t\in\mathbb{R}}$ is inner if $c \in \mathrm{B}^1(G, \mathbb{R})$. Similar statements hold when $\mathrm{C}^*(G, \lambda)$ is replaced by $\mathrm{C}^*_\mathrm{r}(G, \lambda)$. Since K is a subgroupoid of G, the cocycle c can be restricted to a cocycle of K, which we denote by c again (instead of $c|_K$). Since $c_K = 0$, $u_t|_{C_c(K)} = \mathrm{Id}$ for all $t \in \mathbb{R}$ gives a 1-parameter group of automorphisms of $\mathrm{C}^*(K)$. Thus $\mathrm{C}^*(G)$ and $\mathrm{C}^*(K)$ become $\mathbb{R}$-algebras.

Since $K \subseteq G$ is closed, G is a proper right K-space (see Example 3.1.16). λ^{-1} is a right K-invariant family of measures on G. Complete the right $C_c(K, \beta)$-module $C_c(G, \lambda^{-1})$ into a Hilbert $\mathrm{C}^*(K, \beta)$-module, which we denote by $\mathcal{H}(G)$. Proposition 3.6 of [26] shows that for each $t \in \mathbb{R}$ the operator $u_t \colon C_c(G) \to C_c(G)$ defined above extends to a 1-parameter group of unitaries in $\mathbb{B}_{\mathrm{C}^*(K)}(\mathcal{H}(G))$ (or in $\mathbb{B}_{\mathrm{C}^*_r(K)}(\mathcal{H}_r(G))$.

Let H be an open subgroupoid of G such that $G^{(0)} = H^{(0)}$ and let α be a Haar system for H. Furthermore, assume that λ^u is (H, α)-quasi-invariant for each $u \in G^{(0)}$. Let Δ_u be the modular function for the quasi-invariance λ_u. For notational convenience we shall drop the suffix u and simply write Δ. Let H act on G by left multiplication. Then (G, λ, Δ) is a topological correspondence from H to K. The family $\{u_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of automorphisms of $\mathrm{C}^*(H)$, as described previously. We abuse the notation and keep writing u_t for the actions of $\mathbb{R}$ on $C_c(H)$, $\mathrm{C}^*(H)$ or $\mathrm{C}^*_r(H)$.

Since $H, K \subseteq G$, in the computations below the subscripts to the sources and the range maps do not matter a lot.

Proposition 3.5.8. *Let (G, λ), (H, α), c and u_t be as above. Then u_t extends to a 1-parameter group of unitaries in $\mathrm{C}^*(H)$ (or $\mathrm{C}^*_r(H)$). Furthermore, $\mathcal{H}(G)$ (respectively, $\mathcal{H}_r(G)$) is an $\mathbb{R}$-C^*-correspondence from $\mathrm{C}^*(H, \alpha)$ to $\mathrm{C}^*(K, \beta)$. A similar statement holds for the reduced C^*-algebras.*

Proof. The first claim is a direct consequence of Proposition 5.1 in [33]. We check that the conditions in Definition 3.5.3 hold.

Let $f \in C_c(G)$ and $\psi \in C_c(K)$, then the following calculations confirm that Condition (i) in Definition 3.5.3 holds.

$$\begin{aligned}(u_t(f\psi))(\gamma) &= \mathrm{e}^{\mathrm{i}tc(\gamma)} \int f(\gamma\zeta)\psi(\zeta^{-1})\,\mathrm{d}\beta^{s_G(\gamma)}(\zeta)\\ &= \int \mathrm{e}^{\mathrm{i}tc(\gamma\zeta)} f(\gamma\zeta)\,\mathrm{e}^{\mathrm{i}tc(\zeta^{-1})}\psi(\zeta^{-1})\,\mathrm{d}\beta^{s_G(\gamma)}(\zeta)\\ &= (u_t(f)u_t(\psi))(\gamma).\end{aligned}$$

