One-Dimensional Interacting Anyon Gas: Low-Energy Properties and Haldane Exclusion Statistics

M. T. Batchelor,[1] X.-W. Guan,[1] and N. Oelkers[2]

[1]Department of Theoretical Physics, Research School of Physical Sciences and Engineering, and Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia
[2]Department of Mathematics, University of Queensland, Brisbane QLD 4072, Australia

(Received 3 March 2006; published 1 June 2006)

The low-energy properties of the one-dimensional anyon gas with a δ-function interaction are discussed in the context of its Bethe ansatz solution. It is found that the anyonic statistical parameter and the dynamical coupling constant induce Haldane exclusion statistics interpolating between bosons and fermions. Moreover, the anyonic parameter may trigger statistics beyond Fermi statistics for which the exclusion parameter α is greater than one. The Tonks-Girardeau and the weak coupling limits are discussed in detail. The results support the universal role of α in the dispersion relations.

DOI: 10.1103/PhysRevLett.96.210402 PACS numbers: 05.30.−d, 02.30.Ik, 05.30.Pr, 71.10.−w

Anyons, which are used to describe particles with generalized fractional statistics [1,2], are becoming of increasing importance in condensed matter physics [3] and quantum computation [4]. The concept of anyons provides a successful theory of the fractional quantum Hall (FQH) effect [5]. In particular, the signature of fractional statistics has recently been observed in experiments on the elementary excitations of a two-dimensional electron gas in the FQH regime [3]. These developments are seen as promising opportunities for further insight into the FQH effect, quantum computation, superconductivity, and other fundamental problems in quantum physics.

In one dimension, collision is the only way to interchange two particles. Accordingly, interaction and statistics are inextricably related in 1D systems. The 1D Calogero-Sutherland model is seen to obey fractional exclusion statistics [6,7]. In the sense of Haldane exclusion statistics, the 1D interacting Bose gas is equivalent to the ideal gas with generalized fractional statistics [8,9]. We consider an integrable model of anyons with a δ-function interaction introduced and solved by Kundu [10]. Here we obtain the low-energy properties and Haldane exclusion statistics of this 1D anyon gas. We find that the low energies, dispersion relations, and the generalized exclusion statistics depend on both the anyonic statistical and the dynamical interaction parameters. The anyonic parameter not only interpolates between Bose and Fermi statistics, but can trigger statistics beyond Fermi statistics in a super Tonks-Girardeau (TG) gaslike phase.

Bethe ansatz solution.—We consider N anyons with a δ-function interaction in one dimension with Hamiltonian [10]

$$H = \frac{\hbar^2}{2m}\int_0^L dx \partial\Psi^\dagger(x)\partial\Psi(x) + \frac{1}{2}g_{1D}\int_0^L dx \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) \qquad (1)$$

and periodic boundary conditions (BC). Here, m denotes the atomic mass, g_{1D} is the coupling constant, and x is a coordinate in length L. $\Psi^\dagger(x)$ and $\Psi(x)$ are the creation and annihilation operators at point x satisfying the anyonic commutation relations

$$\Psi(x_1)\Psi^\dagger(x_2) = e^{-i\kappa w(x_1,x_2)}\Psi^\dagger(x_2)\Psi(x_1) + \delta(x_1 - x_2),$$
$$\Psi^\dagger(x_1)\Psi^\dagger(x_2) = e^{i\kappa w(x_1,x_2)}\Psi^\dagger(x_2)\Psi^\dagger(x_1). \qquad (2)$$

Here the multistep function $w(x_1, x_2) = -w(x_2, x_1) = 1$ for $x_1 > x_2$, with $w(x, x) = 0$. The coupling constant is determined by $g_{1D} = \hbar^2 c/m$, where the coupling strength c is tuned through an effective 1D scattering length a_{1D} via confinement in experiments. Hereafter we set $\hbar = 2m = 1$ for convenience. We also use a dimensionless coupling constant $\gamma = c/n$ to characterize different physical regimes of the anyon gas, where $n = N/L$ is the linear density.

In contrast to the 1D Bose gas [11], Hamiltonian (1) exhibits both anyonic statistical and dynamical interactions, which can map into a 1D interacting Bose gas with multi-δ-function and momentum-dependent interactions [10]. In Ref. [12] the authors have proposed a way to observe the fractional statistics of anyons in a system of ultracold bosonic atoms in a rapidly rotating trap.

