Measuring Nonlinear Functionals of Quantum Harmonic Oscillator States

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Using only linear interactions and a local parity measurement we show how entanglement can be detected between two harmonic oscillators. The scheme generalizes to measure both linear and nonlinear functionals of an arbitrary oscillator state. This leads to many applications including purity tests, eigenvalue estimation, entropy, and distance measures—all without the need for nonlinear interactions or complete state reconstruction. Remarkably, experimental realization of the proposed scheme is already within the reach of current technology with linear optics.

In the context of quantum communication and computing protocols, measures such as purity, fidelity, and entanglement characterize the performance and nonclassical resources in a physical experiment. Such measures not only provide a link with theoretical models, but also provide standards for defining benchmarks [1]. With continuing technical developments in these areas, it is becoming increasingly necessary to identify practical and efficient schemes to measure such quantities.

One obvious method is to first reconstruct the complete density matrix from a series of measurements using, for example, the well-known procedure of quantum state tomography [2–4]. From this reconstructed density matrix the desired measure can then be computed. Although this technique is realizable, it is not efficient as much more information about the quantum state is obtained than is actually needed.

A more direct method was recently proposed by Filip [5], and expanded upon by others [6–8]. They proposed a specific quantum circuit capable of measuring the nonlinear functional $\text{Tr}(\rho_a^k \rho_b^l)$ of the density matrices ρ_a and ρ_b. The scheme requires as inputs, k and l copies of the states ρ_a and ρ_b, respectively. It was independently shown by Brun [9] that any polynomial function of a state up to degree q can be estimated by a joint measurement on q copies of the system. Amazingly, when the systems are entangled the measurement doubles as an entanglement witness [10], giving a negative value for some entangled states while ensuring a positive value for all separable states.

The circuit, although elegant, unfortunately requires a nontrivial interaction between a control qubit and the k and l copies of each system—the targets. In its simplest form with two targets $k + l = 2$, the interaction must generate a controlled-SWAP operation between the control and the two targets. Such an operation does nothing to the targets if the control is in the logical zero state and applies the unitary swap operation

$$V|\phi_1\rangle|\phi_2\rangle = |\phi_2\rangle|\phi_1\rangle \qquad (1)$$

to the two targets if the control is in the logical one state,

where here $|\phi_1\rangle$ and $|\phi_2\rangle$ are arbitrary target states. Implementing this gate is a challenging requirement for any experimental architecture, especially when the target states are of arbitrary dimension. In the case of a harmonic oscillator where the dimension of the Hilbert space is infinite, implementing this gate using, for example, non-deterministic Knill-Laflamme-Milburn-type gates [11] is only feasible when there is at most one quanta of energy in each oscillator.

In this Letter we propose an alternative to the Filip scheme for harmonic oscillators which, remarkably, requires only *linear* coupling between different oscillators, making it realizable with current technology. Like the Filip proposal, the scheme can measure both linear and nonlinear functionals of the density matrix, while for entangled oscillators the scheme acts as an entanglement witness. We begin by first reviewing some of the properties of the swap operator [6–9], along with its generalization to multiparticles, and show how it relates to a host of relevant information measures.

For the swap operator defined in Eq. (1) it is straightforward to show that for two separable states ρ_a and ρ_b,

$$\langle V \rangle \equiv \text{Tr}_{ab}(\rho_a \otimes \rho_b V) = \text{Tr}(\rho_a \rho_b). \qquad (2)$$

Although this seems like a trivial relation, it suggests that a direct way to measure the overlap between two unknown states is to measure the expectation value of the swap operator $\langle V \rangle$. In the case where both systems are in the same state ρ_i, the expectation value is equivalent to the purity of the system, $\langle V \rangle = \text{Tr}(\rho_i^2)$. From the purity and the overlap one could obtain, for example, the Hilbert-Schmidt distance $\text{Tr}[(\rho_a - \rho_b)^2]$.