If $f, g \in C_c(G)$, then

$$\begin{aligned}\langle u_t(f)\,,u_t(g)\rangle\,(\kappa) &= \int \overline{u_t(f)}(\gamma)u_t(g)(\gamma^{-1}\kappa)\,\mathrm{d}\lambda^{r_K(\kappa)}(\gamma)\\ &= \int \mathrm{e}^{\mathrm{i}tc(\gamma)}\overline{f(\gamma)}\,\mathrm{e}^{\mathrm{i}tc(\gamma^{-1}\kappa)}(\gamma^{-1}\kappa)\,\mathrm{d}\lambda^{r_K(\kappa)}(\gamma)\\ &= \mathrm{e}^{\mathrm{i}tc(\kappa)} \int \overline{f(\gamma)}(\gamma^{-1}\kappa)\,\mathrm{d}\lambda^{r_K(\kappa)}(\gamma)\\ &= \mathrm{e}^{\mathrm{i}tc(\kappa)}\,\langle f\,,g\rangle\,(\kappa)\\ &= u_t(\langle f\,,g\rangle)(\kappa)\end{aligned}$$

Now we check Condition (ii) in the definition. We use a, θ and U instead of $\mathrm{a}_{\mathbb{R}}^{\mathrm{C}^*(H,\alpha)}$, $\theta^{\mathcal{H}}_{\mathrm{C}^*(H,\alpha)}$ and $\mathrm{U}^{\mathcal{H}}_{\mathbb{R}}$, respectively. Let $\phi \in C_c(H)$ and $f \in C_c(G)$, then

$$\begin{aligned}(\theta\,(\mathrm{a}(t)(\phi))\,f)\,(\gamma) &= \int \mathrm{a}(t)(\phi)(\eta)\,f(\eta^{-1}\gamma)\,\Delta^{1/2}(\eta,\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\eta)\\ &= \int \mathrm{e}^{\mathrm{i}tc(\eta)}\phi(\eta)\,f(\eta^{-1}\gamma)\,\Delta^{1/2}(\eta,\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\eta)\\ &= \mathrm{e}^{\mathrm{i}tc(\gamma)} \int \phi(\eta)\,\mathrm{e}^{\mathrm{i}tc(\gamma^{-1}\eta)}f(\eta^{-1}\gamma)\,\Delta^{1/2}(\eta,\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\eta)\\ &= \mathrm{e}^{\mathrm{i}tc(\gamma)} \int \phi(\eta)\,\Big(\mathrm{U}(t)^{-1}(f)\Big)\,(\eta^{-1}\gamma)\,\Delta^{1/2}(\eta,\gamma)\,\mathrm{d}\alpha^{r_G(\gamma)}(\eta)\\ &= \mathrm{e}^{\mathrm{i}tc(\gamma)}\theta(\phi)\,\Big(\mathrm{U}(t)^{-1}(f)\Big)\,(\gamma)\\ &= \mathrm{U}(t)\,\Big(\theta(\phi)\,\mathrm{U}(t)^{-1}(f)\Big)\,(\gamma).\end{aligned}$$

□

We need an equivariant operator on $\mathcal{H}(G)$ which we get from the cocycle c as follows:

Proposition 3.5.9. *Let G, c and K be as above. Then the operator*

$$\begin{aligned}D :&C_c(G) \to C_c(G);\\ D :&f(\gamma) \mapsto c(\gamma)f(\gamma),\end{aligned}$$

is a $C_c(K)$-linear derivation of $C_c(G)$ considered as a bimodule over itself. Moreover, it extends to a self-adjoint regular operator on the $\mathrm{C}^*(K)$*-Hilbert module* $\mathcal{H}(G)$*.*

Proof. Similar to the proof of Proposition 3.8 in [26]. The only difference is that we have to plug-in the adjoining function for the left action. □

Finally, all these pieces are put together in the following theorem:

Theorem 3.5.10. *Let (G, λ) be a second countable locally compact Hausdorff groupoid with a Haar system, let c be a real exact cocycle on G and let H be an open subgroupoid of G such that $H^{(0)} = G^{(0)}$. Let α be a Haar system for H. If for each $e \in G^{(0)}$, the measure λ^e is (H, α)-quasi-invariant, then the operator D in Proposition 3.5.9 makes the $\mathbb{R}$-equivariant correspondence $(\mathcal{H}(G), D)$ into an odd $\mathbb{R}$-equivariant unbounded KK-bimodule from $\mathrm{C}^*(H)$ to $\mathrm{C}^*(K)$.*

