Distillability of Werner states using entanglement witnesses and robust semidefinite programs

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(Received 10 August 2006; published 6 November 2006)

We use robust semidefinite programs and entanglement witnesses to study the distillability of Werner states. We perform exact numerical calculations that show two-undistillability in a region of the state space, which was previously conjectured to be undistillable. We also introduce bases that yield interesting expressions for the *distillability witnesses* and for a tensor product of Werner states with an arbitrary number of copies.

DOI: 10.1103/PhysRevA.74.052306 PACS number(s): 03.67.Mn

I. INTRODUCTION

Maximally entangled states are the main resource in quantum information (QI) processing. Protocols like teleportation of quantum states [1] and entanglement-based quantum cryptography [2], just to cite the two most emblematic, work with perfect fidelity only when maximally entangled bipartite states are available. (As maximally entangled bipartite states are equivalent by local unitary transformations, they are usually referred to as *singlets* in the jargon of QI.) On the other hand, even if one has a source of perfect singlets, the ever present decoherence, due to interactions with the environment, degrades these states to mixed form with reduced entanglement. Bennett *et al.* [3] showed that this practical difficulty could be circumvented by *distilling* singlets from mixed states. This process involves only local quantum operations and classical communication (LOCC) and, in principle, is able to purify any mixed state to a singlet form, given that an arbitrary supply of the former is available. The process works at the expense of sacrificing many of the mixed states in order to concentrate their entanglement to a singlet. Horodecki *et al.* showed [4,5] that only states that violate the Peres criterion [6] can be distilled, i.e., the nonpositivity of the partial transpose (NPT) is a necessary condition for distillability. It was then realized that there are entangled states that are not directly useful in QI processing; they are said to be *bound entangled* [4], and the states with positive partial transpose (PPT) are of this kind. Nevertheless, these states can be *activated* [7] in the sense that used in conjunction with NPT states, they can enhance the fidelity of teleportation. Therefore we have two kinds of entanglement in nature, namely, bound and free. The set of bound entangled states includes all the PPT ones, but it is not known if there are NPT states in this set. It was conjectured by DiVincenzo *et al.* [8] and Dur *et al.* [9] that in fact, there exist bound entangled NPT states.

It can be shown that any bipartite NPT state can be transformed by LOCC to a Werner state [10], keeping the fidelity to the singlet. The process is performed by *twirling* (see Ref. [9], for example) the states through the action of bilocal unitary operations ($U\otimes U$). Thus the study of distillability of arbitrary bipartite states is reduced to the distillability properties of Werner states.

Formally, a bipartite state $[\rho\in\mathcal{B}(\mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{B}})]$ is distillable if and only if there exists a Schmidt rank two pure state $(|\Psi\rangle\in\mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{B}})$ in the Hilbert space in which ρ acts, such that $\langle\Psi|(\rho^{\otimes N})^{T_A}|\Psi\rangle$ is less than zero for some finite integer N [5,8,9], T_A meaning partial transposition. When this condition is verified for some N, ρ is said to be N distillable. In particular, all the bipartite entangled states of the kind qubit-qudit ($2\times d$) are one-distillable [5,11].

In the same fashion that the entanglement of a state can be decided by an *entanglement witness* operator [11], Kraus *et al.* showed that the distillability also can be decided by means of a kind of witness operator [12]. In this paper, we show how to calculate these *distillability witnesses* using robust semidefinite programs (RSDP) [13] and apply it to study one- and two-distillability of Werner states. Starting with some definitions in Sec. II, we revise the RSDP formalism in Sec. III. Sections IV and V present numerical results for the distillability of Werner states in the one- and two-copy cases. In Sec. VI, we derive some interesting expressions for the *distillability witness* and for a tensor product of Werner states with arbitrary N, and then we conclude.

II. DEFINITIONS

The set of nonentangled (separable) states is convex and closed, therefore it follows from the separating hyperplane theorem that there exists a linear functional (*hyperplane*) that separates an entangled state from this set; this results in an entanglement witness [11]. Thus an EW (W) is a Hermitian operator with a nonnegative expectation value for all the separable states, but which can have a negative expectation value for an entangled state. In this case, the state is said to be detected by the EW. An EW, which can be written in the form

$$W = P + Q_1^{T_A} + Q_2^{T_B} + \cdots + Q_N^{T_Z}, \tag{1}$$

with P and Q_i positive operators, is said to be *decomposable* and it is nondecomposable if it cannot be put in this form. Only nondecomposable EWs can detect PPT states. When the EW (hyperplane) is tangent to the separable set, it is said to be optimal (see Ref. [14], for example).