Define a Fock vacuum state $\Psi(x)|0\rangle = 0$ and assign all particle coordinates x_i in an order $x_1 \le x_2 \le \cdots \le x_N$. The N-particle eigenstate is

$$|\Phi\rangle = \int_0^L dx^N e^{-i\kappa N/2}\chi(x_1 \ldots x_N)\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_N)|0\rangle, \qquad (3)$$

where the Bethe ansatz wave function is written as

$$\chi(x_1 \ldots x_N) = e^{-i\kappa/2\sum_{x_i<x_j}^N w(x_i,x_j)} \times \sum_P A(k_{P1}\ldots k_{PN})e^{i(k_{P1}x_1+\cdots+k_{PN}x_N)}. \qquad (4)$$

0031-9007/06/96(21)/210402(4)

Here the sum extends over all $N!$ permutations P. In the N-particle eigenstate the order with which the particles are created incurs the phase factor in the wave function (4). Integration involves the changes of the order in creating particles due to the permutation of coordinates. We easily see that the wave function satisfies the anyonic symmetry $\chi(\cdots x_i \cdots x_j \cdots) = e^{-i\theta}\chi(\cdots x_j \cdots x_i \cdots)$, in which the anyonic phase $\theta = \kappa[\sum_{k=i+1}^{j} w(x_i, x_k) - \sum_{k=i+1}^{j-1} w(x_j, x_k)]$ for $i < j$. We extract a global phase factor $e^{-i\kappa N/2}$ in order to symmetrize the anyonic phase factor in the wave function (4) so that it has $\kappa \to \kappa + 4\pi$ symmetry. The eigenstate still has $\kappa \to \kappa + 2\pi$ symmetry. However, the phase factors in the multivalued wave function (4) are diminished by those from permutations of the particles in the eigenstate $|\Phi\rangle$ such that the integrand in (3) is single valued.

Solving the eigenvalue problem for Hamiltonian (1) reduces to solving the quantum mechanics problem $H_N\chi(x_1 \ldots x_N) = E\chi(x_1 \ldots x_N)$, where

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{1\leq i<j\leq N}\delta(x_i - x_j) \qquad (5)$$

describes the 1D δ-function interacting quantum gas of N anyons confined in a periodic length L.

The $N!$ coefficients $A(k_{P1} \ldots k_{PN})$ are obtained via the two-body scattering relation $A(\ldots k_j, k_i \ldots) = \frac{k_j - k_i + ic'}{k_j - k_i - ic'}A(\ldots k_i, k_j \ldots)$, which follows from the discontinuity condition in the derivative of the wave function and the condition to ensure a continuous probability density with regard to the eigenstate (3). Here the anyonic parameter κ and the dynamical interaction c are inextricably related via the effective coupling constant $c' = c/\cos(\kappa/2)$ [10]. This results in a resonancelike effect in the effective coupling constant c' with respect to the statistical interaction around $\kappa = \pi$, see Fig. 1.

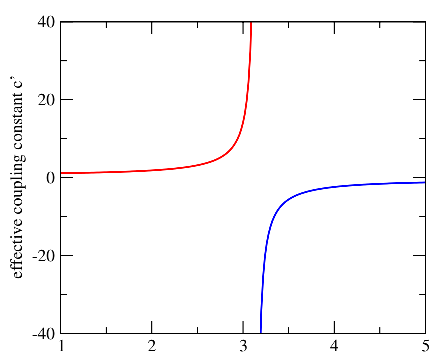

FIG. 1 (color online). The effective coupling constant c' (in units of c) vs the anyonic parameter κ. A key feature of the model is that the anyonic statistical interaction induces a resonancelike behavior where the interaction strength becomes very large.

Applying the periodic BC $\chi(x_1 = 0, x_2 \ldots x_N) = \chi(x_2 \ldots x_N, x_1 = L)$, leads to the eigenvalue $E = \sum_{j=1}^{N} k_j^2$, where the individual quasimomenta k_j satisfy the Bethe ansatz equations (BAE)

$$e^{ik_jL} = -e^{i\kappa(N-1)}\prod_{l=1}^{N}\frac{k_j - k_l + ic'}{k_j - k_l - ic'} \qquad (6)$$

for $j = 1, \ldots, N$. These equations differ slightly from those of Ref. [10]. The Bethe roots k_j are real for $c' > 0$, but may become complex for $c' < 0$. In this way we see that the 1D interacting anyons with periodic BC are equivalent to a 1D δ-function interacting Bose gas with twisted BC, where the interaction strength is tuned via c'.

