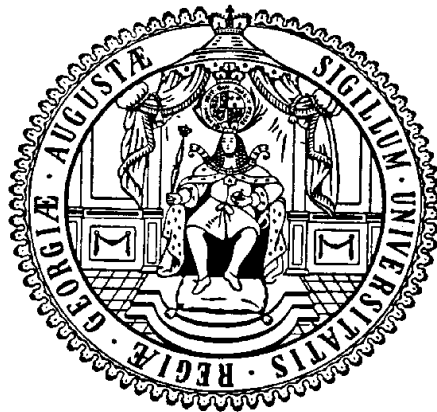


DIOPHANTINE EQUATIONS IN MANY VARIABLES



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ABSTRACT

Let \mathbb{K} denote a p -adic field and $F_1, \dots, F_r \in \mathbb{K}[x_1, \dots, x_n]$ be forms with respective degrees d_1, \dots, d_r . A contemporary version of a conjecture attributed to E. Artin states that F_1, \dots, F_r have a common non-trivial zero whenever

$$n > d_1^2 + \dots + d_r^2.$$

We prove this for a single quintic form (i.e. $r = 1, d_1 = 5$), provided that the cardinality of the residue class field exceeds 9. We also verify the conjecture for a system comprising a cubic and a quadratic form (i.e. $r = 2, d_1 = 3, d_2 = 2$), whenever the residue class field is of characteristic at least 13 and has more than 37 elements.

ZUSAMMENFASSUNG

Seien \mathbb{K} ein p -adischer Körper und $F_1, \dots, F_r \in \mathbb{K}[x_1, \dots, x_n]$ Formen mit dazugehörigen Graden d_1, \dots, d_r . Eine Vermutung, die auf E. Artin zurückgeführt wird, besagt in moderner Fassung, dass F_1, \dots, F_r eine nicht triviale gemeinsame Nullstelle besitzen falls

$$n > d_1^2 + \dots + d_r^2.$$

Wir beweisen dies im Falle einer einzelnen quintischen Form, vorausgesetzt die Kardinalität des Restklassenkörpers ist größer als 9. Zudem verifizieren wir die Vermutung für ein System bestehend aus einer kubischen und einer quadratischen Form, falls der Restklassenkörper mindestens Charakteristik 13 und mehr als 37 Elemente hat.

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1. INTRODUCTION

Let \mathbb{K} be a finite extension of the field \mathbb{Q}_p and $F_1, \dots, F_r \in \mathbb{K}[x_1, \dots, x_n]$ be forms with respective degrees d_1, \dots, d_r . A modern version of an old conjecture attributed to E. Artin states that F_1, \dots, F_r have a common non-trivial zero whenever

$$n > d_1^2 + \dots + d_r^2.$$

Unfortunately, this has only been confirmed for a single quadratic form (Hasse [13]), a single cubic form (Lewis [19]) and a system comprising two quadratic forms (Dem'yanov [9] and independently Birch, Lewis and Murphy [4]). In fact counterexamples are known for many degrees d_1, \dots, d_r . The first counterexample was given by Terjanian [22]. He has found the explicit form

$$\begin{aligned} &G(x_1, x_2, x_3) + G(x_4, x_5, x_6) + G(x_7, x_8, x_9) \\ &+ 4G(x_{10}, x_{11}, x_{12}) + 4G(x_{13}, x_{14}, x_{15}) + 4G(x_{16}, x_{17}, x_{18}) \end{aligned}$$

where

$$G(x_1, x_2, x_3) = \sum_{i=1}^3 x_i^4 - \sum_{0 \leq i < j \leq 3} x_i^2 x_j^2 - x_1 x_2 x_3 (x_1 + x_2 + x_3)$$

that does not have a non-trivial zero over the 2-adic numbers.

Although false in general the conjecture has been partially verified by Ax and Kochen [1]. They showed that for every tuple (d_1, \dots, d_r) and degree η of the field extension \mathbb{K}/\mathbb{Q}_p there exists a positive integer $q_0(d_1, \dots, d_r, \eta)$, such that Artin's Conjecture for forms of degrees d_1, \dots, d_r holds whenever the cardinality q of the residue class field exceeds $q_0(d_1, \dots, d_r, \eta)$. However, little is known about the actual values of $q_0(d_1, \dots, d_r, \eta)$. For a single form of degree d Brown [6] has given a huge, but explicit bound. If we write $a \uparrow b$ for a^b it can be stated as

$$(1) \quad q_0(d, 1) \leq 2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow (d \uparrow (11 \uparrow (4d))))))).$$

This was intended as a neat rather than best possible expression, but optimising the argument is expected to lead to similar magnitudes.

If d is neither composite nor a sum of composite numbers, better bounds are available. Since these hold independently of $[\mathbb{K} : \mathbb{Q}_p]$ we shall omit η in our notion of q_0 . Besides the classical result $q_0(2) = 1$ and $q_0(3) = 1$ this concerns in fact $d = 5, 7, 11$ only. All other degrees are composite or a sum of composite numbers. Leep and Yeomans [18] have shown $q_0(5) \leq 43$ and later this has been improved by Heath-Brown [14]. He proved that a quintic form over \mathbb{Q}_p possesses a non-trivial zero, provided $p \geq 17$. For septic and undecic forms bounds $q_0(7) \leq 883$ and

$q_0(11) \leq 8053$ are due to Wooley [24].

These results use a p -adic minimisation procedure which has been developed by Lewis [19], Birch and Lewis [2] and Laxton and Lewis [16] and ultimately rely on lifting a non-singular zero via Hensel's Lemma.

In this thesis we shall establish the result $q_0(5) \leq 9$. In order to ease notation, we will from now on call a finite extension of the field \mathbb{Q}_p a p -adic field.

Theorem 1. *Let $F(x_1, \dots, x_n)$ be a quintic form with at least $n \geq 26$ variables over a p -adic field \mathbb{K} with residue class field of cardinality $q > 9$. Then there exists a non-zero vector $\mathbf{x} \in \mathbb{K}^n$ with $F(\mathbf{x}) = 0$.*

As we are interested in a zero, we may assume that F is defined over the ring of integers $\mathcal{O}_{\mathbb{K}}$. The proof is based on a p -adic minimisation procedure which has been developed by Schmidt [20] and generalised by Zahid [25]. By applying their technique we may assume that the reduction of F over the residue class field, denoted by $\theta(F)$, is a non-degenerate form with at least $6 + s$ variables, where s is the maximal affine dimension of a vector space on which $\theta(F)$ vanishes.¹ If $\theta(F)$ possesses a non-singular zero, it can be lifted by Hensel's Lemma to a non-trivial zero of F . We recall that a non-singular zero is one which is not a simultaneous zero of the partial derivatives.

We shall use certain properties of quintic forms to choose a suitable subspace and show that it contains a non-singular zero. For $q = 11, 13, 16, 25, 27, 32$ this is accomplished with the help of computer calculations.² The author was able to carry those out on his personal notebook. This, together with the previously mentioned results of Leep and Yeomans and Heath-Brown, yields Theorem 1.

There is numerical evidence to suggest that the imposed constraint on q can be further reduced. Given the current state of technology, it certainly seems doubtful to expect an answer for all q at this stage.

On the other hand there are a number of results in the same vein concerning systems of forms. E. Schuur [21], improving on work of Birch and Lewis [3], has shown that three quadratic forms have a common non-trivial zero if the cardinality of the residue class field exceeds 9. Heath-Brown [15] has established Artin's Conjecture for a system of r quadratic forms, provided $q > (2r)^r$. Recently Zahid [26] has been

¹Precise definitions of "s" and "non-degenerate" can be found at the beginning of Chapter 3.

²For $q = 25, 27$ and 32 a false attempt of proof was made in Theorem 7 of the author's master's thesis.

the first addressing forms of differing degrees by proving $q_0(3, 2) \leq 293$.

We shall establish the following improvement, provided the characteristic of the residue class field is at least 13.

Theorem 2. *Let $C(x_1, \dots, x_n)$ be a cubic and $Q(x_1, \dots, x_n)$ a quadratic form with at least $n \geq 14$ variables over a p -adic field \mathbb{K} with residue class field of cardinality $q > 37$ and characteristic at least 13. Then there exists a non-zero vector $\mathbf{x} \in \mathbb{K}^n$ with $C(\mathbf{x}) = Q(\mathbf{x}) = 0$.*

In order to prove Theorem 2 we follow the path laid by Zahid. By exploiting Schmidt's minimisation procedure Zahid extracts information on the reduction of the system over the residue class field. He then shows that $\theta(C)$ and $\theta(Q)$ have a common non-singular zero, provided a certain absolutely irreducible quartic form $H \in \mathbb{F}_q[x_2, \dots, x_n]$ possesses a non-singular and $q > 5$. Recall that a polynomial over a field is absolutely irreducible if it is irreducible over the algebraic closure of the field. Also note that a common non-singular zero of forms f_1, \dots, f_r over \mathbb{F}_q is a zero \mathbf{x} such that $\nabla f_1(\mathbf{x}), \dots, \nabla f_r(\mathbf{x})$ are linearly independent. By a slicing argument there exists a vector $\xi \in \mathbb{F}_q^{3n-5}$ such that

$$H|_{\xi}(X, Y) := H(\xi_1 + X, \xi_2 + \xi_n X + \xi_{2n-4} Y, \dots, \xi_{n-1} + \xi_{2n-3} X + \xi_{3n-5} Y)$$

remains absolutely irreducible, provided $q > 296$. Using the Lang-Weil Bound he concludes that $H|_{\xi}$, and hence H , has a non-singular zero. The improvement in Theorem 2 is due to a more effective slicing process. Using a quantitative Bertini theorem by Lecerf [17] we obtain ξ as above yet require a much weaker condition on the cardinality of the residue class field, videlicet $q > 37$.

An alternative approach towards Artin's Conjecture is to ask for the minimal number of variables needed to ensure the existence of a non-trivial zero. Brauer [5] has shown that there is a finite non-negative integer $v(d_1, \dots, d_r)$ such that F_1, \dots, F_r possess a common non-trivial zero whenever

$$n \geq v(d_1, \dots, d_r).$$

His proof reduces the problem to diagonal forms, which have been studied extensively (see in particular [8]). Refined subsequent results use quasi-diagonalisation techniques. The best general bound is due to Wooley [23]. For a system comprising r forms of degree d over \mathbb{Q}_p he

showed that $n > (rd^2)^{2^{d-1}}$ suffices.³

For a number of degrees better bounds are available. Firstly, we can extract better estimates from Wooley's proof for specific d . Secondly, Heath-Brown [14] considerably improved the bound for a single quartic form by establishing $v_{\mathbb{Q}_p}(4) \leq 4222$. Here we write $v_{\mathbb{Q}_p}$ to indicate the restriction to forms over \mathbb{Q}_p . His proof has been adapted by Zahid [25] to show $v_{\mathbb{Q}_p}(5) \leq 4562912$.

Heath-Brown's method provides better results if the involved degrees are not multiples of p . The author has developed a variant yielding improved bounds if p does divide the degree (see [11]). As an example, he proves that $v_{\mathbb{Q}_p}(3, 3) \leq 132$ and $v_{\mathbb{Q}_p}(4) \leq 3192$. To prove these bounds the author has by successively choosing and recycling vectors constructed a subspace on which the forms involved are of a special shape. It transpires that this can be done if systems comprising a certain number of quadratic and linear forms have a non-trivial zero. Finally, a version of Hensel's Lemma can be applied.