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To be distillable [4,12], a bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, or a finite tensor product of it $(\rho^{\otimes N})$, must have a projection that is NPT on a four-dimensional subspace, that is, given a Schmidt rank two state,

$$|\Psi\rangle = s_1|e_1 f_1\rangle + s_2|e_2 f_2\rangle, \tag{2}$$

where e_1, e_2 and f_1, f_2 are bases defining bidimensional subspaces in $\mathcal{H}_A^{\otimes N}$ and $\mathcal{H}_B^{\otimes N}$, respectively, and s_1, s_2 are the Schmidt coefficients; ρ is distillable if the inequality

$$\langle\Psi|(\rho^{\otimes N})^{T_A}|\Psi\rangle < 0 \tag{3}$$

is satisfied for some arbitrary $|\Psi\rangle$, of the form of Eq. (2) and some finite integer N.

We use the following parametrization of the Werner states [10]:

$$\rho_w = \frac{I_d + \beta F_d}{d^2 + d\beta}, \tag{4}$$

with $-1 \leqslant \beta \leqslant 1$. ρ_w is separable for $\beta \geqslant -\frac{1}{d}$ and one-distillable for $\beta < -\frac{1}{2}$. F_d is a swap operator for two qudits,

$$F_d = \sum_{i,j=1}^{d} |ij\rangle\langle ji|, \tag{5}$$

and its partial transpose is the bipartite maximally entangled state

$$P_d = \frac{1}{d} F_d^{T_A} = \frac{1}{d}\sum_{i,j=1}^{d} |ii\rangle\langle jj|. \tag{6}$$

I_d is the identity in the space of the two qudits $[\mathrm{Tr}(I_d) = d^2]$.

Equivalently to the inequality (3), Kraus *et al.* showed [12] that the distillability of an arbitrary state ρ [we will consider only the bipartite case $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$] can be decided through the operator

$$W_N = P_2 \otimes (\rho^{T_A})^{\otimes N}, \tag{7}$$

with W_N acting in $(\mathcal{H}_{2A} \otimes \mathcal{H}_{2B}) \otimes (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes N}$, $\mathcal{H}_{2A}$ $(\mathcal{H}_{2B})$ being the Hilbert space of a qubit belonging to A (B). If W_N is not an EW, then $\rho^{\otimes N}$ is N distillable. If W_N is a nondecomposable EW, then the PPT entangled state it detects (π) activates $\rho^{\otimes N}$, i.e., $\rho^{\otimes N} \otimes \pi$ is one-distillable. When W_N is decomposable (and in the case of Werner states it happens to be a positive operator), then $\rho^{\otimes N}$ is undistillable and unactivable. When W_N happens to be an EW, $W_N^{T_A}$ and $W_N^{T_B}$ are optimal EWs.

III. OPTIMAL WITNESSES VIA ROBUST SEMIDEFINITE PROGRAMS

Given a bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we want to determine its optimal entanglement witness W_ρ. We will use the method introduced in Ref. [14], which we briefly describe in the sequel. Let Σ be the set of separable states in $\mathcal{B}(H_A \otimes H_B)$. We define the following set $(\mathcal{W})$ of entanglement witnesses:

$$\mathcal{W} = \{W/W \in \mathcal{B}(H_A \otimes H_B); W^\dagger = W; \mathrm{Tr}(W\sigma) \geqslant 0, \\ \forall\ \sigma \in \Sigma; \mathrm{Tr}(W) = 1. \tag{8}$$

W_ρ is defined by

$$\min_{W\in\mathcal{W}} \mathrm{Tr}(W\rho) = \mathrm{Tr}(W_\rho \rho), \tag{9}$$

and can be determined through the following RSDP:

$$\min_W \mathrm{Tr}(W\rho),$$

$$\text{subject to} \begin{cases} W^\dagger = W, \\ \mathrm{Tr}(W) = 1, \\ \langle\psi_A|W|\psi_A\rangle \geqslant 0\ \forall\ |\psi_A\rangle \in \mathcal{H}_A. \end{cases} \tag{10}$$

This is a NP-hard problem and, in practice, we relax it to a semidefinite program (SDP) by taking a finite number (n) of pure states $(|\psi_A^j\rangle)$ to represent the whole Hilbert space $\mathcal{H}_A$. Thus we replace an infinite number of constraints [cf. last line of Eq. (10)] by a finite set. If $\dim(\mathcal{H}_A)$ is d, the kets $|\psi_A^j\rangle$ can be chosen as a uniformly distributed sample of unit complex vectors $(\vec{c}_i)$ in $\mathcal{C}^d$, and with infinite n this program would yield the exact witness W_ρ. In our calculations, we use an interior point algorithm to solve the SDP [15].