For $\kappa = 0$ the BAE (6) reduce to those of the 1D Lieb-Liniger Bose gas [11]. When $\kappa = \pi$ the BAE characterize free fermions. When $c = 0$ the anyons may collapse into a condensation state with purely anyonic statistical interaction. In general the extra phase factor in the BAE (6), picking up the statistical interaction during the scattering process, shifts the system into higher excitation states, as if there exists a self-sustained Aharonov-Bohm-like flux [13]. The total momentum is $p = N(N-1)\kappa/L + 2d\pi/L$, where d is an arbitrary integer. In minimizing the energy we consider $\kappa(N-1) = \nu \pmod{2\pi}$ in the phase factor with $-\pi \leq \nu \leq \pi$. Each quasimomentum k_j shifts to $k_j + \nu/L$ in the ground state. In the thermodynamic limit, the lowest energy is given by $E = N(n^2 e(\gamma, \kappa) + \nu^2/L^2)$, where $e(\gamma, \kappa) = \frac{\gamma^3}{\lambda^3}\int_{-1}^{1} g(x)x^2dx$. The root density $g(x)$ and the parameter $\lambda = c/Q$, where Q is the cutoff momentum, are determined by Lieb-Liniger-type integral equations

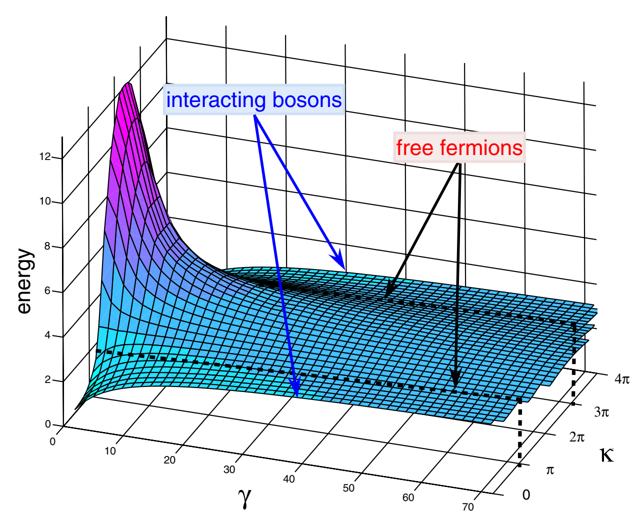

FIG. 2 (color online). The lowest energy $e(\gamma, \kappa)$ in units of n^2 obtained from the integral Eq. (7). For $0 < \kappa < \pi$ the energy curve interpolates between interacting bosons at $\kappa = 0$ and free fermions at $\kappa = \pi$ (dashed line). For $\pi < \kappa < 3\pi$ the effective interaction c' is negative. The super TG gaslike phase is seen in the strong coupling limit $\gamma \gg 1$. For $3\pi < \kappa < 4\pi$ the interpolation is from free fermions to interacting bosons.

$$g(x) = \frac{1}{2\pi} + \frac{\lambda\cos(\kappa/2)}{\pi}\int_{-1}^{1}\frac{g(y)dy}{\lambda^2+\cos^2(\frac{\kappa}{2})(x-y)^2},\qquad \lambda = \gamma\int_{-1}^{1} g(x)dx. \tag{7}$$

Figure 2 shows the energy $e(\gamma, \kappa)$ evaluated numerically from (7) for $\gamma > 0$ and $\kappa \in [0, 4\pi]$.

*Low-energy behavior.—*The low-energy behavior provides significant insight into the nature of the anyonic statistics interaction and the dynamical interaction. In this model the effective coupling constant c' implements the transmutation between statistical and dynamical interactions. In the weak coupling limit $\gamma \ll \cos(\kappa/2)$, the leading term for the lowest energy per particle is obtained from the BAE (6) as $E/N = (N-1)c'/L + \nu^2/L^2$. Here the anyonic statistics shift the energy upwards and the energy increases faster than the ground state energy of the pure Bose gas as γ increases. The fractional statistics are mutual, i.e., the Haldane exclusion parameter discussed below is not a constant.

