Tese apresentada à Divisão de Pós-Graduação do Instituto Tecnológico de Aeronáutica como parte dos requisitos para obtenção do título de Doutor em Ciência no curso de Engenharia Aeronáutica e Mecânica na Área de Mecatrônica e Dinâmica de Sistemas Aeroespaciais.

Carlo Alessandro Zanetti Pece

An Engineering Vector-like Approach to Attitude Kinematics & Nominal Attitude State Tracking Control

Tese aprovada em sua versão final pelos abaixo assinados

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Campo Montenegro
São José dos Campos, SP, Brasil
2002

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ITA, Outubro/2002

Victory is to cleanse your mind of discord within yourself.
That is, fully accomplish what you are here to do.
This is not mere theory. Practice it.
Then you will accept the great power of unity with nature.

Morihei Ueshiba

Abstract

In dealing with rigid body three-dimensional rotational motion, one is inevitably led to face the fact that rotations are not vector quantities. They may, however, be treated as such when the angle of rotation is (very) small. In this context, i.e. the infinitesimal case analysis, the time derivatives of the rotation variables hold simple (sometimes vector-like) relationships to the components of the angular velocity vector. Conventionally, this distinctive characteristic cannot be associated with general moderate-to-large rotations.

In this thesis, it is demonstrated that the kinematical differential relationship between the rotation vector and the angular velocity vector may, in fact, be expressed in terms of a mere time derivative, provided that the angle of rotation is kept within moderate bounds. The key to achieve such simplicity in the kinematical equation (linear attitude kinematics) within moderate angles of rotation is a judicious choice of the basis from which the time derivative is observed. This result is used to advantage within a generalised version of Euler's motion equations to construct a simple control law, which nominally realises both linear attitude tracking and linear angular velocity tracking (nominal linear attitude state tracking), within moderate attitude tracking errors.

The analytical work presented here is unique in the sense that it combines attitude kinematics, dynamics and control in such a way that nominal linearity between the attitude state error variables is achieved within moderate attitude tracking errors. For the first time, an attitude control law explicitly enables the *nominal closed-loop attitude state error dynamics* to be chosen and motivated by useful physical concepts from linear control theory. The text also includes numerical simulations that validate and illustrate the theoretically achieved results.

Resumo

No tratamento do movimento rotacional tridimensional de corpos rígidos é inevitável lidar-se com o fato de que rotações não são quantidades vetoriais. Elas podem, no entanto, ser tratadas como tais quando o ângulo de rotação é (muito) pequeno. Neste contexto, ou seja, o da análise infinitesimal, as derivadas temporais das variáveis de rotação mantêm um relacionamento simples (às vezes mesmo do tipo vetorial) com os componentes do vetor velocidade angular. Convencionalmente, esta distinta característica não pode ser associada a rotações grandes, nem mesmo medianas.

Nesta tese é demonstrado que a relação diferencial entre o vetor rotação e o vetor velocidade angular pode, na realidade, ser expressa em termos de uma simples derivada temporal, desde que o ângulo de rotação seja mantido numa faixa moderada. O artifício permitindo tal simplicidade na equação cinemática (cinemática linear de atitude) com um ângulo de rotação moderado é a escolha criteriosa da base a partir da qual a derivada temporal é observada. Este resultado é utilizado vantajosamente em conjunto com uma versão generalizada das equações de movimento de Euler na construção de uma lei de controle simples. Essa lei realiza, concomitantemente, o rastreamento linear nominal de atitude e o rastreamento linear nominal de velocidade angular (rastreamento linear nominal de estado rotacional), dentro de uma faixa moderada de erro de rastreamento de atitude.

O trabalho analítico apresentado é único no sentido em que este combina cinemática rotacional, dinâmica rotacional e controle de forma tal que linearidade nominal entre as variáveis de erro de estado é atingida mesmo para erros moderados de rastreamento de atitude. Pela primeira vez, uma lei de controle permite explicitamente que a dinâmica de erro de estado rotacional em malha fechada seja escolhida e motivada por conceitos físicos úteis da teoria linear de controle. O texto também inclui simulações numéricas que validam e ilustram os resultados teóricos obtidos.

Acknowledgements

I would like to thank all those who really and truly contributed to the development of this thesis. First and above all, my friend, teacher and supervisor Prof. Dr. Sérgio Frascino Müller de Almeida (ITA, Brazil) for the invaluable support and encouragement during the turbulent course of this research, Dr. Jacques Waldmann (ITA, Brazil), whose ideas enlightened many pertinent points in this thesis, and last but not least, Dr. Jaap H. van Dieën and Dr. Idsard Kingma (Vrije University, Netherlands) for the helpful discussions during the early stages of the project.

It is appropriate to mention here that I began my PhD program under the supervision of Dr. David Wright (Brunel University, England), and that I had financial support from CAPES (Federal Agency for Postgraduate Education, Brazil) at the time. For a number of unfortunate reasons, however, the original program was not successfully completed.

I am very much in debt to ITA (Technological Institute of Aeronautics) for giving me the opportunity to develop my PhD program, and bring this thesis to the present satisfactory state. I am also much obliged to CAPES (Federal Agency for Postgraduate Education, Brazil) and Fundação Casimiro Montenegro Filho (ITA, Brazil) for providing the funds that made this research possible.

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Chapter 1

Introduction

This chapter introduces the reader to the problems faced in this thesis and signposts the corresponding solutions. Comparisons with existing techniques, and a summary of the ensuing chapters are also provided.

The attitude control of a rigid body has long been considered an important and challenging problem. With the advent of the space exploration in the 1950's, this problem took a larger dimension, and has been made since then subject of a substantial body of literature. Nowadays, attitude control is employed in a multitude of dynamical systems, which may be exemplified with aircrafts, robot manipulators, and underwater and space vehicles (see Wen & Kreutz-Delgado, 1991, for a comprehensive review).

The research reported in this thesis is motivated primarily by the recent work of Xing & Parvez (2001), which focuses on the non-linear attitude state tracking (position and velocity tracking) control of a rigid body. The thesis is also closely related to the works of Schaub et al. (2001) and Paielli & Bach (1993), whose approach to attitude control yields a system closed-loop dynamics with linear performance in the attitude tracking error.

This thesis discusses a new linear vector-like approach to the nominal attitude state tracking (position and velocity tracking) control of a rigid body. The approach proposed here incorporates features of the three above-mentioned works, but avoids their main problems, as far as the nominal system is concerned. These problems will be discussed in detail and may be summarised as follows: (a) high control law complexity and nominal closed-loop non-linearity, in the case of Xing & Parvez (2001); and (b) nominal linearity restricted to the attitude tracking error (position only), in the case of Schaub et al. (2001) and Paielli & Bach (1993).

It is convenient, in the name of simplicity, to select right from the outset the body's centroid as the point about which the sum of moments is taken. For this choice, the translational motion can be found from particle dynamics, and the body's rotational motion examined in an independent fashion. Thus, the body's centroid location becomes irrelevant in the context, and will no longer be considered in this thesis (see Pars, 1965, p. 216; Mortensen, 1968; or Whittaker, 1927, p. 127-28).

In order to be scrupulously clear as to the meaning of the terms used, it is useful to emphasise that the rigid body attitude state tracking control problem consists in the specification of an attitude control law, i.e. a torque formulation. The peculiarity of this control law is that it should enforce not only the attitude, but also the angular velocity of the controlled body to approximate the commanded/reference attitude and angular velocity respectively (see Xing & Parvez, 2001).

When the commanded and the actual/body angular motion (path + velocity) are not the same, an error exists and the question arises as to how to specify this error. Evidently, the definition should be suitable for automatic control applications. Since angular velocity is a vector, the angular velocity tracking error is easily specified as the difference between the commanded and the actual angular velocities.

On the other hand, the specification of the attitude tracking error is more involved, and lends itself to many possibilities (see, e.g., Shuster, 1993a). Attitude *cannot* be represented by a single true vector. It is, however, necessary to decide at any instant of time how near the actual attitude is to the desired attitude. In other words, it is necessary to furnish somehow a notion of magnitude and direction for the attitude tracking error. The adopted and most natural choice is, of course, the one suggested by Euler's theorem: the Euler angle/axis variables, whose product results in the so-called rotation vector (see Shuster, 1993a, p. 452).

The definition of the attitude state tracking error variables is an important step and may, in fact, contribute decisively to the design and analysis of the attitude control law. This is because the ease with which a given problem can be solved depends strongly on the underlying mathematical structure. It is therefore desirable to introduce as much structure at the outset as possible.

To the case at hand, the referred mathematical structure is determined by the properties of three-dimensional rotations and corresponding kinematical differential relationships, which are ultimately dependent upon the chosen attitude variables (attitude tracking error variables). Those properties, relationships and implications in the attitude state tracking control problem are analysed throughout the main text and next summarised.

Finite three-dimensional rotations are rigorously represented by orthogonal tensors. They may, however, be *treated* as vectors when the angle of rotation is (very) small. Rotations combine geometrically as though they were vectors only to the first-order of approximation (linear approximation) of the angle of rotation (see Goodman & Warner, 1964, p. 344-46; Argyris, 1982, p. 85; Crouch, 1981, p. 18-20; or Shuster, 1993a, p. 453, 460).

small angle linear approximations	small angle quadratic approximations	small half angle quadratic (moderate) approximations
$\sin\phi \approx \phi$	$\sin\phi \approx \phi$	$\sin\frac{\phi}{2} \approx \frac{\phi}{2}$
$\cos\phi \approx 1$	$\cos\phi \approx 1 - \frac{\phi^2}{2}$	$\cos\frac{\phi}{2} \approx 1 - \frac{\phi^2}{8}$

Although prominently useful, small angle first-order approximations are far too restrictive for a large number of practical/engineering applications. As a consequence, a totally linear approach to attitude control is ordinarily employed only in specific situations, e.g. Kaplan (1976, p. 240-45).

In the infinitesimal case analysis, where first-order approximations are appropriate, the time derivatives of the rotation variables hold simple (sometimes vector-like) relationships to the components of the angular velocity vector (see Hughes, 1986, p. 27-29). Conventionally, this distinctive characteristic cannot be associated with general moderate-to-large rotations (see section 7.4 for more on moderate angle approximations).

A direct consequence of the non-vector/non-linear nature of moderate-to-large rotations is that the well-established and convenient methods from linear control theory cannot, in general, be used in the definition of the system closed-loop attitude state error dynamics. Conventionally, it is difficult to impose system closed-loop qualities whenever the attitude tracking error is larger than just a few degrees. The notable exceptions are stability and homogeneous (unforced) attitude error dynamics (see Xing & Parvez, 2001; Schaub et al., 2001; and Paielli & Bach, 1993).

A control system must not only be stable, but also be reasonably stiff to disturbances. It must respond quickly to commands - position and velocity in the case - and must not require excessive torques, velocities, or power (see Meyer, 1966). These additional qualities are typically difficult to impose when the system closed-loop dynamics is non-linear. The reason is very simple: there are no fundamental principles available to the non-linear case (see Schaub et al., 2001; or Meyer, 1971).

The above-mentioned additional qualities may, however, be partially achieved via feedback linearisation, a technique that has received considerable attention in the last decade or so. This technique transforms the non-linear coupled rotational dynamics into an equivalent linear uncoupled system (see section 1.2 for a brief description). Possibly, the most recent and sophisticated strategy employing the feedback linearisation concept to control the attitude of a rigid body is the one offered by Schaub et al. (2001). The strategy they propose yields linear homogeneous (unforced) closed-loop dynamics in the attitude error without the need to restrict its size.

Although Schaub and his colleagues should be commended for their excellent paper, their control strategy is limited in the sense that it totally ignores its effects on the body's angular velocity. Quality attitude tracking, however, does not necessarily imply quality angular velocity tracking! Conventionally, the relationship between the attitude state tracking error variables is complex whenever the size of the attitude tracking error prevents small angle first-order approximations.

Although restricted to the useful moderate attitude tracking error case, the work presented here solves this deficiency. The proposed control law increases, therefore, the number of qualities that can be imposed to the system nominal closed-loop rotational dynamics. In a single paragraph, the objective of this thesis and corresponding relevance may be stated as follows:

The objective of this thesis is to develop a controller that nominally implements both linear attitude tracking and linear angular velocity tracking, i.e. a controller that implements nominal linear attitude state tracking within *moderate* attitude tracking errors. The relevance of this objective is twofold: (1) the unusual possibility of choosing the system *nominal non-homogeneous closed-loop attitude state error dynamics* by useful physical concepts from linear control theory, and (2) the potential practical/engineering significance of the proposed control law owing to the admissible moderate magnitude of the attitude tracking error vector.

As above established, the domain of validity of the control law should not be restricted to a small / infinitesimal neighbourhood of the commanded (reference) attitude. It should be valid in a moderate one. As a consequence, ordinary linearisation procedures about the target states are precluded (see Hughes, 1986, p. 129). This problem is considerably difficult from an analytical standpoint, since the system describing equations are inherently non-linear, and any attempt to linearise them about the target states is likely to produce meaningless results.

Two pertinent general techniques for dealing with the finite error attitude tracking problem have been found in the literature: the Liapunov's method and the already-mentioned feedback linearisation (see sections 1.1 and 1.2 for a brief description). Nevertheless, the control laws constructed from these two techniques do not customarily realise linear closed-loop attitude state tracking (see Xing & Parvez, 2001; Schaub et al., 2001; Paielli & Bach, 1993, or Wen & Kreutz-Delgado, 1991).

In the analysis of non-linear mechanical systems, a set of fundamental modelling issues surrounds the choice of the coordinates used to describe the system kinematics/dynamics. These decisions have a direct impact on the design and synthesis of (attitude) controllers. This is so because a given physical system may be described by equivalent sets of differential equations, whose degree of non-linearity depends strongly on the selected coordinates (see Junkins, 1997).

The present work proposes a different method of attack on the rigid body attitude state tracking control problem. The method makes full use of the idea discussed in the last paragraph (judicious choice of coordinates) to linearise the attitude kinematics and the corresponding system nominal closed-loop attitude state error dynamics. The only imposed restriction is that the attitude tracking error should be kept within moderate bounds. It is also assumed perfect knowledge of the system parameters and states (nominal case). Robustness is an issue left for future work.

In this thesis, it is analytically demonstrated that the kinematical differential relationship between the rotation vector (equivalent Euler/attitude vector) and the angular velocity vector may, in fact, be expressed in terms of a mere time derivative, provided that the Euler angle of rotation is kept within moderate bounds. The key to achieve such simplicity in the kinematical equation (linear attitude kinematics) within moderate angles of rotation is a judicious choice of the basis from which the time derivative is observed.

This new kinematical result is used to advantage within an also new generalised geometric version of Euler's motion equations. The outcome is a simple control law that nominally realises both linear attitude tracking and linear angular velocity tracking (nominal linear attitude state tracking), within moderate attitude tracking errors. It should be mentioned in passing that the new kinematical result gives *per se* a physical interpretation on quasi-coordinates (see Meirovitch, 1970, p. 139).

Clarity, simplicity and linearity are the order of the day. The experienced reader will notice the clarity with which complex multi-frame formulae are developed throughout the text. This is achieved thanks to a tailor-made explicit notation, which integrates in a single continuum both kinematical and dynamical concepts. It is the author's opinion that the difficulty frequently associated with rotational kinematics and dynamics has a considerable component on the lack of a proper ergonomically designed explicit notation. In fact, the often-advocated minimal notations may even present a hindrance for deeper understanding.

Simplicity and linearity are achieved by questioning the well established form of representation of a few important results in rigid body kinematics and dynamics. There is a widespread tendency of regarding as complete and immutable the works of the great patriarchs of the past, such as Euler, Rodriguez, Hamilton, and Cayley. Nonetheless, the last few decades have seen an explosion of new work, and this way of thinking should be re-evaluated (see Junkins & Shuster, 1993).

This thesis fits within this new work context. The aforementioned novel kinematical result, along with the also novel form of the equations of rotational motion, leads to a *sui generis* linear solution for the attitude state tracking control problem. The corresponding control law discloses what seems to be unique simplicity and linearity in the nominal closed-loop dynamics, within moderate attitude tracking errors.

The nature of, and the linearity achieved between the attitude state tracking error variables make certain a precise interpretation of both results and developed formulation. Feedback of attitude (rotation vector) and angular velocity errors is intuitively analogous to feedback of position and velocity errors to provide stiffness and damping in a linear position controller.

This clear physical equivalence is not observed in the other available control strategies that disclose some kind of linearity in the nominal closed-loop dynamics. Schaub et al. (2001) and Paielli & Bach (1993), for example, define their nominal linear closed-loop error dynamics employing some set of less intuitive attitude variables and respective time derivatives. A better understanding of the system dynamical behaviour can be attained, however, when the control strategy is linear in the attitude states rather than linear in the attitude variables and their time derivatives.

To finalise this section, it should still be mentioned that the development presented is typically analytical and coordinate-free. The former is in accordance with the well-known fact that intuition is not completely reliable in dealing with three-dimensional rotational motion. The latter allows representation of the resulting control law in any coordinate frame, depending only on convenience. Whenever possible, some geometrical mechanism is proposed for additional reinforcement in understanding.

The next two sections give a brief description of the Liapunov's and the feedback linearisation methods when applied to the development of attitude control laws. Closing the chapter, it is also provided an outline of the thesis.

1.1. Liapunov's Method

This method is applied directly to the non-linear rotational dynamics. Basically, the attitude feedback control law is determined by first defining a candidate Liapunov function, and then extracting the corresponding stabilising non-linear control.

The most important step in applying a Liapunov approach to control system design is the selection of the candidate function, which measures the errors from the target states. The selection of the Liapunov function is, however, based on intuition rather than fundamental principles.

Another drawback of the technique is that the resulting system closed-loop dynamics is generally non-linear. As a consequence, very important concepts from linear control theory, such as closed-loop damping and bandwidth, are simply not well defined.

To achieve the desired closed-loop behaviour, the control system designer has to choose between (1) heuristic methods and (2) linearisation of the closed-loop dynamics about the reference motion in order to use linear control theory techniques to pick the feedback gains. It should be noted that whichever method is followed the chosen behaviour will be realised only within a reduced neighbourhood of the reference attitude path.

Hughes (1986, p. 504-10) explains the method. A summary of the main concepts and theorems with examples of application are provided by Roskan (1979, p. 678-82). Further general commentaries and/or applications can be found, among many others, in Schaub et al. (2001), Junkins (1997), Paielli & Bach (1993), Wen & Kreutz-Delgado (1991), Wie & Barba (1985), Debs & Athans (1969), and Mortensen (1968). The last reference presents a particularly elegant and pertinent application of Liapunov's method.

1.2. *Feedback Linearisation*

This technique transforms the non-linear coupled system dynamics into an equivalent linear uncoupled system. This is achieved via non-linear state transformations and feedback laws.

The procedure may be divided into two steps. The first linearisation step reduces the tri-inertial rigid body rotational dynamics into a simpler isoinertial form. The second step reduces the equations even further to a minimal representation in the form of an uncoupled double integrator. The resulting regulator problem can then be easily solved with linear methods.

The transformations involved are exact, as opposed to an ε-close in an ε-neighbourhood, but they arise robustness issues due to the cancellation of the non-linear terms. As a result, adaptive control strategies are often considered. A drawback associated with the full application of this technique is that the corresponding linearising control laws are, potentially, very complex.

Wen & Kreutz-Delgado (1991) describe this technique. The same idea is nicely presented/applied by Bennett et al. (1994, section IV-A). Junkins (1997) draws a number of enlightening commentaries. Slotine & Li (1991, chapter 6) provide a more general description of the method, its use and limitations.

1.3. *Outline of the Thesis*

The chapters of this thesis have been arranged to give the reader an integrated view of the problems faced and the mathematical apparatus needed for their solution. A considerable number of footnotes, appendixes, and figures have been provided. Their intent is twofold: (1) facilitate understanding, and (2) make the thesis accessible to readers with varying degrees of expertise. The appendixes and footnotes are not essential to the main text, and may be disregarded by the more experienced reader. The text is organised as follows:

Chapter 2: Notation

Describes the notation adopted in this thesis and outlines its design process.

Chapter 3: Angular Position

Reviews the issue of rigid body orientation in three-dimensional space.

Chapter 4: Angular Velocity

Develops the expressions for the angular velocity in both direction cosines (orientation coordinate free) and Euler angle/axis variables.

Chapter 5: Equations of Rotational Motion

Presents a few different possibilities, both conventional and generalised geometric (novel), of expressing the equations of motion for a rotating rigid body.

Chapter 6: Nominal Attitude Control Command Law

Defines the form of the attitude control law using the (novel) generalised geometric equations of rotational motion and partial feedback linearisation.

Chapter 7: Nominal Attitude Stability Analysis

Finishes the construction of the control law using a novel kinematical differential relationship between attitude variables and angular velocity, determines the corresponding system nominal closed-loop transfer functions, and analyses the rigid body nominal attitude stability for the assumed control law.

Chapter 8: Kinematical Theorem Numerical Validation

Devises a method of validation for the theoretically achieved nominal results, in particular the novel kinematical relationship, constructs a model in Simulink/Matlab, and illustrates the validation method with numerical examples.

Chapter 9: Discussion and Conclusion

Compares the results of the thesis with the pertinent literature, states the contributions to the field, and proposes a few possible themes for future research based on those results.

Appendix A: Notational Examples

Provides a number of examples of utilisation of the adopted notation.

Appendix B: The Transformation Matrix

Overviews the transformation matrix, its properties and forms of representation.

Appendix C: The Rotation Matrix

Establishes a general (novel) relationship of equivalence between the rotation matrix and the transformation matrix, and derives the finite rotation formula.

Appendix D: Skew-Symmetric Form of the Vector Product

Shows how to represent the vector cross product in algebraic skew-symmetric form.

Appendix E: Invariance of the Antisymmetry Property

Demonstrates the invariance of the matrix antisymmetry property under orthogonal similarity transformations.

Appendix F: Approximated Trigonometric Functions

Exemplifies numerically the relative accuracy of small angle approximations.

Chapter 2

Notation

This chapter describes the notation adopted in the thesis. Firstly, the need for such an unusual notation is clarified. Secondly, the symbols and identifiers utilised in the notation are defined, and their relative location specified. Lastly, lists containing details, formats and examples of all notational elements are provided.

The important issue of notation is given due attention in this thesis. The notation has been designed for easy reading, while still allowing the freedom to interchange between forms (geometric & algebraic) and bases of representation.

Such freedom implies flexibility in formulae manipulation, and is particularly important to this work, since neither the bases nor the direction of the transformations involved are self-evident, making the explicit mention of these necessary.

The notational structure, diagrammatically depicted in figure 1, specifies the relative location of all possible notational elements. Specific quantity representations may include, therefore, only part of the elements depicted in figure 1 (see page 18 for a few illustrative examples).

Appendix A offers a comprehensive list of examples of utilisation of the notation, and may therefore be complementary to its description (the other appendices may also be helpful). The notational elements are defined in section 2.1. Lists containing details, formats and examples of all notational elements are provided in section 2.3.

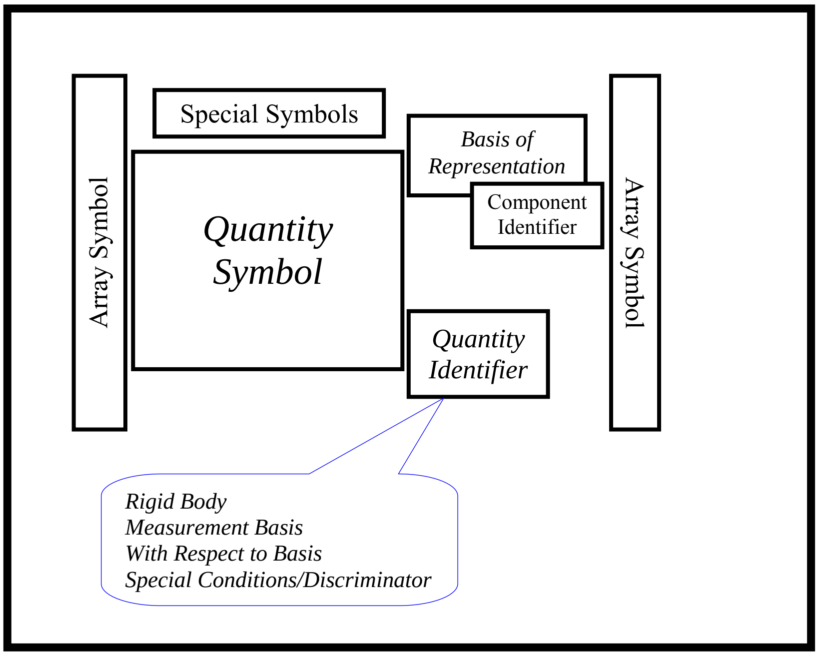


Figure 1: Notational Structure

Two graphical techniques contributed to the achievement of the above-referred ease of reading: notational element disposition and colour coding. The use of both techniques is supported by ergonomic guidelines. They increase informational distinctiveness, therefore minimising the likelihood of perceptual/description errors[1] (see Mayhew, 1992, p. 521 and Lansdale & Ormerod, 1994, p. 133). Next paragraphs describe briefly how these two techniques have been applied in designing the notation.

The *quantity identifier*, the *component identifier* and the *basis of representation* notational elements (see figure 1) have all been located on the same side (right) of the corresponding quantity symbol. This disposition implies immediate and unambiguous identification of the quantity to which each notational element refers, even in a long succession of symbols in complex formulae

[1] Perceptual/description errors are failures in the detection of important information caused by insufficient perceptual cues.

Basis recognition is of fundamental importance to this work. Each of the four employed bases has been assigned a particular colour (see figure 2). This coding conveys the necessary information more vividly, facilitating therefore formulae reading and interpretation. The choice of the colours considered the requirement for visual efficiency.

The selected colours, namely black, red, dark blue and dark green, provide adequate visual contrast over the white background (paper). This selection is recommend by Woodson (1987, p. 235).

Another point considered in the choice of the colours is how spaced they are with respect to the visible spectrum. Colours are easier to discriminate between the further apart they are along the colour spectrum (see Mayhew, 1992, p. 495-96). Given the above contrast requirement, the colours selected comprise a good option.

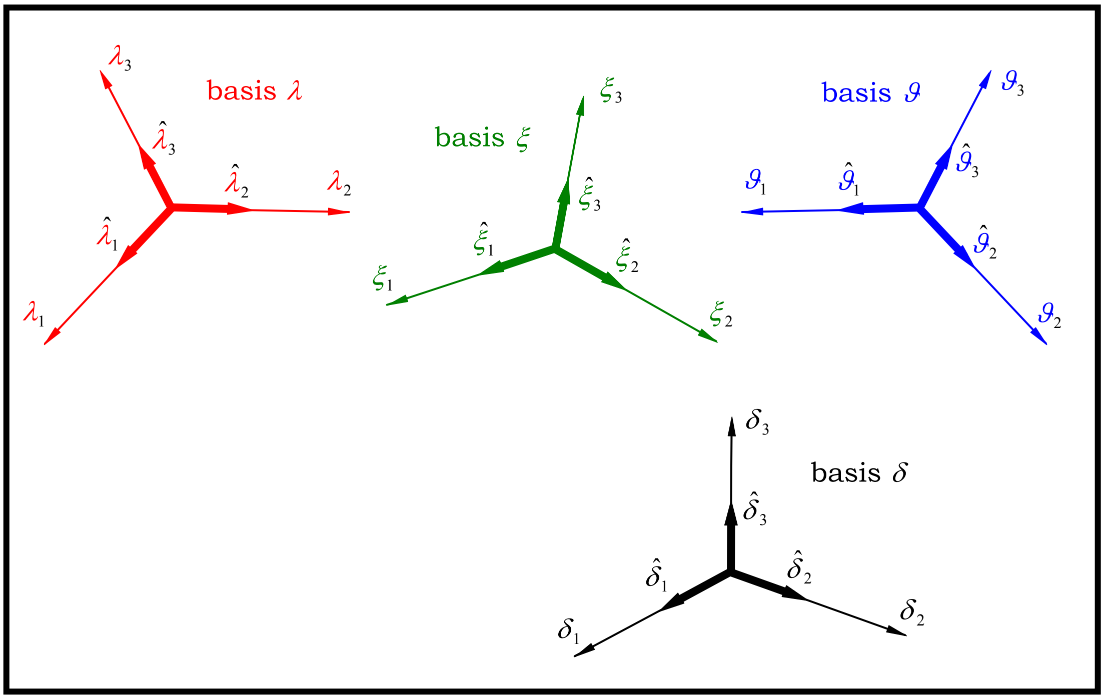


Figure 2: Bases of Representation

The notation as devised emphasises explicitness. In fact, the notation is virtually self-explanatory, facilitating understanding and avoiding lengthy explanations about each term in the formulae. Related quantities and corresponding representations are "built" via simple adaptation of standard symbols, as opposed to the recurrent assignment of new symbols. As a result, the number of symbols utilised in the text is relatively small.

It should be remarked that although the notation is thoroughly comprehensive and consistent, it is concise only in terms of the total number of symbols employed. The symbolic representation of individual quantities is complex when compared to the more conventional methods (generally one letter + one subscript). This drawback is somehow compensated by the notation's improved readability.

Another advantage of the proposed notation is that it stresses the relative nature of kinematic quantities. This is achieved by the definition of these quantities in terms of orthonormal bases (*measurement basis* and *with respect to basis*), without regarding them as inertial or non-inertial frames of coordinates. In the realm of kinematics this distinction is not even necessary, and such consideration is called upon only when dynamic effects have to be considered.

Although the proposed notation may seem peculiar to some, it has been adopted for good reasons. It is awkward (at least) to proceed with formulae manipulation within conventional notations when

(a) The number of bases utilised is greater than two (four in the case of this work);

(b) The time derivatives of the employed quantities are not always observed from the same basis; and

(c) The quantities are not only treated as single entities, but also as arrays of components/elements, which are sometimes individually used.

Hughes (1986, p. 78-79, 522-34) also highlights the complexity of the situation and emphasises the need to choose the notation carefully. He proposes the use of vectrices when dealing with multi-frame problems. Vectrices formalise in an appropriate way the relationship between geometric and algebraic representations (see also Shuster, 1993a, p. 505-07). Although this notational device has a number of virtues, it has been considered unnecessary to the work carried out in this thesis. Thus, for the sake of simplicity, it has not been included in the notation.

Junkins & Turner (1986, p. 6) also recognise the difficulty when facing multi-frame problems. They even pointed out that a very large fraction of errors committed in formulating dynamical equations are of kinematic origin[2]. The notation they adopt is also very explicit. Follows the definitions of the notational elements presented in figure 1, a few illustrative examples of their relative location and utilisation, and the notational element lists.

2.1. Notational Element Definitions

Quantity Symbol - symbol that denotes a physical or mathematical quantity.

Basis of Representation - dextral orthonormal basis along whose axes the quantity is represented.

Array Symbols - symbols that denote the quantity in its algebraic form, i.e the quantity is expressed as an array of elements, one or two-dimensional, which correspond to the particular representation of the quantity in the *basis of representation*.

Component Identifier - one or two-digit number that identifies a single element of the quantity when this is represented in the *basis of representation*.

[2] The engineering note of Churchyard (1972), and the corresponding errata Churchyard (1973), constitute a pertinent example for this difficulty/problem.

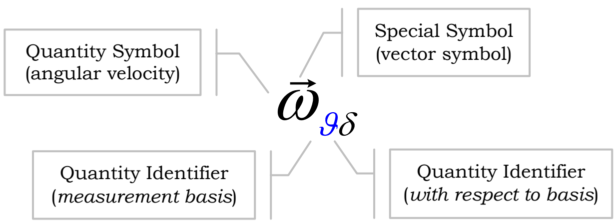
Special Symbols - symbols that specify certain forms of representation of the quantity: the geometric (basis-free, vectors/dyadics), and the algebraic skew-symmetric (basis-dependent, column vectors).

Quantity Identifier - sequence of letters and/or numbers that identifies

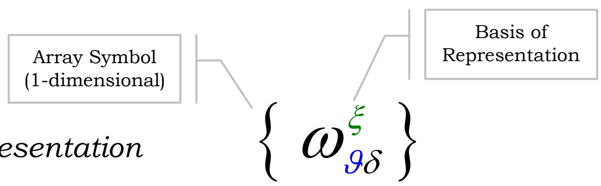
1. The rigid body whose quantity is measured;
2. The *measurement basis* - dextral orthonormal basis where the quantity is measured.
3. The *with respect to basis* - dextral orthonormal basis with respect to where the quantity is measured.
4. The special conditions associated to the quantity, or discriminates the quantity from similar ones.