In Ref. [16], it was shown that W_ρ yields the random robustness $[\mathrm{Rr} = -\mathrm{Tr}(I)\mathrm{Tr}(W_\rho \rho)]$ of ρ, i.e., the minimal amount of mixing with the identity necessary to wash out all the entanglement. Thus, the state $\sigma = [\rho + \mathrm{Rr}I/\mathrm{Tr}(I)]/(1+\mathrm{Rr})$ is in the border between separable and entangled states.

Our main goal is to decide if the operator W_N [cf. Eq. (7)] is an EW. We will do so by determining a state for which W_N could be an optimal EW in the sense of Eq. (9). If we find such a state, we compare its optimal EW with the expression of W_N, and this will tell if W_N is or is not an EW.

IV. ONE-COPY CASE

We will apply the RSDP techniques to calculate optimal EWs [cf. Eq. (10)] to investigate the distillability properties of Werner states in the one-copy case. We will show that the distillability is related to the properties of an EW for a certain family of PPT states.

We want to know if the Hermitian operator $W_1(\beta) \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ defined by

$$W_1(\beta) = P_2 \otimes \rho_w^{T_A}, \tag{11}$$

is an entanglement witness.

We will show that, for $-\frac{1}{2} \leqslant \beta \leqslant -\frac{1}{3}$, this operator is indeed a witness and, for $\beta = -\frac{1}{2}$, it is the optimal witness (W_π) for a certain family of PPT entangled states (π). Our numerical calculations will be restricted to qutrit Werner states $(d=3)$, therefore $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2 \times d = 6$.

We introduce the orthogonal basis

$$B_1 = P_2 \otimes P_3,$$

$$B_2 = P_2 \otimes (I_3 - P_3),$$

$$B_3 = (I_2 - P_2) \otimes P_3,$$

$$B_4 = (I_2 - P_2) \otimes (I_3 - P_3). \quad (12)$$

Note that $B_i/\mathrm{Tr}(B_i)$ is an entangled state in $\mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$, and only B_4 is PPT. In particular, B_1 is the maximally entangled state.

$W_1(\beta)$ can be recast as

$$W_1(\beta) = \frac{1}{d^2 + \beta d}[(1 + \beta d)B_1 + B_2]. \quad (13)$$

In this basis, the states can be written as

$$\rho = \sum_{i=1}^{4} p_i B_i/\mathrm{Tr}(B_i),$$

$$\sum_{i=1}^{4} p_i = 1, \quad p_i \geq 0. \quad (14)$$

Note that the state space is a three-dimensional polytope defined by the constraint $\Sigma_{i=1}^{4} p_i = 1$, $p_i \geq 0$. Optimal witnesses for these states have the form

$$W = \sum_{i=1}^{4} c_i B_i/\mathrm{Tr}(B_i),$$

$$\sum_{i=1}^{4} c_i = 1, \quad c_i \in \Re. \quad (15)$$

Assuming $W_1(\beta)$ is a witness for $-1 \leq \beta \leq -\frac{1}{3}$, we ask for the PPT state (π) it detects the most. It is done through the following SDP:

$$\min_{\pi} \mathrm{Tr}[W_1(\beta)\pi],$$

$$\text{subject to} \begin{cases} \pi^{\dagger} = \pi, \\ \pi \geq 0, \\ \mathrm{Tr}(\pi) = 1, \\ \pi^{T_A} \geq 0. \end{cases} \quad (16)$$

We observe that the optimal solution (π^*) of the above SDP is independent of β and minimizes this other SDP,

$$\min p,$$

$$\text{subject to} \begin{cases} \pi = (1 - p)B_1/\mathrm{Tr}(B_1) + pB_4/\mathrm{Tr}(B_4), \\ \pi^{\dagger} = \pi, \\ \pi \geq 0, \\ \mathrm{Tr}(\pi) = 1, \\ \pi^{T_A} \geq 0. \end{cases} \quad (17)$$

The optimal p is 0.8571 yielding the state π^*. The optimal EW for this state, obtained by means of the RSDP (10), furnishes $\mathrm{Tr}(W_{\pi^*}\pi^*) = -0.0095$.