This method would enormously benefit from better results on systems of quadratics. In contrast, it is not hard to see that $v(2, 2, 2, 2) \leq v(4)$.

This thesis is organised as follows. Chapter 2 gives an account of Schmidt's minimisation procedure on which all our results are based. In Chapter 3 a proof of Theorem 1 on a single quintic is presented. The case of a cubic and a quadratic form is discussed in Chapter 4. The final Chapter 5 provides some of the source code as used in the proof of Theorem 1. Extracts from this thesis have been made available online at [10] and [11]⁴.

³Wooley also gives bounds for the more general situation of forms over p -adic fields.

⁴In [11], this only concerns the introduction.

2. SCHMIDT'S MINIMISATION ARGUMENT

In this chapter we shall introduce a powerful minimisation argument by Schmidt [20]. We will, however, follow a more recent and generalised account by Zahid [26]. His version is the first that extends to systems comprising forms of differing degrees, which allows us to discuss common zeros of a cubic and a quadratic form in Chapter 4. The reader familiar with this argument or chiefly interested in the new ideas involved in the proof of Theorem 1 or Theorem 2 is referred to Chapter 3 or Chapter 4, respectively.

2.1. Statement of the main theorem. Let \mathbb{K} denote a p -adic field with residue class field \mathbb{F}_q and ring of integers $\mathcal{O}_{\mathbb{K}}$. We shall write π for a uniformiser of $\mathcal{O}_{\mathbb{K}}$. Recall that every non-zero $a \in \mathbb{K}$ can be uniquely written as $a = u\pi^l$, where $u \in \mathcal{O}_{\mathbb{K}}$ is a unit and $l \geq 0$, and that its normalised valuation is given by $\nu(a) = l$. Also, note that every non-zero $a \in \mathbb{K}$ has norm $|a| = p^{-\nu(a)}$.

Let $\mathbf{F} = (F_1, \dots, F_r)$ be a system of forms over \mathbb{K} in n variables comprising r forms of degree $d_1 \geq \dots \geq d_r$. As we are interested in a zero, we may assume from now on that F_1, \dots, F_r are defined over $\mathcal{O}_{\mathbb{K}}$.

Before we can state the main theorem we need to introduce a few definitions. Let A be an $n \times n$ -matrix over $\mathcal{O}_{\mathbb{K}}$ and B be an upper-triangular $r \times r$ -matrix with entries

$$(2) \quad (B)_{ij} = \pi^{-c_i} G_{ij}$$

where c_i is a non-negative integer, $G_{ii} = 1$ and otherwise $G_{ij} \in \mathcal{O}_{\mathbb{K}}[x_1, \dots, x_n]$ any form such that

$$\deg(G_{ij}) = \deg(F_i) - \deg(F_j).$$

We say that \mathbf{F} and another system $\hat{\mathbf{F}}$ of r forms over $\mathcal{O}_{\mathbb{K}}$ with n variables are equivalent if there exist A, B as described above, but $c_i = 0$ for all $1 \leq i \leq r$, such that

$$\mathbf{F}(\mathbf{x}) = B\hat{\mathbf{F}}(A\mathbf{x}) \quad \text{and} \quad \nu(\det(A)) = 0.$$

We are now relaxing the last condition and allowing any non-negative integer values for c_i for all $1 \leq i \leq r$. For \mathbf{F} , $\hat{\mathbf{F}}$ and a vector $\Omega = (\omega_1, \dots, \omega_r)$ with positive rational⁵ components we write

$$(3) \quad \mathbf{F} \underset{\Omega}{\sim} \hat{\mathbf{F}}$$

⁵In [26], it is not mentioned that Ω is required to be a vector of rational numbers. This, however, is crucial for the proof of Lemma 1.

if there are A and B as in (2) such that

$$\mathbf{F}(\mathbf{x}) = B\hat{\mathbf{F}}(A\mathbf{x}) \quad \text{and} \quad \sum_{i=1}^r c_i \omega_i - \nu(\det(A)) > 0.$$

If this holds, \mathbf{F} possesses a non-trivial zero if and only if $\hat{\mathbf{F}}$ does. With this key definition at hand we declare a system \mathbf{F} to be Ω -bottomless if there exists an infinite sequence of systems $(\mathbf{F}_i)_i$ such that

$$\mathbf{F} \underset{\Omega}{\succ} \mathbf{F}_1 \underset{\Omega}{\succ} \mathbf{F}_2 \underset{\Omega}{\succ} \mathbf{F}_3 \dots$$

A system that is not Ω -bottomless is said to be Ω -bottomed. An Ω -bottomed system \mathbf{F} is called Ω -reduced if there does not exist a system $\hat{\mathbf{F}}$ such that

$$\mathbf{F} \underset{\Omega}{\succ} \hat{\mathbf{F}}.$$

For any finite set of positive integers $S = \{d_1, \dots, d_s\}$ we set

$$v(S) := v(d_1, \dots, d_s).$$

As defined in the introduction, this is the least integer such that any system comprising forms of degree d_1, \dots, d_s has a non-trivial zero. We are now in the position to state the main result of the minimisation procedure as proved by Zahid.

Theorem 3 (Theorem 2.1, [26]). *Let $S \subset \{d_1, \dots, d_r\}$ denote any subset of cardinality $r-1$ with indexing set I such that $v(S)$ is maximal. Also let $d \in \{d_1, \dots, d_r\} - S$. Then, provided that*

$$(4) \quad n \geq v(S) + d^2,$$

there exists some $\Omega = (\omega_1, \dots, \omega_r) \in \mathbb{Q}_{>0}^r$ such that $\omega_i > d_i$ for each $1 \leq i \leq r$ and such that every Ω -bottomless system \mathbf{F} defined over $\mathcal{O}_{\mathbb{K}}$ has a p -adic zero.

2.2. Proof of Theorem 3. Zahid's proof relies on the fact that every Ω -bottomless system is equivalent to a so-called Ω -special system. It is then shown that any Ω -special system possesses a non-trivial zero. We will follow his account [26] closely.

In order to introduce the notion of an Ω -bottomless system, we associate with a form F_i a unique symmetric multilinear form $M_{F_i}(\mathbf{x}_1, \dots, \mathbf{x}_{d_i})$ such that

$$F_i(\mathbf{x}) = M_{F_i}(\mathbf{x}, \dots, \mathbf{x}).$$

As \mathbb{K} is of characteristic zero, there always exists such a unique form. \mathbf{F} is called Ω -special if there are non-negative integers a_1, \dots, a_n and b_1, \dots, b_r satisfying the relation

$$(5) \quad a_1 + \dots + a_n < \omega_1 b_1 + \dots + \omega_r b_r$$

such that for each tuple of standard basis vectors $\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{d_i}}$ one has

$$(6) \quad M_{F_i}(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{d_i}}) = 0$$

whenever $1 \leq i \leq r$ and

$$(7) \quad a_{j_1} + \dots + a_{j_{d_i}} < b_i.$$

Theorem 4 (Theorem 3.1, [26]). *Every Ω -bottomless system is equivalent to an Ω -special system.*

Instead of proving Theorem 4 now, we shall postpone this for a short while and deduce Theorem 3 first.

2.2.1. *Deriving Theorem 3 from Theorem 4.* Suppose that \mathbf{F} is an Ω -bottomless system. By Theorem 4 we may assume that \mathbf{F} is Ω -special. Let a_1, \dots, a_n and b_1, \dots, b_r be as defined in (5)-(7). For convenience we shall assume that

$$(8) \quad a_1 \leq \dots \leq a_n \quad \text{and} \quad \frac{b_1}{d_1} \leq \dots \leq \frac{b_r}{d_r}.$$

Note that one might have to rearrange the ordering of d_1, \dots, d_r . Let $I \subseteq \{1, \dots, r\}$ be a set indexing $\{F_i\}_{i \in I}$ and $S \subset \{d_1, \dots, d_r\}$ the corresponding set of degrees. If there exists I such that

$$(9) \quad d_i a_{v(S)} < b_i$$

for all $i \notin I$, then relation (7) holds and, consequently, (6) yields

$$M_{F_i}(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{d_i}}) = 0$$

for every $i \notin I$ and $1 \leq j_1, \dots, j_{d_i} \leq v(S)$. Thus, $\{F_i\}_{i \notin I}$ vanishes on a vector space of dimension $v(S)$. By definition $\{F_i\}_{i \in I}$ has a non-trivial zero in this vector space. We conclude that \mathbf{F} possesses a non-trivial zero.

Otherwise, we show that there exists a suitable Ω such that \mathbf{F} is not Ω -special. If (9) fails for all I , then there must for every I be a certain $i \in I$ such that

$$(10) \quad d_i a_{v(S)} \geq b_i.$$

If we write $S_0 = \emptyset$ and $S_i = \{d_1, \dots, d_i\}$ for all $1 \leq i \leq r-1$, then (10) yields

$$(11) \quad d_i a_{v(S_{i-1})} \geq b_i \quad \text{for all } 1 \leq i \leq r.$$

If both a system comprising forms F_1, \dots, F_{i-1} in $v(S_{i-1}) - 1$ variables and a single form F_i in $v(d_i) - 1$ variables, distinct from those of F_1, \dots, F_{i-1} , do not possess a non-trivial zero, then neither does the system F_1, \dots, F_i . In other words

$$v(S_i) - v(S_{i-1}) \geq v(d_i) - 1.$$

Since $v(d_i) \geq d_i^2 + 1$ (see ⁶) we have

$$(12) \quad v(S_i) - v(S_{i-1}) \geq d_i^2.$$

Also note that assumption (4) implies

$$(13) \quad n > d_1^2 + \dots + d_r^2.$$

By (12), the assumed ordering (8) and (13), we have

$$\begin{aligned} a_1 + \dots + a_n &\geq a_1 + \dots + a_{d_1^2} + \dots + a_{d_r^2} + a_n \\ &\geq a_{v(S_0)} d_1^2 + \dots + a_{v(S_{r-1})} d_r^2 + a_n. \end{aligned}$$

Since $a_n \geq (a_1 + \dots + a_r)/r$ this is greater than or equal to

$$\begin{aligned} &\left(d_1^2 + \frac{1}{r}\right) a_{v(S_0)} + \left(d_2^2 + \frac{1}{r}\right) a_{v(S_1)} + \dots + \left(d_r^2 + \frac{1}{r}\right) a_{v(S_{r-1})} \\ &\geq \left(d_1 + \frac{1}{r \max_{1 \leq i \leq r} d_i}\right) b_1 + \dots + \left(d_r + \frac{1}{r \max_{1 \leq i \leq r} d_i}\right) b_r \end{aligned}$$

where the last inequality follows from equation (11). By setting

$$\omega_i := d_i + 1/r \max_{1 \leq i \leq r} (d_i),$$

we have a suitable Ω such that \mathbf{F} is not Ω -special.

Thus we conclude that there is a suitable Ω such that \mathbf{F} has a non-trivial p -adic zero, provided \mathbf{F} is Ω -bottomless.