2.2. *Illustrative Examples*

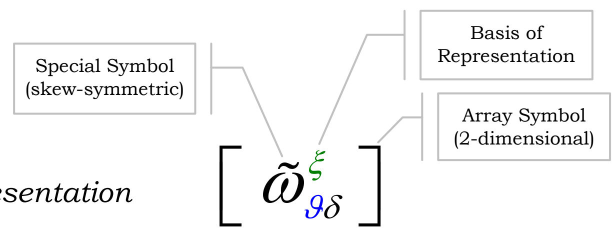
Geometric Representation



Algebraic Column-Vector Representation



Algebraic Skew-Symmetric Representation



2.3. Notational Element Lists

Basis of Representation	***symbols and names***			***orthogonal axes***	***unit vectors***
res. Greek letters lower-case δ, ϑ, λ, ξ italic	δ	delta	black	$\delta_1 , \delta_2 , \delta_3$	$\hat{\delta}_1 , \hat{\delta}_2 , \hat{\delta}_3$
	ϑ	alt. theta	blue	$\vartheta_1 , \vartheta_2 , \vartheta_3$	$\hat{\vartheta}_1 , \hat{\vartheta}_2 , \hat{\vartheta}_3$
	λ	lambda	red	$\lambda_1 , \lambda_2 , \lambda_3$	$\hat{\lambda}_1 , \hat{\lambda}_2 , \hat{\lambda}_3$
	ξ	xi	green	ξ_1 , ξ_2 , ξ_3	$\hat{\xi}_1 , \hat{\xi}_2 , \hat{\xi}_3$

Array Symbols	***notation***	***examples***
1-dimensional array algebraic / column vector	braces	$\left\{ (.)^{(.)} \right\}$
2-dimensional array matrix	brackets	$\left[(.)^{(.)} \right]$

Component Identifier	***notation***	***examples***
components of an 1-dimensional array components of an algebraic/column vector	one digit Times N. Roman regular	$\left\{ (.)^{(.)} \right\} = \begin{Bmatrix} (.)^{(.)_1} \\ (.)^{(.)_2} \\ (.)^{(.)_3} \end{Bmatrix}$
components of a 2-dimensional array elements of a matrix	two digits Times N. Roman regular	$\left[(.)^{(.)} \right] = \begin{bmatrix} (.)^{(.)_{11}} & (.)^{(.)_{12}} & (.)^{(.)_{13}} \\ (.)^{(.)_{21}} & (.)^{(.)_{22}} & (.)^{(.)_{23}} \\ (.)^{(.)_{31}} & (.)^{(.)_{32}} & (.)^{(.)_{33}} \end{bmatrix}$

Special Symbols	***notation***	***examples***
vector	arrow	$\vec{(.)}$
unit vector	hat	$\hat{(.)}$
dyadic	double-arrow	$\vec{\vec{(.)}}$
skew-symmetric representation of a vector	tilde *(inside brackets)*	$\left[\,\tilde{(.)}^{(\cdot)}\right] = \begin{bmatrix} 0 & -(.)^{(\cdot)_3} & (.)^{(\cdot)_2} \\ (.)^{(\cdot)_3} & 0 & -(.)^{(\cdot)_1} \\ -(.)^{(\cdot)_2} & (.)^{(\cdot)_1} & 0 \end{bmatrix}$

Quantity Identifier	***notation***	***examples***
Rigid Body	upper-case letters Times New Roman italic	B
Measurement Basis	reserved Greek letters lower-case δ, ϑ, λ, ξ italic	$\delta, \vartheta, \lambda, \xi$
With Respect to Basis	reserved Greek letters lower-case δ, ϑ, λ, ξ italic	$\delta, \vartheta, \lambda, \xi$
Special Conditions or Discriminators	Times New Roman regular	cm: centre of mass ic: initial condition ext: external torque m: measured/estimated 1st: first-order of approx. 2nd: second-order of approx. 3rd: third-order of approx. C: control torque P: perturbing torque numbers: 1, 2, 3 letters: a, b, c

Quantity Symbol	***notation***	***examples***
angular velocity	reserved Greek letter lower-case ω omega italic	$\vec{\omega}_{\lambda\vartheta}$, $\left\{ \omega^{\xi}_{\lambda\vartheta} \right\}$, $\left[\tilde{\omega}^{\xi}_{\lambda\vartheta} \right]$
Laplace transform of the angular velocity (component)	reserved Greek letter upper-case Ω omega italic	$\mathit{\Omega}^{\xi_1}_{\vartheta\delta}$, $\mathit{\Omega}^{\xi_2}_{\lambda\vartheta}$, $\mathit{\Omega}^{\xi_3}_{\lambda\delta}$
geometric angle	lower-case Greek letter italic φ alt. phi ψ psi θ theta γ gamma	φ, θ, ψ, γ
Euler angle of rotation magnitude of rotation vector	reserved Greek letter lower-case ϕ phi italic	$\phi_{\lambda\vartheta}$, $\phi_{\lambda\delta}$, $\phi_{\xi\vartheta}$
Euler axis of rotation unit vector along axis of rotation	lower-case n Times New Roman italic	$\hat{n}_{\lambda\vartheta}$, $\left\{ n^{\vartheta}_{\lambda\vartheta} \right\}$, $\left[\tilde{n}^{\xi}_{\lambda\vartheta} \right]$
rotation vector	reserved Greek letter lower-case ϕ phi italic	$\vec{\phi}_{\lambda\vartheta}$, $\left\{ \phi^{\xi}_{\lambda\vartheta} \right\}$, $\left[\tilde{\phi}^{\vartheta}_{\lambda\vartheta} \right]$
Laplace transform of the rotation vector (component)	reserved Greek letter upper-case Φ phi italic	$\mathit{\Phi}^{\xi_1}_{\lambda\vartheta}$
generic vector	lower-case r, u, v, w Times New Roman italic	$\vec{r}$, $\vec{u}$, $\left\{ v^{\delta}_{1} \right\}$, $\left[\tilde{w}^{\vartheta}_{2} \right]$
transformation matrix	upper-case T Times New Roman italic	$\left[T^{\lambda}_{\vartheta} \right]$, $\left[T^{\lambda}_{\delta} \right]$, $\left[T^{\delta}_{\xi} \right]$
rotation tensor	upper-case R Times New Roman italic	$\vec{\vec{R}}_{\vartheta\delta}$, $\left[R^{\lambda}_{\vartheta\delta} \right]$
moment of inertia tensor	upper-case I Times New Roman italic	$\vec{\vec{I}}_{\mathrm{Bcm}}$, $\vec{\vec{I}}_{B\vartheta}$, $\left[I^{\xi}_{B\vartheta} \right]$

Quantity Symbol (continuation)	***notation***	***examples***
identity tensor	number 1 Times New Roman italic	$\vec{\vec{1}}$, $[\ 1\ ]$
null tensor	number 0 Times New Roman italic	$\vec{\vec{0}}$, $[\ 0\ ]$
angular momentum	upper-case H Times New Roman italic	$\vec{H}_{\mathrm{Bcm}\delta}$, $\vec{H}_{\mathrm{B}\vartheta\delta}$, $\{\ H^{\delta}_{\mathrm{B}\vartheta\delta}\ \}$
moment of force	upper-case M Times New Roman italic	$\vec{M}_{B\vartheta\ \mathrm{C}}$, $\{\ M^{\xi}_{B\vartheta\ \mathrm{ext}}\ \}$
scalar constant	lower-case italic	k, c, μ_n, ζ

Vector/Array Operations	***notation***	***examples***
cross product	cross-product	$(\vec{\cdot}) \times (\vec{\cdot})$
scalar product	dot-product	$(\vec{\cdot}) \bullet (\vec{\cdot})$ $(\vec{\vec{\cdot}}) \bullet (\vec{\cdot})$
vector magnitude Euclidean norm	double vertical bars	$\left\| (\vec{\cdot}) \right\|$
matrix transpose	superscripted upper-case T Times New Roman italic	$\left[(.)^{(.)} \right]^T$
matrix inverse	superscripted –1 Times New Roman regular	$\left[(.)^{(.)} \right]^{-1}$
vector/dyadic time derivative	vector/dyadic quantity preceded by the time derivative subscripted by the basis symbol, or vector/dyadic quantity over-scripted by the basis symbol *(basis from where the time derivative of the vector/dyadic quantity is observed)*	$\frac{d}{dt_\delta}(\vec{\cdot}) \equiv \overset{\delta}{(\vec{\cdot})}$ $\frac{d}{dt_\vartheta}(\vec{\vec{\cdot}}) \equiv \overset{\vartheta}{(\vec{\vec{\cdot}})}$
array time derivative	array symbol preceded by the time derivative, or array symbol over-scripted by a bullet *(basis implicit)*	$\frac{d}{dt}\left\{ (.)^{(.)} \right\} \equiv \left\{ \overset{\bullet}{(.)}^{(.)} \right\}$ $\frac{d}{dt}\left[(.)^{(.)} \right] \equiv \left[\overset{\bullet}{(.)}^{(.)} \right]$
array time integral	array symbol preceded by the integral symbol and succeeded by the time increment *(basis implicit)*	$\int \left\{ (.)^{(.)} \right\} dt$ $\int \left[(.)^{(.)} \right] dt$

Chapter 3

Angular Position

In this chapter, the issue of rigid body orientation in three-dimensional space is reviewed. The chapter begins with the matter of determining the angular position of a rigid body based on 3D translational data. Subsequently, the discussion is confined to the exploration of the transformation matrix, the rotation matrix, the non-vector nature of finite rotations, and the related variety of orientation methods. The chapter continues commenting on the direction cosine redundancy, and presenting an expression relating the transformation matrix and the 3-1-3 body-fixed set of orientation angles. Some remarks on minimal attitude representations conclude the chapter.

In confronting the general question of how to describe the three-dimensional angular position of a rigid body[3], one realises that all points in the body may be located relative to a coordinate system (basis) fixed in the body. As a consequence, the three-dimensional angular position of a rigid body with respect to a certain frame δ can be described as the orientation, in that frame, of a single basis ϑ attached to the body (see Goldstein, 1980, p. 129; or Nikravesh, 1988, p. 153-54).

The unit vectors associated to that body-fixed basis, namely $\hat{\vartheta}_1$, $\hat{\vartheta}_2$, $\hat{\vartheta}_3$, may be obtained from the coordinates (translational data) of three known non-collinear body points. The corresponding procedure, with slight variations, is described by Osborne & Tolson (1996, p. 3-4), Nikravesh (1988, p. 164-66), and Griffin & French (1991, p. 310-12)

[3] The hypothesis of rigidity basically says that the distance between any two points in the body is unchangeable (see Rosenberg, 1977, p. 61; or Goldstein, 1980, p. 128).

The components of those three unit vectors in basis δ can be readily recognised as the cosines of the angles formed by the unit vectors themselves and the axes of basis δ. Such components are aptly termed direction cosines (see Appendix B).

These nine direction cosines (three components for each unit vector) can now be arranged column-wise forming a 3x3 matrix $\left[T_{\vartheta}^{\delta} \right]$, which gives the relative orientation of the two frames, ϑ and δ. This matrix might be employed, for example, to convert/transform the component resolution of a vector quantity $\vec{v}$ from basis ϑ to basis δ. This matrix is named most commonly in the literature as the transformation matrix[4] (see Appendix B).

$$\left\{ v^{\delta} \right\} = \left[T_{\vartheta}^{\delta} \right] \left\{ v^{\vartheta} \right\} \tag{3.1}$$

Similarly, the inverse transformation matrix $\left[T_{\delta}^{\vartheta} \right]$, i.e. the matrix that converts the resolution of a vector quantity from basis δ to basis ϑ, can be obtained by simply rearranging the direction cosines into a row-wise fashion. The simplicity of this operation stems from the fact that the transformation matrix is orthogonal; thus its inverse equals the transpose[5] (see Appendix B).

$$\left[T_{\vartheta}^{\delta} \right]^{T} = \left[T_{\vartheta}^{\delta} \right]^{-1} = \left[T_{\delta}^{\vartheta} \right] \tag{3.2}$$

The transformation matrix $\left[T_{\vartheta}^{\delta} \right]$ can, in fact, be interpreted in two distinct ways: first as a frame transformation matrix, and second as a rotational operator. In the former interpretation (the one alluded to in the previous paragraphs), the transformation matrix is not dependent on the vector

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[4] Depending on the use, interpretation and/or author's preference, the transformation matrix might be named as direction cosine matrix, rotation matrix, rotational transformation matrix, transformation rotation matrix, orthogonal transformation matrix, matrix of linear vector transformation, orthonormal transformation matrix, attitude matrix, orientation matrix, attitude operator, orthonormal rotational transformation matrix, coordinate transformation matrix and so forth.

[5] This crucial result that inverse equals the transpose holds only for orthogonal matrices (see Arfken & Weber, 1995, p. 187).

quantity being frame-transformed. Similarly, in the latter interpretation, it is not dependent on the vector quantity being rotated (see Appendix C).

This dual interpretation can be used to establish a relationship of equivalence between the transformation matrix and the rotation matrix (coordinate representation of the rotation tensor). There are several lines of approach to deriving this relationship[6]. An elegant one considers Euler's theorem, which states that the general rotation of a rigid body/frame is equivalent to a single rotation about a fixed axis (see Pars, 1965, p. 90-94; Whittaker, 1927, p. 2-3). Following the example of Grubin (1970) and Hughes (1986, p. 17), the unit vector $\hat{n}_{\vartheta\delta}$ along this axis of rotation and the corresponding angle of rotation $\phi_{\vartheta\delta}$ will be called Euler axis/angle variables[7] in this thesis (see Appendix C).

It is interesting to note that the transformation matrix $\left[\, T_{\vartheta}^{\delta} \,\right]$ can be equated to the rotation tensor $\vec{\vec{R}}_{\vartheta\delta}$ whether its basis of representation is the *from basis* ϑ or the *to basis* δ. Equivalently, one could have stated that the transformation matrix $\left[\, T_{\vartheta}^{\delta} \,\right]$ can be equated to the rotation tensor $\vec{\vec{R}}_{\vartheta\delta}$ whether its basis of representation is the *measurement basis* ϑ or the *with respect to basis* δ (see Appendix C).

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\vartheta} \,\right] = \left[\, 1 \,\right] + \sin\phi_{\vartheta\delta} \left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] \tag{3.3}$$

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\delta} \,\right] = \left[\, 1 \,\right] + \sin\phi_{\vartheta\delta} \left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] \tag{3.4}$$

[6] This result is commonly referred to as the Rodriguez Formula (see Shabana, 1994, p. 438-39; Shabana, 1998, p. 31-33; or Rosenberg, 1977, p. 82-84).

[7] One can find in the literature several denominations for this axis, among which are: Euler axis, eigenaxis, unit eigenvector, effective axis of rotation, orientational axis of rotation, principal line, principal axis, equivalent rotation axis, unit equivalent axis and so forth. The denomination of the corresponding angle follows similarly.

The arbitrariness of the basis of representation of the rotation tensor $\vec{\vec{R}}_{\vartheta\delta}$ when equated to the transformation matrix $\left[\ T_{\vartheta}^{\delta}\ \right]$ may, in fact, be extended to any basis ξ whose Euler axis $\hat{n}_{\xi\delta}$ is parallel to $\hat{n}_{\vartheta\delta}$, i.e. $\left\{\ n_{\vartheta\delta}^{\delta}\ \right\} = \left\{\ n_{\xi\delta}^{\xi}\ \right\} = \left\{\ n_{\vartheta\delta}^{\xi}\ \right\}$. Conversely, the corresponding Euler angle $\phi_{\xi\delta}$ has no restrictions, and can therefore assume any value. This gives rise to a third, more general and not normally quoted form for the relationship of equivalence (see Appendix C):

$$\left[\ T_{\vartheta}^{\delta}\ \right] = \left[\ R_{\vartheta\delta}^{\xi}\ \right] = \left[\ 1\ \right] + \sin\phi_{\vartheta\delta}\left[\ \tilde{n}_{\vartheta\delta}^{\xi}\ \right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\ \tilde{n}_{\vartheta\delta}^{\xi}\ \right]\left[\ \tilde{n}_{\vartheta\delta}^{\xi}\ \right] \tag{3.5}$$

The extraction of the Euler angle/axis variables from a given transformation matrix can be found discussed in a number of works, among which are: Hughes (1986, p. 13-14), Gelman (1969), Junkins & Turner (1986, p. 26-28), Craig (1989, 51-53), Paul (1986, 25-32), and Shuster (1993a, 451-52).

3.1. *Alternative Parameterisations and Commentaries*

The direction cosines and the Euler angle/axis variables are only two methods for specifying the spatial angular position of a rigid body/frame. A review of the pertinent literature reveals a wide variety of orientation methods. Important examples used in engineering and technological applications, *inter alia*, are: Euler angles, Bryant angles, Cardan angles, Euler parameters, Rodriguez parameters, quaternions and the various body and space orientation angles.

Possibly, the most comprehensive survey of attitude representation is the one offered by Shuster (1993a). For detailed explanations refer to Hughes (1986, p. 6-30), Kane et al. (1983, p. 1-38), Nikravesh (1988, p. 153-162, 347-52), Junkins & Turner (1986, p. 16-39), Shabana (1994, p. 442-47), Shabana (1998, p. 28-86), Wie (1998, p. 307-20), Adade Filho (2001, p. 39-55), and Wertz (1980, p. 410-20, 758-66). For summaries refer to Trindade & Sampaio (2000), Betsch et al. (1998, p. 275-78), Spring (1986, p. 366-67), and Rooney

(1977). The last reference compares some not mentioned rotational representations used in theoretical physics.

The Euler, Bryant and Cardan angles are, in fact, particular definitions of the more generic classifications body and space orientation angles (see Kane 1983, p. 30-38). All these methods rely upon the idea that the *finite* rotations are performed in precise sequences and/or over specific axes. This is connected to the fact that *finite* rotations do not commute[8] (they do not satisfy the vector parallelogram addition law), and therefore they *cannot* be represented by a single true vector. For general discussions concerning these points, refer to Goldstein (1980, p. 167), Rosenberg (1977, p. 63-65), or Lewis & Ward (1989, p. 305-307); for pertinent examples of utilisation, see Roskam (1979, p. 24-31)[9].

Although the non-vector nature of finite rotations is undeniable and evident, one should realise that by virtue of Euler's theorem any rotation can indeed be parameterised with the components of a single vector defined as $\vec{\phi}_{\vartheta\delta} = \phi_{\vartheta\delta}\hat{n}_{\vartheta\delta}$, and most commonly called rotation vector[10]. In terms of the components of $\vec{\phi}_{\vartheta\delta}$, and its norm $\phi_{\vartheta\delta}$, the transformation/rotation matrix can be obtained directly from equations 3.3 and 3.4:

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\vartheta} \,\right] = \left[\, 1 \,\right] + \left(\frac{\sin\phi_{\vartheta\delta}}{\phi_{\vartheta\delta}}\right)\left[\, \tilde{\phi}_{\vartheta\delta}^{\vartheta} \,\right] + \left(\frac{1-\cos\phi_{\vartheta\delta}}{\phi_{\vartheta\delta}^{\;2}}\right)\left[\, \tilde{\phi}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{\phi}_{\vartheta\delta}^{\vartheta} \,\right] \tag{3.6}$$

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\delta} \,\right] = \left[\, 1 \,\right] + \left(\frac{\sin\phi_{\vartheta\delta}}{\phi_{\vartheta\delta}}\right)\left[\, \tilde{\phi}_{\vartheta\delta}^{\delta} \,\right] + \left(\frac{1-\cos\phi_{\vartheta\delta}}{\phi_{\vartheta\delta}^{\;2}}\right)\left[\, \tilde{\phi}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{\phi}_{\vartheta\delta}^{\delta} \,\right] \tag{3.7}$$

---

[8] Regardless of the fact that finite rotations do not commute, infinitesimal rotations do commute (see Smith, 1982, p. 442-45; or Konopinsky, 1969, p. 234-39).

[9] For a more specialised literature in the subject of finite/large rotations, refer to the seminal work of Argyris (1982).

[10] The terminology and notation concerning rotations found in the literature is *not* universal. Argyris (1982), for example, refers to a finite rotation as a *rotational pseudovector*, while Ibrahimbegovic (1997) refer to the same quantity as a *rotation vector*. The interested reader may refer to Angeles (1997, p. 20-21) for an axiomatic description of vectors, and Goldstein (1980, p. 171-72) for a concise explanation differentiating polar vectors and axial vectors (pseudovectors).

One driving motive for the existence of orientation methods apart from the simple direction cosines is the fact that these nine components of the transformation matrix are not independent (a rigid body has only three rotational degrees of freedom)[11]. There is a minimum set of six equations relating them, which are called collectively conditions of orthogonality. These conditions may be interpreted and stated in several different ways (see Arfken & Weber, 1995, p. 184-87; or Tai, 1997, p. 8-14).

One intuitively appealing way of interpreting the orthogonality conditions is to regard them as the scalar products of the unit vectors of either *from basis* or *to basis*, the columns and rows of the transformation matrix respectively (see Appendix B). An alternative set of orthogonality conditions has been given by Gelman (1968).

In the case of ADAMS[12], which implements the Lagrangian formulation in building its equations of motion, an independent set of parameters is required, rather than the redundant direction cosines[13]. The one such set utilised internally by ADAMS is the 3-1-3 Euler angles[14], i.e. the body-fixed 3-1-3 set of orientation angles (see Blundell, 1997, p. 39; or Wielenga, 1987, p. 2-3).

The relationship between this set of orientation coordinates and the transformation matrix is attained via three successive orthogonal transformations[15], corresponding to the three body-fixed rotations. The derivation of such a matrix can be found in a number of advanced books in the subject, e.g. Goldstein (1980, p. 143-48); Nikravesh (1988, p. 348);

[11] In spatial kinematics, the unconstrained motion of a rigid body is described using six independent coordinates or DOF (degrees of freedom). Three of these DOF represent the translations, and the remaining three the rotations. Therefore, the orientation of a rigid frame can be completely defined in terms of three independent variables (see Shabana, 1998, p. 34; Goldstein, 1980, p. 128-29; or Bottema & Roth, 1979, p. 149).

[12] ADAMS (an acronym for Automatic Dynamic Analysis of Mechanical Systems) is a leading computer software package on the field of mechanical system simulation.

[13] The unduly redundancy of the nine direction cosines demands excessive computational effort when compared to other methods (see Rheinfurth & Wilson, 1991, p. 97; or Shuster, 1993, p. 498-99).

[14] The definition of the Euler Angles is not unique in the literature. Some authors choose the second rotation to be about the y-axis, whereas others consider any body-fixed sequence (see Shuster, 1993, p. 454).

[15] The resulting matrix of a successive product of orthogonal matrices is also orthogonal.

Shabana (1994, p. 363) and Bottema & Roth (1979, p. 153-56). For completeness, the end result is shown below (see figure 3).

$$\left[T_{\vartheta}^{\delta} \right] = \begin{bmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & -\sin\varphi\cos\psi - \cos\varphi\cos\theta\sin\psi & \sin\theta\sin\psi \\ \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & -\sin\theta\cos\psi \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{bmatrix} \quad (3.8)$$

This relationship can be employed either to calculate $\left[T_{\vartheta}^{\delta} \right]$ from the 3-1-3 Euler angles, or to calculate 3-1-3 Euler angles from $\left[T_{\vartheta}^{\delta} \right]$ by solving a set of transcendental equations (see Paul, 1986, p. 65-71; or Adade Filho, 2001, p. 41-44). In fact, any set of orientation coordinates such as Euler angles, Euler parameters, Rodriguez parameters, and so on can be extracted from a given transformation matrix by solving a set of transcendental equations.

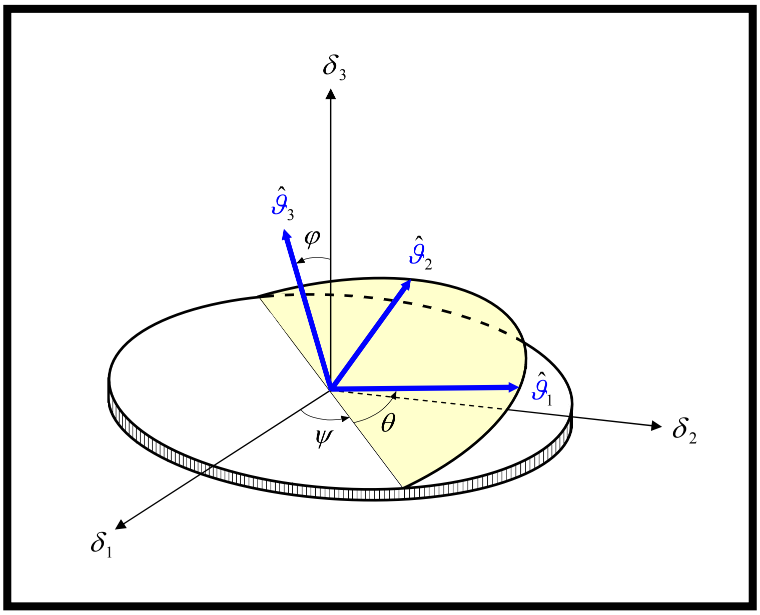

Figure 3: Euler Angles

It should be perceived, however, that the representation of the transformation matrix with Euler angles, equation 3.8, may *not* be well defined (singularities may occur) at certain orientations of the rigid coordinate frame in space. In fact, this disadvantage is not a particular case, but proceeds from any three-variable representation of the transformation matrix. This suggests that a four parameters representation, such as the Euler parameters is sometimes recommended. These points are discussed by many authors, among whom are Stuelpnagel (1964), Klumpp (1976), Shabana (1994, p. 373, 444), Shabana (1998, p. 34, 84-85), Nikravesh (1988, p. 39, 153, 162, 349), Paul (1986, p. 65-70), Pfister (1996), Betsch et al. (1998, p. 278, remark 1), or Rheinfurth & Wilson (1991, p. 82-85, 97-98).

3.2. Remark on Minimal Attitude Representations

Although minimal attitude representations (three-parameter only) inevitably incur singularities, the domain of validity of the parameterisation depends, of course, on the parameterisation itself. Recent research has addressed the problem of singular orientations, and new parameterisations have been proposed. These new parameterisations move the inherent singularity far away from the origin. It is now possible to achieve a globally non-singular minimal attitude parameterisation for all possible ± 360 degrees rotations. The excellent work of Tsiotras et al. (1997) shows in a unified fashion how to generate such representations. Junkins (1997) comments and compares some of the results. Tsiotras (1996) and Crassidis & Markley (1996) use such parameterisations in the construction of attitude control laws.

Chapter 4

Angular Velocity

This chapter focuses on the development of expressions for the angular velocity. Initially, the derivation process considers the transformation matrix as a generic symbol, representing any set of orientation coordinates. The resulting general relationships are subsequently specialised to the case where the orientation coordinates are the Euler angle/axis.

In the previous chapter various methods of defining the orientation of a rigid body/frame in three-dimensional space were examined. Considering the number of methods available and their relative merits, one realises that care must be taken when choosing a parameterisation for finite spatial rotations.

In this manner, it is expedient to regard the transformation matrix as a generic symbol, i.e. avoiding explicit use of any particular set of orientation coordinates, when deriving related kinematic quantities. The virtue of this approach is that it leads to not only more general, but also simpler expressions for the angular velocity *matrix* (skew-symmetric expansion of the angular velocity vector).

These orientation coordinate-free expressions are commonly demonstrated by differentiation of the orthogonality condition of the transformation matrix. The result, a skew-symmetric matrix, is then *defined* as the corresponding angular velocity matrix (see Bottema and Roth, 1979, p. 20-21; Angeles, 1997, p. 83; Shabana, 1994, p. 364-67; Shabana, 1998, p. 85-86; Corben & Stehle, 1994, p. 141-42, Nikravesh, 1988, p. 172-74; Kane et al., 1983, p. 47-48; and Meyer, 1966)[16].

[16] Bradbury (1968, p. 416-22), Junkins & Turner (1986, p. 11-13), Hughes (1986, p. 22-24) and Beggs (1983, p. 55-58) propose different versions for the derivation of the general expressions for the angular velocity matrix.

Although this method of demonstration is fairly simple, it looses track of the intuitive concept of the angular velocity vector. The somewhat unusual and more enlightening demonstration that follows is almost as simple as the one just referred, but it employs the fundamental relationship (transport theorem) between the time derivatives of a vector $\vec{v}$ as seen from two bases ϑ and δ that rotate with respect to each other:

$$\overset{\delta}{\vec{v}} = \overset{\vartheta}{\vec{v}} + \vec{\omega}_{\vartheta\delta} \times \vec{v} \tag{4.1}$$

angular velocity vector of basis ϑ with respect to basis δ

This relationship is, in fact, the result of a geometric argumentation that considers solely the relative angular velocity[17] between the two referred bases (see Smith, 1982, p. 334-36). Much of the formulae development in this thesis makes direct or indirect use of this relationship.

4.1. Angular Velocity Matrix

Consider a vector $\vec{r}$ fixed on basis ϑ, which rotates with respect to basis δ (see figure 4 for an illustration). In this case, the ϑ-observed derivative of $\vec{r}$ is clearly time invariant, thus given simply by

$$\overset{\vartheta}{\vec{r}} = \vec{0} \qquad \Leftrightarrow \qquad \{\dot{r}^{\vartheta}\} = \{0\} \tag{4.2}$$

The differentiation law (transport theorem) reduces in the case to

$$\overset{\delta}{\vec{r}} = \vec{\omega}_{\vartheta\delta} \times \vec{r} \tag{4.3}$$

[17] The angular velocity can be treated as a vector field, and thought of as a property of the corresponding rigid body/frame. The reason substantiating this assertion is that the angular velocity vector is a function of time only, i.e. at a given instant of time it has the same value at all points of the body (see Bradbury, 1968, p. 418, Goodman & Warner, 1964, p. 346; Greenwood, 1965, p. 32; or Roskam, 1979, p. 14).

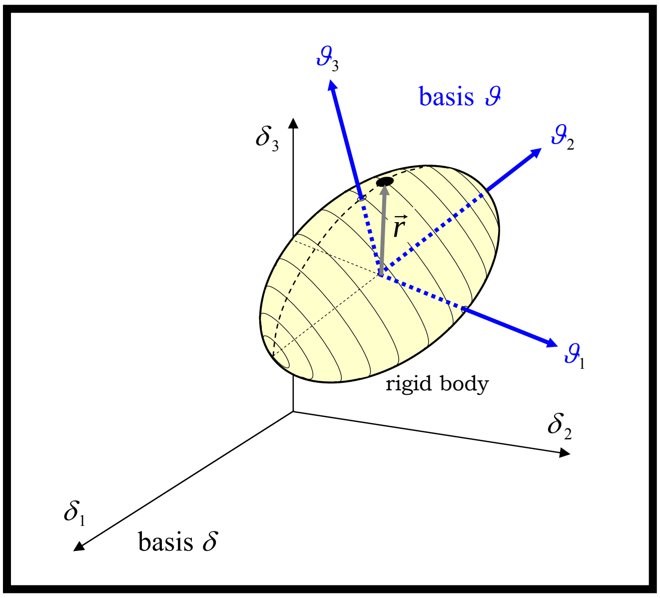

Figure 4: Illustration

As a vector equation, this result has no dependence on the particular basis selected to represent it, and can, therefore, be resolved using any set of base vectors that span the space, for instance

$$\frac{d}{dt_\delta}\left(r^{\delta_1}\hat{\delta}_1 + r^{\delta_2}\hat{\delta}_2 + r^{\delta_3}\hat{\delta}_3 \right) = \left(\omega^{\delta_1}_{\vartheta\delta}\hat{\delta}_1 + \omega^{\delta_2}_{\vartheta\delta}\hat{\delta}_2 + \omega^{\delta_3}_{\vartheta\delta}\hat{\delta}_3 \right)\times\left(r^{\delta_1}\hat{\delta}_1 + r^{\delta_2}\hat{\delta}_2 + r^{\delta_3}\hat{\delta}_3 \right) \tag{4.4}$$

The resolution onto basis δ is particularly convenient in the case since

$$\frac{d}{dt_\delta}\hat{\delta}_1 = \vec{0}$$

$$\frac{d}{dt_\delta}\hat{\delta}_2 = \vec{0} \qquad \hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3 \text{ are constant vectors with respect to basis } \delta$$

$$\frac{d}{dt_\delta}\hat{\delta}_3 = \vec{0}$$

Considering yet that

$$\frac{d}{dt_\delta} r^{\delta_1} = \frac{d}{dt} r^{\delta_1} = \dot{r}^{\delta_1}$$

$$\frac{d}{dt_\delta} r^{\delta_2} = \frac{d}{dt} r^{\delta_2} = \dot{r}^{\delta_2}$$

$$\frac{d}{dt_\delta} r^{\delta_3} = \frac{d}{dt} r^{\delta_3} = \dot{r}^{\delta_3}$$

time derivatives of scalar quantities are independent of the frame of observation

equation 4.4 can be simplified to

$$\dot{r}^{\delta_1}\,\hat{\delta}_1 + \dot{r}^{\delta_2}\,\hat{\delta}_2 + \dot{r}^{\delta_3}\,\hat{\delta}_3 = \left(\omega^{\delta_1}_{\vartheta\delta}\hat{\delta}_1 + \omega^{\delta_2}_{\vartheta\delta}\hat{\delta}_2 + \omega^{\delta_3}_{\vartheta\delta}\hat{\delta}_3\right)\times\left(r^{\delta_1}\,\hat{\delta}_1 + r^{\delta_2}\,\hat{\delta}_2 + r^{\delta_3}\,\hat{\delta}_3\right)$$

which is readily expressible in algebraic format as (see Appendix A for the various forms of representation of the vector cross product)

$$\left\{ \dot{r}^{\delta} \right\} = \left[\tilde{\omega}^{\delta}_{\vartheta\delta} \right]\left\{ r^{\delta} \right\} \tag{4.5}$$

The procedure utilised between equations 4.3 and 4.5, i.e. the resolution of a vector (or vector-dyadic) equation onto the basis of observation of the time derivative, is very useful and will be called upon repeatedly in this text. The key fact to keep in mind is that the vector $\overset{\xi}{\dot{\vec{v}}}$ is the time derivative of $\vec{v}$ as observed from a basis in which $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3$ are fixed.

The geometrical method used in the derivation of equation 4.5 naturally brought forth the already mentioned angular velocity matrix[18]. A second relation for $\left\{ \dot{r}^{\delta} \right\}$ may be attained by time differentiation of the relationship

[18] Skew-symmetric matrix associated with the angular velocity vector, and representing the matrix counterpart of the vector cross product. The tilde placed over the quantity symbol indicates that the components of the associated vector (for instance, the angular velocity vector) are used to generate the skew-symmetric matrix. This convention is fairly common in the literature, and will be used again in this work to represent analogous matrices associated with vectors other than the angular velocity one.

between the δ-resolution and the ϑ-resolution of vector $\vec{r}$, i.e. the transformation equation given by[19]

$$\{ r^{\delta} \} = [T^{\delta}_{\vartheta}] \{ r^{\vartheta} \} \tag{4.6}$$

So, taking the time derivative of both sides of equation 4.6, and remembering that the elements of $\{ r^{\vartheta} \}$ are time invariant (equation 4.2), the next relation emerges

$$\begin{aligned} \{ \dot{r}^{\delta} \} &= [\dot{T}^{\delta}_{\vartheta}] \{ r^{\vartheta} \} + [T^{\delta}_{\vartheta}] \{ \dot{r}^{\vartheta} \} \\ &= [\dot{T}^{\delta}_{\vartheta}] \{ r^{\vartheta} \} \end{aligned} \tag{4.7}$$

Using equation 4.7 in conjunction with equations 4.5 and 4.6, one arrives at the following relation

$$[\dot{T}^{\delta}_{\vartheta}] \{ r^{\vartheta} \} = [\tilde{\omega}^{\delta}_{\vartheta\delta}] [T^{\delta}_{\vartheta}] \{ r^{\vartheta} \}$$

which reduces immediately to

$$[\dot{T}^{\delta}_{\vartheta}] = [\tilde{\omega}^{\delta}_{\vartheta\delta}] [T^{\delta}_{\vartheta}] \tag{4.8}$$

Now, making use of the orthogonality property of the transformation matrix, i.e. $[T^{\delta}_{\vartheta}]^{-1} = [T^{\delta}_{\vartheta}]^{T}$ (see Appendix B), the final result is achieved

$$\begin{aligned} [\tilde{\omega}^{\delta}_{\vartheta\delta}] &= [\dot{T}^{\delta}_{\vartheta}] [T^{\delta}_{\vartheta}]^{T} \\ &= [\dot{T}^{\delta}_{\vartheta}] [T^{\vartheta}_{\delta}] \end{aligned} \tag{4.9}$$

[19] One way of interpreting the transformation matrix is to think of it as relating the “appearance” of the vector in two different bases.

The expansion of this matrix relation leads to two different, but equivalent scalar equations for each component of the angular velocity vector when resolved onto basis δ (*to basis*).

The second resolution of interest is the one in terms of the unit vectors of the *from basis* ϑ. This can be attained by first observing that the matrix property of antisymmetry is invariant under orthogonal similarity transformations[20] (see Appendix E), thus

$$\left[\, \tilde{\omega}_{\vartheta\delta}^{\delta} \,\right] = \left[\, T_{\vartheta}^{\delta} \,\right]\left[\, \tilde{\omega}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, T_{\vartheta}^{\delta} \,\right]^{T} \qquad \Rightarrow \qquad \left[\, \tilde{\omega}_{\vartheta\delta}^{\vartheta} \,\right] = \left[\, T_{\vartheta}^{\delta} \,\right]^{T}\left[\, \tilde{\omega}_{\vartheta\delta}^{\delta} \,\right]\left[\, T_{\vartheta}^{\delta} \,\right]$$

which after substitution of equation 4.9 leads to

$$\begin{aligned} \left[\, \tilde{\omega}_{\vartheta\delta}^{\vartheta} \,\right] &= \left[\, T_{\vartheta}^{\delta} \,\right]^{T}\left[\, \dot{T}_{\vartheta}^{\delta} \,\right] \\ &= \left[\, T_{\delta}^{\vartheta} \,\right]\left[\, \dot{T}_{\vartheta}^{\delta} \,\right] \end{aligned} \tag{4.10}$$

This equation can also be written as

$$\left[\, \dot{T}_{\vartheta}^{\delta} \,\right] = \left[\, T_{\vartheta}^{\delta} \,\right]\left[\, \tilde{\omega}_{\vartheta\delta}^{\vartheta} \,\right] \tag{4.11}$$

4.2. Time Dependence of the Euler Angle/Axis Variables

The angular velocity expressions developed in section 4.1 have been formulated without the use of any particular set of orientation coordinates. So far, the transformation matrix has been considered a collection of direction cosines giving the relative orientation of two coordinate systems.

---

[20] The similarity transformation is an operation that shows how a representation (matrix) that depends on the basis (there are directions associated with the matrix) would change with a change in the basis itself (see Arfken & Weber, 1995, p. 190-92; or Strang, 1988, p. 304-07).

Direction cosines are not the only means – indeed, in many circumstances far from the most useful means – for specifying a rigid rotation (relative orientation). Another possibility, suggested by Euler's theorem, is the specification of the orientation with the axis about which the rotation takes place and the respective angle of rotation, the Euler axis/angle variables. The quantities in this system are, of course, related to the direction cosines, and the relationships between the two systems are given by equations 3.3 and 3.4

These two equations are particularly important to this work, since they comprise the departure points to the derivation of the relationships between the angular velocity and the time development of the orientation when the orientation is parameterised with the Euler angle/axis variables. Such derivations are shown next.