Comparing $\mathrm{Tr}[W_1(\beta)\pi^*]$ with $\mathrm{Tr}(W_{\pi^*}\pi^*)$, we observe that

$$\mathrm{Tr}[W_1(\beta)\pi^*] < \mathrm{Tr}(W_{\pi^*}\pi^*), \quad \forall \ -1 \leq \beta < -\tfrac{1}{2}. \quad (18)$$

These calculations are sufficient to show that $W_1(\beta)$ is not a witness for $\beta < -\frac{1}{2}$, for it gives an expectation value that is lower than that of the optimal EW. On the other hand, for $-\frac{1}{2} < \beta \leq -\frac{1}{3}$, we observe that $\mathrm{Tr}[W_1(\beta)\pi^*] > \mathrm{Tr}(W_{\pi^*}\pi^*)$, and for $\beta = -\frac{1}{2}$, $\mathrm{Tr}[W_1(-\frac{1}{2})\pi^*] = \mathrm{Tr}(W_{\pi^*}\pi^*)$. With W_{π^*} written in the form (15), our calculations converge to $c_1 = -\frac{1}{15}$, $c_2 = \frac{16}{15}$, and $c_3 = c_4 = 0$, which are the parameters of $W_1(-\frac{1}{2})$. Therefore $W_1(-\frac{1}{2})$ is the optimal witness for π^*. This result can be obtained using a large sample of random $|\psi_A^j\rangle$ [cf. Eq. (10)] or through the following deterministic recipe.

Consider the state (σ) defined by

$$\sigma = \frac{\pi^* - \mathrm{Tr}\left[W_1\left(-\frac{1}{2}\right)\pi^*\right]I}{1 - (2d)^2\,\mathrm{Tr}\left[W_1\left(-\frac{1}{2}\right)\pi^*\right]}. \quad (19)$$

If W_{π^*} and $W_1(-\frac{1}{2})$ coincide, they yield the random robustness of π^* [16], and σ is a state in the border between separable and entangled states (viz. Sec. III). Therefore σ contains information about the border of the separable set. For each eigenvector $(|\Psi_k\rangle)$ of σ, we form the state $\rho_A^k = \mathrm{Tr}_B(|\Psi_k\rangle\langle\Psi_k|)$. Then we use the eigenvectors of the $\{\rho_A^k\}$ as a sample of states for the SDP (10). In this case, this recipe yields 216 states, but just 24 already yield the exact result.

Now we want to show that $W_1(\beta)$ is a witness for $-\frac{1}{2} \leq \beta \leq -\frac{1}{3}$. For $\beta = -\frac{1}{2}$, we know it is an optimal witness. Notice that $W_1(-\frac{1}{3})$ is a positive operator. Looking up Eq. (13), it is easy to see that for any state σ and for $\beta_1 < \beta_2$, $\mathrm{Tr}[W_1(\beta_1)\sigma] \leq \mathrm{Tr}[W_1(\beta_2)\sigma]$. In particular, $\mathrm{Tr}[W_1(-\frac{1}{2})\sigma] \leq \mathrm{Tr}[W_1(\beta_2)\sigma]$. If σ is a separable state, $\mathrm{Tr}[W_1(-\frac{1}{2})\sigma] \geq 0$ and therefore $\mathrm{Tr}[W_1(\beta \geq -\frac{1}{2})\sigma] \geq 0$, showing it is an entanglement witness.

All the calculations we have done can be understood more easily by means of Figs. 1 and 2. Figure 1 shows a two-dimensional projection of a three-dimensional plot of the state space. This picture is obtained as follows. We randomly draw 10^6 states ρ. Out of each ρ, we build a border separable state (σ), and a state (ϕ) in the hyperplane defined by $W_1(\beta)$, namely,

$$\sigma = \frac{\rho - \mathrm{Tr}(W_\rho \rho)I}{1 - 36\,\mathrm{Tr}(W_\rho \rho)},$$

$$\phi = \frac{\rho - \mathrm{Tr}[W_1(\beta)\rho]I}{1 - 36\,\mathrm{Tr}[W_1(\beta)\rho]}. \quad (20)$$