In the case where the state is not separable, Eq. (2) is no longer valid. To illustrate this, take, for example, the entangled state $(|\alpha\rangle|\beta\rangle - |\beta\rangle|\alpha\rangle)/\sqrt{2}$, where $|\alpha\rangle$ and $|\beta\rangle$ are orthonormal. From (1) the expectation value of the swap operator is -1 which, in contrast to (2), is negative. This is a example of the separability criteria of Horodecki, Horodecki, and Horodecki [12] which states that a density matrix ρ is entangled iff there exists a Hermitian operator H, an entanglement witness, such that

$$\mathrm{Tr}\,(\rho H) < 0, \tag{3}$$

while for all separable states ρ_{sep},

$$\mathrm{Tr}\,(\rho_{\mathrm{sep}} H) \geq 0. \tag{4}$$

From Eq. (2) and by explicit example we see that the swap operator satisfies Eqs. (3) and (4) making it an example of an entanglement witness. A measurement of $\langle V \rangle$ is said to have witnessed the entanglement when the outcome $\langle V \rangle < 0$. In fact, it can be shown that the swap operator is an *optimal* entanglement witness in the sense that it forms a hyperplane that is tangential to the convex set of separable states [13,14]. That is a separable state ρ'_{sep} exists such that $\mathrm{Tr}[\rho'_{\mathrm{sep}} V] = 0$. Such a state is, for example, $|\alpha\rangle \otimes |\beta\rangle$, where $\langle \alpha | \beta \rangle = 0$.

A generalization of the swap operator to multiple systems is defined as

$$V_N |\phi_1\rangle |\phi_2\rangle \ldots |\phi_N\rangle = |\phi_N\rangle |\phi_1\rangle \ldots |\phi_{N-1}\rangle, \tag{5}$$

which is not Hermitian for $N \geq 3$. For separable states $\rho_{\mathrm{sep}} = \rho_1 \otimes \rho_2 \otimes \ldots \rho_N$ Eq. (2) generalizes to

$$\mathrm{Tr}[\rho_{\mathrm{sep}} V_N] = \mathrm{Tr}[\rho_a \rho_b \ldots \rho_N]. \tag{6}$$

For N identical copies of a state, abbreviated as $\rho^{\otimes N}$, this becomes

$$\mathrm{Tr}[\rho^{\otimes N} V_N] = \mathrm{Tr}(\rho^N) = \sum_i \lambda_i^N, \tag{7}$$

where λ_i is the ith eigenvalue of ρ. We note that for a set of N identical pure states the expectation value is unity, while if the set is not homogeneous the expectation value will be less than unity. This serves as a practical state discrimination test of multiple systems [15].

For a d-dimensional system the spectrum of ρ can be obtained from the $(d-1)$ values of $\mathrm{Tr}(\rho^N)$ for $N = 2, 3, \ldots, d$ [10]. Once the spectrum is known any nonlinear functional of the general form $\mathrm{Tr}[f(\rho)]$ can be computed through the corresponding function of the eigenvalues $\sum_i f(\lambda_i)$. This follows from the fact that the trace is independent of the basis in which the density matrix is expressed. In the case when the system is entangled, knowledge of the spectrum of both the entangled state and the reduced subsystems can be used to test for entanglement through the majorization condition of Nielsen [17,18].

In all of the above cases the desired measure was obtained from a single value: the expectation value of the swap operator. We now introduce a simple experiment to measure $\langle V_N \rangle$ for a system of N harmonic oscillators in an arbitrary state ρ.