The transformation matrix $\left[\, T_{\vartheta}^{\delta} \,\right]$ can always be expressed in terms of $\phi_{\vartheta\delta}$ and $\hat{n}_{\vartheta\delta}$, as implied by Euler's theorem (equation 3.3):

$$\begin{aligned}\left[\, T_{\vartheta}^{\delta} \,\right] &= \left[\, R_{\vartheta\delta}^{\vartheta} \,\right] = \left[\, R^{\vartheta}\left(\phi_{\vartheta\delta}, \hat{n}_{\vartheta\delta}\right) \right] \\ &= \left[\, 1 \,\right] + \sin\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\end{aligned}$$

whose transpose and time derivative are respectively

$$\left[\, T_{\vartheta}^{\delta} \,\right]^{T} = \left[\, 1 \,\right] - \sin\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]$$

$$\begin{aligned}\left[\, \dot{T}_{\vartheta}^{\delta} \,\right] &= \dot{\phi}_{\vartheta\delta}\cos\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] + \sin\phi_{\vartheta\delta}\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\vartheta} \,\right] + \dot{\phi}_{\vartheta\delta}\sin\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] \\ &\quad + \left(1-\cos\phi_{\vartheta\delta}\right)\left(\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\vartheta} \,\right] + \left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\right)\end{aligned}$$

Substitution of these two results into equation 4.10, along with a few cancellations, simplifications and the following identities

$$[\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,][\,\dot{\tilde{n}}^{\vartheta}_{\vartheta\delta}\,][\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,] = [\,0\,] \qquad [\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]^4 = -[\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]^2 \qquad [\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]^3 = -[\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]$$

reduce the resulting expression to

$$[\,\tilde{\omega}^{\vartheta}_{\vartheta\delta}\,] = \dot{\phi}_{\vartheta\delta}\,[\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,] + \sin\phi_{\vartheta\delta}\,[\,\dot{\tilde{n}}^{\vartheta}_{\vartheta\delta}\,] - (1-\cos\phi_{\vartheta\delta})\Big([\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,][\,\dot{\tilde{n}}^{\vartheta}_{\vartheta\delta}\,] - [\,\dot{\tilde{n}}^{\vartheta}_{\vartheta\delta}\,][\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]\Big) \tag{4.12}$$

Observing that the last term in parenthesis is the skew-symmetric representation of a vector cross product (see Appendix D), it follows immediately that

$$\{\,\omega^{\vartheta}_{\vartheta\delta}\,\} = \dot{\phi}_{\vartheta\delta}\,\{\,n^{\vartheta}_{\vartheta\delta}\,\} + \sin\phi_{\vartheta\delta}\,\{\,\dot{n}^{\vartheta}_{\vartheta\delta}\,\} - (1-\cos\phi_{\vartheta\delta})[\,\tilde{n}^{\vartheta}_{\vartheta\delta}\,]\{\,\dot{n}^{\vartheta}_{\vartheta\delta}\,\} \tag{4.13}$$

It is important to observe that several quantities in equation 4.13 are now enclosed with braces (column matrices), which must clearly distinguish them from the corresponding quantities in the anterior equation 4.12 enclosed with brackets (skew-symmetric matrices). In geometric notation, equation 4.13 becomes

$$\vec{\omega}_{\vartheta\delta} = \dot{\phi}_{\vartheta\delta}\,\hat{n}_{\vartheta\delta} + \sin\phi_{\vartheta\delta}\,\overset{\vartheta}{\hat{n}}_{\vartheta\delta} - (1-\cos\phi_{\vartheta\delta})\,\hat{n}_{\vartheta\delta}\times\overset{\vartheta}{\hat{n}}_{\vartheta\delta} \tag{4.14}$$

The δ-resolution counterpart of equation 4.13 can be derived in a similar fashion. The first step is the observation that the relationship of equivalence between the transformation matrix and the rotation matrix can also be expressed as (equation 3.4):

$$\begin{aligned}[\,T^{\delta}_{\vartheta}\,] &= [\,R^{\delta}_{\vartheta\delta}\,] = [\,R^{\delta}(\,\phi_{\vartheta\delta},\hat{n}_{\vartheta\delta}\,)] \\ &= [\,1\,] + \sin\phi_{\vartheta\delta}\,[\,\tilde{n}^{\delta}_{\vartheta\delta}\,] + (1-\cos\phi_{\vartheta\delta})[\,\tilde{n}^{\delta}_{\vartheta\delta}\,][\,\tilde{n}^{\delta}_{\vartheta\delta}\,]\end{aligned}$$

whose transpose and time derivative are respectively

$$\left[\, T_{\vartheta}^{\delta} \,\right]^{T} = \left[\, 1 \,\right] - \sin\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]$$

$$\begin{aligned}\left[\, \dot{T}_{\vartheta}^{\delta} \,\right] = {} & \dot{\phi}_{\vartheta\delta}\cos\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] + \sin\phi_{\vartheta\delta}\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right] + \dot{\phi}_{\vartheta\delta}\sin\phi_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] \\ & + \left(1-\cos\phi_{\vartheta\delta}\right)\left(\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right] + \left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\right)\end{aligned}$$

Substitution of these two results into equation 4.9, along with a few cancellations, simplifications and the following identities

$$\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] = \left[\, 0 \,\right] \qquad \left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]^{4} = -\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]^{2} \qquad \left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]^{3} = -\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]$$

reduce the resulting expression to

$$\left[\, \tilde{\omega}_{\vartheta\delta}^{\delta} \,\right] = \dot{\phi}_{\vartheta\delta}\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right] + \sin\phi_{\vartheta\delta}\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right] + \left(1-\cos\phi_{\vartheta\delta}\right)\left(\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right] - \left[\, \dot{\tilde{n}}_{\vartheta\delta}^{\delta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\right)$$

The matrix identity described in Appendix D may be used once more to rewrite this expression as a column matrix relationship:

$$\left\{\, \omega_{\vartheta\delta}^{\delta} \,\right\} = \dot{\phi}_{\vartheta\delta}\left\{\, n_{\vartheta\delta}^{\delta} \,\right\} + \sin\phi_{\vartheta\delta}\left\{\, \dot{n}_{\vartheta\delta}^{\delta} \,\right\} + \left(1-\cos\phi_{\vartheta\delta}\right)\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left\{\, \dot{n}_{\vartheta\delta}^{\delta} \,\right\} \tag{4.15}$$

In geometric notation, equation 4.15 becomes

$$\vec{\omega}_{\vartheta\delta} = \dot{\phi}_{\vartheta\delta}\,\hat{n}_{\vartheta\delta} + \sin\phi_{\vartheta\delta}\,\overset{\delta}{\hat{n}}_{\vartheta\delta} + \left(1-\cos\phi_{\vartheta\delta}\right)\hat{n}_{\vartheta\delta}\times\overset{\delta}{\hat{n}}_{\vartheta\delta} \tag{4.16}$$

The kinematical differential relationship between the angular velocity and the Euler angle/axis variables is also analysed by Shabana (1998, p. 51-55), Shabana (1994, p. 440), Hughes (1986, p. 24-25), Angeles (1997, p. 89-90) and Meyer (1966, Appendix B). Gelman (1971) provides an alternative derivation for this relationship, along with an interesting geometrical interpretation.

The simple and general relationships for the angular velocity matrix achieved in this chapter, namely 4.9 and 4.10, can be usually found in advanced books of the pertinent literature. Although not so often found, equations 4.14 and 4.16 are also part of the standard literature (see, e.g., Shuster, 1993a, p. 478). There are several lines of approach to deriving and expressing these relationships. Authors utilise methods that are somewhat different from each other. The development proposed here also fits in this context.

Chapter 5

Equations of Rotational Motion

This chapter presents a few different possibilities of expressing the equations of motion for a rotating rigid body. Starting with the basic moment-of-momentum relationship, the derivation of these equations follows a simple, concise and somewhat unusual geometric procedure (basis-free), which evidences the usefulness of the adopted notation and leads to the desired general results.

Last chapter addressed the matter of evaluating the relative angular velocity between two dextral orthonormal bases when the corresponding attitude history is given, equation 4.9 or 4.10. In the derivation process, the transformation matrix was considered as a generic symbol, representing whichever attitude variables.

On the other hand, if the relative angular velocity history is known, the scalar counterparts for either 4.8 or 4.11 can be integrated to find the relative attitude history[21].

Hitherto, only geometric aspects of the rigid body/frame rotational motion have been considered. This chapter introduces expressions relating applied torques and rotational motion, i.e. the dynamical equations of motion for a rotating rigid body.

There are many dynamical formulations available for deriving the motion equations. The two principally employed are the Lagrangian formulation and the Newton-Euler formulation.

[21] Kane (1973) offers an approximate analytical solution to this difficult problem.

In Lagrange's method, the equations of motion are inherently second-order, and well suited for use only when the required generalised coordinates (rotational variables) are independent.

Considering that (a) the choice for rotational variables is often a difficult one, (b) many rotation parameter sets are redundant, and (c) momentum-based differential equations are first-order, it seems that the vector mechanics formulation (Newton-Euler) is a better general choice[22].

In fact, Junkins & Turner (1986, p. 68) and Hughes (1986, p. 39-40, 51-52) conclude that there is no demonstrable advantage of Lagrangian methods over Newton-Euler methods when dealing with a single rigid-body.

Moreover, in vector mechanics the equations of motion are formulated in vector-dyadic terms. This means that the choice of the basis where these equations are represented can be deferred. This possibility makes the Newton-Euler formulation particularly attractive to this work, since it provides a basis-free mathematical apparatus to the problems faced in the next chapters.

So, within the framework of vector mechanics, a formal procedure for obtaining the equations of rotational motion for a rigid body results from employing the basic moment-of-momentum relationship, which states that (see, for example, Smith, 1982, p. 470-72 and Bradbury, 1968, p. 442-43):

> The sum of the moments about the centre of mass of a rigid body due to both external forces and couples equals the time rate of change of the angular momentum taken about the body's centre of mass as measured by observers in an inertial coordinate system.

[22] In the Newton-Euler formulation, the solution of the dynamical differential equations does not require a prior choice of the rotational variables. In this formulation, dynamical and kinematical differential equations are considered separately.

In order to represent symbolically this dynamical relationship within the notation adopted in this text, it is now necessary to specify certain requirements to the bases in use. The first of these requirements is that one of the bases has to be inertial.

The reader should note that the explicit use of this and other soon-to-be-defined requirements is only necessary when dynamical effects have to be considered. If the concern is only with kinematical issues, such considerations are simply unnecessary. Defining δ as the inertial frame, the above stated dynamical relationship may be expressed as

$\delta \triangleq$ inertial

$$\begin{aligned} \sum \vec{M}_{B\text{cm ext}} &= \overset{\delta}{\dot{\vec{H}}}_{B\text{cm}\delta} \\ &= \frac{d}{dt_{\delta}}\left(\vec{\vec{I}}_{B\text{cm}} \bullet \vec{\omega}_{B\delta} \right) \end{aligned} \tag{5.1}$$

moment of inertia tensor of body B taken about its centre of mass

angular velocity vector of body B with respect to basis δ

The moment of inertia tensor (second-moment-of-inertia dyadic) represents the resistance the body offers to changes on its rotational motion. It depends entirely on the body's mass distribution with respect to the point where it is taken. Therefore, it can be understood as a property of the rigid body.

One of the interpretations that can be given to the moment of inertia tensor is that of an (linear) operator, since it assigns a value to the angular momentum vector for any given value of the angular velocity vector. In other words, the angular momentum vector is a *linear vector function* of the angular velocity vector.

The geometric representation (basis-free) of the moment of inertia tensor used above involves the notion and related properties/operations of dyadics in a Cartesian three-dimensional space (orthonormal basis system). An outstanding advantage of using this representation is that the familiar methods of vector manipulation can still be employed. The interested reader may consult Goodbody (1982, p. 152-53), Reddy & Rasmussen (1982, p. 107-21), Meirovitch (1970, p. 126-30, 494-96), or Goldstein (1980, p. 192-98), being the first of these references particularly comprehensive in the subject. The short appendix A of Shuster (1993) may also be helpful.

Due to simplifying considerations, the sum of the moments in equation 5.1 was taken at the body's centre of mass, and so it was the body's moment of inertia tensor. In this case, it is convenient to define the body-fixed frame ϑ as centroidal (located at centre of mass of the body). Recalling yet that the angular velocity vector is not dependent on position (see footnote 17 on page 33), one may write

$$
\begin{aligned}
\overset{\delta}{\vec{H}}_{B\mathrm{cm}\delta} &= \overset{\delta}{\vec{H}}_{B\vartheta\delta} \\
\vec{M}_{B\mathrm{cm\ ext}} &= \vec{M}_{B\vartheta\ \mathrm{ext}} \\
\vec{\vec{I}}_{B\mathrm{cm}} &= \vec{\vec{I}}_{B\vartheta} \\
\vec{\omega}_{B\delta} &= \vec{\omega}_{\vartheta\delta}
\end{aligned}
$$

$\vartheta \triangleq$ body-fixed + centroidal

thus

$$
\begin{aligned}
\sum \vec{M}_{B\vartheta\ \mathrm{ext}} &= \overset{\delta}{\vec{H}}_{B\vartheta\delta} \\
&= \frac{d}{dt_\delta}\left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta}\right)
\end{aligned}
\tag{5.2}
$$

5.1. *Euler's Motion Equations in Body-Fixed Coordinates*

The physical relationship expressed in equation 5.2 holds for any orientation of the basis of representation (basis-free). Although this is palpably true (vector-dyadic equation), one eventually has to resolve it onto some set of specific directions in order to proceed with detailed analysis of the system motion, and evaluate the torque components.

Bearing in mind that the moment of inertia matrix of a rigid body is a constant quantity when expressed along the axes of a body-fixed frame, a considerable simplification will be gained if the basis of observation of the time derivative appearing in equation 5.2 is shifted from the inertial δ basis to the body-fixed ϑ basis.

Making use of the transport theorem, the time derivatives of the angular momentum vector as seen from these two bases (δ and ϑ) can be related as:

$$
\begin{aligned}
\overset{\delta}{\dot{\vec{H}}}_{B\vartheta\delta} &= \overset{\vartheta}{\dot{\vec{H}}}_{B\vartheta\delta} + \vec{\omega}_{\vartheta\delta} \times \vec{H}_{B\vartheta\delta} \\
&= \frac{d}{dt_{\vartheta}}\left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta}\right) + \vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta}\right) \\
&= \overset{\vartheta}{\dot{\vec{\vec{I}}}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta} + \vec{\vec{I}}_{B\vartheta} \bullet \overset{\vartheta}{\dot{\vec{\omega}}}_{\vartheta\delta} + \vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta}\right)
\end{aligned}
$$

Since under the rigid body hypothesis $\overset{\vartheta}{\dot{\vec{\vec{I}}}}_{B\vartheta} = \vec{\vec{0}}$ (time invariant as seen from basis ϑ), the above relationship reduces to

$$
\sum \vec{M}_{B\vartheta\,\text{ext}} = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\vartheta}{\dot{\vec{\omega}}}_{\vartheta\delta} + \vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta}\right) \tag{5.3}
$$

which may be directly resolved onto the body-fixed ϑ basis as

$$
\sum \left\{ M^{\vartheta}_{B\vartheta\,\text{ext}} \right\} = \left[I^{\vartheta}_{B\vartheta} \right] \left\{ \dot{\omega}^{\vartheta}_{\vartheta\delta} \right\} + \left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right] \left[I^{\vartheta}_{B\vartheta} \right] \left\{ \omega^{\vartheta}_{\vartheta\delta} \right\} \tag{5.4}
$$

The scalar expansion of this matrix relation leads to the celebrated Euler's equations of motion[23]. Relationship 5.4 is, in fact, *one* way of expressing the equations of motion for a rotating rigid body. There are several other ways of expressing the motion equations, which might be more advantageous

[23] Most authors refer to the three corresponding scalar equations as the Euler's equations of motion only when the basis of representation is body-fixed *and* principal axes.

depending on the circumstances of the problem. Further examples can be found in Meirovitch (1970, p. 139-40, p. 157-62), and Rheinfurth & Wilson (1991, p. 128-30).

5.2. *Euler's Motion Equations in Inertial Coordinates*

Before heading towards a more general and appropriate relationship to the case at hand, it is worth mentioning a second particular representation of Euler's rotational equations of motion that is sometimes used/referred in textbooks. Firstly, consider once more the transport theorem. The relationship between the δ-observed and the ϑ-observed time derivatives of the angular velocity vector $\vec{\omega}_{\vartheta\delta}$ is given simply by

$$\begin{aligned} \overset{\vartheta}{\vec{\omega}}_{\vartheta\delta} &= \overset{\delta}{\vec{\omega}}_{\vartheta\delta} + \vec{\omega}_{\delta\vartheta} \times \vec{\omega}_{\vartheta\delta} \\ &= \overset{\delta}{\vec{\omega}}_{\vartheta\delta} \end{aligned} \tag{5.5}$$

The reader should note that this result is valid under these exceptional circumstances. The time derivatives of the angular velocity vector as observed from two bases that rotate with respect to each other are equal only when these two bases are the *measurement basis* and the *with respect to basis* of the angular velocity vector itself[24] (compare to Shames, 1998, p. 919; Crouch, 1981, p. 34; and Goodman & Warner, 1964, p. 353-54).

In order to be scrupulously clear as to the meaning of this important result, it should be noted that in general $\{ \dot{\omega}^{\delta}_{\vartheta\delta} \} \neq \{ \dot{\omega}^{\vartheta}_{\vartheta\delta} \}$. Equation 5.5 says only that the vector $\overset{\vartheta}{\vec{\omega}}_{\vartheta\delta}$ can be equated to the vector $\overset{\delta}{\vec{\omega}}_{\vartheta\delta}$. At this point, it is also worth presenting the matrix counterpart of equation 5.5. To derive this expression, consider the transformation equation relating the δ-resolution and the ϑ-resolution of vector $\vec{\omega}_{\vartheta\delta}$:

[24] The same result ensues, of course, if two other bases with the same relative angular velocity are used instead.

$$\{\,\omega^{\delta}_{\vartheta\delta}\,\} = [\,T^{\delta}_{\vartheta}\,]\{\,\omega^{\vartheta}_{\vartheta\delta}\,\} \tag{5.6}$$

Taking the time derivative of both sides of this equation

$$\{\,\dot{\omega}^{\delta}_{\vartheta\delta}\,\} = [\,\dot{T}^{\delta}_{\vartheta}\,]\{\,\omega^{\vartheta}_{\vartheta\delta}\,\} + [\,T^{\delta}_{\vartheta}\,]\{\,\dot{\omega}^{\vartheta}_{\vartheta\delta}\,\}$$

and substituting equation 4.11 into the above result, yields

$$\{\,\dot{\omega}^{\delta}_{\vartheta\delta}\,\} = [\,T^{\delta}_{\vartheta}\,][\,\tilde{\omega}^{\vartheta}_{\vartheta\delta}\,]\{\,\omega^{\vartheta}_{\vartheta\delta}\,\} + [\,T^{\delta}_{\vartheta}\,]\{\,\dot{\omega}^{\vartheta}_{\vartheta\delta}\,\}$$

but $[\,\tilde{\omega}^{\vartheta}_{\vartheta\delta}\,]\{\,\omega^{\vartheta}_{\vartheta\delta}\,\} = \{\,0\,\} \;\Leftrightarrow\; \vec{\omega}_{\vartheta\delta}\times\vec{\omega}_{\vartheta\delta} = \vec{0}$, thus

$$\{\,\dot{\omega}^{\delta}_{\vartheta\delta}\,\} = [\,T^{\delta}_{\vartheta}\,]\{\,\dot{\omega}^{\vartheta}_{\vartheta\delta}\,\} \tag{5.7}$$

which clearly shows that in general $\{\,\dot{\omega}^{\delta}_{\vartheta\delta}\,\} \neq \{\,\dot{\omega}^{\vartheta}_{\vartheta\delta}\,\}$. Having clarified the role of the terms in the formulae, one can proceed by substituting equation 5.5 into equation 5.3, yielding

$$\sum \vec{M}_{B\vartheta\,\mathrm{ext}} = \vec{\vec{I}}_{B\vartheta}\bullet\overset{\delta}{\dot{\vec{\omega}}}_{\vartheta\delta} + \vec{\omega}_{\vartheta\delta}\times\left(\vec{\vec{I}}_{B\vartheta}\bullet\vec{\omega}_{\vartheta\delta}\right) \tag{5.8}$$

which may be resolved directly onto the inertial δ basis

$$\sum\{\,M^{\delta}_{B\vartheta\,\mathrm{ext}}\,\} = [\,I^{\delta}_{B\vartheta}\,]\{\,\dot{\omega}^{\delta}_{\vartheta\delta}\,\} + [\,\tilde{\omega}^{\delta}_{\vartheta\delta}\,][\,I^{\delta}_{B\vartheta}\,]\{\,\omega^{\delta}_{\vartheta\delta}\,\} \tag{5.9}$$

The moment of inertia tensor in relationship 5.9 has been expressed in the inertial δ basis. In this situation, its components, the moments and products of inertia, will evolve continuously as the body rotates. Conversely, the same quantities when expressed in the body-fixed ϑ basis are time independent and can be directly evaluated. In view of this point, it is expedient to find an expression relating these two resolutions.

When the bases of representation are orthonormal, a tensor transforms from one set of components at a certain point (centre of mass in the case) to another set of components at the same point via an orthogonal similarity transformation (see, for example, Arfken & Weber, 1995, p. 192); accordingly

$$\left[I^{\delta}_{B\vartheta} \right] = \left[T^{\delta}_{\vartheta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left[T^{\delta}_{\vartheta} \right]^{T} \tag{5.10}$$

The equivalent form of Euler's rotational equations of motion for a rigid body shown in equation 5.9 is often omitted in the literature. A few authors such as Bradbury (1968, p. 445), Corben & Stehle (1994, p. 149-51), Shabana (1994, p. 415) and Nikravesh (1988, p. 215-219) make use of this representation. Among all the works referred/consulted in this thesis, it is only in Bradbury's book that such representation is actually discussed.

5.3. *Euler's Motion Equations in Arbitrary Coordinates*

For an arbitrary basis of observation ξ, however, the transformation law for the time derivatives of the angular velocity vector is not as simple as it is in equation 5.5. The proper relationship can be attained employing the transport theorem once again:

$$\overset{\vartheta}{\vec{\omega}}_{\vartheta\delta} = \overset{\xi}{\vec{\omega}}_{\vartheta\delta} + \vec{\omega}_{\xi\vartheta} \times \vec{\omega}_{\vartheta\delta} \tag{5.11}$$

which after substitution into equation 5.3 and a little rearrangement yields

$$\sum \vec{M}_{B\vartheta\,\mathrm{ext}} = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\vec{\omega}}_{\vartheta\delta} + \vec{\vec{I}}_{B\vartheta} \bullet \left(\vec{\omega}_{\xi\vartheta} \times \vec{\omega}_{\vartheta\delta} \right) + \vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta} \right) \tag{5.12}$$

This third and more general form of the equations of motion for a rotating rigid body is of paramount importance to this work. It shows how these equations are expressed in geometric terms (basis-free) when the time derivative of the angular velocity vector $\vec{\omega}_{\vartheta\delta}$ is observed from a basis that is not necessarily body-fixed or inertial. To the best of the author's knowledge, the concept of writing the motion equations in this general geometric form has not been previously explored, and is certainly not normally quoted in the literature.

In contrast, the simpler form shown in equation 5.3 can often be found in advanced books of the pertinent literature. Notational conventions usually demand, however, lengthy explanations (sometimes confusing) to clarify the roles of the terms in that equation. The interested reader is referred to Reddy & Rasmussen (1982, p. 109), Rheinfurth & Wilson (1991, p. 70-73), Groesberg (1968, p. 161-62), or Goodbody (1982, p. 199).

In the same way as in the case of equations 5.3 and 5.8 (and also equations 4.3-4.5) equation 5.12 may be directly resolved onto the basis from where the time derivative is observed, accordingly

$$\sum\left\{ M_{B\vartheta\,\text{ext}}^{\xi} \right\} = \left[I_{B\vartheta}^{\xi} \right]\left\{ \dot{\omega}_{\vartheta\delta}^{\xi} \right\} + \left[I_{B\vartheta}^{\xi} \right]\left[\tilde{\omega}_{\xi\vartheta}^{\xi} \right]\left\{ \omega_{\vartheta\delta}^{\xi} \right\} + \left[\tilde{\omega}_{\vartheta\delta}^{\xi} \right]\left[I_{B\vartheta}^{\xi} \right]\left\{ \omega_{\vartheta\delta}^{\xi} \right\} \tag{5.13}$$

where the moment of inertia tensor expressed in the arbitrary basis ξ is given by

$$\left[I_{B\vartheta}^{\xi} \right] = \left[T_{\vartheta}^{\xi} \right]\left[I_{B\vartheta}^{\vartheta} \right]\left[T_{\vartheta}^{\xi} \right]^{T} \tag{5.14}$$

Although the elegant geometric formulation 5.12 has oddly enough never been to the writer's knowledge quoted in the literature, Meirovitch (1970, p. 161) provides an expression for the equations of motion referred to an arbitrary system of axes. His development leads, however, to a Lagrangian form in terms of quasi-coordinates, which still demands lengthy manipulations in order to derive a matricial expression equivalent to 5.13.

The general geometric representation for Euler's equations of motion, namely 5.12, has a further advantage to this work: it explicitly shows the independence of the form of the three equation terms when the basis of observation of the time derivative is changed. Equivalently, one may say that equation 5.12 implicitly shows the independence of the form of the three equation terms when the basis of representation of the corresponding matrix equation 5.13 is changed.

This same point is not straightforwardly perceived in the more conventional approach where the equations of motion are presented in either scalar or matrix forms. The above-discussed independence of the form of the three equation terms will be used to advantage in the design of an attitude controller in the next chapter.

Chapter 6

Nominal Attitude Control Command Law

The present chapter focuses on the design and analysis of a torque controller built upon the analytical apparatus developed so far. As a beginning, a few key points are drawn from the previous chapters and the problem to be examined is clearly stated. A torque formulation, the attitude control command law, is then proposed such that the resulting system of equations governing the rotational motion of the controlled body is nominally uncoupled and linear.

In the previous chapter, it was demonstrated that the equations for the attitude motion of a rigid body in response to external torques have exactly the same form whether all quantities are expressed in the body-fixed ϑ basis or in the inertial δ basis, respectively equations 5.4 and 5.9.

In geometric terms, this is equivalent to stating that the vector-dyadic equation of rotational motion for a rigid body has the same form whether the time derivative is observed from the body-fixed ϑ basis or from the inertial δ basis, respectively equations 5.3 and 5.8.

All these equations were developed under the assumption that the body's centre of mass is the point about which the sum of external moments is taken. This point can in fact be arbitrary, provided that an additional term is included in the originating angular momentum vector equation (see Kaplan, 1976, p. 174-75; or Smith, 1982, p. 470).

In a similar manner, this arbitrariness may be extended to the basis from which the vector time derivative is observed. It does not need to be restricted to the body-fixed ϑ nor the inertial δ bases, provided that an additional term is included in equation 5.3 or 5.8. The implementation of this concept gave rise to equation 5.12.

The appearance of this additional term is intimately connected to the breakdown of the simple transformation law for the time derivative of the angular velocity vector when the basis of observation is arbitrary, i.e. the basis of observation itself might have a nonzero angular velocity with respect to both ϑ and δ (compare equations 5.5 and 5.11).

Throughout this chapter, the focus is on the design and analysis of an attitude control command law making use of equation 5.12. In a single paragraph, the problem may now be stated as follows:

> There is a rotating frame, namely the driver basis λ (reference), whose attitude and angular velocity time histories are not necessarily smooth, nor their details necessarily known beforehand. The rigid body (follower) under consideration should follow the rotational motion of basis λ. The objective is, therefore, to develop a torque formulation (the attitude control command law), which would enable the rigid body to nominally track the (reference) angular motion of basis λ automatically, stably and linearly within moderate attitude tracking errors.

Holding this objective in mind, it is sensible to begin the design process by realising/stressing a couple of important points drawn from the theory enclosed in the preceding chapter:

(a) The equation of rotational motion for a rigid body as given by 5.12 is in geometric form, and the vector time derivative appearing in this equation is observed from an arbitrary basis ξ. This basically means that the scalar expansion of this equation can be done in terms of components resolved in any basis ξ with origin at the body's centre of mass. Nevertheless, the choice for basis ξ will be postponed.

Deferring the choice of the reference frame in which to express a vector-dyadic relationship is simply a matter of keeping one's options open as long as possible. In the case, this option will prove to be a wise strategy.

(b) The terms on the right-hand side of equation 5.12 transform independently. This implies that these terms can be treated as three independent torques, specifically, inertial torques. As a consequence, equation 5.12 can be rewritten in a more convenient fashion as

$$\sum \vec{M}_{B\vartheta\,\mathrm{ext}} - \vec{\vec{I}}_{B\vartheta} \bullet \left(\vec{\omega}_{\xi\vartheta} \times \vec{\omega}_{\vartheta\delta} \right) - \vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta} \right) = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta} \tag{6.1}$$

where the terms on the left-hand side of this equation may be interpreted as the acting/applied torques[25]. For easy of later reference, it is opportune to represent symbolically the two inertial torques transferred to the left-hand side as

$$\vec{M}_{B\vartheta\,\mathrm{P1}} \triangleq -\vec{\vec{I}}_{B\vartheta} \bullet \left(\vec{\omega}_{\xi\vartheta} \times \vec{\omega}_{\vartheta\delta} \right)$$
$$\vec{M}_{B\vartheta\,\mathrm{P2}} \triangleq -\vec{\omega}_{\vartheta\delta} \times \left(\vec{\vec{I}}_{B\vartheta} \bullet \vec{\omega}_{\vartheta\delta} \right)$$

yielding

$$\sum \vec{M}_{B\vartheta\,\mathrm{ext}} + \vec{M}_{B\vartheta\,\mathrm{P1}} + \vec{M}_{B\vartheta\,\mathrm{P2}} = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta} \tag{6.2}$$

6.1. Control Law Selection

A major complication of the problem under examination is that the size of the attitude tracking error should be kept within moderate bounds. In other words, the validity of the control law formulation should not be limited to a small/infinitesimal neighbourhood of the commanded/reference angular

[25] The careful reader will note that this simple rearrangement of terms is *not* some sort of (partial) "application" of D'Alembert's principle (see Rosenberg, 1977, p. 124-25).

motion. It should be valid in a moderate one. Linearisation procedures about the target states are therefore precluded (see Hughes, 1986, p. 129).

In terms of attitude variables, equation 5.12, or its equivalent 6.1, consists of a set of three scalar highly non-linear coupled second-order differential equations. Automatic control theory does not provide exact analytical solutions nor design procedures for such plants (refer to Sidi, 1997, p. 113, 152-53).

In face of the insufficiency of the theory, some less usual method of attack is required. The equation of motion must be somehow transformed into a more easily treatable form if standard automatic control techniques are to be used.

Assuming that the control torque is the dominant external moment acting on body B, i.e. $\sum \vec{M}_{B\vartheta\,\mathrm{ext}} = \vec{M}_{B\vartheta\,\mathrm{C}}$, one may tackle this problem by regarding[26] $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ as computable disturbances (from directly measurable quantities), and then making use of some strategy of compensation.

Because the control law is intended for the tracking of dynamical commands, it accepts feedforward commands. These commands force the controller to respond instantly rather than merely letting it react to the errors.

Of course, in order to stabilise and attain automatic attitude control, the formulation should also contain terms that are function of the attitude tracking error and, for improved stability, terms that are function of the angular velocity error (compare to Sidi, 1997, p. 113). So, within this line of reasoning, a torque (acting at the centre of mass) that would enforce the rigid body B to track the angular motion (path + velocity) of the rotating reference frame λ may be defined as the sum of three terms[27]:

[26] The subscript P stands for perturbing torque (see definition on page 20).

[27] The subscript m stands for measured/estimated (see definition on page 20).

$$\vec{M}_{B\vartheta\,\mathrm{C}} = \vec{M}_{B\vartheta\,\mathrm{C1}} + \vec{M}_{B\vartheta\,\mathrm{C2}} + \vec{M}_{B\vartheta\,\mathrm{C3}} \qquad (6.3)$$

where

$$\vec{M}_{B\vartheta\,\mathrm{C1}} \triangleq \vec{\vec{I}}_{B\vartheta\,\mathrm{m}} \bullet \left(\vec{\omega}_{\xi\vartheta\,\mathrm{m}} \times \vec{\omega}_{\vartheta\delta\,\mathrm{m}} \right) \qquad (6.3a)$$

$$\vec{M}_{B\vartheta\,\mathrm{C2}} \triangleq \vec{\omega}_{\vartheta\delta\,\mathrm{m}} \times \left(\vec{\vec{I}}_{B\vartheta\,\mathrm{m}} \bullet \vec{\omega}_{\vartheta\delta\,\mathrm{m}} \right) \qquad (6.3b)$$

$$\vec{M}_{B\vartheta\,\mathrm{C3}} \triangleq \vec{\vec{K}} \bullet \vec{\phi}_{\lambda\vartheta} + \vec{\vec{C}} \bullet \vec{\omega}_{\lambda\vartheta} \qquad (6.3c)$$

In order to bring the text language closer to the control terminology, a few denominations are now in demand:

λ	driver frame (rotating reference)
ϑ	follower frame (body-fixed + centroidal)
δ	inertial frame
ξ	arbitrary frame
$\vec{\omega}_{\lambda\delta}$	driver's inertial angular velocity vector (reference)
$\vec{\omega}_{\vartheta\delta}$	body's inertial angular velocity vector
$\vec{\omega}_{\lambda\vartheta}$	angular velocity error vector
$\vec{\phi}_{\lambda\vartheta}$	attitude tracking error vector (rotation vector)

The ideas flashed in the last paragraphs raised the question of the feasibility of defining the control law in such a way that the equations governing the rigid body rotational motion would nominally uncouple and linearise. In that being the case, it would be possible to Laplace transform the dynamical equations, thus gaining the important advantage of using linear control theory.

This is in fact not only possible, but also remarkably straightforward to achieve if one (a) fully understands equation 5.12, and (b) assumes that the quantities $\vec{\vec{I}}_{B\vartheta}$, $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ or $\vec{\omega}_{\vartheta\delta}$ can be measured within a relatively high degree of accuracy in some convenient basis.

How this is achievable is the subject now on focus. For the sake of clarity, the discussion has been partitioned. The ensuing concentrates on the first and second control torque terms, $\vec{M}_{B\vartheta\,C1}$ and $\vec{M}_{B\vartheta\,C2}$, and certain engineering realities associated with their particular definitions. The third control torque term, $\vec{M}_{B\vartheta\,C3}$, and the important issue of stability will be subsequently addressed.

6.2. *The Open-Loop Scheme*

What has being effectively done with the proposed control torque formulation is to regard $\vec{M}_{B\vartheta\,P1}$ and $\vec{M}_{B\vartheta\,P2}$ as disturbances of a known nature, whose undesirable effects on the system output are sought to be cancelled. In reality, the compensation for these two terms is only approximately achieved, since the scheme proposed is open-loop[28] (feedforward scheme), and thus relies heavily on the certainty of the parameters/variables in use (see figure 5).

Strictly speaking, it is not proper to use the control terminology without referring to the system *transfer function* block diagram[29]. Be that as it may, such terminology is going to be used in the next paragraphs. The advantage is the evident analogy between the present situation and the technique known in control engineering as disturbance-feedforward control, which the interested reader may consult in the works of Ogata (1997, p. 700-03), Kuo (1995, p. 775-77) and Palm (1998, 592-95). The same principles can also be appreciated in a more sophisticated application in Halyo (1996, p. 1-7).

Further references are Wie (1998, p. 406) and Wie & Lu (1995), two works found in the literature that employ a similar non-linear feedforward scheme in a control logic. In these two recent works, the authors use control torques

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[28] A control system in which the output has no effect on the control decision is called open-loop control system.

[29] Transfer functions are input-output descriptions of the behaviour of a system/subsystem, and may be defined only when the system/subsystem is linear and stationary (see Dorf & Bishop, 1998, p. 48).

to directly counteract $\vec{M}_{B\vartheta\,\mathrm{P2}}$, the inertial torque term due to the centrifugal forces in Euler's motion equation (equation 5.3)[30].

The objective of those authors is to provide a rigid spacecraft (rigid body) with three-axis large-angle rest-to-rest reorientation manoeuvrability about an inertially fixed axis. In the same works, the control logic is subsequently adapted to perform the reorientation manoeuvre about an inertially fixed axis in minimum time, and within the saturation limits of sensors and actuators.