2.2.2. Preparations for the proof of Theorem 4. Suppose that \mathbf{F} and $\hat{\mathbf{F}}$ are systems comprising r forms over $\mathcal{O}_{\mathbb{K}}$. For a vector $\Omega = (\omega_1, \dots, \omega_r)$ with positive rational components and an integer $k \geq 1$ we write

$$\mathbf{F} \underset{\Omega}{\succ}^k \hat{\mathbf{F}},$$

if there are A and B as described in (2) such that

$$(14) \quad \mathbf{F}(\mathbf{x}) = B\hat{\mathbf{F}}(A\mathbf{x}) \quad \text{and} \quad \sum_{i=1}^r c_i \omega_i - \nu(\det(A)) \geq k.$$

⁶ Note that for every positive integer d there exists a form of degree d with d^2 variables which does not have a non-trivial zero. This follows, for instance, from the existence of a central simple division algebra of rank d^2 .

This strengthens the definition of “ $\underset{\Omega}{\succ}$ ” as given in (3). We say that a system of forms \mathbf{F} is Ω -high if for every integer $k \geq 1$ there exists a system $\hat{\mathbf{F}}$ such that

$$\mathbf{F} \underset{\Omega}{\succ}^k \hat{\mathbf{F}}.$$

Lemma 1 (Lemma 3.2, [26]). *If \mathbf{F} is an Ω -bottomless system, then it is Ω -high.*

Proof. Suppose that \mathbf{F}_1 is an Ω -bottomless system. Consequently, there exists an infinite sequence $(\mathbf{F}_i)_{i \geq 2}$ of systems such that

$$\mathbf{F}_1 \underset{\Omega}{\succ} \mathbf{F}_2 \underset{\Omega}{\succ} \mathbf{F}_3 \underset{\Omega}{\succ} \mathbf{F}_4 \underset{\Omega}{\succ} \dots$$

By definition there exist for each $m \geq 1$ matrices A_m and B_m as in (2) such that

$$(15) \quad \mathbf{F}_m(\mathbf{x}) = B_m \mathbf{F}_{m+1}(A_m \mathbf{x}) \quad \text{and} \quad \sum_{i=1}^r c_{i,m} \omega_i - \nu(\det(A_m)) > 0.$$

Let k be an arbitrary positive integer. By (15) and since $\Omega \in \mathbb{Q}_{>0}^r$ there exists a non-negative integer Q such that

$$\sum_{i=1}^r c_{i,m} \omega_i - \nu(\det(A_m)) \geq \frac{1}{Q}$$

for all $1 \leq m \leq kQ$. By setting

$$A := \prod_{m=1}^{kQ} A_m, \quad B := \prod_{m=1}^{kQ} B_m \quad \text{and} \quad c_i := \sum_{m=1}^{kQ} c_{i,m}$$

we have

$$\mathbf{F}_1(\mathbf{x}) = B \mathbf{F}_{kQ}(A \mathbf{x})$$

and

$$\sum_{i=1}^r c_i \omega_i - (\nu(\det(A)) + k) = \sum_{m=1}^{kQ} \left(\sum_{i=1}^r c_{i,m} \omega_i - \nu(\det(A_m)) \right) - k \geq 0.$$

Thus we conclude that

$$\mathbf{F}_1 \underset{\Omega}{\succ}^k \mathbf{F}_{kQ}.$$

□

We shall need two additional and slightly technical lemmas to finish the proof of Theorem 4, but skip their proofs. The reader interested in these is referred to lemmas 8 and 10 as found in [20].

Lemma 2 (Lemma 3.3, [26]). *Let C_1, \dots, C_l and D_1, \dots, D_m be linear forms with integer coefficients in $\mathcal{O}_{\mathbb{K}}$ in the vector $\mathbf{x} = (x_1, \dots, x_n)$. Further, let $\mathbf{x}_1, \mathbf{x}_2, \dots$ be a sequence of vectors such that for all $1 \leq i \leq l$ and $k \geq 1$*

$$C_i(\mathbf{x}_k) \geq 0.$$

Then there exists a subsequence, $\mathbf{y}_1, \mathbf{y}_2, \dots$ say, a constant D and an integer vector \mathbf{a} with

$$C_i(\mathbf{a}) \geq 0$$

for all $1 \leq i \leq l$ such that

$$\lim_{k \rightarrow \infty} D_j(\mathbf{y}_k) = +\infty \quad \text{for } j \text{ with } D_j(\mathbf{a}) > 0$$

and

$$D_j(\mathbf{y}_k) \leq D \quad \text{for } j \text{ with } D_j(\mathbf{a}) \leq 0.$$

For the next lemma, we recall the notion of a lattice. Let $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{K}^n$ be linearly independent vectors, then

$$\Lambda = \{\lambda_1 \mathbf{a}_1 + \dots + \lambda_s \mathbf{a}_s \mid (\lambda_1, \dots, \lambda_s) \in \mathcal{O}_{\mathbb{K}}^s\}$$

is called a lattice with basis $\mathbf{a}_1, \dots, \mathbf{a}_s$. We say that a lattice $\hat{\Lambda}$ is a sub-lattice of Λ if $\hat{\Lambda} \subseteq \Lambda$.

Lemma 3 (Lemma 3.4, [26]). *Suppose that M is a sub-lattice of Λ . Then there exists a basis $\mathbf{u}_1, \dots, \mathbf{u}_s$ of Λ and a basis of $\mathbf{m}_1, \dots, \mathbf{m}_s$ of M such that*

$$\mathbf{m}_1 = \pi^{u_1} \mathbf{u}_1, \dots, \mathbf{m}_s = \pi^{u_s} \mathbf{u}_s,$$

for some non-negative integers u_1, \dots, u_s .

2.2.3. Proof of Theorem 4. Suppose that \mathbf{F} is an Ω -bottomless system over $\mathcal{O}_{\mathbb{K}}$ comprising r forms in n variables. By Lemma 1 we may assume that \mathbf{F} is Ω -high. By definition this means that for every $k \geq 1$ there is an $n \times n$ -matrix A_k over $\mathcal{O}_{\mathbb{K}}$ and an upper-triangular $r \times r$ -matrix B_k with entries

$$(B_k)_{ij} = \pi^{-c_{i,k}} G_{ij,k}$$

where $c_{i,k} \geq 0$, $G_{ii,k} = 1$ and otherwise $G_{ij,k} \in \mathcal{O}_{\mathbb{K}}[x_1, \dots, x_n]$ any form with

$$\deg(G_{ij,k}) = \deg(F_i) - \deg(F_j),$$

and a system \mathbf{F}_k such that

$$(16) \quad \mathbf{F}(\mathbf{x}) = B_k \mathbf{F}_k(A_k \mathbf{x}) \quad \text{and} \quad \sum_{i=1}^r c_{i,k} \omega_i - \nu(\det(A_k)) \geq k.$$

By defining a sequence of forms and taking the limit on a subsequence, we will construct an Ω -special system equivalent to \mathbf{F} . For $1 \leq i \leq r$ set

$$R_{i,k}(\mathbf{x}) := \sum_{j=i}^r G_{ij,k}(\mathbf{x}) F_j(\mathbf{x}).$$

Clearly, we have

$$(17) \quad \pi^{-c_{i,k}} R_{i,k}(\mathbf{x}) \in \mathcal{O}_{\mathbb{K}}$$

for any $\mathbf{x} \in A_k \mathcal{O}_{\mathbb{K}}^n$ and all $1 \leq i \leq r$. Note that $\mathcal{O}_{\mathbb{K}}^n$ is a lattice with sub-lattice $A_k \mathcal{O}_{\mathbb{K}}^n$. Thus, we can apply Lemma 3 and obtain a basis $\mathbf{m}_1, \dots, \mathbf{m}_n$ of $A_k \mathcal{O}_{\mathbb{K}}^n$ and basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of $\mathcal{O}_{\mathbb{K}}^n$ such that

$$(18) \quad \mathbf{m}_1 = \pi^{u_1} \mathbf{u}_1, \dots, \mathbf{m}_n = \pi^{u_n} \mathbf{u}_n$$

for some non-negative integers u_1, \dots, u_n . As \mathbb{K} is of characteristic zero, there exists for every $1 \leq i \leq r$ a unique multilinear form $M_{R_{i,k}}$ associated with $R_{i,k}$. Unfortunately, it does not necessarily follow from (17) that $M_{R_{i,k}}(\mathbf{x}_1, \dots, \mathbf{x}_{d_i}) \in \mathcal{O}_{\mathbb{K}}$ for $\mathbf{x}_1, \dots, \mathbf{x}_{d_i} \in \mathcal{O}_{\mathbb{K}}^n$ and $1 \leq i \leq r$. As $R_{i,k}$ is defined over $\mathcal{O}_{\mathbb{K}}$, there exists, however, $\gamma \in \mathbb{N}_0$, depending on d_1, \dots, d_r , such that

$$\pi^\gamma M_{R_{i,k}}(\mathbf{x}_1, \dots, \mathbf{x}_{d_i}) \in \mathcal{O}_{\mathbb{K}}$$

for all $\mathbf{x}_1, \dots, \mathbf{x}_{d_i} \in \mathcal{O}_{\mathbb{K}}^n$ and $1 \leq i \leq r$. Using the basis vectors $\mathbf{m}_1, \dots, \mathbf{m}_n$ from (18) we have

$$\pi^{-c_{i,k}} M_{R_{i,k}}(\pi^{u_{j_1}} \mathbf{u}_{j_1}, \dots, \pi^{u_{j_{d_i}}} \mathbf{u}_{j_{d_i}}) \in \pi^{-\gamma} \mathcal{O}_{\mathbb{K}}$$

and hence

$$(19) \quad |M_{R_{i,k}}(\mathbf{u}_{j_1}, \dots, \mathbf{u}_{j_{d_i}})| \leq p^{\gamma - (c_{i,k} - u_{j_1} - \dots - u_{j_{d_i}})}$$

for every tuple j_1, \dots, j_{d_i} such that $1 \leq j_1, \dots, j_{d_i} \leq n$. We stress that \mathbf{u}_i, u_i are dependent on k and that (19) holds for all k . Recall that $\mathbf{u}_1, \dots, \mathbf{u}_n$ form a basis of the lattice $\mathcal{O}_{\mathbb{K}}^n$. Since $\mathcal{O}_{\mathbb{K}}^n$ is compact, there exists a convergent subsequence of $(\mathbf{u}_1, \dots, \mathbf{u}_n)_k$ with limit $(\mathbf{a}_1, \dots, \mathbf{a}_n)$. Note that $\mathbf{a}_1, \dots, \mathbf{a}_n$ form again a basis of $\mathcal{O}_{\mathbb{K}}^n$. Moreover, $R_{1,k}, \dots, R_{r,k}$ converge, on a subsequence again, to forms R_1, \dots, R_r , say. There is a linear map σ over $\mathcal{O}_{\mathbb{K}}$, such that

$$\sigma \mathbf{e}_i = \mathbf{a}_i$$

for all $1 \leq i \leq n$. In order to apply Lemma 2 we set for each k in our subsequence

$$(C_1, \dots, C_{n+r})_k := (u_1, \dots, u_n, c_{1,k}, \dots, c_{r,k}).$$

Moreover we put, also for each k ,

$$D(k) := \sum_{i=1}^r c_{i,k} \omega_i - \sum_{i=1}^n u_i$$

and write

$$D_i(k) = c_{i,k} - (u_{j_1} + \dots + u_{j_{d_i}})$$

where $1 \leq j_1, \dots, j_{d_i} \leq n$ and $1 \leq i \leq r$. As

$$\nu(\det(A_k)) = \sum_{i=1}^n u_i$$

we have by (16)

$$\sum_{i=1}^k c_{i,k} \omega_i - \sum_{i=1}^n u_i \geq k$$

and therefore

$$\lim_{k \rightarrow +\infty} D(k) = +\infty.$$

It then follows from Lemma 2 that there is

$$\mathbf{a} = (a_1, \dots, a_n, b_1, \dots, b_r)$$

with a_i for $1 \leq i \leq n$ and b_i for $1 \leq i \leq r$ non-negative integers such that

$$a_1 + \dots + a_n < \omega_1 b_1 + \dots + \omega_r b_r.$$

Moreover Lemma 2 states there is a subsequence such that $D_i(k)$ tends to $+\infty$ for all indices i and j_1, \dots, j_{d_i} , which satisfy

$$a_{j_1} + \dots + a_{j_{d_i}} < b_i.$$

By letting k in equation (19) tend to $+\infty$, we observe that for the same range of indices holds

$$M_{R_i}(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_{d_i}}) = 0.$$

Thus, we obtain with $\hat{\mathbf{F}}(\mathbf{x}) := \mathbf{F}(\sigma \mathbf{x})$ an Ω -special system that is equivalent to \mathbf{F} . This completes the proof of Theorem 4 and ends our exposition of Zahid's account.