The experiment, illustrated in Fig. 1, is conducted in two stages. First the N oscillators evolve under the action of the unitary operator Ω. Following that, a measurement is performed in the energy eigenbasis of each oscillator and the expectation value of an operator D is measured. The specific form of Ω and D is such that
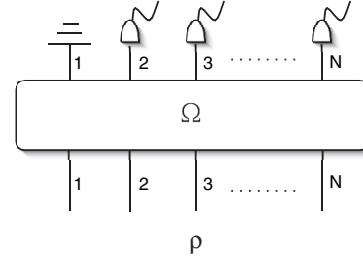

FIG. 1.　Apparatus to measure $\mathrm{Tr}(\rho V_N)$. The evolution Ω generates a discrete Fourier transformation of the N oscillators. Following that an energy measurement of $N - 1$ oscillators is performed from which $\mathrm{Tr}(\rho V_N)$ is estimated.

$$\mathrm{Tr}[\Omega \rho \Omega^\dagger D] = \langle V_N \rangle \tag{8}$$

for all states ρ. Using the cyclic property of the trace this implies

$$V_N = \Omega^\dagger D \Omega. \tag{9}$$

Since, in general, V_N is not Hermitian, it is not strictly possible to associate D in Eqs. (8) and (9) with an observable. Nevertheless, it is possible to express a general operator as a *weighted sum* of positive operator valued measure (POVM) elements as was done in [19]. From the linearity of the trace, the expectation value of the operator corresponds to the same weighted sum of measured probabilities.

From the action of V_N on the basis state in Eq. (5) it can be shown that any operator $\mathcal{O}_j$ acting solely on the jth oscillator transforms as $V_N \mathcal{O}_j V_N^\dagger \rightarrow \mathcal{O}_{j+1}$ (mod N). Of particular importance are the creation and annihilation operators, $a_j^\dagger$ and a_j, respectively, which satisfy the canonical commutation relation $[a_i, a_j^\dagger] = \delta_{i,j}$. Using the Kronecker delta function the transformation of the creation operators can be written as

$$V_N a_j^\dagger V_N^\dagger = \sum_i a_i^\dagger \delta_{i,j+1}. \tag{10}$$

Since the set of creation operators $\{a_j^\dagger\}$ generates an orthonormal basis from the ground state $|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$, Eq. (10), along with the condition

$$V_N |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle, \tag{11}$$

is an equivalent definition of V_N. This is a simple but important result as it shows that the swap operation generates a *linear* and *unitary* transformation of the set of creation and annihilation operators. We now seek a solution to Eq. (9) where the operators Ω and D are of the *same kind*. Specifically, we require

$$\Omega a_j^\dagger \Omega^\dagger = \sum_i a_i^\dagger \Omega_{ij}, \tag{12}$$

$$D a_j^\dagger D^\dagger = \sum_i a_i^\dagger D_{ij}, \tag{13}$$

where the coefficients Ω_{ij} and D_{ij}, to be determined later, are elements of the unitary matrices $\mathbf{\Omega}$ and $\mathbf{D}$, respectively [20].

In general, any such unitary operator U which transforms a set of N creation operators linearly and unitarily,

$$U a_j^\dagger U^\dagger = \sum_i a_i^\dagger U_{ij}, \qquad (14)$$

is an element of the $U(N)$ Lie group. A property of the Lie group is that there corresponds a Hermitian operator H of the general form

$$H = \sum_{ij} \lambda_{ij} a_i^\dagger a_j, \qquad (15)$$

with $\lambda_{ij} = \lambda_{ji}^*$, which generates the group element $U = \exp(-iH)$. Noting that an arbitrary generator H acting on the ground state is identically zero, we can Taylor expand U as powers of H and show that Eq. (11) is satisfied for any unitary U of the Lie group, including the swap operator.

The physical importance of Eq. (12) is that the evolution operator Ω can be implemented with a *linear* coupling Hamiltonian of the form given in Eq. (15). Reck *et al.* have shown how any such multiparticle coupling Hamiltonian can be implemented using a sequence of two particle interactions [21]. In an optics context this corresponds to an array of beam splitters and phase shifters. As for the measurement, our requirement that D be an element of $U(N)$, as well as correspond to a measurement in the energy basis, implies that it must be generated by an operator of the form $H_D = \sum_j \theta_j a_j^\dagger a_j$. Explicitly, we require

$$D = \exp[-i\theta_1 a_1^\dagger a_1] \otimes \exp[-i\theta_2 a_2^\dagger a_2] \otimes \cdots$$
$$\otimes \exp[-i\theta_N a_N^\dagger a_N], \qquad (16)$$

where θ_j are free parameters yet to be determined.