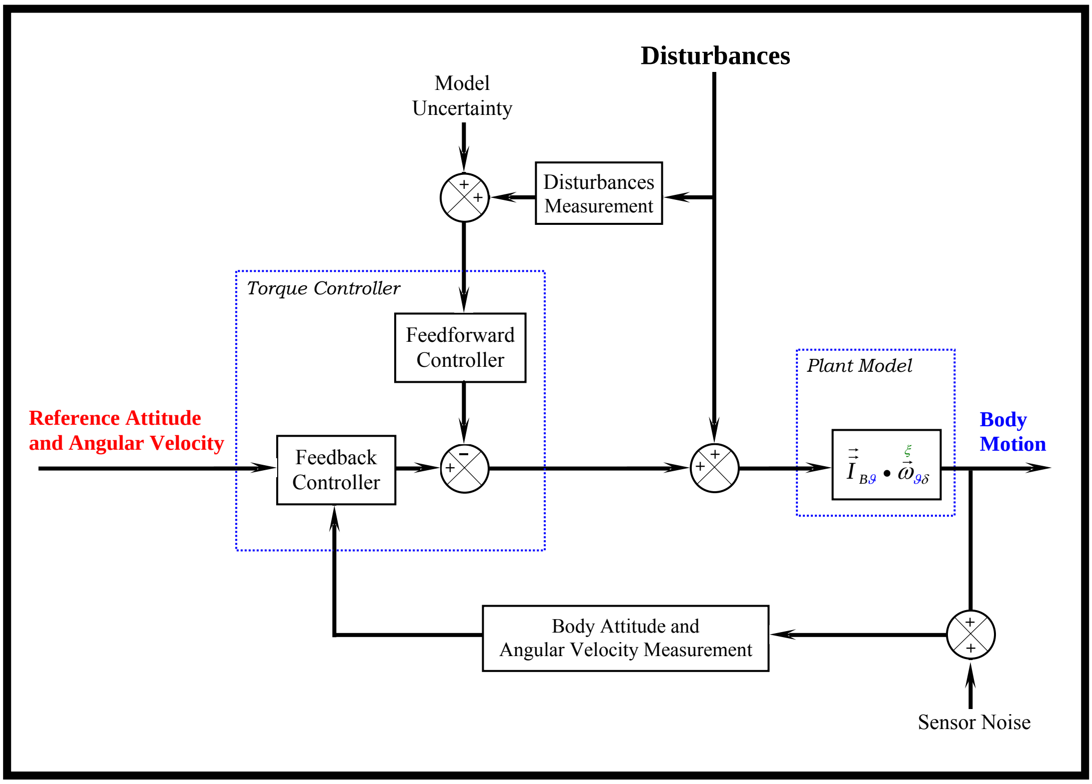


Figure 5: Block Diagram

Although an open-loop scheme may sound powerful in that it is able to eliminate (in practice greatly reduce) the supposedly deleterious effects of

[30] The inertial torque term $\vec{M}_{B\vartheta\,\mathrm{P2}}$ can be found in the literature under different denominations. Rheinfurth & Wilson (1991, p. 69), for example, name it the centrifugal torque. Wie (1998, p. 406) and Wie & Lu (1995), on the other hand, refer to this torque term as the gyroscopic term of Euler's motion equations. A third denomination is the one used by Wen & Kreutz-Delgado (1991), they name it the Coreolis torque term. There are even other denominations: Meyer (1966) call it gyroscopic acceleration, while Wie et al. (1989) gyroscopic coupling torque. The inertial torque term $\vec{M}_{B\vartheta\,\mathrm{P1}}$ has not been found referred in the literature

computable/measurable disturbances before they materialise in the output[31], it has a limited functional accuracy. This limitation is due to the above-mentioned susceptibility of the method to parameter variation. In fact, any drift in the parameter values would result in imperfect compensation, therefore demanding the inclusion of a closed-loop scheme (feedback-loop scheme) in the control system. In the proposed formulation, the closed-loop scheme is provided by the third term, the restoring torque $\vec{M}_{B\vartheta\,\mathrm{C3}}$.

In this work, it is assumed that the perturbing torques $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ can be computed with an accuracy that makes valid the following relations

$$\begin{aligned} \vec{M}_{B\vartheta\,\mathrm{C1}} &= -\vec{M}_{B\vartheta\,\mathrm{P1}} \\ \vec{M}_{B\vartheta\,\mathrm{C2}} &= -\vec{M}_{B\vartheta\,\mathrm{P2}} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \vec{\vec{I}}_{B\vartheta\,\mathrm{m}} &= \vec{\vec{I}}_{B\vartheta} \\ \vec{\omega}_{\xi\vartheta\,\mathrm{m}} &= \vec{\omega}_{\xi\vartheta} \\ \vec{\omega}_{\vartheta\delta\,\mathrm{m}} &= \vec{\omega}_{\vartheta\delta} \end{aligned} \tag{6.4}$$

Within the limits of this assumption, i.e. (near) perfect plant knowledge and state estimation (nominal case), the terms on the left-hand side of equation 6.1 simplify, reducing the equation of motion to

$$\vec{M}_{B\vartheta\,\mathrm{C3}} = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta} \tag{6.5}$$

In this way, the system dynamical equation is greatly simplified. In a practical scenario, however, one should always bear in mind that there is a compromise between the closed-loop gains, the rigid body gyric[32] stability, and the accuracy with which the state variables and parameters used in the open-loop scheme can be actually measured. The secular perturbing torques resulting from the inevitably imperfect compensation for $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ have

[31] A usual feedback control system is inherently reactive, i.e. the corrective action starts only after the output has already been affected (there is no command control when the state has no current errors).

[32] The word gyroscopic is normally used here. Nevertheless, Hughes (1986, p. 511, footnote) points out that this term is defective, and coins the more accurate and appropriate tem gyric.

to be dealt with by the feedback loop. This clearly constrains the choice of individual control parameters, and ultimately restricts the dynamic performance of the whole system.

So, the proposed attitude control command law provides a solution for the problem by separating the objectives for the feedforward control from those related to the feedback control. Accordingly, it becomes possible to compensate perfectly (zero error) for $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ when the system model parameters are perfectly known and the state variables are measured without bias or random noise. On the other hand, the tracking objective is achieved without a corresponding deterioration of the feedback control objectives (no high loop gains), such as noise attenuation, random disturbance accommodation or, particularly, the serious matter of system stability. This last and most important point is going to be discussed in Chapter 7.

6.3. The Closed-Loop Scheme

In the present system, both feedforward and feedback controls are simultaneously in operation. In terms of system dynamical analysis, this concomitant operation splits the difficult problem of solving equation 6.1 into two much simpler ones:

1. The solution of equation 6.5 (feedback control); and
2. The parallel computation of $\vec{M}_{B\vartheta\,\mathrm{C1}}$ and $\vec{M}_{B\vartheta\,\mathrm{C2}}$ (feedforward control).

Taking into consideration that step 2 uses the results of step 1 to generate $\vec{M}_{B\vartheta\,\mathrm{C1}}$ and $\vec{M}_{B\vartheta\,\mathrm{C2}}$, all system dynamical attributes are interconnected with equation 6.5. In fact, the success of the proposed nominal control law depends entirely upon the dynamics of this equation. In terms of the definition 6.3c, equation 6.5 is written as

$$\vec{\vec{K}} \bullet \vec{\phi}_{\lambda\vartheta} + \vec{\vec{C}} \bullet \vec{\omega}_{\lambda\vartheta} = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\vec{\omega}}_{\vartheta\delta} \tag{6.6}$$

In this expression, the quantity $\vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta}$ is equated to a vector-valued function of vectors[33] $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$. The usefulness of this definition, i.e. a linear vector function of $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ rather than the vectors themselves, is that the tensors $\vec{\vec{K}}$ and $\vec{\vec{C}}$ may now be defined in such a way that equation 6.6 would simplify. This is achieved by defining these two tensors as

$$\begin{aligned} \vec{\vec{K}} &= k\,\vec{\vec{I}}_{B\vartheta} \\ \vec{\vec{C}} &= c\,\vec{\vec{I}}_{B\vartheta} \end{aligned} \qquad \Rightarrow \qquad \vec{\vec{I}}_{B\vartheta} \bullet \left(k\,\vec{\phi}_{\lambda\vartheta} + c\,\vec{\omega}_{\lambda\vartheta} \right) = \vec{\vec{I}}_{B\vartheta} \bullet \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta}$$

which reduces the system equation to the following especially simple form[34]

$$k\,\vec{\phi}_{\lambda\vartheta} + c\,\vec{\omega}_{\lambda\vartheta} = \overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta} \tag{6.7}$$

The definitions given to $\vec{\vec{K}}$ and $\vec{\vec{C}}$ have an interesting physical interpretation. To arrive at this interpretation, one should first note that the corrective torque $\vec{M}_{B\vartheta\,C3}$ is provided by a visco-elastic-like connection between the driver frame λ (reference) and the follower frame ϑ (body). The proportionality to the inertia tensor makes this connection anisotropic, i.e. the stiffness of the connection between driver and follower is not necessarily the same in all directions (see figure 6).

The main effect of the above definition is that the gains *k* and *c* can now be chosen regardless of the physical characteristics of the rigid body *B*, i.e. they can be chosen without direct consideration to the body's moment of inertia tensor. Effectively, the simplified feedback control loop - equation 6.7 - sees the body as inertially spherical. This basically means that any rotation, whichever direction this is, can be thought of as a rotation about a principal

[33] For mathematical (tensor) related definitions, refer to Goodbody (1982, p. 66).

[34] It is fundamentally important to note that, in the general case, the kinematical differential relationship between the angular velocity error vector and the rotation error vector is complex and depends on the basis from which the time derivative is observed (compare to equations 4.14 and 4.16).

axis, since any axis is principal. Therefore, the problem of solving equation 6.6 reduces to the simpler isoinertial case.

The reader may find illustrative to compare the flexible connection as above defined to Cauchy's ellipsoid of inertia. This ingenious geometric interpretation of the inertial properties of a given rigid body is, in essence, a plot of the body's moment of inertia as a function of the direction of the axis of rotation. Therefore, it can be helpful in the visualisation of the anisotropic nature of the connection as a function of the direction of the attitude error vector (see Rheinfurth & Wilson, 1991, p. 109-111; Rosenberg, 1977, p. 96-97; Smith, 1982, p. 462-65; or Greenwood, 1965, p. 306-09).

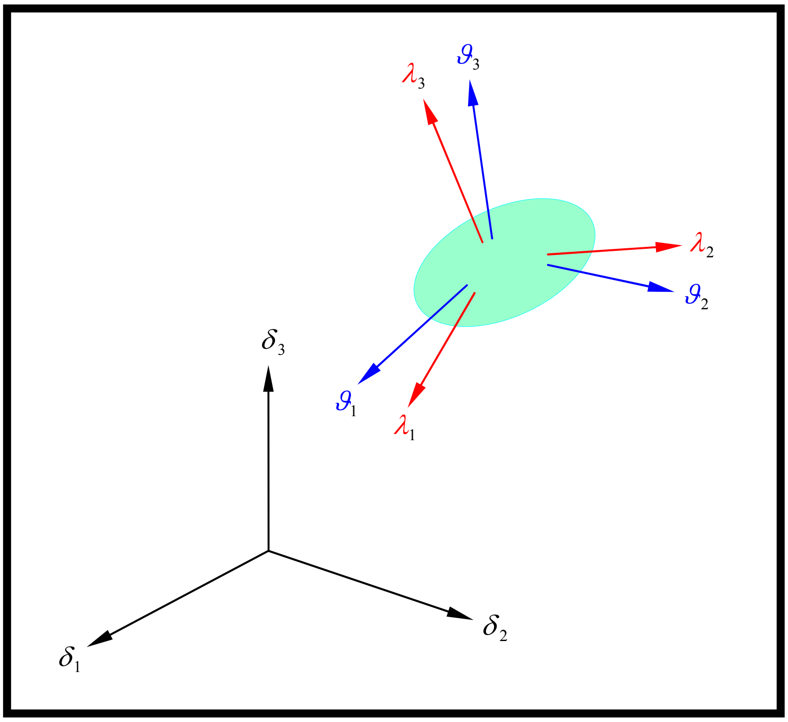


Figure 6: Flexible Connection

Another important characteristic of the proposed closed-loop scheme is that the finite (moderate) angular displacement between the driver and the follower frames has been represented by the corresponding rotation vector, the attitude error vector $\vec{\phi}_{\lambda\vartheta}$. Although no useful distinction can be made between rotation parameter sets when the rotations are infinitesimal, for finite rotations the parameterisation should be done with care (see Hughes, 1986, p. 29-30; or Sidi, 1997, 152-58).

The choice of the rotation vector components as attitude variables has a number advantages, a few of which are:

1. The rotation vector components comprise the most physically significant set of attitude variables one can choose.

2. The set has as many parameters as there are degrees of freedom (a rigid body has three rotational degrees of freedom).

3. For first and second-order analysis the trigonometric functions associated with these variables vanish, and the set becomes suitable for numerical computation (see Hughes, 1986, p. 27, 38/ex. 2.25).

4. The angular path traversed by the controlled body is minimised in manoeuvres about an inertially fixed axis. This is so because equation 6.7 enables the body's rotational motion to be about this inertially fixed axis, i.e. the corresponding attitude error vector axis (compare to Sidi, 1997, p. 155; Klumpp, 1976; Wie et al., 1989; or Wie, 1998, p. 406).

It should be remarked, however, that the use of the rotation vector in the parameterisation of finite rotations does not lead to a universally acceptable solution, since the mapping between this parameterisation and the intrinsic nine-parameter one (orthogonal tensor) ceases to be a bijection when the norm of the rotation vector (Euler angle of rotation) is $2\pi n$, where $n = 1, 2, 3 \ldots$.

In fact, this problem does not arise from a particular situation: it is topologically impossible to have a global three-variable representation of the rotation matrix without singular points (see Ibrahimbegovic et al., 1995; Ibrahimbegovic, 1997; Pfister, 1996; or Stuelpnagel, 1964).

The strategy adopted here for dealing with this difficulty is simply to consider the Euler angle $\phi_{\lambda\vartheta}$ between frames λ and ϑ smaller than 2π radians. This is very much the case, since the size of the attitude error (Euler angle) is hypothesised moderate.

Although the just listed advantages of the parameterisation are highly relevant, the most significant one has not been mentioned yet: the form of the kinematical differential relationship between the attitude error vector $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$, and the corresponding angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$. The form of this relationship is in general complex and depends, of course, on the frame from which the time derivative is observed. This concept is going to be explored in the ensuing chapter in order to simplify the analysis of attitude stability. The definition of the time derivative observer will also define the still pending term $\vec{\omega}_{\xi\vartheta}$ in equation 6.3.

6.4. Remark on the Control Law Definition

The approach of transforming equation 5.12 into 6.7 via control torque is similar to the *first* step of the so-called feedback linearisation method when conventionally applied to the rigid body attitude control problem. The basic difference is that here the linearising control torque (equation 6.3) has a further term in the feedforward path, namely $\vec{M}_{B\vartheta\,C1}$. The worthiness of such apparent additional complexity will become evident shortly in the ensuing chapter.

The feedback linearisation method is normally used with Euler's motion equations, and may be divided into two steps. The first linearising step transforms the tri-inertial rigid body rotational dynamics into a simpler isoinertial form. As above-mentioned, the process is similar to the one used in the transformation of equation 5.12 into 6.7. The second linearising step (not employed here) considers the complex kinematical differential relationship between the body's inertial angular velocity, the driver's inertial angular velocity, and the chosen attitude tracking error variables. It is only after a series of transformations and feedback/feedforward compensations that unforced (homogeneous) closed-loop linearity is achieved in the attitude tracking error dynamics.

Provided that the formulation/parameterisation does not become singular, the three-axis rigid body attitude dynamics is placed via this technique on a simple uncoupled double integrator form in terms of the chosen attitude tracking error variables.

The interested reader may refer to Paielli & Bach (1993), which possibly is the most relevant found reference. The work of Schaub et al. (2001) is also an appropriate one. Further references may be found in section 1.2.

Chapter 7

Nominal Attitude Stability Analysis

This chapter concentrates on the analysis of the nominal rigid body attitude stability for the assumed control law. Firstly, the attitude stability problem is stated. Secondly, the importance of the choice of the arbitrary basis is clarified. Thirdly, the relationship between attitude variables and angular velocity is examined and a novel and simpler form derived. Lastly, the nominal closed-loop transfer functions for the assumed control law are determined and the stability problem analysed. Some remarks on the formulation close the chapter.

Stability is of utmost importance for control systems. Rigid body attitude dynamics and attitude stability, even in the infinitesimal/linear case analysis, is a subject far from straightforward, whose practical and pedagogical relevance has prompted a considerable body of literature. A fairly complete discussion on the subject, exposing the magnitude of its complexity in different guises, can be found in Hughes (1986, p. 93-129).

Basically, there is no general analytical solution for Euler's motion equations, let alone equation 5.13, when arbitrary torques are acting. The presence of the non-linear terms, and the functional dependence of the torque components on body attitude have posed an analytical challenge for centuries.

For most purposes, this highly coupled non-linear set of ordinary differential equations can be integrated only numerically. Nonetheless, there are a few special cases that render analytical progress. Examples can be found in Hughes (1986, p. 124-29), Rimrott (1989, p. 256-62) and Rheinfurth & Wilson (1991, p. 126-39). More recent literature includes the articles of Gick et al. (2000) and Livneh & Wie (1997).

In the particular case in focus, however, matters are greatly simplified with the already discussed attitude control command law. Although at the expense of some added complexity, the proposed control law may provide nominal linear attitude stabilisation if basis ξ is suitably chosen. The development that ensues proves this point via kinematical arguments and elementary linear control theory.

The only analytical approximations utilised, namely $\sin(\phi_{\lambda\vartheta}/2) \approx \phi_{\lambda\vartheta}/2$ and $\cos(\phi_{\lambda\vartheta}/2) \approx 1 - \phi_{\lambda\vartheta}^{2}/8$, are completely plausible within the moderate attitude tracking error hypothesis, making certain a precise interpretation of the results. The underlying idea allowing such simplicity is the use of equation 5.12 in place of 5.3. The inclusion of $\vec{M}_{B\vartheta\,\mathrm{C1}}$ in the feedforward path of the control law has, therefore, a pivotal role in the body's attitude stability.

7.1. Definition Criteria for the Arbitrary Basis

Before the discussion of stability is properly initiated, it is mandatory to assure the understanding of two points. The first point is that the control law has been predicated on the assumption that $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ are computable. This assumption will be normally satisfied as long as $\vec{\omega}_{\xi\vartheta}$ is expressible in terms of the measurable system variables, i.e. $\vec{\phi}_{\lambda\vartheta}$, $\vec{\omega}_{\lambda\vartheta}$ and/or $\vec{\omega}_{\vartheta\delta}$. In practical terms, this means that basis ξ should be defined in such a way that the corresponding $\vec{\omega}_{\xi\vartheta}$ is an amenable function of $\vec{\phi}_{\lambda\vartheta}$, $\vec{\omega}_{\lambda\vartheta}$ and/or $\vec{\omega}_{\vartheta\delta}$.

The second point to be clarified concerns the eventual resolution of the governing system equation in order to proceed with detailed solution for motion and stability. In converting equation 6.7 to its scalar equivalents, one rule is inviolable: every term in a vector (or vector-dyadic) equation must be expressed in the same frame. The selection of the coordinates sometimes proves of crucial importance to both analytical complexity and interpretation of results. Hence, the choice of the frame where to express the governing system equation (basis ξ) may have a drastic effect on the complexity of the analysis of the system dynamical behaviour.

Although the three scalar equations corresponding to the vector equation 6.7 are seemingly uncoupled and linear in the way they stand, they are not, *a priori*, independent (see footnote 34 on page 61)! The attitude state dynamics requires an auxiliary set of kinematical differential equations relating the chosen attitude coordinates, namely the components of the attitude error vector $\vec{\phi}_{\lambda\vartheta}$, and the components of the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$.

The form of these kinematical differential scalar (or matrix) equations depends, of course, on the chosen basis of resolution. Equivalently, if these equations are expressed in geometric notation, one may state that the form of the kinematical differential relationship between $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ will depend on the choice of the basis from which the time derivative is observed. This dependence has already been (partially) examined in section 4.2 for $\vec{\phi}_{\vartheta\delta}$ and $\vec{\omega}_{\vartheta\delta}$, and will be further developed to $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ in sections 7.2 and 7.3.

For now, the important point is the understanding that the form of the kinematical differential relationship between $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ depends on the basis from which the time derivative is observed. In the case, this basis is the to-be-defined ξ. As a consequence, both the governing system equation 6.7 and the corresponding kinematical differential relationship between $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$ should be expressed in ξ coordinates, whichever these may be.

From the two points above discussed, it becomes apparent that the definition of basis ξ influences the control law analysis in at least three distinct ways:

1. It determines how $\overset{\xi}{\dot{\vec{\omega}}}_{\vartheta\delta}$ is evaluated;
2. It defines $\vec{\omega}_{\xi\vartheta}$, and in so doing the form of $\vec{M}_{B\vartheta\,C1}$; and
3. It dictates the form of the kinematical differential relationship between the rotation error vector $\vec{\phi}_{\lambda\vartheta}$ and the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$.

In short, the definition of basis ξ is indeed the cornerstone for the entire formulation. If judiciously chosen, it can facilitate the system dynamical analysis by advantageously dictating the relationship between $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$. Thus, the objective now is to choose a definition for basis ξ in such a way that equation 6.7 is most easily evaluated, and $\vec{\omega}_{\xi\vartheta}$ is expressible as an amenable function of $\vec{\phi}_{\lambda\vartheta}$, $\vec{\omega}_{\lambda\vartheta}$ and/or $\vec{\omega}_{\vartheta\delta}$.

7.2. *Kinematical Differential Relationships*

The vector kinematical differential relationship between the angular velocity vector and the Euler angle/axis variables has already been analysed in section 4.2. This previous analysis considered the cases where the time derivative is observed from either the *measurement basis* ϑ or the *with respect to basis* δ of the rotation vector $\vec{\phi}_{\vartheta\delta} = \phi_{\vartheta\delta}\hat{n}_{\vartheta\delta}$, equations 4.14 and 4.16 respectively. Using these two equations as parent equations, the kinematical differential geometric relationship between $\vec{\omega}_{\lambda\vartheta}$ and $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$, when the time rate of change is observed from bases λ and ϑ, is given respectively by

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\phi_{\lambda\vartheta}\,\overset{\lambda}{\hat{n}}_{\lambda\vartheta} - \left(1-\cos\phi_{\lambda\vartheta}\right)\hat{n}_{\lambda\vartheta}\times\overset{\lambda}{\hat{n}}_{\lambda\vartheta} \tag{7.1}$$

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\phi_{\lambda\vartheta}\,\overset{\vartheta}{\hat{n}}_{\lambda\vartheta} + \left(1-\cos\phi_{\lambda\vartheta}\right)\hat{n}_{\lambda\vartheta}\times\overset{\vartheta}{\hat{n}}_{\lambda\vartheta} \tag{7.2}$$

In order to obtain an expression where the time rate of change is observed from an arbitrary basis ξ, consider the transport theorem:

$$\overset{\vartheta}{\hat{n}}_{\lambda\vartheta} = \overset{\xi}{\hat{n}}_{\lambda\vartheta} + \vec{\omega}_{\xi\vartheta}\times\hat{n}_{\lambda\vartheta}$$

Substituting this relation into 7.2 yields the desired form

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\phi_{\lambda\vartheta}\left(\overset{\xi}{\hat{n}}_{\lambda\vartheta} + \vec{\omega}_{\xi\vartheta}\times\hat{n}_{\lambda\vartheta}\right) + \left(1-\cos\phi_{\lambda\vartheta}\right)\hat{n}_{\lambda\vartheta}\times\left(\overset{\xi}{\hat{n}}_{\lambda\vartheta} + \vec{\omega}_{\xi\vartheta}\times\hat{n}_{\lambda\vartheta}\right) \tag{7.3}$$

Equation 7.3 can now be used to derive the kinematical differential relationship between $\vec{\omega}_{\lambda\vartheta}$ and $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$ when the time derivative is observed from the inertial δ basis[35]. Noting that $\vec{\omega}_{\delta\vartheta} \times \hat{n}_{\lambda\vartheta} = \hat{n}_{\lambda\vartheta} \times \vec{\omega}_{\vartheta\delta}$, and making *momentarily* the arbitrary basis $\xi = \delta$, one easily finds that

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\phi_{\lambda\vartheta}\left(\overset{\delta}{\hat{n}}_{\lambda\vartheta} + \hat{n}_{\lambda\vartheta} \times \vec{\omega}_{\vartheta\delta} \right) + \left(1-\cos\phi_{\lambda\vartheta}\right)\hat{n}_{\lambda\vartheta} \times \left(\overset{\delta}{\hat{n}}_{\lambda\vartheta} + \hat{n}_{\lambda\vartheta} \times \vec{\omega}_{\vartheta\delta} \right) \tag{7.4}$$

In view of the complexity shown by these equations, it is not surprising that the components of the angular velocity vector are generally taken as nonintegrable combinations of the time derivatives of the angular displacements (see Konopinsky, 1969, p. 239; Meirovitch, 1970, p. 139; Shabana, 1994, p. 370; Angeles, 1997, p. 90; Corben & Stehle, 1994, p. 144; and Goldstein, 1980, p. 169, 175 - footnotes).

The nonintegrable relationship between the components of the angular velocity vector and the orientation coordinates takes different format according also to the chosen set of orientation coordinates. A few examples can be found in Shabana (1998, p. 54, 66) and Nikravesh (1988, p. 350, 352). Hughes (1986, p. 22-31) provides a more comprehensive exposition of alternative parameterisations. Kane et al. (1983, p. 427-31) tabulate the kinematical relationships associated with the various body/space orientation angles. Shuster (1993a, p. 477-86) presents briefly the kinematical relations for most of the currently employed sets of orientation coordinates.

Bottema and Roth (1979, p. 154-55) have even shown that it is fundamentally impossible to find rotation quantities expressed in terms of 3-1-3 Euler angles, such that their time derivatives would equal the components of the angular velocity vector. Rimrott (1989, 20-21) and Corben & Stehle (1994, p. 141-42) offer a similar result.

[35] Equations 7.1-7.4 show the strong non-vector nature of three-dimensional rotations.

The same kind of mathematical complexity appears when one attempts to integrate the angular velocity vector with respect to time. Although this integral is a vector quantity and has dimensions of angular displacement, it cannot be directly related to the true finite rotations (see Goodman & Warner, 1964, p. 348; and Greenwood, 1965, p. 365). Even for those cases in which small angle approximations are assumed, there will be a residual angular displacement of geometric origin, which must be considered (see Goodman & Robinson, 1958).

Based upon the results shown and works surveyed in this section, one may conclude that the components of the angular velocity vector cannot be easily treated as just time derivatives of simply definable orientation coordinates. The only exception is, of course, the plane motion case, where the rotational motion is about a fixed axis.

7.3. Changing the Paradigm

The works quoted in section 7.2 have at least two points in common. The first point is evident: all these works recognise the complexity of the kinematical differential relationship between the attitude variables and the angular velocity components. The second point is more subtle: all these works take the time derivative as observed from bases that could *always* be associated with λ, ϑ or δ.

This second and seemingly picayune point is in fact crucial. Stated as it is, it indicates that while the basis of observation of the time derivative is restricted to be λ, ϑ or δ, no easy form is achievable for $\vec{\omega}_{\lambda\vartheta}$, regardless of the chosen parameterisation. In situations like this in which particular choices of a frame of reference lead to terms that are difficult to evaluate, it is advantageous to consider using a different definition to work out the problem, i.e. a change in the paradigm.

Using this approach, the pointed mathematical complexity may be surmounted by defining the orientation of basis ξ as a sort of average orientation of bases λ and ϑ. Symbolically, this definition translates into the following (see figure 7):

$$\vec{\phi}_{\xi\vartheta} = \vec{\phi}_{\lambda\xi} = \frac{1}{2}\vec{\phi}_{\lambda\vartheta} \quad \Rightarrow \quad \begin{array}{l} \hat{n}_{\xi\vartheta} = \hat{n}_{\lambda\xi} = \hat{n}_{\lambda\vartheta} \\ \phi_{\xi\vartheta} = \phi_{\lambda\xi} = \frac{1}{2}\phi_{\lambda\vartheta} \end{array} \tag{7.5}$$

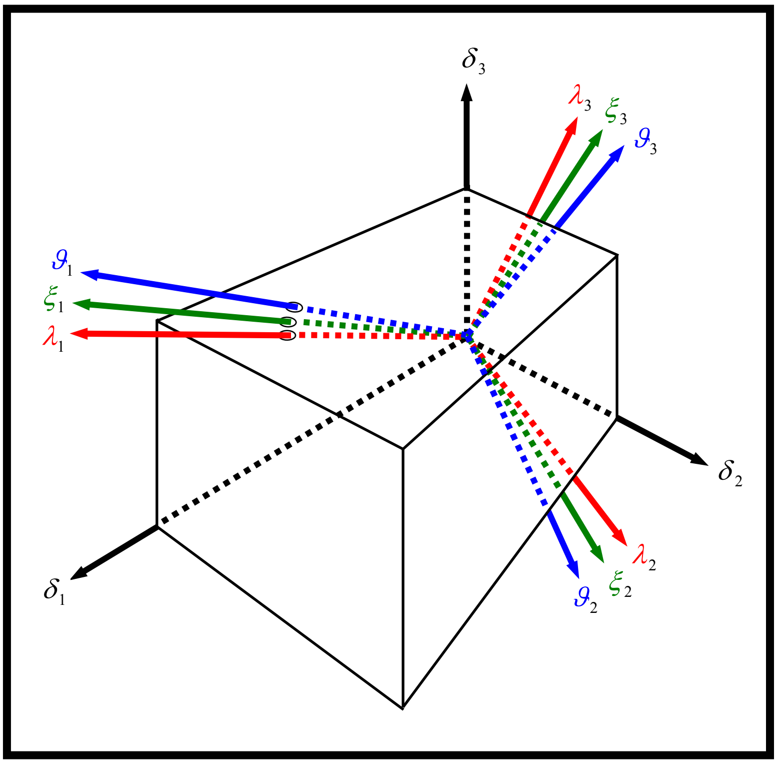


Figure 7: Bases

This somewhat surprising definition for basis ξ was originally envisioned and motivated by the form of equations 7.1 and 7.2, which is exactly the same except for the sign of the cross-product term. This sign change accompanying the time derivative observer change suggests that the cross-product term would nullify if the ξ - observer is oriented as above defined.

Such definition enables indeed further simplification of the kinematical differential relationship between $\vec{\omega}_{\lambda\vartheta}$ and $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$. In order to achieve this simpler relationship, one may first obtain an expression relating $\vec{\omega}_{\xi\vartheta}$ and $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$, and then substitute it into equation 7.3. Taking 7.1 as a parent equation, the relationship between $\vec{\omega}_{\xi\vartheta}$ and $\vec{\phi}_{\xi\vartheta} = \phi_{\xi\vartheta}\hat{n}_{\xi\vartheta}$ is readily obtained as

$$\vec{\omega}_{\xi\vartheta} = \dot{\phi}_{\xi\vartheta}\,\hat{n}_{\xi\vartheta} + \sin\phi_{\xi\vartheta}\,\overset{\xi}{\hat{n}}_{\xi\vartheta} - \left(1-\cos\phi_{\xi\vartheta}\right)\hat{n}_{\xi\vartheta}\times\overset{\xi}{\hat{n}}_{\xi\vartheta} \tag{7.6}$$

Since $\phi_{\xi\vartheta} = \frac{1}{2}\phi_{\lambda\vartheta}$ and $\hat{n}_{\xi\vartheta} = \hat{n}_{\lambda\vartheta}$, equation 7.6 can be rewritten as

$$\vec{\omega}_{\xi\vartheta} = \frac{1}{2}\dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\frac{\phi_{\lambda\vartheta}}{2}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta} - \left(1-\cos\frac{\phi_{\lambda\vartheta}}{2}\right)\hat{n}_{\lambda\vartheta}\times\overset{\xi}{\hat{n}}_{\lambda\vartheta} \tag{7.7}$$

Substitution of the above relation into equation 7.3, along with the following vector identities

$$\hat{n}_{\lambda\vartheta}\bullet\hat{n}_{\lambda\vartheta} = 1 \qquad \hat{n}_{\lambda\vartheta}\bullet\overset{\xi}{\hat{n}}_{\lambda\vartheta} = 0 \qquad \hat{n}_{\lambda\vartheta}\times\hat{n}_{\lambda\vartheta} = \vec{0} \qquad \vec{\omega}_{\xi\vartheta}\times\hat{n}_{\lambda\vartheta} = -\hat{n}_{\lambda\vartheta}\times\vec{\omega}_{\xi\vartheta}$$

$$\hat{n}_{\lambda\vartheta}\times\left(\vec{\omega}_{\xi\vartheta}\times\hat{n}_{\lambda\vartheta}\right) = \left(\hat{n}_{\lambda\vartheta}\bullet\hat{n}_{\lambda\vartheta}\right)\vec{\omega}_{\xi\vartheta} - \left(\hat{n}_{\lambda\vartheta}\bullet\vec{\omega}_{\xi\vartheta}\right)\hat{n}_{\lambda\vartheta} = \vec{\omega}_{\xi\vartheta} - \left(\hat{n}_{\lambda\vartheta}\bullet\vec{\omega}_{\xi\vartheta}\right)\hat{n}_{\lambda\vartheta}$$

$$\hat{n}_{\lambda\vartheta}\times\left(\hat{n}_{\lambda\vartheta}\times\overset{\xi}{\hat{n}}_{\lambda\vartheta}\right) = \left(\hat{n}_{\lambda\vartheta}\bullet\overset{\xi}{\hat{n}}_{\lambda\vartheta}\right)\hat{n}_{\lambda\vartheta} - \left(\hat{n}_{\lambda\vartheta}\bullet\hat{n}_{\lambda\vartheta}\right)\overset{\xi}{\hat{n}}_{\lambda\vartheta} = -\overset{\xi}{\hat{n}}_{\lambda\vartheta}$$

$$\hat{n}_{\lambda\vartheta}\bullet\left(\hat{n}_{\lambda\vartheta}\times\overset{\xi}{\hat{n}}_{\lambda\vartheta}\right) = \left(\hat{n}_{\lambda\vartheta}\times\hat{n}_{\lambda\vartheta}\right)\bullet\overset{\xi}{\hat{n}}_{\lambda\vartheta} = 0$$

reduce the resulting expression to

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + C_1\,\overset{\xi}{\hat{n}}_{\lambda\vartheta} + C_2\,\hat{n}_{\lambda\vartheta}\times\overset{\xi}{\hat{n}}_{\lambda\vartheta}$$

$$C_1 = \sin\frac{\phi_{\lambda\vartheta}}{2} - \cos\phi_{\lambda\vartheta}\sin\frac{\phi_{\lambda\vartheta}}{2} + \sin\phi_{\lambda\vartheta}\cos\frac{\phi_{\lambda\vartheta}}{2}$$

$$C_2 = \cos\frac{\phi_{\lambda\vartheta}}{2} - \cos\phi_{\lambda\vartheta}\cos\frac{\phi_{\lambda\vartheta}}{2} - \sin\phi_{\lambda\vartheta}\sin\frac{\phi_{\lambda\vartheta}}{2}$$

Noting yet that the trigonometric expressions in the two coefficients can be manipulated to

$$C_1 = \sin\frac{\phi_{\lambda\vartheta}}{2} - \cos\phi_{\lambda\vartheta}\sin\frac{\phi_{\lambda\vartheta}}{2} + \sin\phi_{\lambda\vartheta}\cos\frac{\phi_{\lambda\vartheta}}{2} = \sin\frac{\phi_{\lambda\vartheta}}{2} + \sin\left(\phi_{\lambda\vartheta} - \frac{\phi_{\lambda\vartheta}}{2}\right) = 2\sin\frac{\phi_{\lambda\vartheta}}{2}$$

$$C_2 = \cos\frac{\phi_{\lambda\vartheta}}{2} - \cos\phi_{\lambda\vartheta}\cos\frac{\phi_{\lambda\vartheta}}{2} - \sin\phi_{\lambda\vartheta}\sin\frac{\phi_{\lambda\vartheta}}{2} = \cos\frac{\phi_{\lambda\vartheta}}{2} - \cos\left(\phi_{\lambda\vartheta} - \frac{\phi_{\lambda\vartheta}}{2}\right) = 0$$

the resulting expression simplifies to its desired form:

$$\vec{\omega}_{\lambda\vartheta} = \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + 2\sin\frac{\phi_{\lambda\vartheta}}{2}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta} \tag{7.8}$$

In order to obtain a complete and useful description of the system kinematics, it is still necessary to express $\vec{\omega}_{\xi\vartheta}$ in terms of system states, as it was discussed in section 7.1. To proceed towards this objective, one may eliminate $\overset{\xi}{\hat{n}}_{\lambda\vartheta}$ from equation 7.7 in favour of $\vec{\phi}_{\lambda\vartheta}$ and $\vec{\omega}_{\lambda\vartheta}$. This can be easily achieved by first rewriting this equation as

$$\hat{n}_{\lambda\vartheta}\times\hat{n}_{\lambda\vartheta} = \vec{0}$$

$$\begin{aligned}\vec{\omega}_{\xi\vartheta} &= \frac{1}{2}\dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \sin\frac{\phi_{\lambda\vartheta}}{2}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta} - \left(1-\cos\frac{\phi_{\lambda\vartheta}}{2}\right)\hat{n}_{\lambda\vartheta}\times\overset{\xi}{\hat{n}}_{\lambda\vartheta} \\ &= \frac{1}{2}\left(\dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + 2\sin\frac{\phi_{\lambda\vartheta}}{2}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta}\right) - \left(\frac{1-\cos(\phi_{\lambda\vartheta}/2)}{2\sin(\phi_{\lambda\vartheta}/2)}\right)\hat{n}_{\lambda\vartheta}\times\left(\dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + 2\sin\frac{\phi_{\lambda\vartheta}}{2}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta}\right)\end{aligned}$$

and substituting equation 7.8 into the above. This yields the second desired result

$$\vec{\omega}_{\xi\vartheta} = \frac{1}{2}\vec{\omega}_{\lambda\vartheta} - \left(\frac{1-\cos(\phi_{\lambda\vartheta}/2)}{2\sin(\phi_{\lambda\vartheta}/2)}\right)\hat{n}_{\lambda\vartheta}\times\vec{\omega}_{\lambda\vartheta} \tag{7.9}$$

Observing yet the trigonometric identity

$$\frac{1-\cos(\phi_{\lambda\vartheta}/2)}{\sin(\phi_{\lambda\vartheta}/2)} = \frac{2\left(\sin\frac{\phi_{\lambda\vartheta}}{4}\right)^2}{2\,\sin\frac{\phi_{\lambda\vartheta}}{4}\,\cos\frac{\phi_{\lambda\vartheta}}{4}} = \tan\frac{\phi_{\lambda\vartheta}}{4}$$

Equation 7.9 simplifies even further to the following canonical form

$$\vec{\omega}_{\xi\vartheta} = \frac{1}{2}\left(\vec{\omega}_{\lambda\vartheta} - \tan\frac{\phi_{\lambda\vartheta}}{4}\;\hat{n}_{\lambda\vartheta}\times\vec{\omega}_{\lambda\vartheta}\right) \tag{7.10}$$

This last and elegant result is in accordance with the criteria for the definition of basis ξ discussed in section 7.1.