3. ZEROS OF QUINTIC FORMS

In this chapter we shall give a proof of Theorem 1. Firstly, we discuss some preliminary manoeuvres involving the minimisation procedure from Chapter 2, the Chevalley-Waring Theorem and a version of Hensel's Lemma.

3.1. Preliminary manoeuvres. We say that two forms f and $g \in \mathbb{F}_q[x_1, \dots, x_n]$ are equivalent if there exist a non-zero $a \in \mathbb{F}_q$ and $A \in \text{GL}_n(\mathbb{F}_q)$ such that

$$g(\mathbf{x}) = af(A\mathbf{x}).$$

If f and g are equivalent, then f has a non-singular zero if and only if g possesses one. A form f over \mathbb{F}_q is said to be non-degenerate if the number of variables explicit in f is minimal among all forms equivalent to f .

Recall that \mathbb{K} denotes a p -adic field with residue class field \mathbb{F}_q and ring of integers $\mathcal{O}_{\mathbb{K}}$. As previously mentioned, we write π for a uniformiser of $\mathcal{O}_{\mathbb{K}}$.

Let F be a quintic form with at least 26 variables over \mathbb{K} . As we are interested in a zero, we may assume from now on that F is defined over $\mathcal{O}_{\mathbb{K}}$.

We use Schmidt's minimisation procedure to derive some geometric information on the reduction of F over the residue class field. Recall that the reduction of F is denoted by $\theta(F)$. The next lemma extends Proposition 4.3 of [18] in the case of quintic forms. Note that there always exists a non-degenerate form equivalent to $\theta(F)$. In order to ease notation we assume without loss of generality that $\theta(F)$ is non-degenerate.

Lemma 4. *Let F be a quintic form in at least 26 variables over $\mathcal{O}_{\mathbb{K}}$ that does not have a non-trivial zero. Suppose that $\theta(F)$ is a non-degenerate form with m variables x_1, \dots, x_m explicit in $\theta(F)$. Let $s \geq 0$ be an integer such that the form $\theta(F)$ vanishes on an affine s -dimension linear plane contained in the subspace*

$$\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i = 0 \text{ for all } i > m\}.$$

Then $\theta(F)$ must be a non-degenerate form in at least $6 + s$ variables.

Proof. If F is bottomless we conclude by Theorem 3 that F has a non-trivial zero. Thus, F must be Ω -bottomed and consequently we may assume that F is ω -reduced for some $\omega > 5$.

Suppose that $\theta(F)$ is non-degenerate with variables x_1, \dots, x_m explicit

in $\theta(F)$ and $m \leq 5 + s$. After a change of variables we may assume that $\theta(F)$ vanishes on the subspace

$$\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i = 0 \text{ for all } 1 \leq i \leq 5\}.$$

If $\theta(F)$ vanishes on this space, but is not the zero polynomial, then it has a non-singular zero. Consequently, $\theta(F)$ has a non-trivial zero by Hensel's Lemma (see Lemma 6 below).

Otherwise, every monomial of $\theta(F)$ has at least one of x_1, \dots, x_5 as a factor. Let $A \in \mathcal{O}_{\mathbb{K}}^{n \times n}$ be such that

$$(A)_{ij} = \begin{cases} \pi & \text{for } 1 \leq i \leq 5 \text{ and } i = j \\ 1 & \text{for } 6 \leq i \leq n \text{ and } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Then $\pi^{-1}F(A\mathbf{x})$ has coefficients in $\mathcal{O}_{\mathbb{K}}$ and we conclude

$$F \succ \pi^{-1}F(A\mathbf{x}).$$

□

Note that for every degree larger than five there are non-zero forms which are vanishing everywhere, but do not possess a non-singular zero. An example is the form

$$(x_1x_2^3 + x_1^3x_2)x_1x_2 \cdots x_l \in \mathbb{F}_2[x_1, \dots, x_l].$$

The next lemma shows in particular that $s \geq 1$. For a system comprising forms f_1, \dots, f_r over \mathbb{F}_q we shall denote the set of its projective zeros by $Z(f_1, \dots, f_r)$.

Lemma 5 (Chevalley-Warning Theorem). *Let f_1, \dots, f_r be forms of degree d_1, \dots, d_r over \mathbb{F}_q in n variables. If $n > d_1 + \dots + d_r$ we have*

$$|Z(f_1, \dots, f_r)| \geq \frac{q^{n-d_1-\dots-d_r} - 1}{q - 1}.$$

A proof of this classical result can be found in [7]. Lemmas 4 and 5 yield the following consequence.

Corollary 1. *Let F be a quintic form in at least 26 variables over $\mathcal{O}_{\mathbb{K}}$ that does not have a non-trivial zero. Let s be as defined in Lemma 4. We then have*

$$|Z(\theta(F))| \geq \frac{q^{s+1} - 1}{q - 1}.$$

A zero of $\theta(F)$ is not sufficient for a non-trivial zero of F , instead we require a non-singular zero. Once we have found one, we can apply the version of Hensel's Lemma given below.

Lemma 6 (Hensel's Lemma). *Let $F \in \mathcal{O}_{\mathbb{K}}[x_1, \dots, x_n]$. If $\theta(F)$ has a non-singular zero, then F has a non-trivial zero in \mathbb{K}^n .*

For a discussion of Hensel's Lemma see [12], for example.

3.2. Proof of Theorem 1. Suppose that F is a quintic form with $n \geq 26$ variables over a p -adic field \mathbb{K} with residue class field of cardinality $q > 9$. We denote the linear span of vectors $\mathbf{v}_1, \dots, \mathbf{v}_l \in \mathbb{F}_q^n$ by $\langle \mathbf{v}_1, \dots, \mathbf{v}_l \rangle$.

By Lemma 4 we may assume that $\theta(F)$ is a non-degenerate form in at least $6 + s$ variables, where s is the maximal affine dimension of a linear subspace of $Z(f)$. More precisely, s is the maximal non-negative integer among those defined in Lemma 4. For ease of notation we shall write f for the reduction $\theta(F)$.

Suppose that f does not have a non-singular zero. We show that there are at least four linearly independent zeros

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \in Z(f) \text{ such that } \langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f)$$

for all $1 \leq i < j \leq 4$. Hence the form

$$g(x_1, x_2, x_3, x_4) := f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3 + x_4\mathbf{z}_4)$$

must be of a certain shape. In particular, certain coefficients of g do not vanish. We then prove the existence of a non-singular zero of g , contrary to our assumption. This is achieved by considering successively larger subspaces of $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \rangle$ and sieving out forms possessing non-singular zeros.

As a first step, we prove that there are five distinct non-zero vectors

$$\mathbf{z}_1, \dots, \mathbf{z}_5 \in Z(f)$$

such that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly independent and f does not vanish on any plane spanned by two vectors of one of the quadruples

$$\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_i\} \quad \text{where } i = 4, 5.$$

In order to establish this, we begin by showing that there are three distinct subspaces $V_1, V_2, V_3 \subseteq Z(f)$ of maximal dimension and two zeros $\mathbf{z}_1, \mathbf{z}_2 \in Z(f)$ such that

$$\mathbf{z}_1, \mathbf{z}_2 \notin \bigcup_{i=1}^3 V_i \quad \text{and} \quad \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \not\subseteq Z(f).$$

Secondly, we prove the existence of a third zero $\mathbf{z}_3 \in V_3 \setminus (V_1 \cup V_2)$ such that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly independent. Thirdly, we show that there is a fourth zero $\mathbf{z}_4 \in V_2 \setminus V_1$ completing the first quadruple and finally, we will choose a fifth zero $\mathbf{z}_5 \in V_1$ completing the second quadruple.

For convenience, we first state a basic lemma and give the details of the argument outlined afterwards.

Lemma 7 ([18, Lemma 5.1]). *Let f be a quintic form over \mathbb{F}_q possessing two distinct non-trivial zeros \mathbf{z}_1 and \mathbf{z}_2 . Then f either has a non-singular zero or*

$$f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2) = c_{12}x_1^3x_2^2 + c_{21}x_2^3x_1^2$$

and $c_{12}c_{21} = 0$. If, in addition, $|\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap Z(f)| \geq 3$, then $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2)$ either possesses a non-singular zero or is the zero polynomial.

Proof. We write

$$f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2) = a_1x_1^5 + b_{12}x_1^4x_2 + c_{12}x_1^3x_2^2 + c_{21}x_2^3x_1^2 + b_{21}x_2^4x_1 + a_2x_2^5.$$

We may assume that \mathbf{z}_1 and \mathbf{z}_2 are singular zeros and hence

$$f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2) = (c_{12}x_1 + c_{21}x_2)x_1^2x_2^2.$$

If $c_{12}c_{21} \neq 0$ then $(-c_{21}, c_{12})$ is a non-singular zero and otherwise $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap Z(f) = \{\mathbf{z}_1, \mathbf{z}_2\}$ or $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \subseteq Z(f)$. \square

Since f has at least 6 variables, Lemma 5 yields a non-trivial zero and thus we may assume $s \geq 1$. By Corollary 1 we have

$$(20) \quad |Z(f)| > \frac{4(q^s - 1)}{q - 1},$$

provided $q \geq 4$. Thus we can pick four distinct subspaces

$$V_1, V_2, V_3, V_4 \subseteq Z(f)$$

such that V_i is of maximal dimension for $1 \leq i \leq 4$. By equation (20) we can choose an additional zero $\mathbf{z}_1 \in Z(f) \setminus \bigcup_{i=1}^4 V_i$. We set $S_3 := \bigcup_{i=1}^3 V_i$ and show that there exists a vector $\mathbf{z}_2 \in V_4 \setminus S_3$ such that $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \not\subseteq Z(f)$. Suppose by the contrary that

$$(21) \quad \text{for all } \mathbf{z} \in V_4 \setminus S_3 \text{ we have } \langle \mathbf{z}_1, \mathbf{z} \rangle \subseteq Z(f).$$

If $V_4 \cap S_3 = \{0\}$, then (21) contradicts the maximality of V_4 and otherwise we shall argue as follows. Let $\mathbf{s} \in V_4 \cap S_3$ be arbitrary. As V_4 is distinct from S_3 we can choose a non-zero vector $\mathbf{v} \in V_4 \setminus S_3$ and consider the projective line $L_{\mathbf{s}} := \langle \mathbf{v}, \mathbf{s} \rangle$. Since $\mathbf{v} \notin S_3$, the projective line $L_{\mathbf{s}}$ can not contain two vectors of V_i for each $1 \leq i \leq 3$. Thus the intersection $L_{\mathbf{s}} \cap S_3$ contains at most three non-zero points. On the other hand, since $q \geq 5$, there are at least three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in L_{\mathbf{s}}$ not contained in S_3 . It follows from our assumption (21) that $\langle \mathbf{z}_1, \mathbf{p}_i \rangle \subseteq Z(f)$ for all $1 \leq i \leq 3$.