These states are rewritten in the zero trace basis (I, G_1, G_2, G_3),

$$G_1 = 8B_1 - B_2,$$

$$G_2 = 8B_3 - B_4,$$

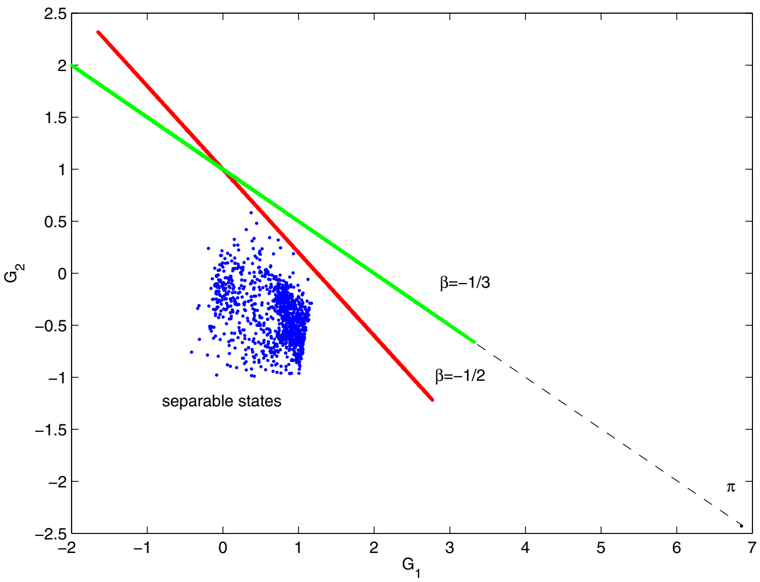

FIG. 1. (Color online) A projection of the state space showing the set of separable states [just border states—cf. Eq. (20)], the planes $W_1(\beta)$ and the PPT state π for which $W_1(-\frac{1}{2})$ is the optimal witness. The planes separate the state π from the separable set.

$$G_3 = -3(B_1 + B_2) + B_3 + B_4, \tag{21}$$

and their coefficients are plotted. We clearly see that the planes $W_1(\beta)$ have a common axis, which is parallel to G_3. In the picture, we can also see the state π^* [cf. Eqs. (16) and (17)], which is in the plane $W_1(-\frac{1}{3})$. Notice that the plane $W_1(-\frac{1}{2})$ is tangent to a face of the polytope defined by the separable states. It can be clearly seen in Fig. 2, which is the three-dimensional picture. Figure 2 also clearly illustrates the concept of optimal entanglement witness, i.e., a hyperplane tangent to the separable set. Notice that in Fig. 1, we have a family of nonoptimal EWs, the planes for $-\frac{1}{2} < \beta \leqslant -\frac{1}{3}$. Therefore, these calculations show that the Werner states are one-undistillable for $-\frac{1}{2} \leqslant \beta \leqslant -\frac{1}{3}$.

Of course the one-undistillability of Werner states is not a novelty [8,9]. The interesting result here is the technique to decide if the Kraus-Lewenstein-Cirac operator (7) [12] is an EW. The strategy was to compare the candidate to EW with the optimal EW of a certain PPT state, which can be obtained with arbitrary precision by means of the RSDP (10), and to show that the candidate operator converges to this EW. In this sense our calculations are exact, leaving no room to doubt if the candidate operator is or is not an EW. This technique extends straightforwardly to higher dimensions. The other interesting result is that we obtain a family of PPT entangled states $\pi(p)$, $0.8571 \leqslant p < 1$ [cf. Eq. (17)], which activate the Werner states in the interval $-\frac{1}{2} \leqslant \beta < -\frac{1}{3}$, i.e., $\rho_w \otimes \pi(p)$ is one-distillable. In Ref. [17], similar results about the activation of Werner states were also obtained.

V. TWO-COPY CASE

Now we will apply the techniques we have developed in the one-copy case to study the distillability of Werner states in the two-copy case. We will determine the optimal EW by means of the RSDP (10) for a family of PPT states. This will show that the Werner states which are one-undistillable are also two-undistillable.

The calculations for the two-copy case mirror the one-copy case and we arrive at analogous conclusions. The orthogonal basis to expand the witnesses and states is

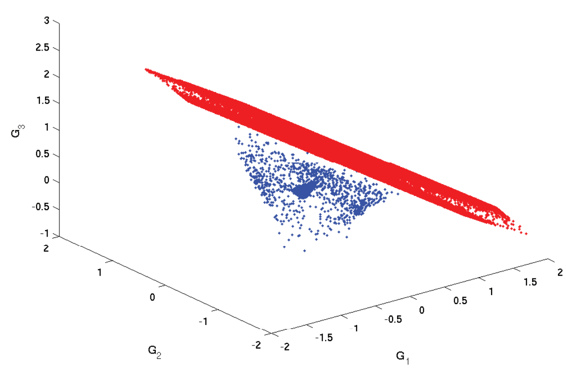

FIG. 2. (Color online) A three-dimensional picture of the state space showing the set of separable states [just border states—cf. Eq. (20)], which is a polytope, and the plane $W_1(-\frac{1}{2})$ sitting on the polytope. $W_1(-\frac{1}{2})$ is an optimal entanglement witness, i.e., a plane that is tangent to the separable set.