7.4. Moderate Angle Approximations

The trigonometric functions sine and cosine are expandable as infinite alternating sums of ascending powers of the angle, namely γ. Such expansions converge for any value of γ, and can be obtained by direct application of the Maclaurin's series (see Kreyszig, 1999, p. 751-57; or Arfken & Weber, 1995, p. 313-19):

$$\sin\gamma = \gamma - \frac{\gamma^3}{3!} + \frac{\gamma^5}{5!} - \frac{\gamma^7}{7!} + -\ldots \tag{7.11}$$

$$|\gamma| < \infty$$

$$\cos\gamma = 1 - \frac{\gamma^2}{2!} + \frac{\gamma^4}{4!} - \frac{\gamma^6}{6!} + -\ldots \tag{7.12}$$

When the angle γ is small and measured in radians, these series converge quite quickly, and the following approximations are adequate for many engineering applications

$$\sin\gamma \approx \gamma \tag{7.13}$$

small angle second-order approximations

$$\cos\gamma \approx 1 - \frac{\gamma^2}{2} \tag{7.14}$$

In fact, these approximations are surprisingly good up to relatively large angles, and are known as the small angle second-order (quadratic) approximations (see Appendix F for pertinent numerical examples). The same designation "small angle" is given, however, to the first-order (linear) approximations often used in attitude kinematics and attitude dynamics. These linear approximations are generally taken as infinitesimal rotations and may be expressed as

$$\sin\gamma \approx \gamma \tag{7.15}$$

small angle first-order approximations

$$\cos\gamma \approx 1 \tag{7.16}$$

In order to avoid confusion, and considering the corresponding relative accuracy with respect to the size of the angle of rotation, it has been decided to refer in this work to the following small *half* angle second-order approximations as moderate angle approximations:

$$\sin\frac{\gamma}{2} \approx \frac{\gamma}{2} \tag{7.17}$$

small half angle second-order approximations ⇔ moderate angle approximations

$$\cos\frac{\gamma}{2} = 1 - \frac{\gamma^2}{8} \tag{7.18}$$

Making $\gamma = \phi_{\lambda\vartheta}$ and substituting these formulae into equations 7.8 and 7.9, yields the remarkable results:

$$\vec{\omega}_{\lambda\vartheta} \approx \dot{\phi}_{\lambda\vartheta}\,\hat{n}_{\lambda\vartheta} + \phi_{\lambda\vartheta}\,\overset{\xi}{\hat{n}}_{\lambda\vartheta} = \overset{\xi}{\vec{\phi}}_{\lambda\vartheta} \qquad \Rightarrow \qquad \left\{\omega^{\xi}_{\lambda\vartheta}\right\} \approx \left\{\dot{\phi}^{\xi}_{\lambda\vartheta}\right\} \tag{7.19}$$

$$\vec{\omega}_{\xi\vartheta} \approx \frac{1}{2}\vec{\omega}_{\lambda\vartheta} - \frac{1}{8}\vec{\phi}_{\lambda\vartheta}\times\vec{\omega}_{\lambda\vartheta} \tag{7.20}$$

These two results are kinematic in nature. The information they convey is *per se* valid within the moderate angle hypothesis. Their utilisation demands, of course, flexibility on the choice of the basis of observation of the time derivative. It cannot be emphasised too strongly that this flexibility has been made possible in the case only because equation 5.12 was chosen in place of the simpler Eulerian form, equation 5.3.

If one had chosen at the onset Euler's motion equations, and followed steps analogous to those taken in the design of the control law, the resulting governing equation would characterise just another complicated way of stating the problem! The kinematical differential relationship between $\vec{\omega}_{\lambda\vartheta}$ and $\vec{\phi}_{\lambda\vartheta} = \phi_{\lambda\vartheta}\hat{n}_{\lambda\vartheta}$ would then be given by equation 7.2. Consequently, such an approach would not represent a relevant reduction in the complexity of the solution to the problem.

The inclusion of $\vec{M}_{B\vartheta\,\mathrm{C1}}$ in the feedforward path of the control law is, therefore, the fundamental step towards a simple solution to the nominal rigid body stability problem. It enables the shift of the basis of observation of the time derivative, which has been used to reduce the somewhat complicated kinematics of finite (moderate) rotations to congruity with simple vector-like operations. Observed from basis ξ and within moderate angles of rotation, the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$ is just the time derivative of the attitude error vector $\vec{\phi}_{\lambda\vartheta}$, equation 7.19.

7.5. *Nominal Transfer Functions and Stability of Motion*

The chief advantage of shifting the basis of observation of the time derivative from basis ϑ to basis ξ is that the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$ can now be directly integrated to obtain the attitude error vector $\vec{\phi}_{\lambda\vartheta}$, within moderate attitude tracking errors. From equation 7.19, it readily follows that

$$\int \left\{ \omega^{\xi}_{\lambda\vartheta} \right\} dt \approx \left\{ \phi^{\xi}_{\lambda\vartheta} \right\} + \left\{ \phi^{\xi}_{\lambda\vartheta\,\mathrm{ic}} \right\} \tag{7.21}$$

For the purpose of evaluating nominal stability, $\left\{ \phi^{\xi}_{\lambda\vartheta\,\mathrm{ic}} \right\}$ can be taken as zero. Thus, resolving equation 6.7 onto basis ξ, and using 7.21, yields

$$k\int \left\{ \omega^{\xi}_{\lambda\vartheta} \right\} dt + c\left\{ \omega^{\xi}_{\lambda\vartheta} \right\} \approx \left\{ \dot{\omega}^{\xi}_{\vartheta\delta} \right\} \tag{7.22}$$

Since $\left\{ \omega^{\xi}_{\lambda\vartheta} \right\} = \left\{ \omega^{\xi}_{\lambda\delta} \right\} - \left\{ \omega^{\xi}_{\vartheta\delta} \right\}$, equation 7.22 can be rewritten as

$$k\int \left\{ \omega^{\xi}_{\vartheta\delta} \right\} dt + c\left\{ \omega^{\xi}_{\vartheta\delta} \right\} + \left\{ \dot{\omega}^{\xi}_{\vartheta\delta} \right\} \approx k\int \left\{ \omega^{\xi}_{\lambda\delta} \right\} dt + c\left\{ \omega^{\xi}_{\lambda\delta} \right\} \tag{7.23}$$

The expansion of this matrix relation leads to three linear uncoupled constant-coefficient second-order differential equations. Thus, the three-axis non-linear rigid body attitude dynamics has been transformed into an equivalent linear set of three one-axis second-order dynamical equations. Such simplified dynamical equations representing the nominal closed-loop system can be analysed uni-dimensionally with the convenient transfer function[36] approach.

The system's transfer function is obtained by taking the Laplace transform[37], with zero initial conditions, of the differential equation describing the system itself. Using arbitrarily the first vector component (observe the subscript 1), the Laplace transform of equation 7.23 is

$$\begin{aligned} &\mathcal{L}\left(k\int \omega^{\xi_1}_{\vartheta\delta}\, dt + c\,\omega^{\xi_1}_{\vartheta\delta} + \dot{\omega}^{\xi_1}_{\vartheta\delta} \right) \approx \mathcal{L}\left(k\int \omega^{\xi_1}_{\lambda\delta}\, dt + c\,\omega^{\xi_1}_{\lambda\delta} \right) \\ \Rightarrow\quad &\frac{k}{s}\,\Omega^{\xi_1}_{\vartheta\delta} + c\,\Omega^{\xi_1}_{\vartheta\delta} + s\,\Omega^{\xi_1}_{\vartheta\delta} \approx \frac{k}{s}\,\Omega^{\xi_1}_{\lambda\delta} + c\,\Omega^{\xi_1}_{\lambda\delta} \end{aligned} \tag{7.24}$$

---

[36] The transfer function of a linear time invariant system (constant parameter) is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. The transfer function of a system represents the relationship describing the dynamics of the system under consideration (see Dorf & Bishop, 1998, p. 48).

[37] The Laplace transform is a transformation of a function $f(t)$ from the time domain into the complex frequency domain yielding $F(s)$ (see Dorf & Bishop, 1998, p. 42).

From which the nominal closed-loop transfer function is easily found

$$G_{\Omega}(s) = \frac{\Omega^{\xi 1}_{\vartheta\delta}}{\Omega^{\xi 1}_{\lambda\delta}} \approx \frac{cs + k}{s^2 + cs + k} \tag{7.25}$$

The same relationship (same k and c) holds if the second or third vector component is used in place of the first one, giving rise to the ratios $\Omega^{\xi 2}_{\vartheta\delta} / \Omega^{\xi 2}_{\lambda\delta}$ and $\Omega^{\xi 3}_{\vartheta\delta} / \Omega^{\xi 3}_{\lambda\delta}$, respectively. It is opportune to rearrange[38] algebraically the transfer function $G_{\Omega}(s)$ so that it could be expressed in a more general and standardised notation for second-order systems:

$$G_{\Omega}(s) = \frac{\Omega^{\xi 1}_{\vartheta\delta}}{\Omega^{\xi 1}_{\lambda\delta}} \approx \frac{2\zeta\mu_n s + \mu_n^2}{s^2 + 2\zeta\mu_n s + \mu_n^2} \tag{7.26}$$

where $\mu_n = \sqrt{k}$ system's natural frequency

$\zeta = \dfrac{c}{2\sqrt{k}}$ system's damping ratio

One of the conveniences of transfer functions is that they enable the manipulation of the system's model to obtain expressions for several quantities of interest. In the case, there are three such quantities: the above-defined ratio $\Omega^{\xi 1}_{\vartheta\delta} / \Omega^{\xi 1}_{\lambda\delta}$ and the error ratios $\Phi^{\xi 1}_{\lambda\vartheta} / \Omega^{\xi 1}_{\lambda\delta}$ and $\Omega^{\xi 1}_{\lambda\vartheta} / \Omega^{\xi 1}_{\lambda\delta}$.

These three ratios provide the means to analyse the effects of a specified[39] $\omega^{\xi 1}_{\lambda\delta}$ on the body's rotational motion, $\omega^{\xi 1}_{\vartheta\delta}$, $\phi^{\xi 1}_{\lambda\vartheta}$ and $\omega^{\xi 1}_{\lambda\vartheta}$ respectively. The ratio $\Phi^{\xi 1}_{\lambda\vartheta} / \Omega^{\xi 1}_{\lambda\delta}$ may be obtained by first taking the Laplace transform with zero initial conditions of equation 7.21 (first component)

[38] The standard control notation employs the Greek letter ω to denote frequency. Since this symbol has already been adopted to represent the angular velocity, μ is used instead.

[39] For the purpose of system analysis and design, the input signal is generally specified as a simple function of time, such as step, ramp, or sinusoidal (test input signal).

$$\mathcal{L}\left(\phi_{\lambda\vartheta}^{\xi_1}\right) \approx \mathcal{L}\left(\int \omega_{\lambda\vartheta}^{\xi_1} dt\right) \quad \Rightarrow \quad \Phi_{\lambda\vartheta}^{\xi_1} = \frac{\Omega_{\lambda\vartheta}^{\xi_1}}{s} \tag{7.27}$$

and then proceeding as follows:

$$\Phi_{\lambda\vartheta}^{\xi_1} = \frac{\Omega_{\lambda\vartheta}^{\xi_1}}{s} = \frac{1}{s}\left(\Omega_{\lambda\delta}^{\xi_1} - \Omega_{\vartheta\delta}^{\xi_1}\right) = \Omega_{\lambda\delta}^{\xi_1}\left(\frac{1-G_{\Omega}(s)}{s}\right)$$

Substitution of equation 7.26 into the above relationship, along with a slight rearrangement, yields the first/position error ratio describing the system's nominal non-homogeneous closed-loop attitude state error dynamics

$$G_{\Phi_E}(s) = \frac{\Phi_{\lambda\vartheta}^{\xi_1}}{\Omega_{\lambda\delta}^{\xi_1}} \approx \frac{s}{s^2 + 2\zeta\mu_n s + \mu_n^2} \tag{7.28}$$

Using once again equation 7.27 and substituting it into 7.28, yields the second/velocity desired error ratio describing the system's nominal non-homogeneous closed-loop attitude state error dynamics

$$G_{\Omega_E}(s) = \frac{\Omega_{\lambda\vartheta}^{\xi_1}}{\Omega_{\lambda\delta}^{\xi_1}} \approx \frac{s^2}{s^2 + 2\zeta\mu_n s + \mu_n^2} \tag{7.29}$$

The attitude stability for such linear time invariant system (equation 7.28) can be investigated via the Routh-Hurwitz criterion (see Dorf & Bishop, 1998, p. 299-301; or Bishop, 1997, p. 105-111). The technique provides an answer to the question of stability by considering the characteristic equation of the system (system's transfer function denominator). For the system under consideration, second-order type, the requirement for stability is simply that the coefficients of the characteristic equation must be all positive or all negative. Given that the coefficients of the characteristic equation $k = \mu_n^2$ and $c = 2\zeta\mu_n$ are scalars and supposedly positive, the stability of the nominal closed-loop system is guaranteed.

After establishing that the nominal closed-loop system is stable, the next logical step would be the design of the control system, i.e. the selection of the control parameters μ_n and ζ so that the three one-axis control systems about the axes of basis ξ would have the desired dynamic characteristics. Keeping in mind the saturation limits of sensors and actuators, this choice can be made at the discretion of the control-system designer, depending on which performance criteria are most important in a given circumstance.

This issue is, however, out of the scope of this thesis. The transfer functions of interest are second-order with numerator dynamics (equations 7.26, 7.28 and 7.29). The design procedures for such control systems are well documented in the literature. The interested reader may refer to Palm (1998, p. 194-96, 522-25, 566-69), Dorf & Bishop (1998, p. 240-45, 716-17), Mutambara (1999, p. 188-93), or Clark (1962, p. 112-25).

In any case, the decision on which test input signal, design criteria or performance index to use is ultimately dependent upon the form of the input the system will be most frequently subjected to under normal operation (see Ogata, 1997, p. 134-35). This most frequent system input has been regarded as a general angular velocity profile (driver's inertial angular velocity). Further specification for system mission and system input signal would be necessary for proper selection of the control parameters.

Thus, given the assumptions employed so far, namely:

1. The control torque can be considered the dominant external moment, i.e. $\sum \vec{M}_{B\vartheta\,\mathrm{ext}} = \vec{M}_{B\vartheta\,\mathrm{C}}$;
2. The system states and parameters can be measured in such a way that the perturbing torques $\vec{M}_{B\vartheta\,\mathrm{P1}}$ and $\vec{M}_{B\vartheta\,\mathrm{P2}}$ are either nullified or well approximated by $\vec{M}_{B\vartheta\,\mathrm{C1}}$ and $\vec{M}_{B\vartheta\,\mathrm{C2}}$ respectively (nominal case); and
3. The attitude tracking error $\phi_{\lambda\vartheta}$ is kept small enough to enable moderate angle approximations.

it is now possible to analyse, design and optimise the attitude control system with linear methods along the axes of basis ξ.

7.6. Formulation Remarks

Remark 1

The control torque - equation 6.3 - has been defined as an algebraic (no time derivatives) vector-dyadic relationship. As a consequence, it can be directly resolved onto any basis, not necessarily ξ. Good candidate bases are, of course, the ones in which $\vec{\vec{I}}_{B\vartheta}$ is a constant quantity.

The most obvious option is a basis that is centroidal and principal axis. This option would normally imply the simplest form for the control torque components, since in this case $\vec{\vec{I}}_{B\vartheta}$ is represented by a diagonal matrix. In terms of components resolved along the axes of basis ϑ and using equation 7.20, the control torque can be written as

$$\begin{aligned} &\left\{ M^{\vartheta}_{B\vartheta\,C} \right\} = \left\{ M^{\vartheta}_{B\vartheta\,C1} \right\} + \left\{ M^{\vartheta}_{B\vartheta\,C2} \right\} + \left\{ M^{\vartheta}_{B\vartheta\,C3} \right\} \\ &\quad \left\{ M^{\vartheta}_{B\vartheta\,C1} \right\} = -\frac{1}{2}\left[I^{\vartheta}_{B\vartheta} \right]\left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right]\left(\left[\mathit{1} \right] - \frac{1}{4}\left[\tilde{\phi}^{\vartheta}_{\lambda\vartheta} \right] \right)\left\{ \omega^{\vartheta}_{\lambda\vartheta} \right\} \\ &\quad \left\{ M^{\vartheta}_{B\vartheta\,C2} \right\} = \left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left\{ \omega^{\vartheta}_{\vartheta\delta} \right\} \\ &\quad \left\{ M^{\vartheta}_{B\vartheta\,C3} \right\} = \left[I^{\vartheta}_{B\vartheta} \right]\left(k\left\{ \phi^{\vartheta}_{\lambda\vartheta} \right\} + c\left\{ \omega^{\vartheta}_{\lambda\vartheta} \right\} \right) \end{aligned} \tag{7.30}$$

or, more compactly,

$$\begin{aligned} &\left\{ M^{\vartheta}_{B\vartheta\,C} \right\} = \left[I^{\vartheta}_{B\vartheta} \right]\left(k\left\{ \phi^{\vartheta}_{\lambda\vartheta} \right\} + \left[P^{\vartheta} \right]\left\{ \omega^{\vartheta}_{\lambda\vartheta} \right\} \right) + \left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left\{ \omega^{\vartheta}_{\vartheta\delta} \right\} \\ &\quad \left[P^{\vartheta} \right] = c\left[\mathit{1} \right] - \frac{1}{2}\left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right] + \frac{1}{8}\left[\tilde{\omega}^{\vartheta}_{\vartheta\delta} \right]\left[\tilde{\phi}^{\vartheta}_{\lambda\vartheta} \right] \end{aligned} \tag{7.31}$$

The torque terms may now be studied in order to determine their relative contribution. Depending on the expected input data, i.e. the inertial angular velocity of the driver frame $\vec{\omega}_{\lambda\delta}$, some of these terms might prove to be negligible when compared to the others. In that being the case, simplifications are possible.

There are several instances in which the inertial torque term $\vec{M}_{B\vartheta\,\mathrm{P2}}$ due to the centrifugal forces is assumed negligible when compared to the total control torque (see, Byers & Vadali, 1993; or Wie et al., 1989).

Remark 2

The parameterisation of the error in attitude between the driver frame λ and the follower frame ϑ has been made in terms of the corresponding rotation vector $\vec{\phi}_{\lambda\vartheta}$. The idea of using a vector-like parameterisation (three parameter only) for finite rotations is not new, and has been exploited, for example, in the recent papers of Aicardi et al. (2000), Ibrahimbegovic (1997), Ibrahimbegovic et al. (1995).

Nevertheless, it is in the strapdown inertial literature where this parameterisation is most used. Examples are: Waldmann (2001), Savage (1998a), Savage (1998b), Savage (1998c), Musoff & Murphy (1995), Ignagni (1994), Jiang & Lin (1992), Jiang (1991), Lee et al. (1990), Ignagni (1990), Miller (1983) , Nazaroff (1979) , Bortz (1971) and Jordan (1969).

Parameterising a rotation with the components of the rotation vector has a number of virtues, some of which have been commented in section 6.3. To this work, the most notable one is the possibility of reducing the somewhat complicated kinematics of finite (moderate) rotations to congruity with simple vector-like operations, equation 7.19.

Although equation 7.19 is in the desired form, i.e. in terms of the rotation vector $\vec{\phi}_{\lambda\vartheta}$, the analytical work in producing it undertook an indirect route. The rotation variables employed were $\hat{n}_{\lambda\vartheta}$ and $\phi_{\lambda\vartheta}$, rather than $\vec{\phi}_{\lambda\vartheta}$ and $\phi_{\lambda\vartheta}$ (see sections 4.2, 7.2, 7.3 and 7.4). It is possible, however, to attain expressions for the transformation matrix and the angular velocity in terms of the rotation vector. Equations 3.6 and 3.7 exemplify the former.

The expressions for the transformation matrix when parameterised with the rotation vector components and its norm can be found discussed in Argyris (1982, 85-88) and Stuelpnagel (1964). Accordingly, discussions concerning the derivation of expressions for the angular velocity when parameterised with the rotation vector can be found in Ibrahimbegovic (1997), Ibrahimbegovic et al. (1995), Pfister (1996), Ignagni (1994), Shuster (1993b). and Bortz (1971).

Considering that both the transformation matrix and the angular velocity can be expressed in terms of the rotation vector components and its norm, it should be possible to take the direct route to develop equation 7.19. Nevertheless, this approach should also imply more involved derivations. Since the end result (equation 7.19) must be the same, such an approach has not been attempted here.

Chapter 8

Kinematical Theorem Numerical Validation

This chapter focuses on the numerical validation and illustration of the main theoretically achieved results of this thesis, particularly the kinematical theorem. Firstly, a procedure for this result validation is devised. Secondly, the detailed implementation of the proposed procedure into Simulink/Matlab is described. Thirdly, numerical examples illustrating the validation procedure are provided.

This chapter describes procedures for validating the main theoretically achieved results of this thesis, particularly the kinematical theorem, equation 7.8, when the angle of rotation (attitude tracking error) is kept within moderate bounds. The idea is to test approximation 7.19 when utilised within equation 6.7 via a model built in Simulink. It is assumed, of course, perfect plant knowledge and state estimation (nominal case).

One way of testing approximation 7.19 is by comparing the components of $\left\{ \omega_{\vartheta\delta}^{\xi} \right\}$, the ξ- resolution of the follower's inertial angular velocity vector, when evaluated via two different methods: the integral method and the derivative method.

In the integral method, $\left\{ \omega_{\vartheta\delta}^{\xi} \right\}$ is evaluated via straightforward time integration of equation 7.22, i.e. this method assumes valid the linear relationship between the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$ and the rate of change of the attitude error vector $\vec{\phi}_{\lambda\vartheta}$ when observed from basis ξ (equation 7.19).

In the derivative method, $\left\{ \omega_{\vartheta\delta}^{\xi} \right\}$ is evaluated via the conventional algebra of rotations. This method involves, therefore, time differentiation of the corresponding transformation matrix.

The integral and the derivative methods should produce very closely related results as long as the attitude error $\phi_{\lambda\vartheta}$ is kept within moderate bounds. It should be noticed, however, that the derivative method calls for some kind of numerical derivative (noisy operation), whereas the integral method is encoded as an integrating only loop. This point is discussed in subsection 8.2.3.

The details of the two methods are explained throughout section 8.1, along with the topology of the Simulink model. Section 8.2 numerically illustrates the procedure of validation as above devised.

8.1. Simulink Model

The Simulink model is composed of three main elements: feedback loop, subsystem-1 and subsystem-2, which are described in subsections 8.1.1, 8.1.2 and 8.1.3 respectively.

The model also contains two scopes, which can be seen on the top part of figure 8. The left-hand side scope compares $\left\{ \omega_{\vartheta\delta}^{\xi} \right\}$ when produced in the feedback loop (integral method), and when produced in the subsystems (derivative method). The right-hand side scope monitors the corresponding attitude error $\phi_{\lambda\vartheta}$. The curves displayed by these scopes, i.e. the time histories for $\omega_{\vartheta\delta}^{\xi_1}$, $\omega_{\vartheta\delta}^{\xi_2}$, $\omega_{\vartheta\delta}^{\xi_3}$ and $\phi_{\lambda\vartheta}$, have been named validation curves.

The symbols employed in the Simulink model - figures 8, 9, and 10 - are recognised as follows:

Arb_k	input port 1 (subsystem-1)	$\{\phi^{\xi}_{\lambda\vartheta}\}$
Wrg_r	input port 2 (subsystem-1)	$\{\omega^{\lambda}_{\lambda\delta}\}$
Wbg_k	output port 1 (subsystem-1)	$\{\omega^{\xi}_{\vartheta\delta}\}$
Wrg_k	output port 2 (subsystem-1)	$\{\omega^{\xi}_{\lambda\delta}\}$
Trk	input port 1 (subsystem-2)	$[T^{\xi}_{\lambda}]$
Wrb_k	output port 1 (subsystem-2)	$\{\omega^{\xi}_{\lambda\vartheta}\}$

8.1.1. Feedback Loop

The feedback loop is the main element in the Simulink model. Basically, it solves equation 7.22, which can be represented as a closed-loop feedback control system (see figure 8). The input for this loop is the driver's inertial angular velocity when expressed in ξ coordinates, $\{\omega^{\xi}_{\lambda\delta}\}$.

There are two integrators in the loop. The right-hand side integrator outputs $\{\phi^{\xi}_{\lambda\vartheta}\}$, while the left-hand side integrator outputs $\{\omega^{\xi}_{\vartheta\delta}\}$. For the sake of simplicity, the initial conditions $\{\phi^{\xi}_{\lambda\vartheta\,\mathrm{ic}}\}$ and $\{\omega^{\xi}_{\lambda\vartheta\,\mathrm{ic}}\}$ have been set to zero. This means that the driver frame λ and the follower frame ϑ are initially aligned and have the same initial angular velocity.

The output of the right-hand side integrator $\{\phi^{\xi}_{\lambda\vartheta}\}$, along with the model's reference signal $\{\omega^{\lambda}_{\lambda\delta}\}$ (From Workspace block), is fed into subsystem-1 for further processing.

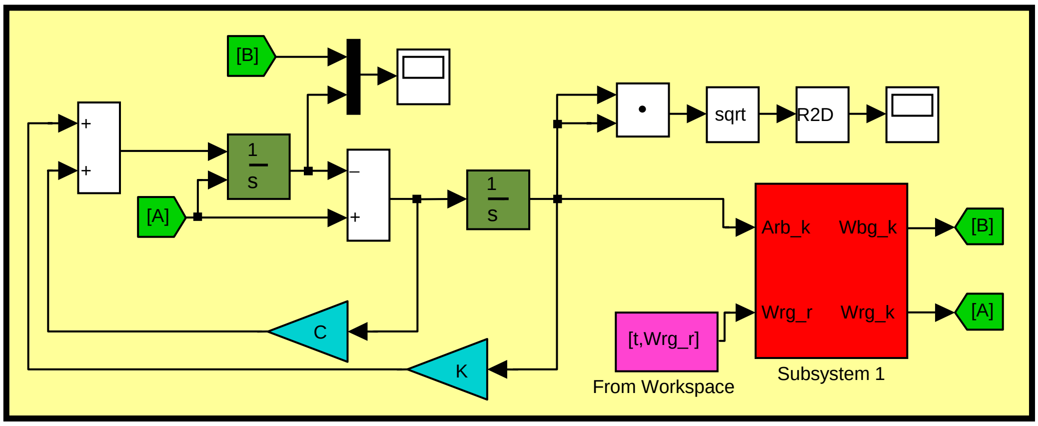


Figure 8: Feedback Loop

8.1.2. Subsystem One

In the preceding subsection, the central issue was the solution of equation 7.22 and the production of the components of the attitude error vector when expressed in ξ coordinates, $\left\{ \phi^{\xi}_{\lambda\vartheta} \right\}$. These objectives were fulfilled with the feedback loop implemented as a Simulink model, assuming that the quantity $\left\{ \omega^{\xi}_{\lambda\delta} \right\}$ is available for integration.

The main goal here is the parallel computation of $\left\{ \omega^{\xi}_{\lambda\delta} \right\}$ employing the results of the feedback loop and the model's input signal $\left\{ \omega^{\lambda}_{\lambda\delta} \right\}$. This is achieved using a two-step procedure:

(1) estimate the elements of $\left[T^{\xi}_{\lambda} \right]$ via a routine that employs the attitude error vector components as output from the feedback loop, and then

(2) use this matrix to transform the representation of the model's input signal $\left\{ \omega^{\lambda}_{\lambda\delta} \right\}$ from basis λ to basis ξ, that is

$$\left\{ \omega^{\xi}_{\lambda\delta} \right\} = \left[T^{\xi}_{\lambda} \right] \left\{ \omega^{\lambda}_{\lambda\delta} \right\} \tag{8.1}$$

The transformation matrix $\left[\, T^{\xi}_{\lambda} \,\right]$ can be obtained as follows

$$\begin{aligned} \left[\, T^{\xi}_{\lambda} \,\right] &= \left[\, R^{\xi}_{\lambda\xi} \,\right] = \left[\, R^{\xi}\left(\, \phi_{\lambda\xi}, \hat{n}_{\lambda\xi} \,\right)\right] = \left[\, R^{\xi}\left(\, \tfrac{1}{2}\phi_{\lambda\vartheta}, \hat{n}_{\lambda\vartheta} \,\right)\right] \\ &= \left[\, 1 \,\right] + \sin\frac{\phi_{\lambda\vartheta}}{2}\left[\, \tilde{n}^{\xi}_{\lambda\vartheta} \,\right] + \left(1 - \cos\frac{\phi_{\lambda\vartheta}}{2}\right)\left[\, \tilde{n}^{\xi}_{\lambda\vartheta} \,\right]\left[\, \tilde{n}^{\xi}_{\lambda\vartheta} \,\right] \end{aligned} \tag{8.2}$$

where equations 3.4 and 7.5 have been used. Subsystem-1 is mostly devoted to the implementation of equations 8.1 and 8.2.

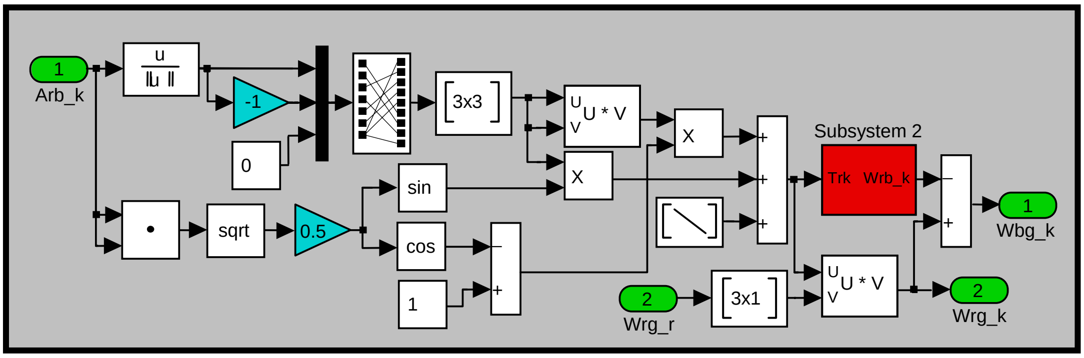


Figure 9: Subsystem One

8.1.3. Subsystem Two

Subsystem-2 evaluates the components of the angular velocity error vector resolved onto basis ξ $(\omega^{\xi_1}_{\lambda\vartheta}, \omega^{\xi_2}_{\lambda\vartheta}, \omega^{\xi_3}_{\lambda\vartheta})$ from the rotation matrix $\left[\, R^{\xi}\left(\, \tfrac{1}{2}\phi_{\lambda\vartheta}, \hat{n}_{\lambda\vartheta} \,\right)\right]$ and its time derivative. There are different ways of achieving this result. One possibility is by first expanding $\left\{\, \omega^{\xi}_{\lambda\vartheta} \,\right\}$ into a skew-symmetric matrix and then employing equations 4.9 and 4.10 as parent equations to produce the following expression:

$$\begin{aligned} \left[\, \tilde{\omega}^{\xi}_{\lambda\vartheta} \,\right] &= \left[\, \tilde{\omega}^{\xi}_{\lambda\xi} \,\right] + \left[\, \tilde{\omega}^{\xi}_{\xi\vartheta} \,\right] \\ &= \left[\, \dot{T}^{\xi}_{\lambda} \,\right]\left[\, T^{\xi}_{\lambda} \,\right]^{T} + \left[\, T^{\vartheta}_{\xi} \,\right]^{T}\left[\, \dot{T}^{\vartheta}_{\xi} \,\right] \end{aligned} \tag{8.3}$$

The relationship between $\left[T^{\xi}_{\lambda} \right]$ and $\left[R^{\xi}\left(\tfrac{1}{2}\phi_{\lambda\vartheta}, \hat{n}_{\lambda\vartheta} \right) \right]$ has already been established in equation 8.2. The relationship between the $\left[T^{\vartheta}_{\xi} \right]$ and $\left[R^{\xi}\left(\tfrac{1}{2}\phi_{\lambda\vartheta}, \hat{n}_{\lambda\vartheta} \right) \right]$ can be obtained in a similar fashion:

$$\begin{aligned} \left[T^{\vartheta}_{\xi} \right] &= \left[R^{\xi}_{\xi\vartheta} \right] = \left[R^{\xi}\left(\phi_{\xi\vartheta}, \hat{n}_{\xi\vartheta} \right) \right] = \left[R^{\xi}\left(\tfrac{1}{2}\phi_{\lambda\vartheta}, \hat{n}_{\lambda\vartheta} \right) \right] \\ &= \left[1 \right] + \sin\frac{\phi_{\lambda\vartheta}}{2}\left[\tilde{n}^{\xi}_{\lambda\vartheta} \right] + \left(1 - \cos\frac{\phi_{\lambda\vartheta}}{2} \right)\left[\tilde{n}^{\xi}_{\lambda\vartheta} \right]\left[\tilde{n}^{\xi}_{\lambda\vartheta} \right] \end{aligned} \tag{8.4}$$

where equations 3.3 and 7.5 have been used. Subsystem-2 (figure 10) implements equation 8.3 and extracts $\left\{ \omega^{\xi}_{\lambda\vartheta} \right\}$ from the resulting $\left[\tilde{\omega}^{\xi}_{\lambda\vartheta} \right]$.

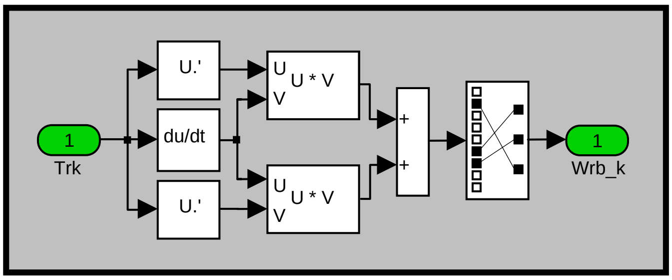


Figure 10: Subsystem Two

In order to attain $\left\{ \omega^{\xi}_{\vartheta\delta} \right\}$ via this alternative (and longer) route, it is still necessary to subtract the output of subsystem-2, $\left\{ \omega^{\xi}_{\lambda\vartheta} \right\}$, from the quantity $\left\{ \omega^{\xi}_{\lambda\delta} \right\}$ produced in subsystem-1:

$$\left\{ \omega^{\xi}_{\vartheta\delta} \right\} = \left\{ \omega^{\xi}_{\lambda\delta} \right\} - \left\{ \omega^{\xi}_{\lambda\vartheta} \right\} \tag{8.5}$$

The implementation of this equation can be seen in the far right side of figure 9. The result is then sent to the left-hand side scope for comparison.

8.2. Numerical Study

For the purpose of illustration of the validation procedure, the attitude of the driver frame λ relative to the inertial frame δ is prescribed in terms of Euler 3-2-1 angles. The kinematical differential relationship between these attitude variables and the angular velocity is, consequently, also prescribed.

The table that follows shows (a) the three rotations as explicit functions of the time, (b) the corresponding time derivatives, and (c) the corresponding angular velocity components when expressed along the axes of the driver frame λ.