To derive the specific form of the matrix elements Ω_{ij} and the coefficients θ_j we substitute Eq. (9) into (10) and derive, with the help of (12) and (13), the *matrix* equation

$$\mathbf{V}_N = \mathbf{\Omega}^\dagger \mathbf{D} \mathbf{\Omega}, \qquad (17)$$

where $[\mathbf{V}_N]_{ij} := \delta_{i,j+1}$. We note that the commutation relation between the creation and annihilation operators implies

$$\exp(-i\phi a_j^\dagger a_j) a_k^\dagger \exp(i\phi a_j^\dagger a_j) = a_k^\dagger \exp(-i\phi \delta_{j,k}), \quad (18)$$

from which it is straightforward to show from (16) and (13) that the matrix $\mathbf{D}$ is diagonal with elements $D_{jj} = \exp(-i\theta_j)$. The matrix $\mathbf{\Omega}$ in (17) then is such that it diagonalizes $\mathbf{V}_N$. Using standard techniques we can solve Eq. (17) to give

$$\Omega_{ij} = \omega^{ij}/\sqrt{N}, \qquad (19)$$

$$D_{jj} = \omega^{j-1}, \qquad (20)$$

where $\omega = \exp(i2\pi/N)$ is the Nth root of unity. We see that the matrix $\mathbf{\Omega}$ is a discrete Fourier transformation and defines the action of the unitary operator Ω. The unknown phases in (16) are found from (20) to be

$$\theta_j = 2\pi(j-1)/N, \qquad j = 1, 2, \ldots, N, \qquad (21)$$

and characterize the operator D in (16). To express this in terms of POVM elements we rewrite the energy operator $a_j^\dagger a_j$ in the energy basis of the jth oscillator as $\sum_{n_j} n_j |n_j\rangle \times \langle n_j|$. With this (16) becomes

$$D = \sum_{\{n_i\}} w_1(n_1) \ldots w_N(n_N) |n_1 \ldots n_N\rangle\langle n_1 \ldots n_N|, \quad (22)$$

where

$$w_j(n_j) = \exp(-i\theta_j n_j) \qquad (23)$$

is a complex weighting coefficient. In the context of a measurement, the projector $|n_1 \ldots n_N\rangle\langle n_1 \ldots n_N|$ is associated with the joint measurement outcome $(n_1, n_2, \ldots, n_N)$, which is interpreted as the outcome n_1 occurring at the first oscillator, n_2 at the second oscillator, and so on. The probability of observing the event $(n_1, n_2, \ldots, n_N)$ is given by the overlap of the state with the associated projector. For the state $\Omega \rho \Omega^\dagger$ the joint probability is

$$\Pr(n_1, n_2, \ldots, n_N) = \mathrm{Tr}(\Omega \rho \Omega^\dagger |n_1 \ldots n_N\rangle\langle n_1 \ldots n_N|). \quad (24)$$

From (22) and (24) the expectation value $\mathrm{Tr}(\Omega \rho \Omega^\dagger D)$ can be expressed as a (complex) weighted sum of measured probabilities,

$$\mathrm{Tr}(\Omega \rho \Omega^\dagger D) = \sum_{\{n_i\}} w_1(n_1) w_2(n_2) \ldots w_N(n_N)$$
$$\times \Pr(n_1, n_2, \ldots, n_N), \qquad (25)$$

which is, by definition, the expectation value $\langle V_N \rangle$.

To illustrate the practicality of this apparatus we will briefly discuss the simplest case which is when $N = 2$. The circuit diagram of this experiment is illustrated in Fig. 2.