Lemma 8. *Let f be a quintic form over \mathbb{F}_q without a non-singular zero, L a projective line, \mathbf{z} a non-zero point not on L and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in L$ three distinct non-zero points. Assume that*

$$\langle \mathbf{p}_i, \mathbf{z} \rangle \subseteq Z(f) \quad \text{for all } 1 \leq i \leq 3.$$

Then $\langle L, \mathbf{z} \rangle \subseteq Z(f)$.

Proof. Let $\mathbf{x} \in \langle L, \mathbf{z} \rangle$ and $\mathbf{x} \notin \bigcup_{i=1}^3 \langle \mathbf{p}_i, \mathbf{z} \rangle$. There exists a projective line H in $\langle L, \mathbf{z} \rangle$ through \mathbf{x} that does not contain \mathbf{z} . Since we have assumed that $\mathbf{x} \notin \langle \mathbf{p}_i, \mathbf{z} \rangle$ and $\langle \mathbf{p}_i, \mathbf{z} \rangle$ has co-dimension 1 in $\langle L, \mathbf{z} \rangle$, the line H intersects $\langle \mathbf{p}_i, \mathbf{z} \rangle$ in exactly one point \mathbf{s}_i , say, for each $1 \leq i \leq 3$. Since $\bigcap_{i=1}^3 \langle \mathbf{p}_i, \mathbf{z} \rangle = \mathbf{z}$ and $\mathbf{z} \notin H$, we conclude that there are at least three distinct points, namely \mathbf{s}_i for $1 \leq i \leq 3$, in H that are contained in $Z(f)$. By Lemma 7 we have $H \subseteq Z(f)$ and hence $\mathbf{x} \in Z(f)$. We conclude that $\langle L, \mathbf{z} \rangle \subseteq Z(f)$. \square

By applying Lemma 8 we have $\langle \mathbf{z}_1, V_4 \rangle \subseteq Z(f)$, contrary to the maximality of the dimension of V_4 . We conclude that there are three non-identical subspaces $V_1, V_2, V_3 \subseteq Z(f)$ of maximal dimension and two zeros $\mathbf{z}_1, \mathbf{z}_2 \notin \bigcup_{i=1}^3 V_i$ such that

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap Z(f) = \{\mathbf{z}_1, \mathbf{z}_2\}.$$

As mentioned above we shall proceed by proving the existence of a third vector $\mathbf{z}_3 \in V_3 \setminus (V_1 \cup V_2)$ with the property $\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f)$ for all $1 \leq i < j \leq 3$. Suppose by the contrary that for every $\mathbf{z} \in V_3 \setminus (V_1 \cup V_2)$ at least one of the following holds

$$(22) \quad \langle \mathbf{z}, \mathbf{z}_1 \rangle \subseteq Z(f) \quad \text{or} \quad \langle \mathbf{z}, \mathbf{z}_2 \rangle \subseteq Z(f).$$

We set $S_2 := V_1 \cup V_2$ for shorter notation and shall argue that we may assume $S_2 \cap V_3 = \{0\}$. Suppose there exists at least one non-zero vector $\mathbf{s} \in S_2 \cap V_3$. We then pick a vector $\mathbf{v} \in V_3 \setminus S_2$ and define for any vector $\mathbf{s} \in S_2 \cap V_3$ the projective line $L_{\mathbf{s}} := \langle \mathbf{s}, \mathbf{v} \rangle$. We show that

$$(23) \quad \langle L_{\mathbf{s}}, \mathbf{z}_1 \rangle \subseteq Z(f) \quad \text{or} \quad \langle L_{\mathbf{s}}, \mathbf{z}_2 \rangle \subseteq Z(f).$$

Since $\mathbf{v} \notin S_2$, neither two vectors of the subspace V_1 nor two of the subspace V_2 can be contained in $L_{\mathbf{s}}$. Thus there are at least 5 projective points in $L_{\mathbf{s}} \setminus S_2$, provided $q \geq 6$. By our assumption (22) there are three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ among them such that $\langle \mathbf{p}_i, \mathbf{z}_k \rangle \subseteq Z(f)$ for all $1 \leq i \leq 3$ and a certain $1 \leq k \leq 2$. Equation (23) then follows from Lemma 8 and thus, we have that for every $\mathbf{z} \in V_3$ at least one of the following holds

$$(24) \quad \langle \mathbf{z}, \mathbf{z}_1 \rangle \subseteq Z(f) \quad \text{or} \quad \langle \mathbf{z}, \mathbf{z}_2 \rangle \subseteq Z(f).$$

Lemma 9. *Let f be a quintic form over \mathbb{F}_q without a non-singular zero, $V \subseteq Z(f)$ an m -dimensional subspace where $m \geq 2$ and $\mathbf{z}_1, \dots, \mathbf{z}_k$ non-trivial zeros not contained in V . We assume $q \geq 2k$ and that there exists for any projective plane $W \subseteq V$ of co-dimension 1 an index $i \in \{1, \dots, k\}$ such that $\langle W, \mathbf{z}_i \rangle \subseteq Z(f)$. Then there exists an index $i \in \{1, \dots, k\}$ such that*

$$\langle V, \mathbf{z}_i \rangle \subseteq Z(f).$$

Proof. We write $[x_1 : \dots : x_m]$ for a projective point in V . Since $m \geq 2$ we can define the following subspaces

$$W_{(a,b)} := \{[x_1 : \dots : ax_{m-1} : bx_{m-1}] \mid x_i \in \mathbb{F}_q \text{ for } 1 \leq i \leq m\}$$

for $(a, b) \in (\{1\} \times \mathbb{F}_q) \cup \{(0, 1)\}$.

Since $q \geq 2k$ there are at least $2k + 1$ subspaces $W_{(a,b)}$. Thus we may assume that there are at least three subspaces, W_1, W_2, W_3 say, among these and a zero $\mathbf{z} \in \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ such that

$$\langle W_i, \mathbf{z} \rangle \subseteq Z(f) \quad \text{for } 1 \leq i \leq 3.$$

We shall complete the proof of this lemma by following Leep and Yeomans [[18], Lemma 5.3]. For W_1, W_2, W_3 as above, we have

$$(25) \quad \langle W_i, \mathbf{z} \rangle \cap \langle W_j, \mathbf{z} \rangle = \langle W_i \cap W_j, \mathbf{z} \rangle,$$

$$(26) \quad \langle W_i, \mathbf{z} \rangle \cap \langle W_j, \mathbf{z} \rangle = \bigcap_{i=1}^3 \langle W_i, \mathbf{z} \rangle$$

for any $1 \leq i < j \leq 3$. We notice that for equation (25) we have for each pair $i \neq j$ with $\langle W_i, \mathbf{z} \rangle$ and $\langle W_j, \mathbf{z} \rangle$ two non-identical m -dimensional planes and that $\langle W_i \cap W_j, \mathbf{z} \rangle$ is an $m - 1$ dimensional plane. Equation (26) follows from (25) and the fact that

$$W_i \cap W_j = \bigcap_{i=1}^3 W_i \quad \text{for distinct } i, j.$$

Let \mathbf{x} be a point in $\langle V, \mathbf{z} \rangle \setminus \bigcup_{i=1}^3 \langle W_i, \mathbf{z} \rangle$. We observe that $\bigcap_{i=1}^3 W_i$ has co-dimension 2 in V . Thus, we conclude by (25) and (26) that $\bigcap_{i=1}^3 \langle W_i, \mathbf{z} \rangle$ has co-dimension 2 in $\langle V, \mathbf{z} \rangle$. Hence we can choose a projective line H through the point \mathbf{x} that does not intersect with $\bigcap_{i=1}^3 \langle W_i, \mathbf{z} \rangle$. Since $\mathbf{x} \notin \langle W_i, \mathbf{z} \rangle$ and $\langle W_i, \mathbf{z} \rangle$ has co-dimension 1 in $\langle V, \mathbf{z} \rangle$, we conclude that there exists for each i a point $\mathbf{p}_i \in \langle W_i, \mathbf{z} \rangle \cap H$. Since $\langle W_i, \mathbf{z} \rangle \subseteq Z(f)$ and H does not intersect $\bigcap_{i=1}^3 \langle W_i, \mathbf{z} \rangle$ there are at least three distinct non-trivial zeros of f on H . Thus we conclude by Lemma 7 that $\langle V, \mathbf{z} \rangle \subseteq Z(f)$. \square

We apply Lemma 9 to (24) and thus, we have

$$\langle V_3, \mathbf{z}_1 \rangle \subseteq Z(f) \quad \text{or} \quad \langle V_3, \mathbf{z}_2 \rangle \subseteq Z(f).$$

However, this contradicts the maximality of the dimension of V_3 . Moreover, the vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly independent, since by Lemma 7 there are at most two zeros on the projective line $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. Thus we have found three linearly independent vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i < j \leq 3.$$

We show that there exists a fourth vector $\mathbf{z}_4 \in V_2 \setminus V_1$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i < j \leq 4.$$

Suppose by the contrary that for all $\mathbf{z} \in V_2 \setminus V_1$ at least one of the following holds

$$(27) \quad \langle \mathbf{z}, \mathbf{z}_1 \rangle \subseteq Z(f), \quad \langle \mathbf{z}, \mathbf{z}_2 \rangle \subseteq Z(f) \quad \text{or} \quad \langle \mathbf{z}, \mathbf{z}_3 \rangle \subseteq Z(f).$$

We shall argue that there is no loss of generality if we assume $V_1 \cap V_2 = \{0\}$. As there exists a point $\mathbf{v} \in V_2 \setminus V_1$ we consider for any vector $\mathbf{s} \in V_2 \cap V_1$ the plane $L_{\mathbf{s}} := \langle \mathbf{s}, \mathbf{v} \rangle$. We show that

$$\langle L_{\mathbf{s}}, \mathbf{z}_1 \rangle \subseteq Z(f), \quad \langle L_{\mathbf{s}}, \mathbf{z}_2 \rangle \subseteq Z(f) \quad \text{or} \quad \langle L_{\mathbf{s}}, \mathbf{z}_3 \rangle \subseteq Z(f).$$