$$B_1 = P_2 \otimes P_3 \otimes P_3,$$

$$B_2 = P_2 \otimes [(I_3 - P_3) \otimes P_3 + P_3 \otimes (I_3 - P_3)],$$

$$B_3 = P_2 \otimes \ (I_3 - P_3) \otimes (I_3 - P_3)],$$

$$B_4 = (I_2 - P_2) \otimes P_3 \otimes P_3,$$

$$B_5 = (I_2 - P_2) \otimes [(I_3 - P_3) \otimes P_3 + P_3 \otimes (I_3 - P_3)],$$

$$B_6 = (I_2 - P_2) \otimes \ (I_3 - P_3) \otimes (I_3 - P_3)]. \quad (22)$$

Again, if normalized, these are entangled states, with B_1 the maximally entangled state and B_6 the only PPT state. The state space is a five-dimensional polytope defined by

$$\rho = \sum_{i=1}^{6} p_i B_i / \mathrm{Tr}(B_i),$$
$$\sum_{i=1}^{6} p_i = 1, \quad p_i \geqslant 0. \quad (23)$$

$W_2(\beta) = P_2 \otimes (\rho_w^{T_A})^{\otimes 2}$ reads

$$W_2(\beta) = \left(\frac{1}{d^2 + \beta d}\right)^2 [(1 + \beta d)^2 B_1 + (1 + \beta d) B_2 + B_3]. \quad (24)$$

Using a SDP analogous to Eq. (16), we obtain that the PPT state (π) most detected by $W_2(\beta)$ has the coefficients (0.0278, 0.2222, 0, 0, 0.0833, 0.6667). We also obtain that $W_2(-\frac{1}{2}) = W_\pi$ and $\mathrm{Tr}(W_\pi \pi) = -0.0019$. We can derive different families of PPT states for which $W_2(-\frac{1}{2})$ is optimal. A particularly interesting one reads

$$\tilde{\pi} = (1 - p)\frac{B_2}{\mathrm{Tr}(B_2)} + p\frac{B_6}{\mathrm{Tr}(B_6)}, \quad (25)$$

with p exactly the same as in the one-copy case, namely, 0.8571. For this state we have $\mathrm{Tr}[W_2(-\frac{1}{2})\tilde{\pi}] = -0.0013$.

As our calculations show that $W_2(-\frac{1}{2})$ is an optimal witness, $W_2(\beta)$ is an entanglement witness for $-\frac{1}{2} \leqslant \beta \leqslant -\frac{1}{3}$. Notice that these calculations are exact and show that qutrit Werner states are two-undistillable in this interval of β. On the other hand, they can be activated by the families of PPT entangled states detected by $W_2(-\frac{1}{2})$. The best results so far showed two-undistillability analytically [8,9] in the region $-0.417 \leqslant \beta \leqslant -\frac{1}{3}$, and provided numerical evidence in $-\frac{1}{2} \leqslant \beta \leqslant -0.417$.

We note that our calculations have been made in a laptop with 1 GByte of memory, and we used the symmetry of the Werner states to reduce the parameters in the optimization problem. In a larger computer, maybe the three-copy case could be handled, but the four-copy case would need more than 5 GBytes just to load the basis set (28). One could try to explore the symmetry to reduce the matrices' size, but in the face of the constraints in the robust SDP [cf. last line of Eq. (10)], which is also the most memory consuming part of the calculations, it is far from trivial.

VI. W_N OPERATOR

Although we are computationally limited to calculations for the two-copy case, we would like to understand the properties of the operator W_N when the number of copies (N) increases, hoping to shed light on the general problem. Indexing the copies, we can write explicitly

$$W_N = P_2 \otimes \rho_{w1}^{T_A} \otimes \rho_{w2}^{T_A} \otimes \cdots \otimes \rho_{wN}^{T_A}. \quad (26)$$

Note that it is normalized $[\mathrm{Tr}(W_N) = 1]$ and tracing out the Nth copy yields the operator for $N-1$ copies,

$$\mathrm{Tr}_N(W_N) = W_{N-1}. \quad (27)$$

Now we introduce a basis set, which allows us to write W_N as a polynomial, generalizing Eqs. (13) and (24).