From Eqs. (12) and (19) the required unitary transformation is of the form

$$\Omega \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} \Omega^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}. \qquad (26)$$

This specific unitary is generated by an interaction Hamiltonian of the form $i\kappa(a_1^\dagger a_2 - a_1 a_2^\dagger)$ applied for a
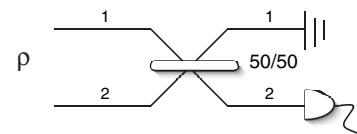

FIG. 2.   Apparatus to measure $\mathrm{Tr}(\rho V_2)$. No measurement is performed on oscillator one, while detector two measures the average value of the parity. The two oscillators interact via a 50/50 coupling device.

time $\pi/(4\kappa)$, where κ is the interaction strength. In optics, where the two oscillators are realized by different spatial modes of the quantized field, for example, this can be achieved with a simple 50-50 beam splitter. Following the interaction a measurement is performed in the energy basis of each oscillator and the probability $\Pr(n_1, n_2)$ is observed after repeated trials. To extract $\langle V_2 \rangle$ the distribution is weighted against the coefficients $w_1(n_1) = 1$ and $w_2(n_2) = (-1)^{n_2}$ given by (23) and (21). The result is

$$\langle V_2 \rangle = \sum_{n_2} (-1)^{n_2} \Pr(n_2), \tag{27}$$

where $\Pr(n_2) = \sum_{n_1} \Pr(n_1, n_2)$, and is independent of n_1, the outcome of the measurement on oscillator one. To simplify matters experimentally only the distribution $\Pr_+ = \sum_{n_2} \Pr(2n_2)$ and $\Pr_- = 1 - \Pr_+$ of the second oscillator need be measured. The expectation value $\langle V_2 \rangle$ can then be obtained from the difference $\Pr_+ - \Pr_-$, which is known as the average parity and corresponds to the zero of the Wigner function [22]. Similar simplifications reside in the measurement of $\langle V_N \rangle$ for higher values of N. It is a surprising result that such a range of meaningful measures can be obtained from an experiment that requires no non-linearity, no interferometers, just a linear interaction and a single parity measurement.

We note that higher order moments of the swap operator $\langle (V_N)^k \rangle$ can also be obtained from the general apparatus illustrated in Fig. 1. This can be seen by writing $(V_N)^k = \Omega^\dagger D^k \Omega$. Physically, this corresponds to the same evolution Ω of the state ρ followed by a measurement of D^k which can be written as

$$D^k = \exp[-ik\theta_1 a_1^\dagger a_1] \otimes \exp[-ik\theta_2 a_2^\dagger a_2] \otimes \cdots$$
$$\otimes \exp[-ik\theta_N a_N^\dagger a_N]. \tag{28}$$

Repeating the calculation it is seen that this corresponds to measuring the same probability distribution $\Pr(n_1, n_2 \ldots n_N)$, however now the distribution is weighting by the functions $w_j(n_j, k) = [w_j(n_j)]^k$.

The measurement procedure introduced here generalizes to measure the expectation value $\mathrm{Tr}(\rho U)$ of *any* unitary operator U that transforms the creation operators of N harmonic oscillators linearly. The key is that the associated unitary matrix $\mathbf{U}$ can always be diagonalized as $\mathbf{\Omega}^\dagger \mathbf{D} \mathbf{\Omega}$ where both $\mathbf{\Omega}$ and $\mathbf{D}$ are associated with the unitary evolution Ω and the measurement D through Eqs. (12) and (13), respectively. The expectation value $\mathrm{Tr}(\rho U)$ is then given by the weighted sum of probabilities as in (25), where the specific values of the phases θ_j are determined from the diagonal matrix $\mathbf{D}$.

In conclusion, we have introduced a procedure to directly measure the quantity $\mathrm{Tr}(\rho U)$ for any unitary operator that is an element of the $U(N)$ Lie group. The procedure removes the experimentally challenging step of entangling an ancilla to the N systems which was present in

previous proposals. The result is a more practical procedure involving only linear interactions between N oscillators and local energy measurements. In the case where the unitary operator is the N-particle swap operator, the measurement corresponds to a range of useful measures depending on the input state, including the purity, the fidelity, the overlap, an entanglement witness, and generalized nonlinear functionals.

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