The experimental realization of the TG gas [14] has shed further light on the quantum nature of 1D many-body systems. In particular, on the fermionization of bosons in one dimension, which can be experimentally realized via tuning the interaction strength. The generalized exclusion statistics vary from Bose statistics to Fermi statistics during the fermionization process. This may provide opportunities to investigate generalized exclusion statistics in future experiments. In the TG regime, i.e., $\gamma \gg 1$, the anyonic statistical interaction may trigger another regime in which the density-density correlations are more strongly correlated than in the TG gas, namely, the super TG gas [15,16]. Here this super TG gaslike phase is seen to be stable because there exists a large kinetic energy inherited from the TG phase as the anyonic parameter κ is tuned smoothly from $\kappa < \pi$ to $\kappa > \pi$. In this way, the hard core behavior of the particles with Fermi-like pressure prevents the collapse of the super TG phase. The statistics-induced super TG phase ($\pi < \kappa < 3\pi$) appears only in the strong coupling limit. It may become unstable as the interaction strength becomes weaker due to the appearance of bound states. In general the anyonic parameter κ implements a range of different statistical phases, from the Bose gas to the TG gas, from the free Fermi phase to the super TG phase.

In the TG regime the lowest energy per particle is $E_0/N \approx \frac{\pi^2}{3L^2}(N^2-1)[1-4\gamma^{-1}\cos(\kappa/2)] + \nu^2/L^2$ with the impenetrable fermionic distribution $\{\pm k_{2m}, m = 1, \ldots, (N-1)/2\}$ for odd N, where $k_l = \frac{l\pi}{L} \times [1-2\gamma^{-1}\cos(\kappa/2)] + \nu/L$. In the thermodynamic limit and at zero temperature, the last term in E_0/N can be ignored compared to the kinetic and interaction energies. Now consider the effect of the anyonic statistical interaction on the linear dispersion relation for the lowest excitation. The elementary lowest excitation is obtained by moving the largest quasimomentum k_N from the Dirac sea to $k_N + p$. To $O(p^2)$, the low-lying excitation close to the Fermi point

$$E = E_0 + p\left[\frac{2(N-1)\pi}{L}\left(1 - \frac{4\cos(\kappa/2)}{\gamma}\right) + \frac{2\nu}{L}\right] \tag{8}$$

follows from the discrete BAE (6). The dispersion remains linear, with sound velocity $v_c = v_F[1 - 4\gamma^{-1}\cos(\kappa/2)]$ as $p \to 0$ in the thermodynamic limit. In the above equation, the term $2p\nu/L$ is irrelevant. Here the Fermi velocity $v_F = 2\pi n$. The finite-size corrections to the lowest energy in the thermodynamic limit for strong coupling are directly given by $E_0(N, L) - Le_0^{(\infty)} = -\frac{\pi C v_c}{6L} + O(1/L^2)$ with central charge $C = 1$.

The thermodynamic BAE (TBA) is the key equation for understanding Haldane exclusion statistics of the model. Following the Yang-Yang approach [17], the TBA and the thermodynamic potential are given by

$$\epsilon(k) = \epsilon^0(k) - \mu - \frac{T}{2\pi}\int_{-\infty}^{\infty} dk'\theta'(k-k')\ln(1+e^{-\epsilon(k')/T}), \tag{9}$$

$$\Omega = -\frac{T}{2\pi}\int_{-\infty}^{\infty} dk \ln(1+e^{-\epsilon(k)/T}). \tag{10}$$

Here T is the temperature, $\epsilon(k)$ is the dressed energy, $\theta'(x) = \frac{2c\cos(\kappa/2)}{c^2+\cos^2(\kappa/2)x^2}$, and $\epsilon^0(k) = (k+\nu/L)^2$. In general this TBA result is only valid for the case $c' > 0$. We also consider the TBA to be valid for the super TG gas phase, when bound states do not form.