Proof. Given $\phi \in C_c(H)$, we use Proposition 3.5.9 to see that $[D, \phi]g = D(\phi) * g$ for all $g \in C_c(G)$. Hence using the same proposition for each ϕ, we can see that the commutator $[D, \phi]$ is bounded.

Next we show that $\phi(1 + DD^*)^{-1}$ has a $\mathrm{C}^*(K)$-compact resolvent. $\phi(1 + DD^*)^{-1}$ acts on $g \in C_c(G)$ as:

$$\phi(1 + DD^*)^{-1} \circ g(\omega) = \int_G f(\gamma)(1 + c^2(\gamma^{-1}\omega))^{-1}\, \Delta(\gamma, \gamma^{-1}\omega)\, \mathrm{d}\alpha^{r(\omega)}(\gamma).$$

Here Δ is the adjoining function.

The action of K on G is free and proper. Using the standard theory of Morita equivalence we can see that $\mathrm{C}^*(G \ltimes G/K) = \mathbb{K}_{\mathrm{C}^*(K)}(\mathcal{H}(G))$. The action of $\Psi \in C_c(G \ltimes G/K)$ on g is given by

$$\Psi g(\omega) = \int_{G \ltimes G/K} \Psi(\gamma', [\gamma''])g(\gamma'^{-1}\omega)\, \mathrm{d}\lambda_2(\gamma', [\gamma'']),$$

where λ_2 is the Haar system on $G \ltimes G/K$ induced by λ on G. Since $G \ltimes G/K$ is a transformation groupoid, this Haar system can be found easily and the entries in the previous equation can be simplified. After the simplification, the equation becomes:

$$\Psi g(\omega) = \int_G \Psi(\gamma, [\gamma^{-1}\omega])g(\gamma^{-1}\omega)\, \mathrm{d}\lambda^{r(\omega)}(\gamma).$$

Note that, since the measure on G is $G \ltimes G/K$-invariant, the adjoining function is the constant 1. Looking at the simplified version of the action of a compact operator, it is enough if for each $\phi \in C_c(H) \subseteq C_c(G)$ the kernel

$$k_\phi(\gamma, [\omega]) := \phi(\gamma)(1 + c^2(\omega))^{-1}\, \Delta(\gamma, \omega)$$

is a norm limit of elements in $C_c(G \ltimes G/K)$. In the rest of the proof, we write k instead of k_ϕ.

The cocycle c induces a homomorphism $\bar{c}\colon G/K \to \mathbb{R}$. Lemma 3.5.7 identifies G/K with its image in $G^{(0)} \times \mathbb{R}$. Using these facts, for $n \in \mathbb{N}$ define subsets K_n of G/K as

$$K_n = r_G(\mathrm{supp}(\phi) \times \mathbb{R}) \cap \bar{c}^{-1}([-n, n]).$$

Then $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ is an increasing covering of $r_G(\mathrm{supp}(\phi) \times \mathbb{R}) \cap G/K$ by compact sets. We can assume that the image of c is not a bounded subset of $\mathbb{R}$ and that $K_n \neq K_{n+1}$ for any n. If $K_n = K_{n+1}$ for some n, then just renumber them. Since the action of K on G is free and proper, we can take functions $e_n \in C(G/K, [0, 1])$ such that $e_n = 1$ on K_n and 0 outside K_{n+1}. Define

$$k^n(\gamma, [\omega]) := e_n[\omega]k(\gamma, [\omega]).$$

We show that the sequence $\{k^n\}$ is a Cauchy sequence in the I-norm. Then the sequence will be Cauchy for the reduced C*-norm and the full C*-norm as well. Let $n > m$ be natural numbers.