Attitude Variables	Body 3-2-1 orientation angles
First Rotation - φ	$\varphi = \sin 3t \cos 5t$ $\dot{\varphi} = 3\cos 3t \cos 5t - 5\sin 3t \sin 5t$
Second Rotation - θ	$\theta = 0.4\pi \sin 5t$ $\dot{\theta} = 2\pi \cos 5t$
Third Rotation - ψ	$\psi = 0.5\cos 5t\,(0.1+\sin 3t)^3$ $\dot{\psi} = -2.5\sin 5t\,(0.1+\sin 3t)^3 + 4.5\cos 5t \cos 3t\,(0.1+\sin 3t)^2$
Angular Velocity	$\omega_{\lambda\delta}^{\lambda_1} = -\dot{\varphi}\sin\theta + \dot{\psi}$ $\omega_{\lambda\delta}^{\lambda_2} = \dot{\varphi}\cos\theta \sin\psi + \dot{\theta}\cos\psi$ $\omega_{\lambda\delta}^{\lambda_3} = \dot{\varphi}\cos\theta \cos\psi - \dot{\theta}\sin\psi$ (adapted from Kane et al., 1983, p. 428, Body-three: 3-2-1)

The time functions defining the Body 3-2-1 orientation angles φ, θ, ψ are inspired in the work of Junkins (1997). The corresponding angular velocity components ($\omega_{\lambda\delta}^{\lambda_1}$, $\omega_{\lambda\delta}^{\lambda_2}$, $\omega_{\lambda\delta}^{\lambda_3}$) seem generic enough to justify their utilisation as validation curves. The time histories for these components are depicted in figures 11-13.

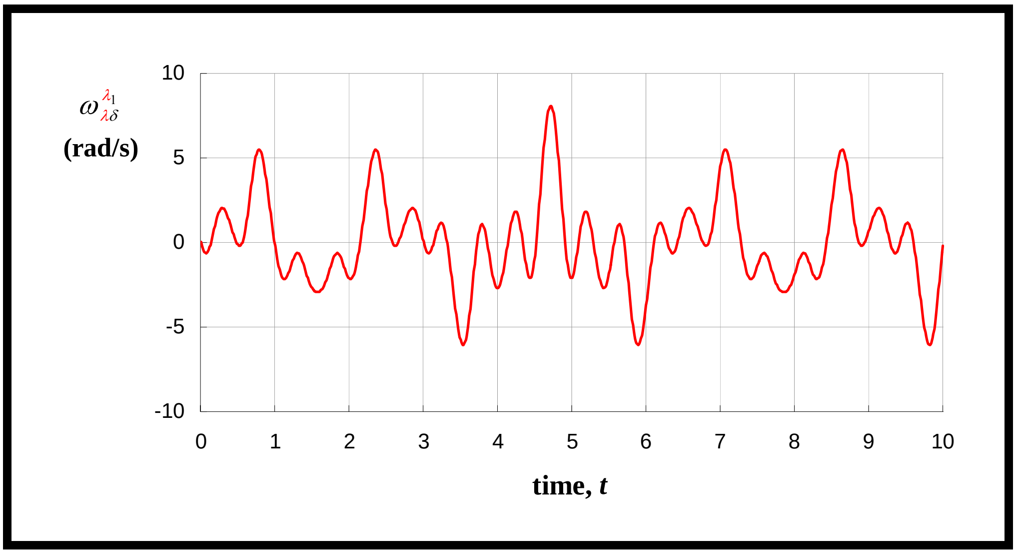


Figure 11: Driver's Inertial Angular Velocity – first component

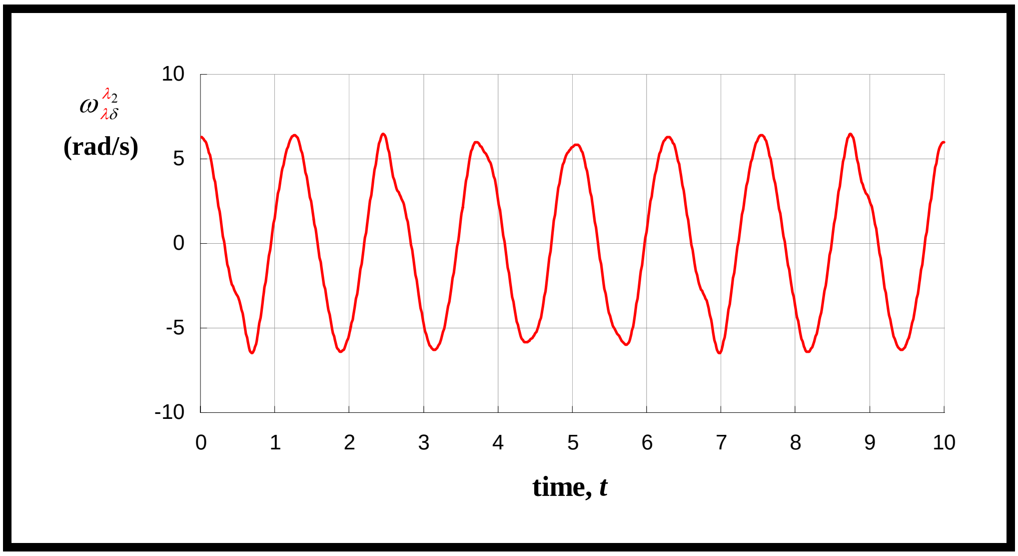


Figure 12: Driver's Inertial Angular Velocity – second component

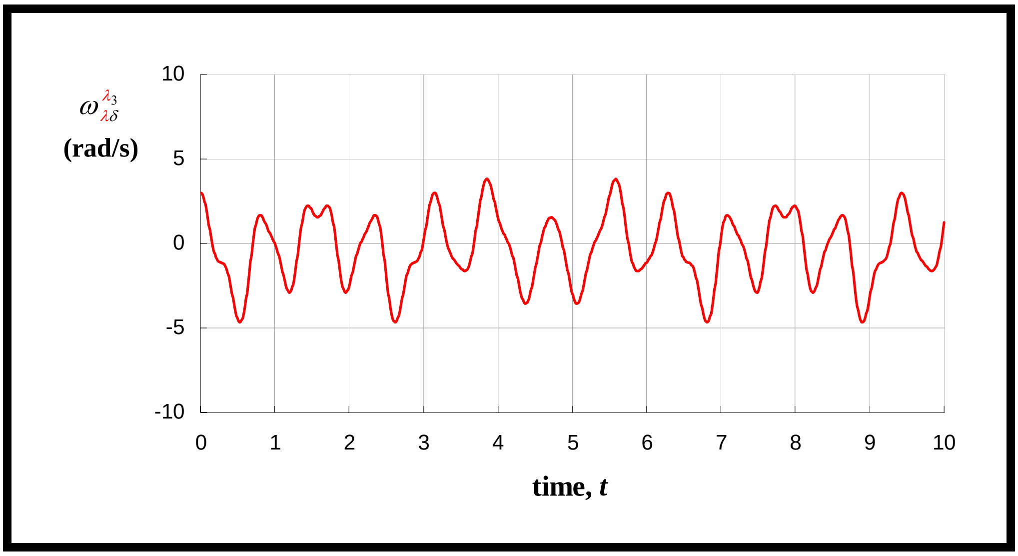


Figure 13: Driver's Inertial Angular Velocity – third component

The M-file[40] used to generate the input data for the Simulink model, along with a number of commentaries, is presented in page 94. In terms of system parameter definitions, two cases have been considered: the underdamped ($\zeta = 0.7$, $\mu_n = 10$) and the overdamped ($\zeta = 1.6$, $\mu_n = 10$). The model's reference input ($\omega_{\lambda\delta}^{\lambda_1}$, $\omega_{\lambda\delta}^{\lambda_2}$, $\omega_{\lambda\delta}^{\lambda_3}$) for these two cases is used exactly as prescribed (noise-free, $n = 0$, see M-file).

The results for the underdamped and the overdamped cases are presented in subsections 8.2.1 and 8.2.2 respectively. Subsection 8.2.3 considers the case in which the input signal is not noise-free ($n = 1$, see M-file), and the corresponding effects on the validation method.

[40] M-files are files that contain Matlab language code. They can be functions that accept and produce output, or, as in this case, they can also be scripts that execute a series of Matlab statements (see The MathWorks, 1999, p. 10-2).

```
% This M-file generates the input data for the Simulink model
% The solver options are the default ones.

clear                          % clears the workspace
freq = 1000;                   % data frequency
sim_time = 10;                 % stop time (simulation parameter)
t = [0:1/freq:sim_time]';      % output times (simulation parameter)

% Control parameters
fn = 10;          % system's natural frequency
zeta = 1.6;       % system's damping ratio (0.7 or 1.6)
K = fn^2;         % stiffness coefficient
C = 2*zeta*fn;    % damping coefficient

% Artificial noise
n = 0;          % "1" turns noise on, "0" turns noise off
noise = n*0.3*(1-2*rand(sim_time*freq+1,3));

% First rotation and time derivative
phi = sin(3*t).*cos(5*t);
Dphi = 3*cos(3*t).*cos(5*t) - 5*sin(3*t).*sin(5*t);

% Second rotation and time derivative
theta = 0.4*pi*sin(5*t);
Dtheta = 2*pi*cos(5*t);

% Third rotation and time derivative
A = 0.1 + sin(3*t);
psi = 0.5*cos(5*t).*(A.^3);
Dpsi = 4.5*cos(3*t).*cos(5*t).*(A.^2) -2.5*sin(5*t).*(A.^3);

% Driver's inertial angular velocity.
% Components along the axes of the driver frame.
% The orientation angles are assumed Euler 3-2-1.
w1 = -Dphi.*sin(theta)           + Dpsi ;
w2 =  Dphi.*cos(theta).*sin(psi) + Dtheta.*cos(psi);
w3 =  Dphi.*cos(theta).*cos(psi) - Dtheta.*sin(psi);
Wrg_r = [w1 w2 w3] + noise;

% Cleaning
clear Dphi Dtheta Dpsi psi phi theta w1 w2 w3 fn zeta n A noise
```

8.2.1. Numerical Results for the Underdamped Case

The results presented next show that, for the nominal system, there is no appreciable difference between the integral method and the derivative method for the parameters ($\zeta = 0.7$, $\mu_n = 10$, $n = 0$), scale, and data frequency (1000/s) utilised in the simulation of the underdamped case.

Figures 14-16 depict the time histories for the three components of the body's inertial angular velocity vector when resolved onto basis ξ ($\omega_{\vartheta\delta}^{\xi_1}$, $\omega_{\vartheta\delta}^{\xi_2}$, $\omega_{\vartheta\delta}^{\xi_3}$), and evaluated by both methods. One validation curve is superimposed onto the other (for each component), even though the corresponding attitude tracking error time history $\phi_{\lambda\vartheta}$ shows values as high as 25 degrees (figure 17).

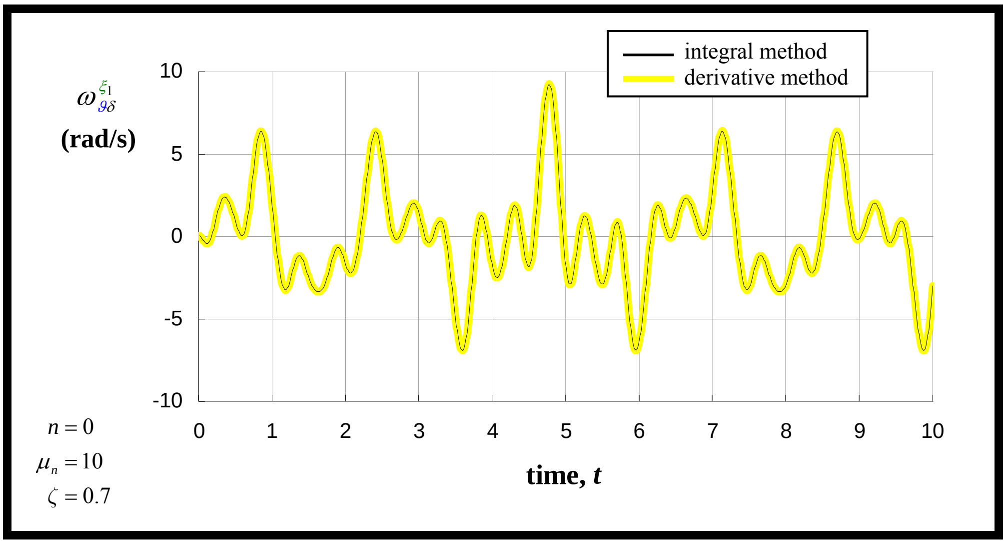


Figure 14: Validation Curves – first component, underdamped case

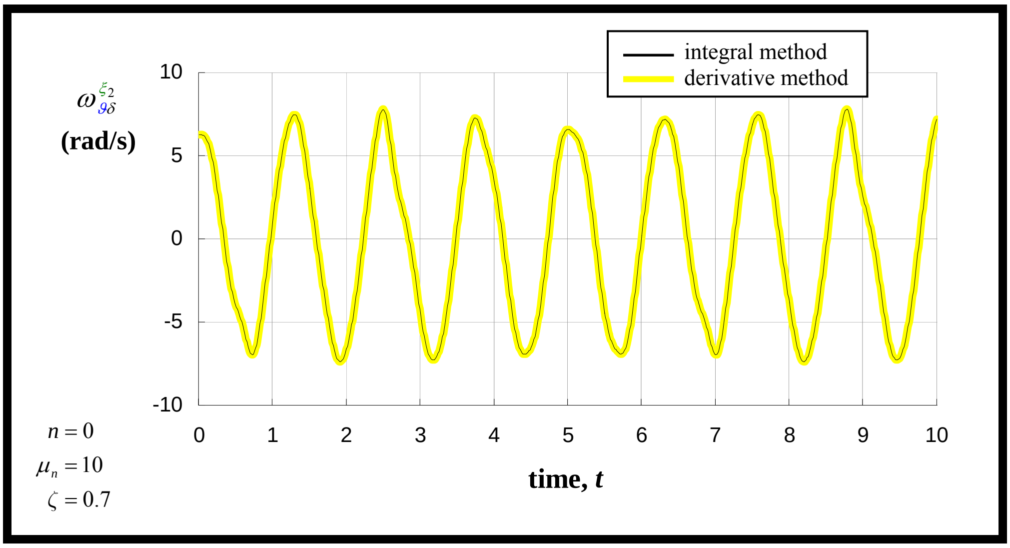


Figure 15: Validation Curves – second component, underdamped case

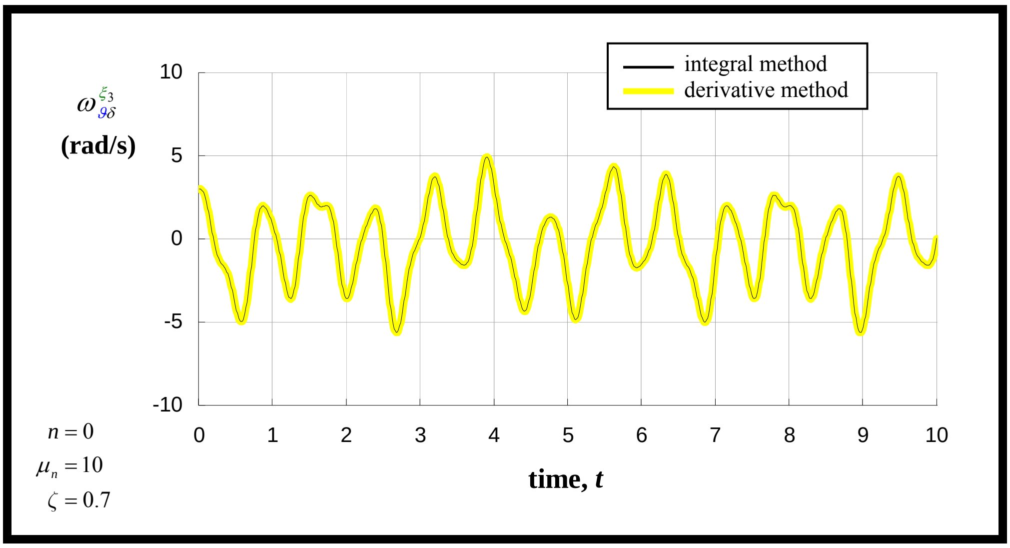


Figure 16: Validation Curves – third component, underdamped case

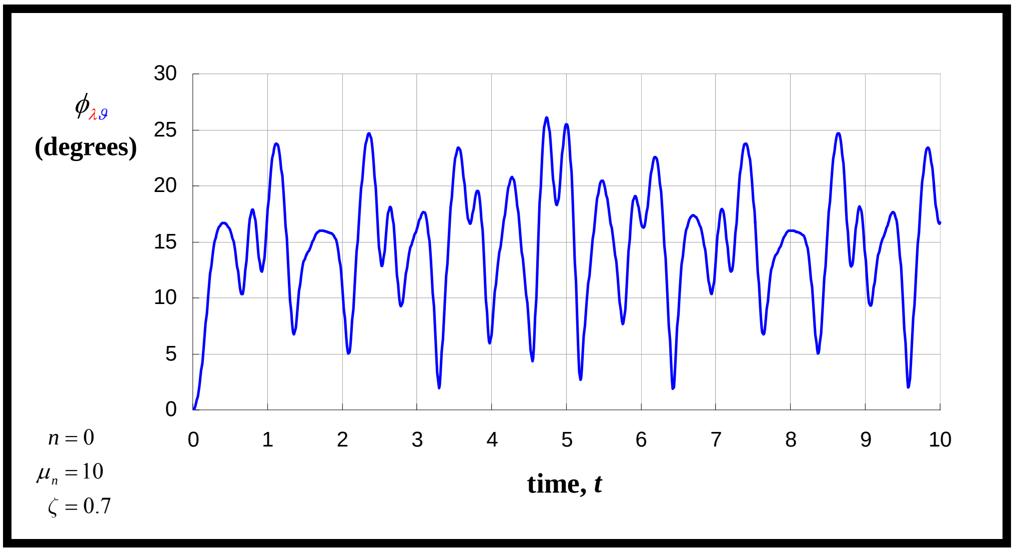


Figure 17: Validation Curves – attitude error, underdamped case

8.2.2. Numerical Results for the Overdamped Case

The results presented next show that, for the nominal system, there is no appreciable difference between the integral method and the derivative method for the parameters ($\zeta = 1.6$, $\mu_n = 10$, $n = 0$), scale, and data frequency (1000/s) utilised in the simulation of the overdamped case.

Figures 18-20 depict the time histories for the three components of the body's inertial angular velocity vector when resolved onto basis ξ ($\omega_{\vartheta\delta}^{\xi_1}$, $\omega_{\vartheta\delta}^{\xi_2}$, $\omega_{\vartheta\delta}^{\xi_3}$), and evaluated by both methods. One validation curve is also superimposed onto the other (for each component) in this case, whereas the corresponding attitude tracking error time history $\phi_{\lambda\vartheta}$ is less than 15 degrees (see figure 21).

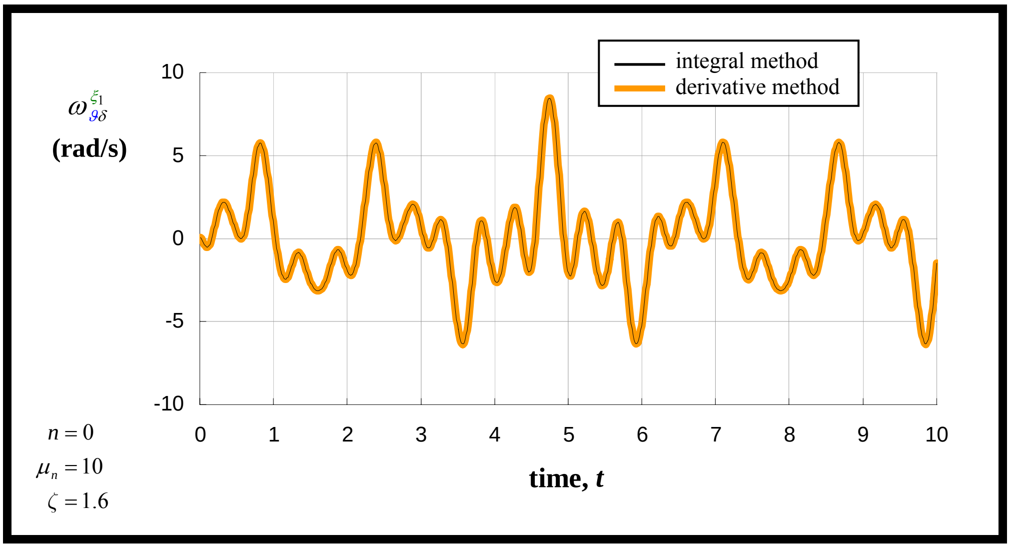


Figure 18: Validation Curves – first component, overdamped case

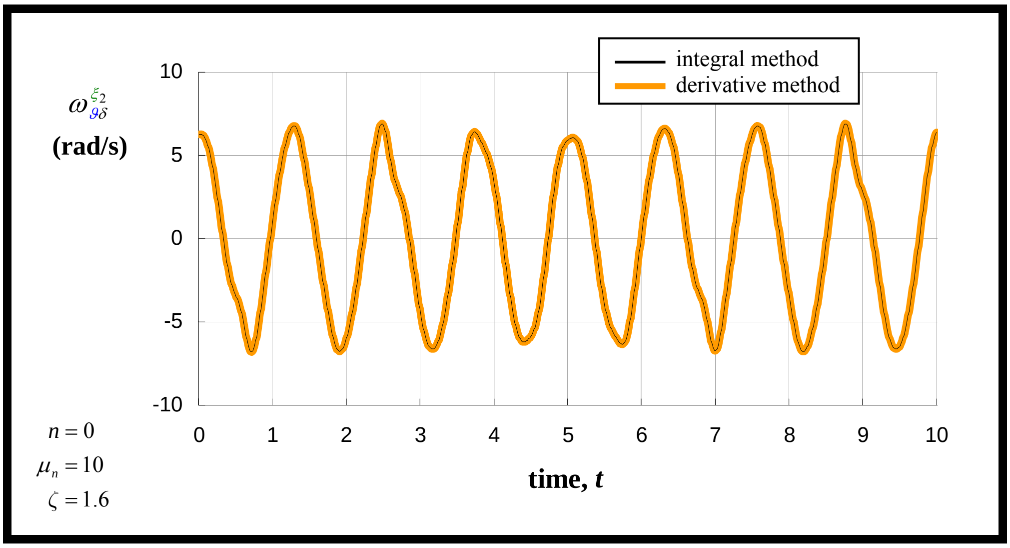


Figure 19: Validation Curves – second component, overdamped case

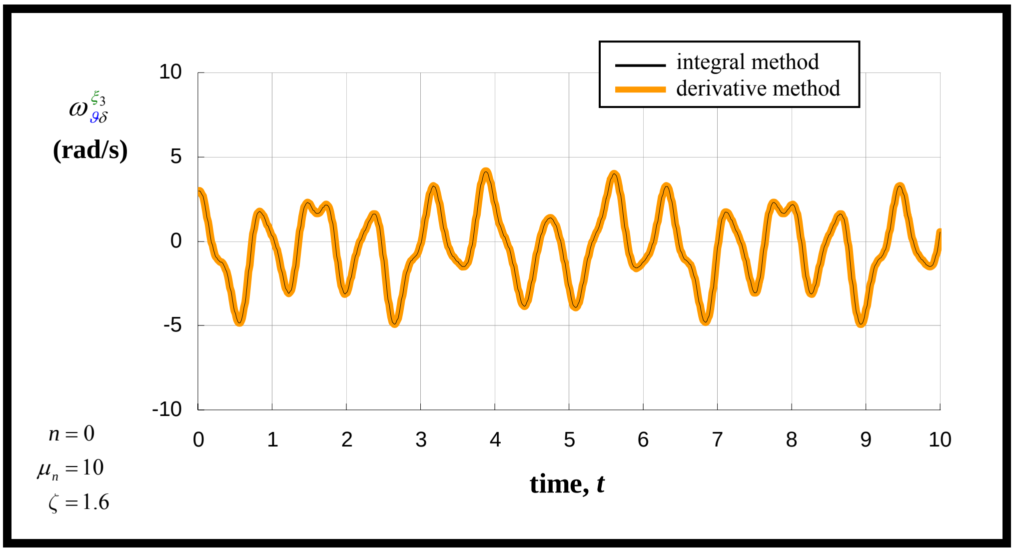


Figure 20: Validation Curves – third component, overdamped case

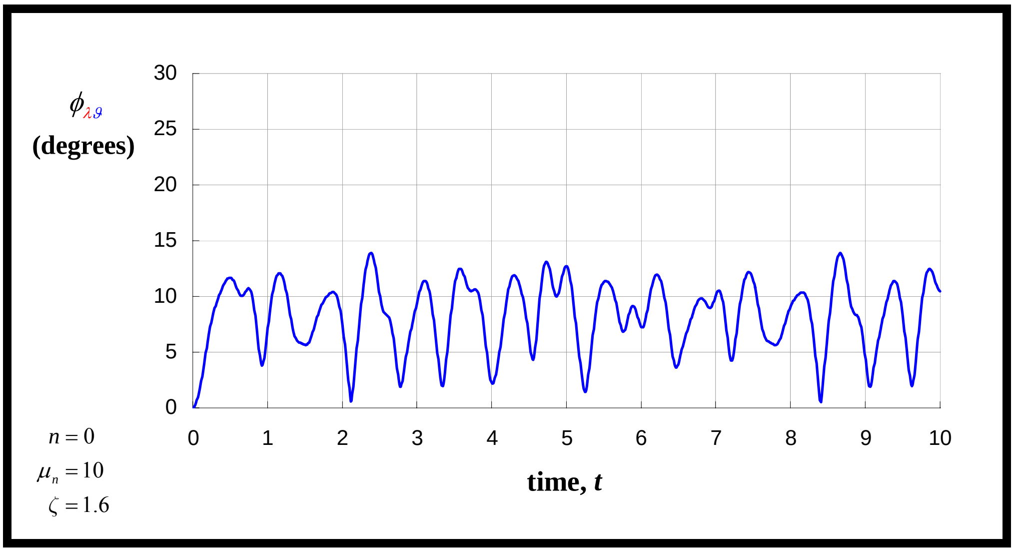


Figure 21: Validation Curves – attitude error, overdamped case

8.2.3. *Effect of Noise in the Input Data*

The integral method does not have recourse to numerical derivatives in determining the time history for $\{\omega_{\vartheta\delta}^{\xi}\}$. Differentiation of a signal always decrease the signal-to-noise ratio, and should be avoided whenever possible. Integrators, on the other hand, increase this important ratio.

This decrease of the signal-to-noise ratio observed in operations of differentiation is related to the fact that noise generally fluctuates more rapidly than the commanded signal (see, for example, Kaplan, 1976, p. 218; or Ogata, 1997, p. 813).

The derivative method, however, does have recourse to numerical derivatives in determining the time history for $\{\omega_{\vartheta\delta}^{\xi}\}$. It is expected, therefore, that any noise content in the model's input signal $\{\omega_{\lambda\delta}^{\lambda}\}$ would jeopardise the output of the derivative method. As a consequence, any noise content in the input signal should also jeopardise the validation procedure as devised.

Although the frequency of input data is high (1000/s), a considerable difference in the output signal quality can be appreciated when noise is present in the input signal. This difference would be much larger if higher derivatives (e.g. angular acceleration) were considered. Figure 22 shows a detail of figure 20 when noise is artificially added to the input signal ($n=1$, see M-file).

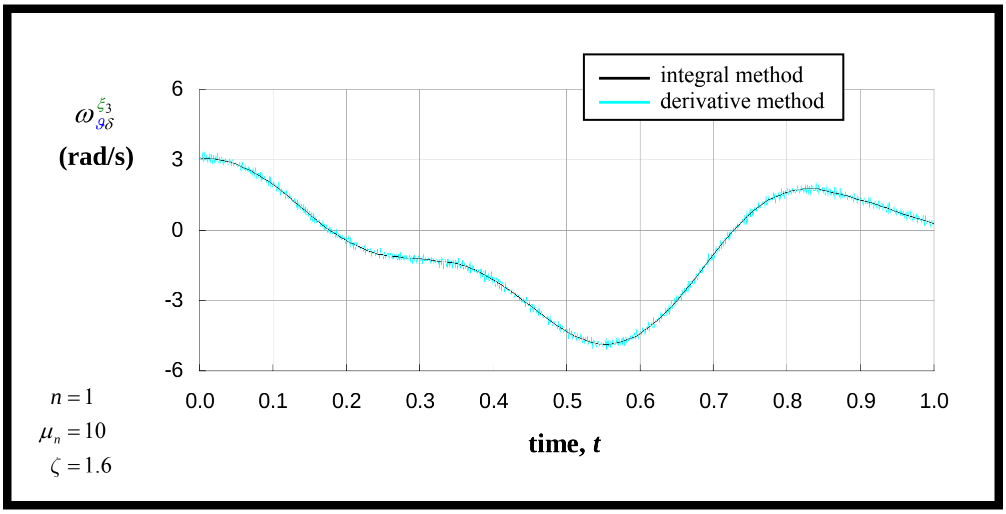


Figure 22: Noise Effect – third component, overdamped case (detail)

8.2.4. Numerical Study Conclusion

So far as the nominal system is concerned, the results achieved in subsections 8.2.1 and 8.2.2 showed an excellent agreement between the time histories for $\left\{ \omega_{\vartheta\delta}^{\xi} \right\}$ when this quantity is evaluated via the integral method and the derivative method. Even when noise is present in the input signal (subsection 8.2.3), the analysis showed a strong bound between the validation curves.

These numerical results reinforce the validity of the linear approximation between the angular velocity error vector $\vec{\omega}_{\lambda\vartheta}$ and the time rate of change of the attitude tracking error vector $\vec{\phi}_{\lambda\vartheta}$ when observed from basis ξ (equation 7.19), provided that the attitude tracking error $\phi_{\lambda\vartheta}$ is kept within moderate bounds. These results also reinforce the validity of equation 7.22 within moderate attitude errors, and the consequent possibility of using linear control theory in the study of the nominal rigid body attitude state tracking control problem.

Chapter 9

Discussion and Conclusion

This chapter closes the thesis. Initially, an overview of the attitude control problem and some of its peculiarities are given. Subsequently, the analytical development undertaken in this thesis is compared with the pertinent literature, and the corresponding contributions to the field clearly stated. Finally, a few possible themes for future research based on the results of the thesis are also proposed.

Attitude kinematics, dynamics and control have matured rapidly over the past few decades. In spite of the significant advances, there still is plenty of possibility for new development. The pertinent literature still bursts with a constant production of papers and related material. This thesis is just one more example of this continuing progress.

Undoubtedly, the dynamical behaviour of rotating bodies is a fascinating and challenging subject. The fact that finite rotations do not obey the vector parallelogram addition law poses distinct difficulties. The most notable of them is that the angular velocity of a rigid body/frame cannot in general be simply integrated to give the corresponding attitude (see section 7.2).

Vector mechanics makes possible to circumvent this difficulty in a straightforward manner. In this formulation, dynamical and kinematical differential equations are considered separately. The structure of the system of equations has a cascade form: the control input drives the angular velocity via the dynamical motion equations, and the resulting angular velocity drives the attitude parameters via the kinematical equations. Conventionally, there is no direct connection between the control input and the attitude parameters (see section 7.1, or footnote 22 on page 43 in this thesis; alternatively, the reader may refer to Tsiotras, 1996).

The analytical development undertaken in the previous chapters showed that this cascade interconnection of the system of equations is not strictly necessary. Within moderate angle rotations, the angular velocity can indeed be directly integrated to obtain the corresponding attitude (equation 7.21). To the best of the author's knowledge, this result has not been previously quoted in the literature. The ample number of books, articles and other material surveyed has not generated any parallel to this kinematical result.

The proposed control law (equation 7.30 or 7.31) has been constructed from this kinematical result and the general geometric form of the equations of motion for a rotating rigid body (equation 5.12). The anticipation of using the kinematical result 7.19 is what suggested to the author in the first place the need for a general geometric form of the equations of motion. Similarly, it also imparted the form of the linearising control law (equation 6.3).

The general geometric form of the equations of rotational motion 5.12 has neither been found in the literature. Hence, it constitutes the second major original contribution of this work. Since both the kinematical and the dynamical differential equations used do not appear to have been previously published in the open literature, it is not surprising that the corresponding linearising control law discloses what seems to be a unique form. This form is remarkably simple and results in what also seems to be a unique nominal system closed-loop dynamics.

Stability, and consequently the domain of validity of the nominal formulation, has been established only for moderate attitude tracking errors. Within this domain, the proposed control law nominally realises both linear attitude tracking and linear angular velocity tracking, i.e. it realises nominal linear attitude state tracking (section 7.5). This seems to be in marked contrast to the other available control strategies, and it is in accordance with the objectives posed in the introduction of the thesis (Chapter 1). The proposed control law constitutes, therefore, the third major contribution of this work.

Although less important when compared to the other three original results, the adopted notation (Chapter 2) also constitutes an original contribution to the field. There is a clear need for standardisation, and minimal notations simply do not suffice in many situations. The ergonomically designed explicit notation developed and adopted in this thesis comprises a step forward in this direction.

The very recent work of Xing & Parvez (2001) has perhaps the closest connection to this thesis. The controller they propose also implements attitude state tracking, but the resulting nominal closed-loop dynamics is highly non-linear and their nominal control law much more complex than the one proposed here. This non-linearity of the system dynamics tends to make it difficult to specify certain important system requirements, such as closed-loop damping and bandwidth. These quantities are not well defined when the system dynamics is non-linear.

The proposed control law realises *nominal linear non-homogeneous closed-loop attitude state error dynamics* (section 7.5). Although limited to the useful moderate attitude tracking error case, this characteristic contrasts sharply with the available literature. The published works reporting some kind of linearity in the nominal closed-loop dynamics propose control laws that realise *linear homogeneous (unforced) closed-loop attitude error dynamics* only. Among these works are Schaub et al. (2001), Bennett et al. (1994, section IV-A), Paielli & Bach (1993), Wen & Kreutz-Delgado (1991) and Dwyer (1984).

The nominal control laws these works report vary substantially in complexity, performance, parameterisation and domain of validity. Nevertheless, all of them are position control only, and are constructed via the same two-step procedure:

(1) prescribe a linear stable homogeneous (unforced) closed-loop dynamics for the attitude error (regulation of the attitude tracking error), and then

(2) compute the nominal non-linear control law that enforces the prescribed dynamics via a feedback linearisation (-like) approach.

The procedure adopted in this thesis to construct the control law is fairly similar to the one above (Chapter 6). However, the proposed control law along with the kinematical result 7.19 enables one to prescribe *nominal linear non-homogeneous closed-loop attitude state error dynamics* in place of simply nominal linear attitude tracking error regulation. The use of the kinematical result 7.19 is the stepping-stone to achieve that.

It should be remarked that the discussed advantages and validity of the proposed control law apply to the moderate attitude tracking error case only. In essence, the proposed control law trades domain of validity with analytical simplicity and linearity. The moderate attitude tracking error condition should, however, be amenable to a large number of practical/engineering applications.

Stability in the controller's non-linear region (large attitude tracking error case) has not been analysed. Such an analysis is left for future work. The issue of robustness in face of inertia and/or state uncertainty is also left for future work. This issue is important and should therefore be addressed before a practical implementation of the proposed control law is attempted. Examples of other possible themes for future research based on the results of this thesis are:

(a) similar analytical development using other orientation parameterisations;

(b) examination of other definitions for the arbitrary basis;

(c) in depth ergonomic analysis and development of the notation;

(d) extension of the theory to the multi-body case; and

(e) examination of other potential applications, such as inertial navigation systems, robotics and non-linear dynamical beam theory.

Appendix A

Notational Examples

The aim of this appendix is to exemplify the utilisation of the notation adopted in this thesis.