Since $q \geq 7$ there are at least 7 projective points in $L_{\mathbf{s}}$ not contained in V_1 . Thus, by (27) there are three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ among them such that $\langle \mathbf{p}_i, \mathbf{z}_k \rangle \subseteq Z(f)$ for all $1 \leq i \leq 3$ and a certain $1 \leq k \leq 3$. By Lemma 8, we have that for every $\mathbf{z} \in V_2$ at least one of the following holds

$$(28) \quad \langle \mathbf{z}, \mathbf{z}_1 \rangle \subseteq Z(f), \quad \langle \mathbf{z}, \mathbf{z}_2 \rangle \subseteq Z(f) \quad \text{or} \quad \langle \mathbf{z}, \mathbf{z}_3 \rangle \subseteq Z(f).$$

It then follows in conjunction with Lemma 9 that

$$\langle V_2, \mathbf{z}_1 \rangle \subseteq Z(f), \quad \langle V_2, \mathbf{z}_2 \rangle \subseteq Z(f) \quad \text{or} \quad \langle V_2, \mathbf{z}_3 \rangle \subseteq Z(f).$$

However, any of those contradicts the maximality of the dimension of V_2 and hence we may assume the existence of a vector $\mathbf{z}_4 \in V_2 \setminus V_1$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i < j \leq 4.$$

We show that there exists a fifth vector $\mathbf{z}_5 \in V_1$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_5 \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i \leq 3.$$

Suppose by the contrary that for all $\mathbf{z} \in V_1$ at least one of the conditions in equation (27) holds. By Lemma 9 this implies

$$\langle V_1, \mathbf{z}_1 \rangle \subseteq Z(f), \quad \langle V_1, \mathbf{z}_2 \rangle \subseteq Z(f) \quad \text{or} \quad \langle V_1, \mathbf{z}_3 \rangle \subseteq Z(f).$$

However, any of these contradicts the maximality of the dimension of V_1 and thus we conclude that there is a vector $\mathbf{z}_5 \in V_1$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_5 \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i \leq 3.$$

In summary, we have shown that there are two quadruples of zeros,

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \quad \text{and} \quad \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_5,$$

such that f does not vanish on any two-dimensional plane spanned by two zeros of one quadruple. Moreover, we know that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly independent. We will now estimate the number of zeros of f in $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle$.

Lemma 10. *Let f be a quintic form over \mathbb{F}_q with three linearly independent zeros $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in Z(f)$ such that $\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f)$ for all $1 \leq i < j \leq 3$. Then the following holds.*

If $q \geq 17$, then f has a non-singular zero. If $11 \leq q < 17$, it possesses a non-singular zero or $|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle \cap Z(f)| = 3$ holds. If $q < 11$ it has a non-singular zero or $|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle \cap Z(f)| \leq 4$ holds.

The last inequality is sharp. For instance,

$$2x_1^3x_2^2 + 2x_1^3x_3^2 + 4x_2^3x_3^2 + x_1x_2x_3(5x_1^2 + 6x_2^2 + 2x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)$$

is a form over \mathbb{F}_7 possessing exactly four singular zeros, namely

$$\langle (1, 0, 0) \rangle, \langle (0, 1, 0) \rangle, \langle (0, 0, 1) \rangle, \langle (1, 6, 2) \rangle.$$

Proof. Suppose that f does not have a non-singular zero. Thus we can write $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3)$ as

$$x_1x_2x_3Q(x_1, x_2, x_3) + \sum_{1 \leq i < j \leq 3} c_{ij}x_i^3x_j^2 + c_{ji}x_j^3x_i^2$$

where $Q(x_1, x_2, x_3)$ is a quadratic form. By applying Lemma 7 to any two variables of $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3)$ we have $c_{ij}c_{ji} = 0$ for all $1 \leq i < j \leq 3$. Since f does not vanish on any of the projective lines $\langle \mathbf{z}_i, \mathbf{z}_j \rangle$ with $1 \leq i < j \leq 3$, we have either

$$c_{ij} \neq 0 \quad \text{or} \quad c_{ji} \neq 0 \quad \text{for all } 1 \leq i < j \leq 3.$$

Hence, we see after permuting the variables that $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3)$ takes one of the following shapes

$$t_1(x_1, x_2, x_3) = c_{12}x_1^3x_2^2 + c_{13}x_1^3x_3^2 + c_{23}x_2^3x_3^2 + x_1x_2x_3Q(x_1, x_2, x_3),$$

$$t_2(x_1, x_2, x_3) = c_{12}x_1^3x_2^2 + c_{31}x_3^3x_1^2 + c_{23}x_2^3x_3^2 + x_1x_2x_3Q(x_1, x_2, x_3),$$

where $Q(x_1, x_2, x_3)$ is a quadratic form and c_{12}, c_{13}, c_{23} and c_{31} are all non-zero coefficients.

It has been proved by Leep and Yeomans [18] using the Lang-Weil

Bound that $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3)$ has always a non-singular zero, provided $q \geq 43$. Heath-Brown [14] has extended this to prime values of $q \geq 17$.

Similarly, we show by computer calculations that f has a non-singular zero for $q = 25, 27, 32$. In each case there are, after an appropriate rescaling of both, the forms t_1, t_2 and the variables, just 6 degrees of freedom. A computer program can verify the existence of a non-singular zero for each form t_1 , respectively each form t_2 , by successively testing points in \mathbb{F}_q^3 .

If $q < 17$ it can be checked by an analogous computer calculation that t_1 and t_2 either possess a non-singular zero or that the bound on $|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle \cap Z(f)|$ holds. \square

Lemma 10 establishes Theorem 1, provided $q \geq 17$. Moreover, it shows that not both quadruples $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ and $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_5$ can consist of linearly dependent vectors. Thus we may assume, after renaming, that we have linearly independent vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_j \rangle \not\subseteq Z(f) \quad \text{for all } 1 \leq i < j \leq 4.$$

We write $f(x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + x_3\mathbf{z}_3 + x_4\mathbf{z}_4)$ as

$$(29) \quad \sum_{i \neq j} a_{ij} x_i^3 x_j^2 + \sum_{\substack{k \neq i, j \\ i < j}} b_{ijk} x_i x_j x_k^3 + \sum_{\substack{i \neq j, k \\ j < k}} c_{ijk} x_i x_j^2 x_k^2 + \sum_{\substack{l \neq i, j, k \\ i < j < k}} d_{ijkl} x_i x_j x_k x_l^2,$$

where $1 \leq i, j, k \leq 4$. By applying Lemma 7 and since f does not vanish on any of the projective lines $\langle \mathbf{z}_i, \mathbf{z}_j \rangle$, we conclude that for each pair (i, j) with $i \neq j$ exactly one of a_{ij} and a_{ji} is zero. It then follows that, after a permutation of the variables, the form (29) can take only four different shapes. If we write h for

$$a_{23} x_2^3 x_3^2 + a_{24} x_2^3 x_4^2 + a_{34} x_3^3 x_4^2 + \sum_{\substack{k \neq i, j \\ i < j}} b_{ijk} x_i x_j x_k^3 + \sum_{\substack{i \neq j, k \\ j < k}} c_{ijk} x_i x_j^2 x_k^2 + \sum_{\substack{l \neq i, j, k \\ i < j < k}} d_{ijkl} x_i x_j x_k x_l^2$$

those are

$$\begin{aligned} g_1 &:= a_{12} x_1^3 x_2^2 + a_{13} x_1^3 x_3^2 + a_{14} x_1^3 x_4^2 + h, \\ g_2 &:= a_{12} x_1^3 x_2^2 + a_{31} x_3^3 x_1^2 + a_{14} x_1^3 x_4^2 + h, \\ g_3 &:= a_{12} x_1^3 x_2^2 + a_{13} x_1^3 x_3^2 + a_{41} x_4^3 x_1^2 + h, \\ g_4 &:= a_{21} x_2^3 x_1^2 + a_{13} x_1^3 x_3^2 + a_{41} x_4^3 x_1^2 + h. \end{aligned}$$

As indicated it has been checked on a computer that each of those forms has a non-singular zero, provided $9 < q \leq 16$. We briefly describe the assembling process.

Along the way, we have already excluded, via Lemma 7, all forms that have a non-singular zero on one of the projective lines $\langle \mathbf{z}_i, \mathbf{z}_j \rangle$ for some $1 \leq i < j \leq 4$. Furthermore, we know from the proof of Lemma 10 all forms which do not have a non-singular zero in one of the subspaces

$$\langle \mathbf{z}_i, \mathbf{z}_j, \mathbf{z}_k \rangle \quad \text{for some } 1 \leq i < j < k \leq 4.$$

Note that g_1, g_2, g_3 and g_4 restricted to such a subspace are, after permuting the variables, equal to t_1 or t_2 as stated in the proof of Lemma 10. The computer programs for g_1, g_2, g_3 and g_4 are analogous. Suppose g_s for some $1 \leq s \leq 4$ is one of these cases. We save the rearranged coefficients of those forms of shape t_1 , respectively t_2 , without a non-singular zero in four multidimensional arrays

$$A_{ijk}[\star, \star] \quad \text{where } 1 \leq i < j < k \leq 4$$

such that they represent the coefficients of g_s restricted to the subspace $\langle \mathbf{z}_i, \mathbf{z}_j, \mathbf{z}_k \rangle$. Thus, every set of coefficients of the form $g_s|_{\langle \mathbf{z}_i, \mathbf{z}_j, \mathbf{z}_k \rangle}$ without a non-singular zero corresponds to a line $A_{ijk}[r, \star]$.

We use these data to construct all remaining forms by combining data in these arrays and four additional degrees of freedom. Let r_{ijk} denote the r_{ijk} -th line of $A_{ijk}[\star, \star]$ for $1 \leq i < j < k \leq 4$. The non-negative integers $r_{123}, r_{124}, r_{134}, r_{234}$, provided the corresponding lines are compatible with respect to the coefficients they share, determine a form

$$C(r_{123}, r_{124}, r_{134}, r_{234})$$

in four variables, x_1, x_2, x_3, x_4 say, with each monomial in at most three variables. Thus any relevant form of shape g_s can be written as

$$\begin{aligned} & C(r_{123}, r_{124}, r_{134}, r_{234}; a, b, c, d) \\ &= C(r_{123}, r_{124}, r_{134}, r_{234}) + x_1 x_2 x_3 x_4 (a x_1 + b x_2 + c x_3 + d x_4). \end{aligned}$$

For all admissible $r_{123}, r_{124}, r_{134}, r_{234}$ and for all $a, b, c, d \in \mathbb{F}_q$ we then search for a non-singular zero $(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^4$ of

$$C(r_{123}, r_{124}, r_{134}, r_{234}; a, b, c, d)$$

by trying points successively. To do this efficiently, one can rescale both the forms and variables. For instance, rescale g_1, g_2, g_3 such that

$$a_{12} = 1, \quad a_{23} = 1, \quad a_{34} = 1$$

and g_4 such that

$$a_{21} = 1, \quad a_{23} = 1, \quad a_{34} = 1.$$

It is easier to choose a rescaling that is compatible with the one used in Lemma 10 (and hence with the data in the arrays $A_{ijk}[\star, \star]$). Besides these considerations, we put a general effort on implementing the algorithm efficiently.