Define the following basis of orthogonal projectors for the N-copy case:

$$B_1^N = P_2 \otimes P_d^{\otimes N},$$

$$B_{j+1}^N = P_2 \otimes \frac{1}{(N-j)!j!}\sum_{i=1}^{N!} \hat{\mathcal{P}}_i [P_d^{\otimes (N-j)} \otimes (I_d - P_d)^{\otimes j}],$$

$$B_{N+1}^N = P_2 \otimes (I_d - P_d)^{\otimes N}, \quad (28)$$

with $B_{j+1}^N \in \mathcal{B}(H_A \otimes H_B)$. The $\hat{\mathcal{P}}_i$ permute the elements in the tensor product, yielding an expression that is totally symmetric under exchange of any P_d and $(I_d - P_d)$. In this basis, W_N has the diagonal representation

$$W_N = \sum_{j=0}^{N} \lambda_{j+1} B_{j+1}^N,$$

$$\lambda_{j+1} = \frac{(1 + d\beta)^{N-j}}{(d^2 + d\beta)^N}. \quad (29)$$

We will show the correctness of this expression by induction. Note first that

$$\mathrm{Tr}(B_{j+1}^N) = \binom{N}{j}(d^2 - 1)^j, \quad (30)$$

where $\binom{N}{j}$ is the binomial coefficient. The basis for $N+1$ copies is related to the N-copy basis through the recurrence relation

$$B_{j+1}^N \otimes P_d + B_j^N \otimes (I_d - P_d) = B_{j+1}^{N+1}. \quad (31)$$

If we normalize the basis (28)

$$b_{j+1}^N \equiv \frac{B_{j+1}^N}{\binom{N}{j}(d^2 - 1)^j}, \quad (32)$$

we can rewrite Eq. (29) as

$$W_N = \frac{1}{(d^2 + d\beta)^N}\sum_{j=0}^{N}\binom{N}{j}(1 + d\beta)^{N-j}(d^2 - 1)^j b_{j+1}^N. \quad (33)$$

Now it is easy to see that the trace of Eq. (33) is 1;

$$\mathrm{Tr}(W_N) = \frac{\sum_{j=0}^{N} \binom{N}{j} (1+d\beta)^{N-j}(d^2-1)^j}{(d^2+d\beta)^N} = 1. \tag{34}$$

To finish the proof of the correctness of Eq. (29), we build W_{N+1} by appending ρ_w to W_N;

$$W_{N+1} = W_N \otimes \rho_w^{T_A} = \frac{1}{(d^2+d\beta)^{N+1}} \sum_{j=0}^{N} (1+d\beta)^{N-j} B_{j+1}^N \otimes [(I_d - P_d) + (1+d\beta)P_d]. \tag{35}$$

Splitting this sum in two parts and redefining the index in the second sum as $j+1=k$, we obtain

$$W_{N+1} = \frac{1}{(d^2+d\beta)^{N+1}} \left\{ \sum_{j=0}^{N} (1+d\beta)^{N+1-j} B_{j+1}^N \otimes P_d + \sum_{k=1}^{N+1} (1+d\beta)^{N+1-k} B_k^N \otimes (I_d - P_d) \right\}. \tag{36}$$

Writing out explicitly the terms for $j=0$ and $k=N+1$, we arrive at

$$W_{N+1} = \frac{1}{(d^2+d\beta)^{N+1}} \left\{ (1+d\beta)^{N+1} B_1^N \otimes P_d + \sum_{j=1}^{N} (1+d\beta)^{N+1-j} [B_{j+1}^N \otimes P_d + B_j^N \otimes (I_d - P_d)] + B_{N+1}^N \otimes (I_d - P_d) \right\}. \tag{37}$$

Finally, using the recurrence relation (31) and the basis definition (28), we obtain the desired result,

$$W_{N+1} = \frac{1}{(d^2+d\beta)^{N+1}} \left\{ (1+d\beta)^{N+1} B_1^{N+1} + \sum_{j=1}^{N} (1+d\beta)^{N+1-j} B_{j+1}^{N+1} + B_{N+2}^{N+1} \right\} = \frac{1}{(d^2+d\beta)^{N+1}} \sum_{j=0}^{N+1} (1+d\beta)^{N+1-j} B_{j+1}^{N+1}. \tag{38}$$