Component Resolution of Vectors

(a) Geometric Representation

$$\begin{aligned}\vec{v} &= v^{\lambda_1}\,\hat{\lambda}_1 + v^{\lambda_2}\,\hat{\lambda}_2 + v^{\lambda_3}\,\hat{\lambda}_3 \\ &= v^{\delta_1}\,\hat{\delta}_1 + v^{\delta_2}\,\hat{\delta}_2 + v^{\delta_3}\,\hat{\delta}_3\end{aligned}$$

(b) Algebraic Representation

$$\left\{ v^{\lambda} \right\} = \begin{Bmatrix} v^{\lambda_1} \\ v^{\lambda_2} \\ v^{\lambda_3} \end{Bmatrix} \qquad \left\{ v^{\delta} \right\} = \begin{Bmatrix} v^{\delta_1} \\ v^{\delta_2} \\ v^{\delta_3} \end{Bmatrix} \qquad \text{(column vector)}$$

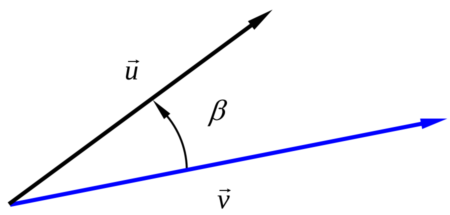
$$\left[\tilde{v}^{\lambda} \right] = \begin{bmatrix} 0 & -v^{\lambda_3} & v^{\lambda_2} \\ v^{\lambda_3} & 0 & -v^{\lambda_1} \\ -v^{\lambda_2} & v^{\lambda_1} & 0 \end{bmatrix} \qquad \left[\tilde{v}^{\delta} \right] = \begin{bmatrix} 0 & -v^{\delta_3} & v^{\delta_2} \\ v^{\delta_3} & 0 & -v^{\delta_1} \\ -v^{\delta_2} & v^{\delta_1} & 0 \end{bmatrix} \qquad \text{(skew-symmetric)}$$

Vector Magnitude/Length

$$\begin{aligned}\| \vec{u} \| = u &= \sqrt{\left(u^{\lambda_1} \right)^2 + \left(u^{\lambda_2} \right)^2 + \left(u^{\lambda_3} \right)^2} \\ &= \sqrt{\left(u^{\vartheta_1} \right)^2 + \left(u^{\vartheta_2} \right)^2 + \left(u^{\vartheta_3} \right)^2} \\ &= \sqrt{\left(u^{\delta_1} \right)^2 + \left(u^{\delta_2} \right)^2 + \left(u^{\delta_3} \right)^2}\end{aligned}$$

Vector Scalar Product

$$\vec{u} \bullet \vec{v} = \| \vec{u} \| \| \vec{v} \| \cos(\vec{u}, \vec{v}) = u\, v \cos\beta$$
$$= u^{\lambda_1} v^{\lambda_1} + u^{\lambda_2} v^{\lambda_2} + u^{\lambda_3} v^{\lambda_3}$$
$$= u^{\vartheta_1} v^{\vartheta_1} + u^{\vartheta_2} v^{\vartheta_2} + u^{\vartheta_3} v^{\vartheta_3}$$
$$= u^{\delta_1} v^{\delta_1} + u^{\delta_2} v^{\delta_2} + u^{\delta_3} v^{\delta_3}$$



Vector Cross Product

(a) Geometric Representation

$$\vec{w} = \vec{u} \times \vec{v}$$
$$= \left(u^{\lambda_2} v^{\lambda_3} - u^{\lambda_3} v^{\lambda_2} \right) \hat{\lambda}_1 + \left(u^{\lambda_3} v^{\lambda_1} - u^{\lambda_1} v^{\lambda_3} \right) \hat{\lambda}_2 + \left(u^{\lambda_1} v^{\lambda_2} - u^{\lambda_2} v^{\lambda_1} \right) \hat{\lambda}_3$$

(b) Algebraic Representation

$$\{ w^{\lambda} \} = [\tilde{u}^{\lambda}] \{ v^{\lambda} \}$$
$$= \begin{bmatrix} 0 & -u^{\lambda_3} & u^{\lambda_2} \\ u^{\lambda_3} & 0 & -u^{\lambda_1} \\ -u^{\lambda_2} & u^{\lambda_1} & 0 \end{bmatrix} \begin{Bmatrix} v^{\lambda_1} \\ v^{\lambda_2} \\ v^{\lambda_3} \end{Bmatrix} = \begin{Bmatrix} u^{\lambda_2} v^{\lambda_3} - u^{\lambda_3} v^{\lambda_2} \\ u^{\lambda_3} v^{\lambda_1} - u^{\lambda_1} v^{\lambda_3} \\ u^{\lambda_1} v^{\lambda_2} - u^{\lambda_2} v^{\lambda_1} \end{Bmatrix} \quad \text{(column vector)}$$

$$[\tilde{w}^{\lambda}] = [\tilde{u}^{\lambda}][\tilde{v}^{\lambda}] - [\tilde{v}^{\lambda}][\tilde{u}^{\lambda}]$$
$$= \begin{bmatrix} 0 & -\left(u^{\lambda_1} v^{\lambda_2} - u^{\lambda_2} v^{\lambda_1} \right) & u^{\lambda_3} v^{\lambda_1} - u^{\lambda_1} v^{\lambda_3} \\ u^{\lambda_1} v^{\lambda_2} - u^{\lambda_2} v^{\lambda_1} & 0 & -\left(u^{\lambda_2} v^{\lambda_3} - u^{\lambda_3} v^{\lambda_2} \right) \\ -\left(u^{\lambda_3} v^{\lambda_1} - u^{\lambda_1} v^{\lambda_3} \right) & u^{\lambda_2} v^{\lambda_3} - u^{\lambda_3} v^{\lambda_2} & 0 \end{bmatrix} \quad \text{(skew-symmetric)}$$

Component Resolution of Unit Vectors

(a) Geometric Representation

$$\begin{aligned}\hat{\lambda}_1 &= 1\,\hat{\lambda}_1 + 0\,\hat{\lambda}_2 + 0\,\hat{\lambda}_3 \\ &= \lambda_1^{\delta_1}\,\hat{\delta}_1 + \lambda_1^{\delta_2}\,\hat{\delta}_2 + \lambda_1^{\delta_3}\,\hat{\delta}_3\end{aligned}$$

$$\begin{aligned}\hat{\lambda}_3 &= 0\,\hat{\lambda}_1 + 0\,\hat{\lambda}_2 + 1\,\hat{\lambda}_3 \\ &= \lambda_3^{\delta_1}\,\hat{\delta}_1 + \lambda_3^{\delta_2}\,\hat{\delta}_2 + \lambda_3^{\delta_3}\,\hat{\delta}_3\end{aligned}$$

$$\begin{aligned}\hat{\lambda}_2 &= 0\,\hat{\lambda}_1 + 1\,\hat{\lambda}_2 + 0\,\hat{\lambda}_3 \\ &= \lambda_2^{\delta_1}\,\hat{\delta}_1 + \lambda_2^{\delta_2}\,\hat{\delta}_2 + \lambda_2^{\delta_3}\,\hat{\delta}_3\end{aligned}$$

$$\begin{aligned}\hat{n}_{\lambda\vartheta} &= n_{\lambda\vartheta}^{\lambda_1}\,\hat{\lambda}_1 + n_{\lambda\vartheta}^{\lambda_2}\,\hat{\lambda}_2 + n_{\lambda\vartheta}^{\lambda_3}\,\hat{\lambda}_3 \\ &= n_{\lambda\vartheta}^{\delta_1}\,\hat{\delta}_1 + n_{\lambda\vartheta}^{\delta_2}\,\hat{\delta}_2 + n_{\lambda\vartheta}^{\delta_3}\,\hat{\delta}_3\end{aligned}$$

(b) Algebraic Representation

$$\{\lambda_1^{\lambda}\} = \begin{Bmatrix}1\\0\\0\end{Bmatrix} \qquad \{\lambda_2^{\lambda}\} = \begin{Bmatrix}0\\1\\0\end{Bmatrix} \qquad \{\lambda_3^{\lambda}\} = \begin{Bmatrix}0\\0\\1\end{Bmatrix} \qquad \{n_{\lambda\vartheta}^{\lambda}\} = \begin{Bmatrix}n_{\lambda\vartheta}^{\lambda_1}\\ n_{\lambda\vartheta}^{\lambda_2}\\ n_{\lambda\vartheta}^{\lambda_3}\end{Bmatrix}$$

(column vector)

$$\{\lambda_1^{\delta}\} = \begin{Bmatrix}\lambda_1^{\delta_1}\\ \lambda_1^{\delta_2}\\ \lambda_1^{\delta_3}\end{Bmatrix} \qquad \{\lambda_2^{\delta}\} = \begin{Bmatrix}\lambda_2^{\delta_1}\\ \lambda_2^{\delta_2}\\ \lambda_2^{\delta_3}\end{Bmatrix} \qquad \{\lambda_3^{\delta}\} = \begin{Bmatrix}\lambda_3^{\delta_1}\\ \lambda_3^{\delta_2}\\ \lambda_3^{\delta_3}\end{Bmatrix} \qquad \{n_{\lambda\vartheta}^{\delta}\} = \begin{Bmatrix}n_{\lambda\vartheta}^{\delta_1}\\ n_{\lambda\vartheta}^{\delta_2}\\ n_{\lambda\vartheta}^{\delta_3}\end{Bmatrix}$$

$$[\tilde{\lambda}_1^{\lambda}] = \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix} \qquad [\tilde{\lambda}_2^{\lambda}] = \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$$

$$[\tilde{\lambda}_3^{\lambda}] = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} \qquad [\tilde{\lambda}_1^{\delta}] = \begin{bmatrix}0&-\lambda_1^{\delta_3}&\lambda_1^{\delta_2}\\ \lambda_1^{\delta_3}&0&-\lambda_1^{\delta_1}\\ -\lambda_1^{\delta_2}&\lambda_1^{\delta_1}&0\end{bmatrix}$$

(skew-symmetric)

$$[\tilde{n}_{\lambda\vartheta}^{\lambda}] = \begin{bmatrix}0&-n_{\lambda\vartheta}^{\lambda_3}&n_{\lambda\vartheta}^{\lambda_2}\\ n_{\lambda\vartheta}^{\lambda_3}&0&-n_{\lambda\vartheta}^{\lambda_1}\\ -n_{\lambda\vartheta}^{\lambda_2}&n_{\lambda\vartheta}^{\lambda_1}&0\end{bmatrix} \qquad [\tilde{n}_{\lambda\vartheta}^{\delta}] = \begin{bmatrix}0&-n_{\lambda\vartheta}^{\delta_3}&n_{\lambda\vartheta}^{\delta_2}\\ n_{\lambda\vartheta}^{\delta_3}&0&-n_{\lambda\vartheta}^{\delta_1}\\ -n_{\lambda\vartheta}^{\delta_2}&n_{\lambda\vartheta}^{\delta_1}&0\end{bmatrix}$$

Matrix Elements

$$\left[R^{\vartheta}_{\vartheta\delta} \right] = \begin{bmatrix} R^{\vartheta_{11}}_{\vartheta\delta} & R^{\vartheta_{12}}_{\vartheta\delta} & R^{\vartheta_{13}}_{\vartheta\delta} \\ R^{\vartheta_{21}}_{\vartheta\delta} & R^{\vartheta_{22}}_{\vartheta\delta} & R^{\vartheta_{23}}_{\vartheta\delta} \\ R^{\vartheta_{31}}_{\vartheta\delta} & R^{\vartheta_{32}}_{\vartheta\delta} & R^{\vartheta_{33}}_{\vartheta\delta} \end{bmatrix}$$

(Rotation Matrix)

$$\left[\tilde{v}^{\delta} \right] = \begin{bmatrix} \tilde{v}^{\delta_{11}} & \tilde{v}^{\delta_{12}} & \tilde{v}^{\delta_{13}} \\ \tilde{v}^{\delta_{21}} & \tilde{v}^{\delta_{22}} & \tilde{v}^{\delta_{23}} \\ \tilde{v}^{\delta_{31}} & \tilde{v}^{\delta_{32}} & \tilde{v}^{\delta_{33}} \end{bmatrix} = \begin{bmatrix} 0 & -v^{\delta_3} & v^{\delta_2} \\ v^{\delta_3} & 0 & -v^{\delta_1} \\ -v^{\delta_2} & v^{\delta_1} & 0 \end{bmatrix}$$

(Skew-Symmetric Matrix)

$$\left[T^{\vartheta}_{\lambda} \right] = \begin{bmatrix} T^{\vartheta_{11}}_{\lambda} & T^{\vartheta_{12}}_{\lambda} & T^{\vartheta_{13}}_{\lambda} \\ T^{\vartheta_{21}}_{\lambda} & T^{\vartheta_{22}}_{\lambda} & T^{\vartheta_{23}}_{\lambda} \\ T^{\vartheta_{31}}_{\lambda} & T^{\vartheta_{32}}_{\lambda} & T^{\vartheta_{33}}_{\lambda} \end{bmatrix} = \begin{bmatrix} \lambda^{\vartheta_1}_1 & \lambda^{\vartheta_1}_2 & \lambda^{\vartheta_1}_3 \\ \lambda^{\vartheta_2}_1 & \lambda^{\vartheta_2}_2 & \lambda^{\vartheta_2}_3 \\ \lambda^{\vartheta_3}_1 & \lambda^{\vartheta_3}_2 & \lambda^{\vartheta_3}_3 \end{bmatrix}$$

(Transformation Matrix)

Basis Transformation

$$\left\{ v^{\delta} \right\} = \left[T^{\delta}_{\lambda} \right] \left\{ v^{\lambda} \right\}$$

$\left[T^{\delta}_{\lambda} \right]$ transformation matrix *from basis λ to basis δ*

$$\left\{ v^{\lambda} \right\} = \left[T^{\lambda}_{\delta} \right] \left\{ v^{\delta} \right\}$$

$\left[T^{\lambda}_{\delta} \right]$ transformation matrix *from basis δ to basis λ*

$$\left[I^{\delta}_{B\vartheta} \right] = \left[T^{\delta}_{\vartheta} \right] \left[I^{\vartheta}_{B\vartheta} \right] \left[T^{\delta}_{\vartheta} \right]^T$$

orthogonal similarity transformation

$$\left[\tilde{\omega}^{\vartheta}_{\lambda\vartheta} \right] = \left[T^{\vartheta}_{\lambda} \right] \left[\tilde{\omega}^{\lambda}_{\lambda\vartheta} \right] \left[T^{\vartheta}_{\lambda} \right]^T$$

invariance of the matrix antisymmetry property under an orthogonal similarity transformation

Vector and Array Time Derivatives

$$\overset{\delta}{\vec{r}} = \overset{\lambda}{\vec{r}} + \vec{\omega}_{\lambda\delta} \times \vec{r} \qquad \Leftrightarrow \qquad \frac{d}{dt_{\delta}}\vec{r} = \frac{d}{dt_{\lambda}}\vec{r} + \vec{\omega}_{\lambda\delta} \times \vec{r}$$

$$\begin{aligned} \left\{ \dot{r}^{\delta} \right\} &= \frac{d}{dt}\left(\left[T^{\delta}_{\lambda} \right]\left\{ r^{\lambda} \right\} \right) \\ &= \left[\dot{T}^{\delta}_{\lambda} \right]\left\{ r^{\lambda} \right\} + \left[T^{\delta}_{\lambda} \right]\left\{ \dot{r}^{\lambda} \right\} \end{aligned}$$

$$\begin{aligned} \left[\dot{I}^{\delta}_{B\vartheta} \right] &= \frac{d}{dt}\left(\left[T^{\delta}_{\vartheta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left[T^{\delta}_{\vartheta} \right]^{T} \right) \\ &= \left[\dot{T}^{\delta}_{\vartheta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left[T^{\delta}_{\vartheta} \right]^{T} + \left[T^{\delta}_{\vartheta} \right]\left[\dot{I}^{\vartheta}_{B\vartheta} \right]\left[T^{\delta}_{\vartheta} \right]^{T} + \left[T^{\delta}_{\vartheta} \right]\left[I^{\vartheta}_{B\vartheta} \right]\left[\dot{T}^{\delta}_{\vartheta} \right]^{T} \end{aligned}$$

Array Time Integrals

$$\int \left\{ r^{\delta} \right\} dt = \int \begin{Bmatrix} r^{\delta_1} \\ r^{\delta_2} \\ r^{\delta_3} \end{Bmatrix} dt = \begin{Bmatrix} \int r^{\delta_1}\, dt \\ \int r^{\delta_2}\, dt \\ \int r^{\delta_3}\, dt \end{Bmatrix}$$

$$\int \left[\tilde{\omega}^{\xi}_{\lambda\vartheta} \right] dt = \int \begin{bmatrix} 0 & -\omega^{\xi_3}_{\lambda\vartheta} & \omega^{\xi_2}_{\lambda\vartheta} \\ \omega^{\xi_3}_{\lambda\vartheta} & 0 & -\omega^{\xi_1}_{\lambda\vartheta} \\ -\omega^{\xi_2}_{\lambda\vartheta} & \omega^{\xi_1}_{\lambda\vartheta} & 0 \end{bmatrix} dt = \begin{bmatrix} 0 & -\int \omega^{\xi_3}_{\lambda\vartheta}\, dt & \int \omega^{\xi_2}_{\lambda\vartheta}\, dt \\ \int \omega^{\xi_3}_{\lambda\vartheta}\, dt & 0 & -\int \omega^{\xi_1}_{\lambda\vartheta}\, dt \\ -\int \omega^{\xi_2}_{\lambda\vartheta}\, dt & \int \omega^{\xi_1}_{\lambda\vartheta}\, dt & 0 \end{bmatrix}$$

Appendix B

The Transformation Matrix

The aim of this appendix is to derive some forms of representation of the transformation matrix and demonstrate its orthogonality.

Consider the component resolution of a geometric vector $\vec{v}$ onto two dextral orthonormal bases λ and ϑ

$$\vec{v} = v^{\lambda_1}\hat{\lambda}_1 + v^{\lambda_2}\hat{\lambda}_2 + v^{\lambda_3}\hat{\lambda}_3 \tag{B.1}$$

$$= v^{\vartheta_1}\hat{\vartheta}_1 + v^{\vartheta_2}\hat{\vartheta}_2 + v^{\vartheta_3}\hat{\vartheta}_3 \tag{B.2}$$

The unit vectors of basis λ, namely $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$, can also be resolved in a number of ways, for instance onto basis ϑ

$$\hat{\lambda}_1 = \lambda_1^{\vartheta_1}\hat{\vartheta}_1 + \lambda_1^{\vartheta_2}\hat{\vartheta}_2 + \lambda_1^{\vartheta_3}\hat{\vartheta}_3$$

$$\hat{\lambda}_2 = \lambda_2^{\vartheta_1}\hat{\vartheta}_1 + \lambda_2^{\vartheta_2}\hat{\vartheta}_2 + \lambda_2^{\vartheta_3}\hat{\vartheta}_3$$

$$\hat{\lambda}_3 = \lambda_3^{\vartheta_1}\hat{\vartheta}_1 + \lambda_3^{\vartheta_2}\hat{\vartheta}_2 + \lambda_3^{\vartheta_3}\hat{\vartheta}_3$$

Substitution of these three relationships into expression B.1 (vector resolution onto basis λ), followed by a little rearrangement yields

$$\vec{v} = \left(\lambda_1^{\vartheta_1}v^{\lambda_1} + \lambda_2^{\vartheta_1}v^{\lambda_2} + \lambda_3^{\vartheta_1}v^{\lambda_3}\right)\hat{\vartheta}_1 + \left(\lambda_1^{\vartheta_2}v^{\lambda_1} + \lambda_2^{\vartheta_2}v^{\lambda_2} + \lambda_3^{\vartheta_2}v^{\lambda_3}\right)\hat{\vartheta}_2 + \left(\lambda_1^{\vartheta_3}v^{\lambda_1} + \lambda_2^{\vartheta_3}v^{\lambda_2} + \lambda_3^{\vartheta_3}v^{\lambda_3}\right)\hat{\vartheta}_3$$

Comparing the above result with expression B.2 (vector resolution onto basis ϑ), one readily concludes that

$$v^{\vartheta_1} = \lambda_1^{\vartheta_1} v^{\lambda_1} + \lambda_2^{\vartheta_1} v^{\lambda_2} + \lambda_3^{\vartheta_1} v^{\lambda_3}$$

$$v^{\vartheta_2} = \lambda_1^{\vartheta_2} v^{\lambda_1} + \lambda_2^{\vartheta_2} v^{\lambda_2} + \lambda_3^{\vartheta_2} v^{\lambda_3}$$

$$v^{\vartheta_3} = \lambda_1^{\vartheta_3} v^{\lambda_1} + \lambda_2^{\vartheta_3} v^{\lambda_2} + \lambda_3^{\vartheta_3} v^{\lambda_3}$$

The last three equations can be conveniently represented in algebraic notation as

$$\begin{Bmatrix} v^{\vartheta_1} \\ v^{\vartheta_2} \\ v^{\vartheta_3} \end{Bmatrix} = \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix} \begin{Bmatrix} v^{\lambda_1} \\ v^{\lambda_2} \\ v^{\lambda_3} \end{Bmatrix} \quad \Leftrightarrow \quad \{ v^{\vartheta} \} = \left[\{ \lambda_1^{\vartheta} \} \ \{ \lambda_2^{\vartheta} \} \ \{ \lambda_3^{\vartheta} \} \right] \{ v^{\lambda} \}$$

The quantity that appears in brackets, a matrix, is clearly converting (transforming) the component resolution of vector $\vec{v}$ *from basis* λ *to basis* ϑ, and is most commonly denominated transformation matrix.

Observing yet that the elements of this matrix are nothing more than the components of the unit vectors of basis λ when resolved onto basis ϑ (arranged column-wise), it is expedient to symbolise it as $\left[T_{\lambda}^{\vartheta} \right]$ within the notational conventions adopted in this thesis. Therefore

$$\{ v^{\vartheta} \} = \left[T_{\lambda}^{\vartheta} \right] \{ v^{\lambda} \} \qquad \text{where} \qquad \left[T_{\lambda}^{\vartheta} \right] = \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix}$$

This important matrix can, in fact, be represented in several ways. Three of these are directly obtainable from the definition of scalar product. Recalling that (a) unit vectors have length one, and (b) the orthogonal projection of a vector along another vector equals the vector length times the cosine of the angle between them, one has

$$
\begin{aligned}
\hat{\lambda}_1 \bullet \hat{\vartheta}_1 &= \left\| \hat{\lambda}_1 \right\| \left\| \hat{\vartheta}_1 \right\| \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) \\
&= 1 * 1 * \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) = \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) \\
&= \left(\left\| \hat{\lambda}_1 \right\| \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) \right) \left\| \hat{\vartheta}_1 \right\| = \left(\lambda_1^{\vartheta_1} \right) * 1 = \lambda_1^{\vartheta_1} \\
&= \left(\left\| \hat{\vartheta}_1 \right\| \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) \right) \left\| \hat{\lambda}_1 \right\| = \left(\vartheta_1^{\lambda_1} \right) * 1 = \vartheta_1^{\lambda_1}
\end{aligned}
$$

Employing the same procedure to the other pair combinations of unit vectors, one finds the following set of equalities

$$
\begin{aligned}
\hat{\lambda}_1 \bullet \hat{\vartheta}_1 &= \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_1 \right) = \lambda_1^{\vartheta_1} = \vartheta_1^{\lambda_1} \\
\hat{\lambda}_1 \bullet \hat{\vartheta}_2 &= \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_2 \right) = \lambda_1^{\vartheta_2} = \vartheta_2^{\lambda_1} \\
\hat{\lambda}_1 \bullet \hat{\vartheta}_3 &= \cos\left(\hat{\lambda}_1 , \hat{\vartheta}_3 \right) = \lambda_1^{\vartheta_3} = \vartheta_3^{\lambda_1} \\
\hat{\lambda}_2 \bullet \hat{\vartheta}_1 &= \cos\left(\hat{\lambda}_2 , \hat{\vartheta}_1 \right) = \lambda_2^{\vartheta_1} = \vartheta_1^{\lambda_2} \\
\hat{\lambda}_2 \bullet \hat{\vartheta}_2 &= \cos\left(\hat{\lambda}_2 , \hat{\vartheta}_2 \right) = \lambda_2^{\vartheta_2} = \vartheta_2^{\lambda_2} \\
\hat{\lambda}_2 \bullet \hat{\vartheta}_3 &= \cos\left(\hat{\lambda}_2 , \hat{\vartheta}_3 \right) = \lambda_2^{\vartheta_3} = \vartheta_3^{\lambda_2} \\
\hat{\lambda}_3 \bullet \hat{\vartheta}_1 &= \cos\left(\hat{\lambda}_3 , \hat{\vartheta}_1 \right) = \lambda_3^{\vartheta_1} = \vartheta_1^{\lambda_3} \\
\hat{\lambda}_3 \bullet \hat{\vartheta}_2 &= \cos\left(\hat{\lambda}_3 , \hat{\vartheta}_2 \right) = \lambda_3^{\vartheta_2} = \vartheta_2^{\lambda_3} \\
\hat{\lambda}_3 \bullet \hat{\vartheta}_3 &= \cos\left(\hat{\lambda}_3 , \hat{\vartheta}_3 \right) = \lambda_3^{\vartheta_3} = \vartheta_3^{\lambda_3}
\end{aligned}
$$

which can now be used to construct the aforementioned representations of the transformation matrix:

$$\left[T_{\lambda}^{\vartheta} \right] = \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix}$$

representation in terms of the components of the unit vectors of the *from basis* λ

$$= \begin{bmatrix} \vartheta_1^{\lambda_1} & \vartheta_1^{\lambda_2} & \vartheta_1^{\lambda_3} \\ \vartheta_2^{\lambda_1} & \vartheta_2^{\lambda_2} & \vartheta_2^{\lambda_3} \\ \vartheta_3^{\lambda_1} & \vartheta_3^{\lambda_2} & \vartheta_3^{\lambda_3} \end{bmatrix}$$

representation in terms of the components of the unit vectors of the *to basis* ϑ

$$= \begin{bmatrix} \hat{\lambda}_1 \bullet \hat{\vartheta}_1 & \hat{\lambda}_2 \bullet \hat{\vartheta}_1 & \hat{\lambda}_3 \bullet \hat{\vartheta}_1 \\ \hat{\lambda}_1 \bullet \hat{\vartheta}_2 & \hat{\lambda}_2 \bullet \hat{\vartheta}_2 & \hat{\lambda}_3 \bullet \hat{\vartheta}_2 \\ \hat{\lambda}_1 \bullet \hat{\vartheta}_3 & \hat{\lambda}_2 \bullet \hat{\vartheta}_3 & \hat{\lambda}_3 \bullet \hat{\vartheta}_3 \end{bmatrix}$$

representation in terms of the scalar products of the unit vectors

$$= \begin{bmatrix} \cos\left(\hat{\lambda}_1, \hat{\vartheta}_1\right) & \cos\left(\hat{\lambda}_2, \hat{\vartheta}_1\right) & \cos\left(\hat{\lambda}_3, \hat{\vartheta}_1\right) \\ \cos\left(\hat{\lambda}_1, \hat{\vartheta}_2\right) & \cos\left(\hat{\lambda}_2, \hat{\vartheta}_2\right) & \cos\left(\hat{\lambda}_3, \hat{\vartheta}_2\right) \\ \cos\left(\hat{\lambda}_1, \hat{\vartheta}_3\right) & \cos\left(\hat{\lambda}_2, \hat{\vartheta}_3\right) & \cos\left(\hat{\lambda}_3, \hat{\vartheta}_3\right) \end{bmatrix}$$

representation in terms of the cosines of the angles between the unit vectors *(direction cosines)*

The last representation of the transformation matrix evidences why it is also referred to as the direction cosine matrix.

Possibly, the most significant property of the transformation matrix is the orthogonality, i.e. the inverse of this matrix equals its transpose:

$$\left[T_{\lambda}^{\vartheta} \right]^{-1} = \left[T_{\lambda}^{\vartheta} \right]^{T} \tag{B.3}$$

There are many ways of demonstrating this property. A simple and intuitively plausible one is by first considering the scalar product of the unit vectors of the *from basis* λ with themselves (scalar product of two columns of the transformation matrix), and the corresponding component resolutions onto the *to basis* ϑ.

These unit vectors are, by hypothesis, mutually perpendicular (dextral orthonormal basis), which leads to the following relationships

$$\hat{\lambda}_1 \bullet \hat{\lambda}_1 = 1 \quad \Rightarrow \quad \left(\lambda_1^{\vartheta_1}\right)^2 + \left(\lambda_1^{\vartheta_2}\right)^2 + \left(\lambda_1^{\vartheta_3}\right)^2 = 1$$

$$\hat{\lambda}_2 \bullet \hat{\lambda}_2 = 1 \quad \Rightarrow \quad \left(\lambda_2^{\vartheta_1}\right)^2 + \left(\lambda_2^{\vartheta_2}\right)^2 + \left(\lambda_2^{\vartheta_3}\right)^2 = 1$$

$$\hat{\lambda}_3 \bullet \hat{\lambda}_3 = 1 \quad \Rightarrow \quad \left(\lambda_3^{\vartheta_1}\right)^2 + \left(\lambda_3^{\vartheta_2}\right)^2 + \left(\lambda_3^{\vartheta_3}\right)^2 = 1$$

$$\hat{\lambda}_1 \bullet \hat{\lambda}_2 = 0 \quad \Rightarrow \quad \lambda_1^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_2^{\vartheta_3} = 0$$

$$\hat{\lambda}_1 \bullet \hat{\lambda}_3 = 0 \quad \Rightarrow \quad \lambda_1^{\vartheta_1}\lambda_3^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_3^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_3^{\vartheta_3} = 0$$

$$\hat{\lambda}_3 \bullet \hat{\lambda}_2 = 0 \quad \Rightarrow \quad \lambda_3^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_3^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_3^{\vartheta_3}\lambda_2^{\vartheta_3} = 0$$

These six relationships can now be employed to demonstrate that the transformation matrix is indeed orthogonal. Recalling that by definition the inverse of a matrix is the one that multiplied by the matrix equals the identity matrix, $\left[\, 1 \,\right] = \left[T_{\lambda}^{\vartheta} \right]^{-1} \left[T_{\lambda}^{\vartheta} \right]$, one may simply replace the inverse by the transpose in this relationship, and carry out the appropriate multiplications and substitutions, that is

$$\left[\, 1 \,\right] = \left[T_{\lambda}^{\vartheta} \right]^{-1} \left[T_{\lambda}^{\vartheta} \right] = \left[T_{\lambda}^{\vartheta} \right]^{T} \left[T_{\lambda}^{\vartheta} \right]$$

$$= \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix}^{T} \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_1^{\vartheta_2} & \lambda_1^{\vartheta_3} \\ \lambda_2^{\vartheta_1} & \lambda_2^{\vartheta_2} & \lambda_2^{\vartheta_3} \\ \lambda_3^{\vartheta_1} & \lambda_3^{\vartheta_2} & \lambda_3^{\vartheta_3} \end{bmatrix} \begin{bmatrix} \lambda_1^{\vartheta_1} & \lambda_2^{\vartheta_1} & \lambda_3^{\vartheta_1} \\ \lambda_1^{\vartheta_2} & \lambda_2^{\vartheta_2} & \lambda_3^{\vartheta_2} \\ \lambda_1^{\vartheta_3} & \lambda_2^{\vartheta_3} & \lambda_3^{\vartheta_3} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\lambda_1^{\vartheta_1}\right)^2 + \left(\lambda_1^{\vartheta_2}\right)^2 + \left(\lambda_1^{\vartheta_3}\right)^2 & \lambda_1^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_2^{\vartheta_3} & \lambda_1^{\vartheta_1}\lambda_3^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_3^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_3^{\vartheta_3} \\ \lambda_1^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_2^{\vartheta_3} & \left(\lambda_2^{\vartheta_1}\right)^2 + \left(\lambda_2^{\vartheta_2}\right)^2 + \left(\lambda_2^{\vartheta_3}\right)^2 & \lambda_3^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_3^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_3^{\vartheta_3}\lambda_2^{\vartheta_3} \\ \lambda_1^{\vartheta_1}\lambda_3^{\vartheta_1} + \lambda_1^{\vartheta_2}\lambda_3^{\vartheta_2} + \lambda_1^{\vartheta_3}\lambda_3^{\vartheta_3} & \lambda_3^{\vartheta_1}\lambda_2^{\vartheta_1} + \lambda_3^{\vartheta_2}\lambda_2^{\vartheta_2} + \lambda_3^{\vartheta_3}\lambda_2^{\vartheta_3} & \left(\lambda_3^{\vartheta_1}\right)^2 + \left(\lambda_3^{\vartheta_2}\right)^2 + \left(\lambda_3^{\vartheta_3}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.E.D

The definition of the inverse of the transformation matrix used above was the *left* one. The *right* inverse is equally valid. In fact, both are the same, and the existence of one implies the other (see Apostol, 1997, p. 154). The relationship making use of the right inverse, i.e. $\left[\ 1\ \right] = \left[\ T_{\lambda}^{\vartheta}\right]\left[\ T_{\lambda}^{\vartheta}\right]^{-1}$, can be demonstrated in a similar fashion by considering the scalar product of the unit vectors of the *to basis* ϑ (instead of the *from basis* λ), and the *second* presented representation of the transformation matrix, i.e. the one in terms of the components of the unit vectors of the *to basis* ϑ.

There certainly are other ways of presenting orthogonal transformation matrices and their properties. The one utilised here meets the objectives of the appendix within the formalism and level of generality common to this thesis. The interested reader may refer to Arfken & Weber (1995, p. 181-94), or Kreyszig (1999, p. 382-84) for other approaches. An alternative, but equivalent set of orthogonality conditions has been given by Gelman (1968). Junkins & Turner (1986, p. 10) offers an elegant demonstration of the orthogonality property of transformation matrices.

Appendix C

The Rotation Matrix

In this appendix the relationship of equivalence between the rotation matrix and the transformation matrix is established. Subsequently, the finite rotation formula and the angle/axis representation of the transformation matrix are derived.

The orientation of an orthonormal basis with respect to another one may always be described by a transformation matrix. In many circumstances, however, the direction cosines (elements of the transformation matrix) are not the most suitable means, and alternative possibilities should be considered. One such possibility, suggested by Euler's theorem, is the specification of the orientation with the axis about which the equivalent rotation takes place (Euler axis of rotation), and the respective angle of rotation (Euler angle of rotation). The quantities in this system are, of course, related to the direction cosines, and the relationships between the two systems are now to be examined.

Active and Passive Points of View

A change in orientation of a vector quantity when operated by $\left[T_{\vartheta}^{\delta} \right]$ may be interpreted from two points of view: the passive and the active. In the passive point of view, $\left[T_{\vartheta}^{\delta} \right]$ may be thought as relating the components of a single undisturbed vector in two coordinate systems, i.e. $\left[T_{\vartheta}^{\delta} \right]$ is functioning as a frame transformation matrix. Conversely, in the active point of view, $\left[T_{\vartheta}^{\delta} \right]$ may be thought as relating two vectors of same length expressed in only one coordinate system, i.e. $\left[T_{\vartheta}^{\delta} \right]$ is functioning as a rotational operator.

The algebra is the same when either of the two points of view is followed. Figures 23 and 24, and accompanying derivations depict these ideas when the rotation is about the δ_3 axis. Further discussion on the subject can be found in Shuster (1993a, p. 494-95), Nikravesh (1988, p. 157-58), Arfken & Weber (1995, p. 190-92), Konopinsky (1969, p. 265-66) and Bottema & Roth (1979, p. 13).