The source code of the C++ program may be found in the appendix. The data and the source code used in the assembling process are available at [10]. This completes the proof of Theorem 1.

Note that apart from the computer checks we have not used any assumption other than $q > 5$. For $q = 8, 9$ it is likely that one can also find by a computer search a non-singular zero of every form of the shapes g_1, g_2, g_3 and g_4 . Whereas the case $q = 7$ seems more doubtful than $q = 8, 9$, one can easily find counterexamples, for instance of shape g_1 , for $q = 5$ using the same algorithm.

4. SYSTEMS COMPRISING A CUBIC AND A QUADRATIC FORM

We give a condensed account of Zahid's work [26] and incorporate a slicing result by Lecerf to prove Theorem 2.

Recall that \mathbb{K} denotes a p -adic field with residue class field \mathbb{F}_q and ring of integers $\mathcal{O}_{\mathbb{K}}$. As previously mentioned, we write π for a uniformiser of $\mathcal{O}_{\mathbb{K}}$. The h -invariant of a system comprising forms $F_1, \dots, F_r \in \mathcal{O}_{\mathbb{K}}[\mathbf{x}]$, denoted by $h(F_1, \dots, F_r)$, is the smallest non-negative integer h such that we can write

$$F_i(\mathbf{x}) = L_1 H_{1,i}(\mathbf{x}) + \dots + L_h H_{h,i}(\mathbf{x}) \pmod{\pi}$$

where L_1, \dots, L_h are linear and $H_{1,i}, \dots, H_{h,i}$ suitable forms over \mathbb{F}_q and this holds for all $1 \leq i \leq r$.

Suppose that $q > 37$ and \mathbb{F}_q has characteristic at least 13. Let $C(x_1, \dots, x_n)$ be a cubic and $Q(x_1, \dots, x_n)$ a quadratic form over \mathbb{K} with $n \geq 14$ variables. As we are interested in a zero, we may assume from now on that C and Q are defined over $\mathcal{O}_{\mathbb{K}}$.

By Schmidt's minimisation procedure (see Theorem 3) we may assume that the system comprising C and Q is (ω_1, ω_2) -reduced for some $\omega_1 > 3$ and $\omega_2 > 2$. Zahid then shows that

$$(30) \quad h(Q) > 2, \quad h(C - LQ) > 3 \quad \text{and} \quad h(C - LQ, Q) > 5$$

for all linear forms $L(\mathbf{x}) \in \mathcal{O}_{\mathbb{K}}[\mathbf{x}]$. These inequalities can be proved similarly to Lemma 4.

We denote by f and g the reduction of C and Q , respectively, over the residue class field. Using the Chevalley-Waring Theorem (Lemma 5) and the Lang-Weil Bound (Lemma 12 below) he then shows that there exists a non-zero vector $\mathbf{e} \in \mathbb{F}_q^n$ such that

$$f(\mathbf{e}) = g(\mathbf{e}) = 0 \quad \text{and} \quad \nabla g(\mathbf{e}) \neq 0.$$

Thus we may consider instead the equivalent system

$$\begin{aligned} f(\mathbf{x}) &= x_1 f_2(\mathbf{x}_3) + f_3(\mathbf{x}_2) \\ g(\mathbf{x}) &= x_1 x_2 + g_2(\mathbf{x}_3) \end{aligned}$$

where $\mathbf{x}_l = (x_l, x_{l+1}, \dots, x_n)$.

Following Zahid we show that there exists $\mathbf{x}_2 \in \mathbb{F}_q^{n-1}$ such that $x_2 \neq 0$ and $(-x_2^{-1} g_2(\mathbf{x}_3), \mathbf{x}_2)$ is a common non-singular zero of f and g . Recall that a common non-singular zero of f and g is a zero \mathbf{x} such that $\nabla f(\mathbf{x})$ and $\nabla g(\mathbf{x})$ are linearly independent. By applying the following variant

of Hensel's Lemma we then find a common non-trivial zero of C and Q .

Lemma 11 (Hensel's Lemma for systems of forms). *Let F_1, \dots, F_r be a system of forms over $\mathcal{O}_{\mathbb{K}}$. Suppose that $\theta(F_1), \dots, \theta(F_r)$ possess a common non-singular zero. Then F_1, \dots, F_r have a common non-trivial zero.*

Suppose first that $f_2 = 0$. It is enough to show that there exists a non-singular zero $\mathbf{x}_2 \in \mathbb{F}_q^{n-1}$ of f_3 such that $x_2 \neq 0$. If $\deg_{x_2} f_3(\mathbf{x}_2) < 3$, this is relatively straightforward, provided $q > 5$. If $\deg_{x_2} f_3(\mathbf{x}_2) = 3$, it is straightforward to find a non-singular zero $\mathbf{a} \in \mathbb{F}_q^{n-1}$ of f_3 . If $a_2 = 0$, then Zahid sets $\mathbf{e}_3 := \mathbf{a}$ and chooses an additional vector $\mathbf{e}_4 \in \mathbb{F}_q^{n-1}$ such that

$$f_3(X\mathbf{e}_2 + Y\mathbf{e}_3 + Z\mathbf{e}_4) := cX^3 + Y^2u(X, Z) + Yv(X, Z) + w(X, Z) + dZ^3$$

where $c, d \neq 0$, $u(X, 1) \neq 0$ and $\deg_X w \leq 2$, provided $q > 5$. If this form is absolutely irreducible he uses the following variant of the Lang-Weil Bound to obtain a non-singular zero of f_3 such that $x_2 \neq 0$, provided $q > 3$.

Lemma 12 (Lemma 5.3, [26]). *Let N be the number of non-singular zeros of an absolutely irreducible polynomial of degree d with two variables, defined over \mathbb{F}_q . Then N satisfies*

$$N \geq q + 1 - \frac{1}{2}(d-1)(d-2)[2\sqrt{q}].$$

If $f_3(X\mathbf{e}_2 + Y\mathbf{e}_3 + Z\mathbf{e}_4)$ is not absolutely irreducible, it must factor as a product of a linear form over \mathbb{F}_q and a quadratic form. It then follows from a short argument that we can find \mathbf{x}_2 as claimed.

Consequently, we may assume that $f_2 \neq 0$. Zahid then defines

$$H(\mathbf{x}_2) = x_2 f_3(\mathbf{x}_2) - (f_2 g_2)(\mathbf{x}_3)$$

and by exploiting (30) establishes that H is absolutely irreducible. Moreover, he shows that $(-x_2^{-1}g_2(\mathbf{x}_3), \mathbf{x}_2)$ is a common non-singular zero of f and g , provided that \mathbf{x}_2 is a non-singular zero of H such that $x_2 \neq 0$. For $\xi \in \mathbb{F}_q^{3n-5}$ we define the sliced polynomial

$$H|_{\xi}(X, Y) := H(\xi_1 + X, \xi_2 + \xi_n X + \xi_{2n-4} Y, \dots, \xi_{n-1} + \xi_{2n-3} X + \xi_{3n-5} Y).$$

We use the following variant of a result of Lecerf (see Corollary 8, [17]) to show that there exists a slice such that $H|_{\xi}$ is absolutely irreducible.

Lemma 13. *Let $F \in \mathbb{F}_q[x_2, \dots, x_n]$ be an absolutely irreducible polynomial of degree d . Assume that \mathbb{F}_q has characteristic at least $d(d-1)+1$ and $q > 3d(d-1)+1$. Then there exists a slice $\xi \in \mathbb{F}_q^{3n-5}$ such that the sliced polynomial $F|_\xi$ is absolutely irreducible.*

By Lemma 13 there exists a slice $\xi \in \mathbb{F}_q^{3n-5}$ such that $H|_\xi$ is absolutely irreducible, provided \mathbb{F}_q has characteristic at least 13 and $q > 37$. The x_2 -component of $H|_\xi$ can not be identically zero, as this would contradict the fact that $H|_\xi$ is absolutely irreducible. Finally, we apply the Lang-Weil Bound as stated in Lemma 12 to show that there exists a non-singular zero of H such that $x_2 \neq 0$. This completes the proof.

Given there exists a suitable slice, the Lang-Weil Bound, as stated in Lemma 12, yields a non-singular zero as long as $q \geq 37$. If one could make sure that the sliced polynomial has one or more singular zeros, then its genus would decrease and the condition on q could be weakened.

5. APPENDIX

The following C++ program serves as an example for the computer programs used in Chapter 3. For improved readability much of the data in the arrays A123, A124 and A134 has been removed. The data as well as the program source code can be accessed at [10].

```

////////////////////////////////////
//
// This program verifies that all forms of shape g_1
// have a non-singular zero, provided p=13.
//
// Note:
// i) g_1 restricted to three variables is
// always equal to t_1.
//
// ii) Programs for g_2,g_3,g_4 are analogous.
// Necessary data is available from
// data_for_p13_shape_t_1.txt and
// data_for_p13_shape_t_2.txt at the location
// mentioned above.
//
// iii) p=11 is analogous and data is provided
// in data_for_p11_shape_t_1.txt,
// data_for_p11_shape_t_2.txt at the location
// mentioned above. Notice that for p=11 it is
// recommended to optimise the computations below
// with regard to speed. This is not included here
// as it would make this source file more
// difficult to read.
//
// iv) q=16 this case has very few forms of the
// shape t_1 and t_2 that do not possess a non-
// singular zero. Thus, one can verify this case
// easily by using e.g. pari/gp
// (http://pari.math.u-bordeaux.fr).
//
////////////////////////////////////

#include <iostream>
#include <fstream>

using namespace std;
int const p=13;

//sets array length needed to store data

```

```

int const r123=48;
int const r124=576;
int const r134=576;

//log-file
ofstream cfile;

// Returns for coefficients (A[i][0],A[i][1],A[i][2],
//A[i][3],A[i][4],A[i][5],A[i][6],A[i][7],A[i][8],
//A[j][1],A[j][2],A[j][3],A[j][4],A[j][5],A[j][6],
//A[j][7],A[j][8],A[k][1],A[k][3],A[k][4],A[k][5],
//A[k][6],A[k][7],A[k][8],A[l][3],A[l][4],A[l][5],
//A[l][6],A[l][7],A[l][8],d_1,d_2,d_3,d_4) the
//value 1 if (x_1,x_2,x_3,x_4) is a non-singular zero
// and 0 otherwise.

int nonsingular(int A123_i_0,int A123_i_1,int A123_i_2,
int A123_i_3,int A123_i_4,int A123_i_5,int A123_i_6,
int A123_i_7,int A123_i_8,int A123_j_1,int A123_j_2,
int A123_j_3,int A123_j_4,int A123_j_5,int A123_j_6,
int A123_j_7,int A123_j_8,int A134_k_1,int A134_k_3,
int A134_k_4,int A134_k_5,int A134_k_6,int A134_k_7,
int A134_k_8,int A124_l_3,int A124_l_4,int A124_l_5,
int A124_l_6,int A124_l_7,int A124_l_8,int d_1,int d_2,
int d_3,int d_4,int x_1,int x_2,int x_3,int x_4)
{

//first partial derivative
int A=3*A123_i_0*x_1*x_1*x_2*x_2+3*A123_i_1*x_1*x_1*x_3*x_3
+x_2*x_3*(x_1*x_1*A123_i_3+x_1*x_2*A123_i_6+
x_1*x_3*A123_i_7+x_2*x_2*A123_i_4+x_2*x_3*A123_i_8+
x_3*x_3*A123_i_5)+x_1*x_2*x_3*(2*x_1*A123_i_3+x_2*A123_i_6
+x_3*A123_i_7)+3*A134_k_1*x_1*x_1*x_4*x_4+
x_3*x_4*(x_1*x_1*A134_k_3+x_1*x_3*A134_k_6+x_1*x_4*A134_k_7
+x_3*x_3*A134_k_4+x_3*x_4*A134_k_8+x_4*x_4*A134_k_5)+
x_1*x_3*x_4*(2*x_1*A134_k_3+x_3*A134_k_6+x_4*A134_k_7)+
x_2*x_4*(x_1*x_1*A124_l_3+x_1*x_2*A124_l_6+x_1*x_4*A124_l_7
+x_2*x_2*A124_l_4+x_2*x_4*A124_l_8+x_4*x_4*A124_l_5)+
x_1*x_2*x_4*(2*x_1*A124_l_3+x_2*A124_l_6+x_4*A124_l_7)+
d_1*x_1*x_2*x_3*x_4+
(d_1*x_1+d_2*x_2+d_3*x_3+d_4*x_4)*x_2*x_3*x_4;

//second partial derivative
int B=2*A123_i_0*x_1*x_1*x_1*x_2+3*A123_i_2*x_2*x_2*x_3*x_3
+x_1*x_3*(x_1*x_1*A123_i_3+x_1*x_2*A123_i_6+