$$\begin{aligned}
\|k^n - k^m\| &= \sup_{[\omega]\in G/K} \int_{G\ltimes G/K} |k^n(\gamma,[\omega]) - k^m(\gamma,[\omega])| \,\mathrm{d}\lambda_2^{[\omega]}(\gamma,[\omega]) \\
&= \sup_{[\omega]\in G/K} \int_G |k^n(\gamma,[\omega]) - k^m(\gamma,[\omega])| \,\mathrm{d}\lambda^{r_G(\omega)}(\gamma) \\
&= \sup_{[\omega]\in G/K} \int_G |(e_n - e_m)[\omega]\ \phi(\gamma)(1+c^2[\omega])^{-1}\,\Delta(\gamma,\omega)| \,\mathrm{d}\lambda^{r(\omega)}(\gamma) \\
&\le \frac{m}{1+m^2} \sup_{[\omega]\in G/K} \int_G |f(\gamma)| \,\mathrm{d}\lambda^{r_G(\omega)}(\gamma) \\
&< \frac{1}{m}\|f\|_{I,r}
\end{aligned}$$

Here $\|\|_{I,r}$ is the right I-norm. One can work with λ^{-1} and prove a similar result for the left I-norm and finally conclude that

$$\|k^n - k^m\| < \frac{1}{m}\|f\|_I.$$

By construction $\phi(1+D^2)^{-1}\Delta$ is the limit of this sequence.

It is clear that D is a generator of the $\mathbb{R}$-action on $\mathcal{H}(G)$. Hence D commutes with the $\mathbb{R}$-action, so this KK-cycle is $\mathbb{R}$-equivariant. □

Proposition 3.5.11. *Assume that we have the same data as in Theorem 3.5.10 and the same hypotheses. A similar result as in Theorem 3.5.10 holds for $(\mathcal{H}_r(G), D_r)$ from $C^*_r(H)$ to $C^*_r(K)$.*

We developed this theorem keeping Example 3.1.7 and Example 3.1.13 in mind. Hence transformation groupoids of group actions provide good examples of this theorem.

Example 3.5.12. In Example 3.1.7, if K is the kernel of a homomorphism from G to $\mathbb{R}$ and $H \subseteq G$ is open, then we get an element of $\mathrm{KK}^{\mathbb{R}}(C^*(H), C^*(K))$. If G is a discrete group, then H can be any subgroup of G.

A concrete example is: let $S^1 \rtimes_\theta \mathbb{Z}$ be the groupoid corresponding to the noncommutative 2-torus A_θ for an irrational $\theta \in \mathbb{R}$. Then the projection map onto the second factor $\pi_2\colon S^1 \ltimes_\theta \mathbb{Z} \to \mathbb{Z}$ is an $\mathbb{R}$-valued cocycle. For $n \in \mathbb{N}$, this projection gives us an element of $\mathrm{KK}^{\mathbb{R}}(C^*(S^1 \rtimes_\theta n\mathbb{Z}), C_0(S^1))$.

Example 3.5.13. Now we generalise the previous example using Example 3.1.13. Let G, H, K and X be as In Example 3.1.13. Recall that $\mathbf{G} = X \rtimes G, \mathbf{H} = X \rtimes H$ and $\mathbf{K} = X \rtimes K$. If $\mathbf{H} \subseteq \mathbf{G}$ is open and $\mathbf{K} = \ker(c)$ for a homomorphism from $\mathbf{K} \to \mathbb{R}$, then we get an element of $\mathrm{KK}^{\mathbb{R}}(C^*(\mathbf{H}), C^*(\mathbf{K}))$.

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Publications

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Conference Talks and visits

1. Presented work at
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2. Visiting fellow at Université de Orléans, March 2013.

3. Invited speaker at
   Quantum Geometry and Matter 2013 (QGM13), Trieste, Italy,

April 2013. Topic of talk was *A notion of Fell bundles morphisms and a 2-category of Fell bundles*.

4. A short talk at
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Achievements and scholarships

1. Erasmus Mundus *EuroIndia* scholarship for doctoral studies in Germany, 2010.
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Teaching

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