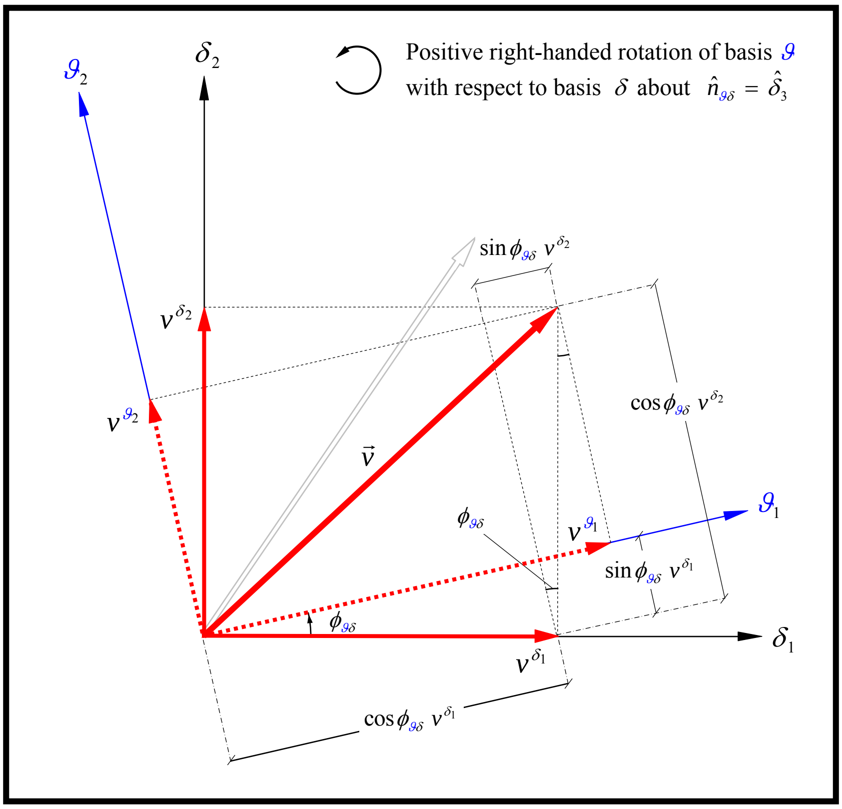


Figure 23: Passive point of view

$$v^{\vartheta_1} = \cos\phi_{\vartheta\delta}\ v^{\delta_1} + \sin\phi_{\vartheta\delta}\ v^{\delta_2}$$
$$v^{\vartheta_2} = -\sin\phi_{\vartheta\delta}\ v^{\delta_1} + \cos\phi_{\vartheta\delta}\ v^{\delta_2}$$
$$v^{\vartheta_3} = v^{\delta_3}$$

$$\begin{Bmatrix} v^{\vartheta_1} \\ v^{\vartheta_2} \\ v^{\vartheta_3} \end{Bmatrix} = \begin{bmatrix} \cos\phi_{\vartheta\delta} & \sin\phi_{\vartheta\delta} & 0 \\ -\sin\phi_{\vartheta\delta} & \cos\phi_{\vartheta\delta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v^{\delta_1} \\ v^{\delta_2} \\ v^{\delta_3} \end{Bmatrix}$$

$$\left\{ v^{\vartheta} \right\} = \left[T^{\vartheta}_{\delta} \right] \left\{ v^{\delta} \right\}$$

$$v^{\delta_1} = \cos\phi_{\vartheta\delta}\ v^{\vartheta_1} - \sin\phi_{\vartheta\delta}\ v^{\vartheta_2}$$
$$v^{\delta_2} = \sin\phi_{\vartheta\delta}\ v^{\vartheta_1} + \cos\phi_{\vartheta\delta}\ v^{\vartheta_2}$$
$$v^{\delta_3} = v^{\vartheta_3}$$

$$\begin{Bmatrix} v^{\delta_1} \\ v^{\delta_2} \\ v^{\delta_3} \end{Bmatrix} = \begin{bmatrix} \cos\phi_{\vartheta\delta} & -\sin\phi_{\vartheta\delta} & 0 \\ \sin\phi_{\vartheta\delta} & \cos\phi_{\vartheta\delta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v^{\vartheta_1} \\ v^{\vartheta_2} \\ v^{\vartheta_3} \end{Bmatrix}$$

$$\left\{ v^{\delta} \right\} = \left[T^{\delta}_{\vartheta} \right] \left\{ v^{\vartheta} \right\}$$

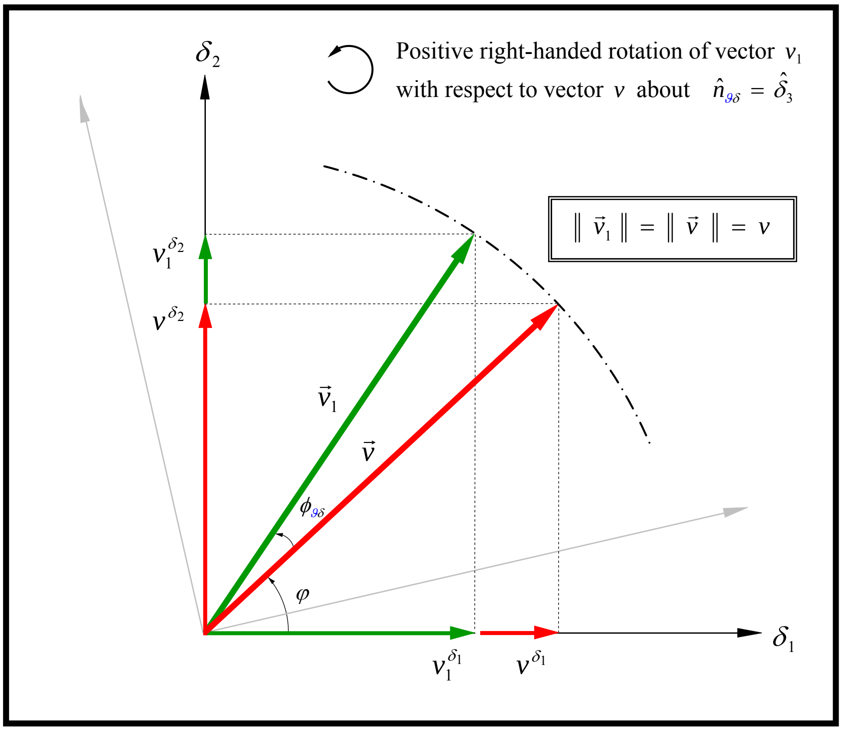


Figure 24: Active point of view

$$v_1^{\delta_1} = v\cos(\phi_{\vartheta\delta}+\varphi) = \cos\phi_{\vartheta\delta}\,(v\cos\varphi) - \sin\phi_{\vartheta\delta}\,(v\sin\varphi) = \cos\phi_{\vartheta\delta}\,v^{\delta_1} - \sin\phi_{\vartheta\delta}\,v^{\delta_2}$$

$$v_1^{\delta_2} = v\sin(\phi_{\vartheta\delta}+\varphi) = \sin\phi_{\vartheta\delta}\,(v\cos\varphi) + \cos\phi_{\vartheta\delta}\,(v\sin\varphi) = \sin\phi_{\vartheta\delta}\,v^{\delta_1} + \cos\phi_{\vartheta\delta}\,v^{\delta_2}$$

$$v_1^{\delta_3} = v_1^{\delta_3}$$

$$\begin{Bmatrix} v_1^{\delta_1} \\ v_1^{\delta_2} \\ v_1^{\delta_3} \end{Bmatrix} = \begin{bmatrix} \cos\phi_{\vartheta\delta} & -\sin\phi_{\vartheta\delta} & 0 \\ \sin\phi_{\vartheta\delta} & \cos\phi_{\vartheta\delta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v^{\delta_1} \\ v^{\delta_2} \\ v^{\delta_3} \end{Bmatrix} \quad \Leftrightarrow \quad \{ v_1^{\delta} \} = [R_{\vartheta\delta}^{\delta}] \{ v^{\delta} \}$$

Hence, by comparing the results for the passive and active points of view one concludes that

$$[T_{\vartheta}^{\delta}] = [R_{\vartheta\delta}^{\delta}] = [R^{\delta} (\phi_{\vartheta\delta}, \hat{n}_{\vartheta\delta})] \tag{C.1}$$

The foregoing has shown that the linear relations that give the components of a fixed vector on a rotating frame may be reinterpreted as giving the components of a rotating vector on a fixed frame. These linear relations are in precisely the same form when the matrix operating the change in the frame orientation is defined as the transformation from the rotating frame ϑ (*from basis*) to the original/primary frame δ (*to basis*). Corresponding results hold for the three-dimensional case (see figure 25).

The dual interpretation for the transformation matrix is very fortunate within the context of this discussion. It is now possible to obtain an equivalent expression for the transformation matrix by direct employment of the active point of view. This will give rise to the desired relationship between the direction cosine and the Euler angle/axis representations.

The Finite Rotation Formula

The derivation of an equation for the finite displacement of a point of a rigid body turning about a fixed line is not only a challenging exercise, but it also has a broad practical applicability. As a consequence, its construction has attracted the interest of many writers and one can find in the pertinent literature several versions and proofs of the rotation formula.

Without troubling to list any of the early works by Euler, Chasles and others responsible for the classical origins of the theory of rigid body motion (these writers used scalar methods), the following list gives an idea of the plethora of available literature related to the derivation of the finite rotation formula:

Authors who used mainly vector procedures[41]**:**

Shabana (1998, p. 29-31)	Shabana (1994, p. 438-40)
Williams (1996, p. 688-90)	Goldstein (1980, p. 164-65)
Angeles (1997, p. 30-33)	Konopinsky (1969, p. 234-36)
Mathews (1976)	Lewis & Ward (1989, p. 304-07)
Kozik (1976)	Bottema & Roth (1979, p. 56-60)
Grubin (1970)	Rosenberg (1977, p. 63-64, 82-84)
Grubin (1962)	Nikravesh (1988, p. 158-59)
Beatty (1963)	Junkins & Turner (1986, p. 13-15,26-28)
Argyris (1982, p. 87-88)	Rheinfurth & Wilson (1991, p. 86-87)
Torkamani (1998)	Battin (1987, p. 86-89)
Amirouche (1992, p. 22-26)	Shuster (1993a, p. 450-51)

Authors who used mainly matrix procedures:

Hughes (1986, p. 10-13)	Craig (1989, p. 51-53)
Smith (1982, p. 436-38)	Paul (1986, p. 25-31)
Wie (1998, p. 312–15)	

Classical references:

Whittaker (1927, p. 8)	provides a scalar version for the problem
Pars (1965, p. 95-97)	provides three separate proofs illustrating various lines of approach

Reviews:

Beatty (1977) - until the end of the 1970s

Cheng & Gupta (1989)

Argyris & Poterasu (1993, p. 22-25)

Rooney (1977)

[41] The development of the rotation formula proposed by Grubin (1962) and Battin (1987, p. 86-88) differs quite substantially from the ones proposed by the other authors, since it is based on the solution of a differential vector equation.

For ease of understanding and completeness, a commonly used formal proof that establishes the rotation formula by a straightforward geometrical argument is briefly recapitulated below:

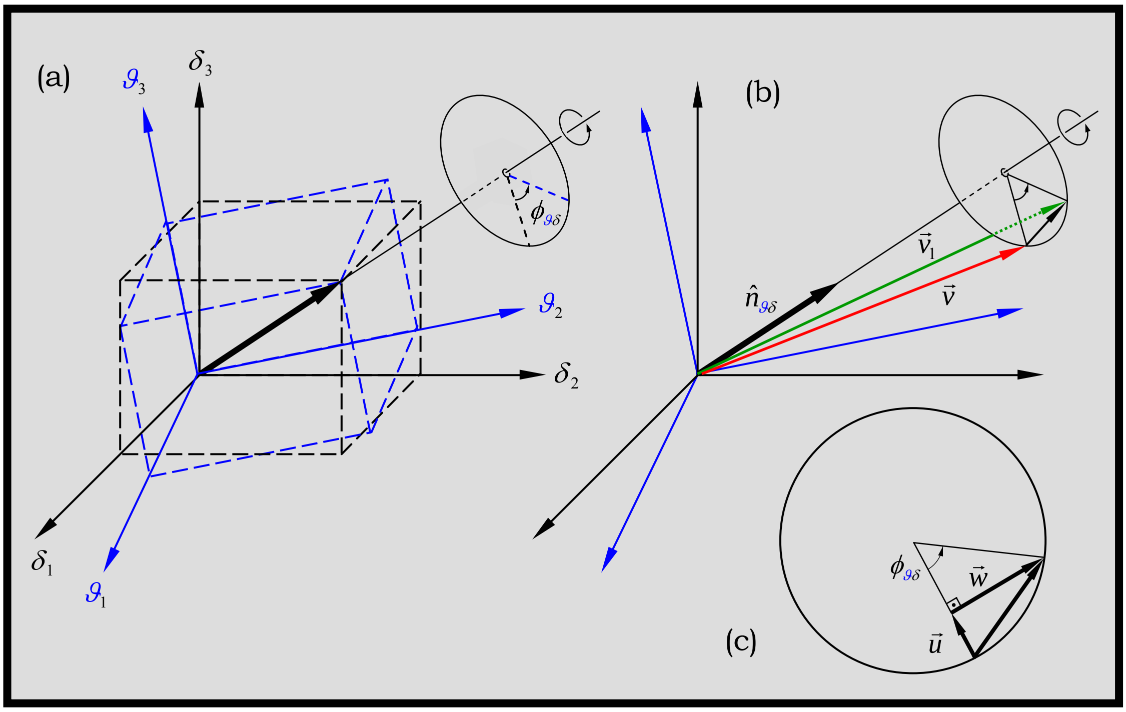


Figure 25: Diagrams for the Rotation Formula

(a) Coordinate system rotation (positive right-handed – anticlockwise)
(b) Vector diagram for derivation of the finite rotation formula
(c) Plane normal to the axis of rotation – geometric relations

Referring to the above figure, the following proceeds

$$\vec{v}_1 = \vec{v} + \vec{w} + \vec{u}$$
$$= \vec{v} + \sin\phi_{\vartheta\delta}\ \hat{n}_{\vartheta\delta}\times\vec{v} + \left(1-\cos\phi_{\vartheta\delta}\right)\hat{n}_{\vartheta\delta}\times\left(\hat{n}_{\vartheta\delta}\times\vec{v}\right)$$

transformed/ rotated vector

Euler angle of rotation of basis ϑ with respect to basis δ

Euler axis of rotation of basis ϑ with respect to basis δ

original vector

In Cartesian three-dimensional space, this geometrical relationship may be rewritten as (compare to Shuster, 1993a, p. 503-05; Ibrahimbegovic, 1997; Ibrahimbegovic et al., 1995; and Pfister, 1996)

$$\vec{v}_1 = \vec{\vec{1}} \bullet \vec{v} + \sin\phi_{\vartheta\delta} \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \bullet \vec{v} + (1-\cos\phi_{\vartheta\delta}) \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \bullet \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \bullet \vec{v}$$

$$= \left(\vec{\vec{1}} + \sin\phi_{\vartheta\delta} \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) + (1-\cos\phi_{\vartheta\delta}) \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \bullet \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \right) \bullet \vec{v}$$

The quantity enclosed in the outer parenthesis transforms the vector $\vec{v}$ into the vector $\vec{v}_1$. This tensor must represent, consequently, a finite rotation of arbitrary magnitude $\phi_{\vartheta\delta}$ and direction $\hat{n}_{\vartheta\delta}$:

$$\vec{\vec{R}}_{\vartheta\delta} = \vec{\vec{1}} + \sin\phi_{\vartheta\delta} \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) + (1-\cos\phi_{\vartheta\delta}) \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \bullet \left(\hat{n}_{\vartheta\delta} \times \vec{\vec{1}} \right) \tag{C.2}$$

Relation C.2 is *one* of the various forms of the finite rotation formula[42]. As a vector-dyadic equation, it has no dependence on the particular coordinate system selected to represent it, and can, therefore, be resolved using any Cartesian set of base vectors. For instance, expressing the corresponding quantities onto basis δ (*with respect to basis*) and using matrix algebraic notation, one may write:

$$\left[R^{\delta}_{\vartheta\delta} \right] = \left[1 \right] + \sin\phi_{\vartheta\delta} \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] + \left(1-\cos\phi_{\vartheta\delta} \right) \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] \tag{C.3}$$

As already discussed, there is a relationship of equivalence between the rotation matrix $\left[R^{\delta}_{\vartheta\delta} \right]$ and the transformation matrix $\left[T^{\delta}_{\vartheta} \right]$, equation C.1:

$$\left[T^{\delta}_{\vartheta} \right] = \left[R^{\delta}_{\vartheta\delta} \right] = \left[1 \right] + \sin\phi_{\vartheta\delta} \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] + \left(1-\cos\phi_{\vartheta\delta} \right) \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] \left[\tilde{n}^{\delta}_{\vartheta\delta} \right] \tag{C.4}$$

[42] There is some disagreement in the literature about who the first contributor to the derivation of the finite rotation formula was (see Goldstein, 1980, p. 165, footnote). This formula is usually referred to as the Rodriguez Formula (see Shabana, 1994, p. 438-40; or Shabana, 1998, p. 31), but Cheng & Gupta (1989) claim, based on historical sources, that it should be ascribed to Leonhard Euler (1707-1783) and, therefore, called Euler's Finite Rotation Formula (see also Shuster, 1993, p. 451, 496).

This is the desired angle/axis representation of the transformation matrix. The reader should note in the symbolism of the rotation matrix the presence of a superscript denoting the basis where the rotation tensor has been resolved (*basis of representation*).

As shown above, a finite three-dimensional rotation may be rigorously described by a tensor (dyadic), i.e. a basis-free form of representation. The rotation matrix is the coordinate representation of such a tensor. When the rotation matrix is equated to the transformation matrix as in equation C.4, the basis of representation of the rotation tensor is dictated by the transformation matrix itself (compare to Angeles, 1997, p. 30-33, 48-51).

It is interesting to note that the above-mentioned basis of representation is not unique. This point is conventionally shown considering a single alternative, but equally important resolution of the rotation tensor $\vec{\vec{R}}_{\vartheta\delta}$: the one in terms of components along the axes of the *measurement basis* ϑ, as it is demonstrated in the ensuing argumentation.

According to Euler's theorem, any vector lying along the axis of rotation remains invariant under the rotation. For that reason, the components of the rotation vector possess a fascinating characteristic: they must at any time coincide in both original and rotated coordinate systems. This statement can be verified in several different ways; one possibility is as follows (compare to Angeles, 1997, p. 27; Nikravesh, 1988, p. 160; Shabana, 1994, p. 22; Ibrahimbegovic et al., 1995; or Grubin, 1970, p. 1262)

$$\begin{aligned}
\{ \phi^{\vartheta}_{\vartheta\delta} \} &= [\, T^{\delta}_{\vartheta} \,]^{-1} \{ \phi^{\delta}_{\vartheta\delta} \} \\
&= [\, R^{\delta}_{\vartheta\delta} \,]^{T} \{ \phi^{\delta}_{\vartheta\delta} \} \\
&= \{ \phi^{\delta}_{\vartheta\delta} \} - \phi_{\vartheta\delta} \sin\phi_{\vartheta\delta} [\, \tilde{n}^{\delta}_{\vartheta\delta} \,] \{ n^{\delta}_{\vartheta\delta} \} + \phi_{\vartheta\delta} (1 - \cos\phi_{\vartheta\delta}) [\, \tilde{n}^{\delta}_{\vartheta\delta} \,][\, \tilde{n}^{\delta}_{\vartheta\delta} \,] \{ n^{\delta}_{\vartheta\delta} \} \\
&= \{ \phi^{\delta}_{\vartheta\delta} \}
\end{aligned}$$

where the orthogonality property $\left[\, T_{\delta}^{\vartheta} \,\right] = \left[\, T_{\vartheta}^{\delta} \,\right]^{-1} = \left[\, T_{\vartheta}^{\delta} \,\right]^{T}$;

the relationship $\left\{\, \phi_{\vartheta\delta}^{\delta} \,\right\} = \phi_{\vartheta\delta} \left\{\, n_{\vartheta\delta}^{\delta} \,\right\}$;

the identity $\left[\, \tilde{n}_{\vartheta\delta}^{\delta} \,\right]\left\{\, n_{\vartheta\delta}^{\delta} \,\right\} = \left\{\, 0 \,\right\}$; and

equation C.4 have all been used.

With this property in mind, it becomes evident that $\left\{\, n_{\vartheta\delta}^{\delta} \,\right\} = \left\{\, n_{\vartheta\delta}^{\vartheta} \,\right\}$, which is a intuitively plausible result. As a consequence, the relationship of equivalence, equation C.4, can be alternatively expressed as

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\vartheta} \,\right] = \left[\, 1 \,\right] + \sin\phi_{\vartheta\delta} \left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] + \left(1 - \cos\phi_{\vartheta\delta} \right) \left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\vartheta} \,\right] \qquad \text{(C.5)}$$

The arbitrariness of the basis of representation of the rotation tensor $\vec{\vec{R}}_{\vartheta\delta}$ when equated to the transformation matrix $\left[\, T_{\vartheta}^{\delta} \,\right]$ may, in fact, be extended to any basis ξ whose Euler axis $\hat{n}_{\xi\delta}$ is parallel to $\hat{n}_{\vartheta\delta}$, i.e. $\left\{\, n_{\vartheta\delta}^{\delta} \,\right\} = \left\{\, n_{\xi\delta}^{\xi} \,\right\} = \left\{\, n_{\vartheta\delta}^{\xi} \,\right\}$. Conversely, the corresponding Euler angle $\phi_{\xi\delta}$ has no restrictions, and can therefore assume any value. Thus, it follows a more general form for the relationship of equivalence:

$$\left[\, T_{\vartheta}^{\delta} \,\right] = \left[\, R_{\vartheta\delta}^{\xi} \,\right] = \left[\, 1 \,\right] + \sin\phi_{\vartheta\delta} \left[\, \tilde{n}_{\vartheta\delta}^{\xi} \,\right] + \left(1 - \cos\phi_{\vartheta\delta} \right) \left[\, \tilde{n}_{\vartheta\delta}^{\xi} \,\right]\left[\, \tilde{n}_{\vartheta\delta}^{\xi} \,\right] \qquad \text{(C.6)}$$

This third and more general form for the relationship of equivalence, where the basis of representation of the rotation tensor is not necessarily its own *measurement basis* ϑ, nor its own *with respect to basis* δ, has not been found in the literature.

N.B. One can find works in the literature where the authors do not differentiate between transformation and rotation matrices. There is, however, a significant conceptual difference between them. As defined here, a transformation matrix is a collection of direction cosines relating the orientation of two frames of coordinates. On the other hand, a rotation matrix (coordinate representation of the rotation tensor) is an operator that transforms/rotates vector quantities. Clearly, only the latter involves a tensorial transformation. Therefore, it is only when interpreted from the active point of view that a transformation matrix may be called a tensor (see Arfken & Weber, 1995, p. 192).

Skew-Symmetric Form of the Vector Product

The aim of this appendix is to show how to represent the vector cross product in algebraic skew-symmetric form.

Consider two vectors $\vec{u}$ and $\vec{v}$, their cross product $\vec{w}$, and the corresponding component resolutions onto a dextral orthonormal basis λ:

$$\vec{u} = u^{\lambda_1}\hat{\lambda}_1 + u^{\lambda_2}\hat{\lambda}_2 + u^{\lambda_3}\hat{\lambda}_3$$
$$\vec{v} = v^{\lambda_1}\hat{\lambda}_1 + v^{\lambda_2}\hat{\lambda}_2 + v^{\lambda_3}\hat{\lambda}_3$$

$$\begin{aligned}\vec{w} &= \vec{u}\times\vec{v} \\ &= \left(u^{\lambda_2}v^{\lambda_3} - u^{\lambda_3}v^{\lambda_2}\right)\hat{\lambda}_1 + \left(u^{\lambda_3}v^{\lambda_1} - u^{\lambda_1}v^{\lambda_3}\right)\hat{\lambda}_2 + \left(u^{\lambda_1}v^{\lambda_2} - u^{\lambda_2}v^{\lambda_1}\right)\hat{\lambda}_3\end{aligned}$$

The associated skew-symmetric matrices for these three vectors are respectively

$$\left[\tilde{u}^{\lambda}\right] = \begin{bmatrix} 0 & -u^{\lambda_3} & u^{\lambda_2} \\ u^{\lambda_3} & 0 & -u^{\lambda_1} \\ -u^{\lambda_2} & u^{\lambda_1} & 0 \end{bmatrix}$$

$$\left[\tilde{v}^{\lambda}\right] = \begin{bmatrix} 0 & -v^{\lambda_3} & v^{\lambda_2} \\ v^{\lambda_3} & 0 & -v^{\lambda_1} \\ -v^{\lambda_2} & v^{\lambda_1} & 0 \end{bmatrix}$$

$$\left[\tilde{w}^{\lambda}\right] = \begin{bmatrix} 0 & -\left(u^{\lambda_1}v^{\lambda_2} - u^{\lambda_2}v^{\lambda_1}\right) & u^{\lambda_3}v^{\lambda_1} - u^{\lambda_1}v^{\lambda_3} \\ u^{\lambda_1}v^{\lambda_2} - u^{\lambda_2}v^{\lambda_1} & 0 & -\left(u^{\lambda_2}v^{\lambda_3} - u^{\lambda_3}v^{\lambda_2}\right) \\ -\left(u^{\lambda_3}v^{\lambda_1} - u^{\lambda_1}v^{\lambda_3}\right) & u^{\lambda_2}v^{\lambda_3} - u^{\lambda_3}v^{\lambda_2} & 0 \end{bmatrix}$$

By direct matrix multiplication, one obtains

$$\left[\tilde{u}^{\lambda}\right]\left[\tilde{v}^{\lambda}\right]=\begin{bmatrix}0 & -u^{\lambda_3} & u^{\lambda_2}\\ u^{\lambda_3} & 0 & -u^{\lambda_1}\\ -u^{\lambda_2} & u^{\lambda_1} & 0\end{bmatrix}\begin{bmatrix}0 & -v^{\lambda_3} & v^{\lambda_2}\\ v^{\lambda_3} & 0 & -v^{\lambda_1}\\ -v^{\lambda_2} & v^{\lambda_1} & 0\end{bmatrix}$$

$$=\begin{bmatrix}-\left(u^{\lambda_3}v^{\lambda_3}+u^{\lambda_2}v^{\lambda_2}\right) & u^{\lambda_2}v^{\lambda_1} & u^{\lambda_3}v^{\lambda_1}\\ u^{\lambda_1}v^{\lambda_2} & -\left(u^{\lambda_3}v^{\lambda_3}+u^{\lambda_1}v^{\lambda_1}\right) & u^{\lambda_3}v^{\lambda_2}\\ u^{\lambda_1}v^{\lambda_3} & u^{\lambda_2}v^{\lambda_3} & -\left(u^{\lambda_2}v^{\lambda_2}+u^{\lambda_1}v^{\lambda_1}\right)\end{bmatrix}$$

and

$$\left[\tilde{v}^{\lambda}\right]\left[\tilde{u}^{\lambda}\right]=\begin{bmatrix}0 & -v^{\lambda_3} & v^{\lambda_2}\\ v^{\lambda_3} & 0 & -v^{\lambda_1}\\ -v^{\lambda_2} & v^{\lambda_1} & 0\end{bmatrix}\begin{bmatrix}0 & -u^{\lambda_3} & u^{\lambda_2}\\ u^{\lambda_3} & 0 & -u^{\lambda_1}\\ -u^{\lambda_2} & u^{\lambda_1} & 0\end{bmatrix}$$

$$=\begin{bmatrix}-\left(u^{\lambda_3}v^{\lambda_3}+u^{\lambda_2}v^{\lambda_2}\right) & u^{\lambda_1}v^{\lambda_2} & u^{\lambda_1}v^{\lambda_3}\\ u^{\lambda_2}v^{\lambda_1} & -\left(u^{\lambda_3}v^{\lambda_3}+u^{\lambda_1}v^{\lambda_1}\right) & u^{\lambda_2}v^{\lambda_3}\\ u^{\lambda_3}v^{\lambda_1} & u^{\lambda_3}v^{\lambda_2} & -\left(u^{\lambda_2}v^{\lambda_2}+u^{\lambda_1}v^{\lambda_1}\right)\end{bmatrix}$$

From which immediately follows that

$$\left[\tilde{u}^{\lambda}\right]\left[\tilde{v}^{\lambda}\right]-\left[\tilde{v}^{\lambda}\right]\left[\tilde{u}^{\lambda}\right]=\begin{bmatrix}0 & -\left(u^{\lambda_1}v^{\lambda_2}-u^{\lambda_2}v^{\lambda_1}\right) & u^{\lambda_3}v^{\lambda_1}-u^{\lambda_1}v^{\lambda_3}\\ u^{\lambda_1}v^{\lambda_2}-u^{\lambda_2}v^{\lambda_1} & 0 & -\left(u^{\lambda_2}v^{\lambda_3}-u^{\lambda_3}v^{\lambda_2}\right)\\ -\left(u^{\lambda_3}v^{\lambda_1}-u^{\lambda_1}v^{\lambda_3}\right) & u^{\lambda_2}v^{\lambda_3}-u^{\lambda_3}v^{\lambda_2} & 0\end{bmatrix}$$

Comparing this result with the skew-symmetric expansion of vector $\vec{w}$, i.e. $\left[\tilde{w}^{\lambda}\right]$, it becomes evident that

$$\left[\tilde{w}^{\lambda}\right]=\left[\tilde{u}^{\lambda}\right]\left[\tilde{v}^{\lambda}\right]-\left[\tilde{v}^{\lambda}\right]\left[\tilde{u}^{\lambda}\right]$$

The above expression shows, therefore, how to represent the vector cross product in algebraic skew-symmetric form.

Q.E.D.

This very same identity can be found in Nikravesh (1988, p. 25, eq. 2.52) and Argyris (1982, p. 99, eq. 72). It is also proposed as an exercise/problem by Shabana (1998, p. 358, ex. 11) and Shabana (1994, p. 79, ex. 21), and introduced in Beggs (1983, p. xv) and Shuster (1993a, p. 446).

Invariance of the Antisymmetry Property

The aim here is to demonstrate that the matrix antisymmetry property is invariant under orthogonal similarity transformations.

Consider the following vector cross product and corresponding resolutions onto the dextral orthonormal bases λ and ϑ

$$\vec{w} = \vec{u} \times \vec{v} \quad \Rightarrow \quad \begin{aligned} \{ w^{\lambda} \} &= [\tilde{u}^{\lambda}] \{ v^{\lambda} \} \qquad (E.1) \\ \{ w^{\vartheta} \} &= [\tilde{u}^{\vartheta}] \{ v^{\vartheta} \} \qquad (E.2) \end{aligned}$$

Pre-multiplying by $[T^{\vartheta}_{\lambda}]$ both sides of equation E.1 (resolution of the vector product onto basis λ), one obtains

$$\begin{aligned} [T^{\vartheta}_{\lambda}] \{ w^{\lambda} \} &= [T^{\vartheta}_{\lambda}] [\tilde{u}^{\lambda}] \{ v^{\lambda} \} \\ &= [T^{\vartheta}_{\lambda}] [\tilde{u}^{\lambda}] [1] \{ v^{\lambda} \} \\ &= [T^{\vartheta}_{\lambda}] [\tilde{u}^{\lambda}] [T^{\vartheta}_{\lambda}]^{T} [T^{\vartheta}_{\lambda}] \{ v^{\lambda} \} \end{aligned}$$

where the property of orthogonality of the transformation matrix has been used, $[T^{\vartheta}_{\lambda}]^{T} [T^{\vartheta}_{\lambda}] = [1]$.

Observing yet that

$\{ w^{\vartheta} \} = [T^{\vartheta}_{\lambda}] \{ w^{\lambda} \}$ and that $\{ v^{\vartheta} \} = [T^{\vartheta}_{\lambda}] \{ v^{\lambda} \}$

it immediately follows

$$\left[T_{\lambda}^{\vartheta} \right]\left\{ w^{\lambda} \right\} = \left[T_{\lambda}^{\vartheta} \right]\left[\tilde{u}^{\lambda} \right]\left[T_{\lambda}^{\vartheta} \right]^{T}\left[T_{\lambda}^{\vartheta} \right]\left\{ v^{\lambda} \right\} \quad \Leftrightarrow \quad \left\{ w^{\vartheta} \right\} = \left[T_{\lambda}^{\vartheta} \right]\left[\tilde{u}^{\lambda} \right]\left[T_{\lambda}^{\vartheta} \right]^{T}\left\{ v^{\vartheta} \right\}$$

Comparison of this last result and equation E.2 (resolution of the vector product onto basis ϑ) leads to the desired relationship

$$\left[\tilde{u}^{\vartheta} \right] = \left[T_{\lambda}^{\vartheta} \right]\left[\tilde{u}^{\lambda} \right]\left[T_{\lambda}^{\vartheta} \right]^{T}$$

This relationship clearly shows that the matrix property of antisymmetry is invariant under orthogonal similarity transformations.

Q.E.D.

Alternative demonstrations can be found in Nikravesh (1988, p. 172, eq. 6.89); Meirovitch (1970, p. 109-10) and Crouch (1981, p. 23). It is also proposed as an exercise by Arfken & Weber (1995, p. 193, ex. 3.3.11) and Shabana (1998, p. 87, ex. 11), and introduced in Shuster (1993a, p. 466).

Approximated Trigonometric Functions

The aim of this appendix is to provide the reader with numerical-based examples for the relative accuracy of small angle approximations.

Table F1: Sine (definitions)

$$\varphi = \text{angle (radians)}$$

$$s = \sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - + \ldots$$

$$\text{1st order: } s_{1st} = \varphi \quad \Rightarrow \quad \text{error}(\%) = 100\,\frac{(s_{1st} - s)}{s}$$

$$\text{3rd order: } s_{3rd} = \varphi - \frac{\varphi^3}{6} \quad \Rightarrow \quad \text{error}(\%) = 100\,\frac{(s_{3rd} - s)}{s}$$

Table F2: Cosine (definitions)

$$\psi = \text{angle (radians)}$$

$$c = \cos\psi = 1 - \frac{\psi^2}{2!} + \frac{\psi^4}{4!} - + \ldots$$

$$\text{1st order: } c_{1st} = 1 \quad \Rightarrow \quad \text{error}(\%) = 100\,\frac{(c_{1st} - c)}{c}$$

$$\text{2nd order: } c_{2nd} = 1 - \frac{\psi^2}{2} \quad \Rightarrow \quad \text{error}(\%) = 100\,\frac{(c_{2nd} - c)}{c}$$

Table F1: Sine

angle		sine				
degrees	radians	exact	1st order	error (%)	3rd order	error (%)
0	0.0000	0.0000	0.0000	0.00%	0.0000	0.00%
1	0.0175	0.0175	0.0175	0.01%	0.0175	0.00%
2	0.0349	0.0349	0.0349	0.02%	0.0349	0.00%
3	0.0524	0.0523	0.0524	0.05%	0.0523	0.00%
4	0.0698	0.0698	0.0698	0.08%	0.0698	0.00%
5	0.0873	0.0872	0.0873	0.13%	0.0872	0.00%
6	0.1047	0.1045	0.1047	0.18%	0.1045	0.00%
7	0.1222	0.1219	0.1222	0.25%	0.1219	0.00%
8	0.1396	0.1392	0.1396	0.33%	0.1392	0.00%
9	0.1571	0.1564	0.1571	0.41%	0.1564	0.00%
10	0.1745	0.1736	0.1745	0.51%	0.1736	0.00%
11	0.1920	0.1908	0.1920	0.62%	0.1908	0.00%
12	0.2094	0.2079	0.2094	0.73%	0.2079	0.00%
13	0.2269	0.2250	0.2269	0.86%	0.2249	0.00%
14	0.2443	0.2419	0.2443	1.00%	0.2419	0.00%
15	0.2618	0.2588	0.2618	1.15%	0.2588	0.00%
16	0.2793	0.2756	0.2793	1.31%	0.2756	-0.01%
17	0.2967	0.2924	0.2967	1.48%	0.2924	-0.01%
18	0.3142	0.3090	0.3142	1.66%	0.3090	-0.01%
19	0.3316	0.3256	0.3316	1.86%	0.3255	-0.01%
20	0.3491	0.3420	0.3491	2.06%	0.3420	-0.01%
21	0.3665	0.3584	0.3665	2.27%	0.3583	-0.02%
22	0.3840	0.3746	0.3840	2.50%	0.3745	-0.02%
23	0.4014	0.3907	0.4014	2.74%	0.3906	-0.02%
24	0.4189	0.4067	0.4189	2.99%	0.4066	-0.03%
25	0.4363	0.4226	0.4363	3.25%	0.4225	-0.03%
26	0.4538	0.4384	0.4538	3.52%	0.4382	-0.04%
27	0.4712	0.4540	0.4712	3.80%	0.4538	-0.04%
28	0.4887	0.4695	0.4887	4.09%	0.4692	-0.05%
29	0.5061	0.4848	0.5061	4.40%	0.4845	-0.06%
30	0.5236	0.5000	0.5236	4.72%	0.4997	-0.07%

Table F2: Cosine

angle		cosine				
degrees	**radians**	**exact**	**1st order**	**error (%)**	**2nd order**	**error (%)**
0	0.0000	1.0000	1.0000	0.00%	1.0000	0.00%
1	0.0175	0.9998	1.0000	0.02%	0.9998	0.00%
2	0.0349	0.9994	1.0000	0.06%	0.9994	0.00%
3	0.0524	0.9986	1.0000	0.14%	0.9986	0.00%
4	0.0698	0.9976	1.0000	0.24%	0.9976	0.00%
5	0.0873	0.9962	1.0000	0.38%	0.9962	0.00%
6	0.1047	0.9945	1.0000	0.55%	0.9945	0.00%
7	0.1222	0.9925	1.0000	0.75%	0.9925	0.00%
8	0.1396	0.9903	1.0000	0.98%	0.9903	0.00%
9	0.1571	0.9877	1.0000	1.25%	0.9877	0.00%
10	0.1745	0.9848	1.0000	1.54%	0.9848	0.00%
11	0.1920	0.9816	1.0000	1.87%	0.9816	-0.01%
12	0.2094	0.9781	1.0000	2.23%	0.9781	-0.01%
13	0.2269	0.9744	1.0000	2.63%	0.9743	-0.01%
14	0.2443	0.9703	1.0000	3.06%	0.9701	-0.02%
15	0.2618	0.9659	1.0000	3.53%	0.9657	-0.02%
16	0.2793	0.9613	1.0000	4.03%	0.9610	-0.03%
17	0.2967	0.9563	1.0000	4.57%	0.9560	-0.03%
18	0.3142	0.9511	1.0000	5.15%	0.9507	-0.04%
19	0.3316	0.9455	1.0000	5.76%	0.9450	-0.05%
20	0.3491	0.9397	1.0000	6.42%	0.9391	-0.07%
21	0.3665	0.9336	1.0000	7.11%	0.9328	-0.08%
22	0.3840	0.9272	1.0000	7.85%	0.9263	-0.10%
23	0.4014	0.9205	1.0000	8.64%	0.9194	-0.12%
24	0.4189	0.9135	1.0000	9.46%	0.9123	-0.14%
25	0.4363	0.9063	1.0000	10.34%	0.9048	-0.17%
26	0.4538	0.8988	1.0000	11.26%	0.8970	-0.20%
27	0.4712	0.8910	1.0000	12.23%	0.8890	-0.23%
28	0.4887	0.8829	1.0000	13.26%	0.8806	-0.27%
29	0.5061	0.8746	1.0000	14.34%	0.8719	-0.31%
30	0.5236	0.8660	1.0000	15.47%	0.8629	-0.36%

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