```

```

x_1*x_3*A123_i_7+x_2*x_2*A123_i_4+x_2*x_3*A123_i_8+
x_3*x_3*A123_i_5)+x_1*x_2*x_3*(x_1*A123_i_6+2*x_2*A123_i_4+
x_3*A123_i_8)+3*A123_j_1*x_2*x_2*x_4*x_4+
x_3*x_4*(x_2*x_2*A123_j_3+x_2*x_3*A123_j_6+x_2*x_4*A123_j_7
+x_3*x_3*A123_j_4+x_3*x_4*A123_j_8+x_4*x_4*A123_j_5)+
x_2*x_3*x_4*(2*x_2*A123_j_3+x_3*A123_j_6+x_4*A123_j_7)+
x_1*x_4*(x_1*x_1*A124_l_3+x_1*x_2*A124_l_6+x_1*x_4*A124_l_7
+x_2*x_2*A124_l_4+x_2*x_4*A124_l_8+x_4*x_4*A124_l_5)+
x_1*x_2*x_4*(x_1*A124_l_6+2*x_2*A124_l_4+x_4*A124_l_8)+
d_2*x_1*x_2*x_3*x_4+
(d_1*x_1+d_2*x_2+d_3*x_3+d_4*x_4)*x_1*x_3*x_4;

```

```
//third partial derivative
```

```

int C=2*A123_i_1*x_1*x_1*x_1*x_3+2*A123_i_2*x_2*x_2*x_2*x_3
+x_1*x_2*(x_1*x_1*A123_i_3+x_1*x_2*A123_i_6+
x_1*x_3*A123_i_7+x_2*x_2*A123_i_4+x_2*x_3*A123_i_8+
x_3*x_3*A123_i_5)+x_1*x_2*x_3*(x_1*A123_i_7+x_2*A123_i_8+
2*x_3*A123_i_5)+3*A123_j_2*x_3*x_3*x_4*x_4+
x_2*x_4*(x_2*x_2*A123_j_3+x_2*x_3*A123_j_6+
x_2*x_4*A123_j_7+x_3*x_3*A123_j_4+x_3*x_4*A123_j_8+
x_4*x_4*A123_j_5)+x_2*x_3*x_4*(x_2*A123_j_6+2*x_3*A123_j_4+
x_4*A123_j_8)+x_1*x_4*(x_1*x_1*A134_k_3+x_1*x_3*A134_k_6+
x_1*x_4*A134_k_7+x_3*x_3*A134_k_4+x_3*x_4*A134_k_8+
x_4*x_4*A134_k_5)+x_1*x_3*x_4*(x_1*A134_k_6+2*x_3*A134_k_4+
x_4*A134_k_8)+d_3*x_1*x_2*x_3*x_4+
(d_1*x_1+d_2*x_2+d_3*x_3+d_4*x_4)*x_1*x_2*x_4;

```

```
//fourth partial derivative
```

```

int D=2*A123_j_1*x_2*x_2*x_2*x_4+2*A123_j_2*x_3*x_3*x_3*x_4
+x_2*x_3*(x_2*x_2*A123_j_3+x_2*x_3*A123_j_6+
x_2*x_4*A123_j_7+x_3*x_3*A123_j_4+x_3*x_4*A123_j_8+
x_4*x_4*A123_j_5)+x_2*x_3*x_4*(x_2*A123_j_7+x_3*A123_j_8+
2*x_4*A123_j_5)+2*A134_k_1*x_1*x_1*x_1*x_4+
x_1*x_3*(x_1*x_1*A134_k_3+x_1*x_3*A134_k_6+
x_1*x_4*A134_k_7+x_3*x_3*A134_k_4+x_3*x_4*A134_k_8+
x_4*x_4*A134_k_5)+x_1*x_3*x_4*(x_1*A134_k_7+x_3*A134_k_8+
2*x_4*A134_k_5)+x_1*x_2*(x_1*x_1*A124_l_3+x_1*x_2*A124_l_6+
x_1*x_4*A124_l_7+x_2*x_2*A124_l_4+x_2*x_4*A124_l_8+
x_4*x_4*A124_l_5)+x_1*x_2*x_4*(x_1*A124_l_7+x_2*A124_l_8+
2*x_4*A124_l_5)+d_4*x_1*x_2*x_3*x_4
+(d_1*x_1+d_2*x_2+d_3*x_3+d_4*x_4)*x_1*x_2*x_3;

```

```
if ((A)%p!=0 ){return 1;};
```

```
if ((B)%p!=0 ){return 1;};
```

```
if ((C)%p!=0 ){return 1;};
```

```

if ((D)%p!=0 ){return 1;};

return 0;
}

int main () {

// file would have saved the message that counterexamples
//have been found
cfile.open ("shape_g_1.txt");

//We rescale those coefficient in front
//of  $x_1^3x_2^2$ ,  $x_2^3x_3^2$ ,  $x_3^3x_4^2$  in  $g_1$ .

//Lines in array represent  $g_1|_{\{y_1,y_2,y_3\}}$ 
//without a non-singular zero.

//We use the compatible rescaling
//ie.  $1x_1^3x_2^2$ ,  $1x_2^3x_3^2$  in  $t_1$ 

int A123[r123][9]={
{1,1,1,2,7,11,2,2,5},
...
};

//We use the compatible rescaling ie.  $1x_1^3x_2^2$  in  $t_1$ 

int A124[r124][9]={
{1,1,1,2,7,11,2,2,5},
...
};

//We use the compatible rescaling ie.  $1x_2^3x_3^2$  in  $t_1$ 

int A134[r134][9]={
{1,1,1,2,7,11,2,2,5},
...
};

//Note that  $A234 = A123$ .

//indicates whether a non-singular zero has been found
int found=0;

//Loops used in the assembling process:

```

```

for (int i = 0; i < r123; i++)
{
//note: "123=234"
for (int j = 0; j < r123; j++)
{
//compatibility condition for shared coefficients
if(A123[i][2]==A123[j][0]){
// gives some indication about waiting time
cout << i<<" ," <<j <<"\n";
for (int k = 0; k < r134; k++)
{
//compatibility condition
if(A123[j][2]==A134[k][2] && A123[i][1]==A134[k][0]){
for (int l = 0; l < r124; l++)
{
//compatibility condition
if(A123[i][0]==A124[l][0] && A124[l][1]==A134[k][1]
&& A124[l][2]==A123[j][1] ){

//Additional degrees of freedom:
for (int d_1 = 0; d_1 < p; d_1++)
{
for (int d_2 = 0; d_2 < p; d_2++)
{
for (int d_3 = 0; d_3 < p; d_3++)
{
for (int d_4 = 0; d_4 < p; d_4++)
{
found=0;
//running through all points in  $\{1\}x(F_q-0)^3$ :
for (int x_1 = 1; x_1 < 2; x_1++)
{
for (int x_2 = 1; x_2 < p; x_2++)
{
for (int x_3 = 1; x_3 < p; x_3++)
{
for (int x_4 = 1; x_4 < p; x_4++)
{

//tests form for a zero
if((A123[i][0]*x_1*x_1*x_1*x_2*x_2+
A123[i][1]*x_1*x_1*x_1*x_3*x_3+
A123[i][2]*x_2*x_2*x_2*x_3*x_3+
x_1*x_2*x_3*(A123[i][3]*x_1*x_1+A123[i][4]*x_2*x_2+
A123[i][5]*x_3*x_3+A123[i][6]*x_1*x_2+A123[i][7]*x_1*x_3+

```

```

A123[i][8]*x_2*x_3) //123
+A123[j][1]*x_2*x_2*x_2*x_4*x_4+A123[j][2]*x_3*x_3*x_3*x_4*x_4+
x_2*x_3*x_4*(A123[j][3]*x_2*x_2+A123[j][4]*x_3*x_3+
A123[j][5]*x_4*x_4+A123[j][6]*x_2*x_3+A123[j][7]*x_2*x_4+
A123[j][8]*x_3*x_4) //234
+A134[k][1]*x_1*x_1*x_1*x_4*x_4+x_1*x_3*x_4*(A134[k][3]*x_1*x_1+
A134[k][4]*x_3*x_3+A134[k][5]*x_4*x_4+A134[k][6]*x_1*x_3+
A134[k][7]*x_1*x_4+A134[k][8]*x_3*x_4) //134
+x_1*x_2*x_4*(A124[l][3]*x_1*x_1+A124[l][4]*x_2*x_2+
A124[l][5]*x_4*x_4+A124[l][6]*x_1*x_2+A124[l][7]*x_1*x_4+
A124[l][8]*x_2*x_4) //124
+(d_1*x_1 +d_2*x_2 +d_3*x_3 +d_4*x_4)*x_1*x_2*x_3*x_4)%p==0){

//test whether this zero is non-singular
if (nonsingular(A123[i][0],A123[i][1],A123[i][2],A123[i][3],
A123[i][4],A123[i][5],A123[i][6],A123[i][7],A123[i][8],
A123[j][1],A123[j][2],A123[j][3],A123[j][4],A123[j][5],
A123[j][6],A123[j][7],A123[j][8],A134[k][1],A134[k][3],
A134[k][4],A134[k][5],A134[k][6],A134[k][7],A134[k][8],
A124[l][3],A124[l][4],A124[l][5],A124[l][6],A124[l][7],
A124[l][8],d_1,d_2,d_3,d_4,x_1,x_2,x_3,x_4)==1)
{
  found=1;x_1=p-1; x_2=p-1;x_3=p-1;x_4=p-1;
}
}}}}

//If a counterexample had occurred, this event would
have been logged in shapeg_1.txt
if (found==0)
{
  cfile <<"A counterexample has been found.";
  cfile.flush();
  return 0;
}
}}}}
}}}}}}

cfile.close();

return 0;
}

```

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