Once the correctness of Eq. (29) is proved, we highlight an interesting property of the W_N operator for Werner states. If $|1+d\beta| < |d^2+d\beta|$, then all the eigenvalues of $W_N(\beta)$ go to zero when N tends to infinity, for any $-1 \leqslant \beta \leqslant 1$. This is an expected property. If $W_N(-\frac{1}{2})$ is an EW for some N, it is necessarily optimal, and if properly normalized, it furnishes the random robustness (Rr) for a family of entangled states (π), i.e., $\mathrm{Rr} = -(2d)^{2N}\mathrm{Tr}[W_N(-\frac{1}{2})\pi]$ [16], and we see that the entanglement, as measured by the random robustness, increases with N.

We can also work out an expression analogous to Eq. (29) for a tensor product of Werner states;

$$\rho_w^{\otimes N} = \frac{(I_d + \beta F_d)^{\otimes N}}{(d^2+d\beta)^N}. \tag{39}$$

We construct the following basis set, which has the same algebraic structure of Eq. (28), although it is not orthogonal and only the last element is a positive operator:

$$A_1^N = f_d^{\otimes N},$$
$$A_{j+1}^N = \hat{S}[f_d^{\otimes(N-j)} \otimes (I_d - f_d)^{\otimes j}],$$
$$A_{N+1}^N = (I_d - f_d)^{\otimes N}, \tag{40}$$

with $A_{j+1}^N \in \mathcal{B}(H_A \otimes H_B)$, and $f_d \equiv F_d/d$. Note that this basis is obtained by discarding P_2 in Eq. (28) and by taking the partial transpose of P_d. $\hat{S}$ is the symmetrizer $\left(\hat{S} \equiv \frac{1}{(N-j)!j!}\Sigma_{i=1}^{N!}\hat{\mathcal{P}}_i\right)$ appearing in Eq. (28). In this basis, $\rho_w^{\otimes N}$ reads

$$\rho^{\otimes N} = \frac{1}{(d^2+d\beta)^N} \sum_{j=0}^{N} (1+d\beta)^{N-j} A_{j+1}^N. \tag{41}$$

Note that A_{N+1}^N is a fully separable positive operator. In particular, if we take $\beta = -\frac{1}{d}$, then

$$\rho^{\otimes N} = \frac{A_{N+1}^N}{\mathrm{Tr}(A_{N+1}^N)} = \frac{A_{N+1}^N}{(d^2-1)^N}. \tag{42}$$

Normalizing the basis

$$a_{j+1}^N \equiv \frac{A_{j+1}^N}{\mathrm{Tr}(A_{j+1}^N)} = \frac{A_{j+1}^N}{\binom{N}{j}(d^2-1)^j}, \tag{43}$$

$\rho^{\otimes N}$ reads

$$\rho^{\otimes N} = \frac{1}{(d^2+d\beta)^N} \sum_{j=0}^{N} \binom{N}{j} (1+d\beta)^{N-j}(d^2-1)^j a_{j+1}^N. \tag{44}$$

Then, for $\beta = -\frac{1}{d}$, $\rho^{\otimes N} = a_{N+1}$. When we take N to infinity, the binomial coefficients in Eq. (44) with $j \neq 0$ and $j \neq N$ dominate the sum, but nothing special seems to occur.

VII. CONCLUSIONS

We have done exact numerical calculations, which show the two-undistillability of qutrit Werner states in the region $-\frac{1}{2} \leqslant \beta \leqslant -\frac{1}{3}$. We have shown that $W_N(-\frac{1}{2})$ is an optimal entanglement witness (for $N=1,2$), and this is our certificate of one and two undistillability. We have derived families of PPT entangled states, which activate the one- and two-undistillable Werner states. We have introduced a basis of orthogonal projectors to expand $W_N(\beta)$, which shows that this operator is a polynomial in $(1+d\beta)$, and it acts in a state space, which is a polytope. The eigenvalues of $W_N(\beta)$ tend to zero when we consider infinite many copies of Werner states and it is a property related to the random robustness of the states it detects. We have also introduced a basis that provides a simple polynomial expression for a tensor product of Werner states.

ACKNOWLEDGMENTS

R.O.V. is grateful to Paulo H. Souto Ribeiro and Fernando G. S. L. Brandão for their criticism and suggestions on the manuscript. R.O.V. also acknowledges the financial support provided by the Brazilian agencies CAPES, FAPEMIG, CNPQ, and Instituto do Milênio de Informação Quântica MCT/CNPq.

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