*Haldane exclusion statistics.—*The crucial point of Haldane exclusion statistics is that the number of available single-particle states of species i, denoted by d_i, depends on the number of other species $\{N_j\}$ when adding one particle of the ith species to the system while keeping the boundary conditions unchanged [2]. Following Refs. [8,18], we define

$$d_i(\{N_j\}) = G_i^0 - \sum_j \alpha_{ij}N_j. \tag{11}$$

Here $G_i^0 = d_i(\{0\})$ is the number of available single-particle states with no particles present in the system, called the bare number of available single-particle states. Haldane [2] defined the fractional statistical interactions α_{ij} through a linear relation $\Delta d_i/\Delta N_j = -\alpha_{ij}$ with total number of particles $N = \sum_j N_j$. As remarked in Ref. [8], this definition allows different species to refer to identical particles with different momenta. The total energy is given by $E = \sum_i N_i\epsilon_i$, where ϵ_i is the energy of a particle of species i. For the ideal gas with no mutual statistics $\alpha_{ij} = \alpha\delta_{ij}$. The statistical distribution is then given by $n_i = 1/(e^{\bar{\epsilon}_i/T} + \alpha)$, where the function $\bar{\epsilon}_i$ satisfies [8]

$$\bar{\epsilon}_i + T(1-\alpha)\ln(1+e^{\bar{\epsilon}_i/T}) = \epsilon_i - \mu. \tag{12}$$

The exclusion statistics are clearly seen from this relation.

For instance, $\alpha = 0$ and $\alpha = 1$ are Bose and Fermi statistics, respectively.

In order to apply Haldane statistics to the anyon model (1), we take a similar approach as that used for the interacting Bose gas [8]. At zero temperature there are no holes in the ground state. Excitations arise from moving quasimomenta out of the Dirac sea. Particle excitations leave d_i holes in a momentum interval Δk_i. Thus all accessible states in Δk_i are $D_i = L(\rho + \rho_{\rm h})\Delta k_i$. ρ and $\rho_{\rm h}$ are the density of occupied states and the density of holes in interval Δk_i, respectively. For an arbitrary state, the BAE (6) become

$$\rho + \rho_{\rm h} = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk' \theta'(k - k')\rho(k'). \quad (13)$$

On the other hand, from the definition d_i in (11), we have $G_i^0 = \Delta k_i L/2\pi$, so substituting d_i into $D_i = d_i + N_i - 1$ in the thermodynamic limit gives

$$\rho + \rho_{\rm h} = \frac{1}{2\pi} + \int_{-\infty}^{\infty} [\delta(k, k') - \alpha(k, k')]\rho(k')dk'. \quad (14)$$

Comparison of (13) and (14) thus gives [19]

$$\alpha_{ij} := \alpha(k, k') = \delta(k, k') - \frac{1}{2\pi}\theta'(k - k'). \quad (15)$$

It is clearly seen that the leading order of the off-diagonal contribution to $\alpha(k, k')$ is proportional to $(k - k')^2/c^3$ at low temperatures as $c \to \infty$. Here we require that the dynamical interaction c overwhelms the thermal fluctuations. It follows that the exclusion statistics $\alpha(k, k') \approx \alpha\delta(k, k')$ are independent of the quasimomenta at low temperatures. From (14) and the root distributions for the ground state, we thus find the Haldane exclusion statistics parameter $\alpha \approx 1 - 2\gamma^{-1}\cos(\kappa/2)$. The above relations suggest that $\alpha = -\Delta\rho_h(k)/\Delta\rho(k)$. The meaning of α is that one particle excitation is accompanied by α hole excitations. It is interesting to note that the super TG phase corresponds to exclusion statistics with $\alpha > 1$ as $\pi < \kappa < 3\pi$. From the roots of the BAE (6) we have the relation $\Delta k_i = k_{i+1} - k_i = \frac{2\pi}{L}(\alpha + l)$, where l is a positive integer for an arbitrary state. This relation was also noticed in the study of exclusion statistics in the Calogero-Sutherland model [6,7] and provides further evidence for its universality [20]. Further, the quasiparticle dispersion relation can be expressed as $E - E_0 \approx (p^2 + 2k_{\rm F}p)\alpha^2$ as $p \to 0$, where $k_{\rm F} = n\pi$ is the Fermi momentum.

It is clear to see from (12) that as $T \to 0$, $n(k) \approx 1/\alpha$ if $\epsilon^0(k) \leq \mu_0$, where $\mu_0 \approx k_F^2\alpha^2$ is the chemical potential at $T = 0$. Also $n(k) = 0$ if $\epsilon^0(k) > \mu_0$. These results coincide with the BAE result for $\rho(k)$ via the relation $\rho(k) = g^0(k)n(k)$ with $g^0(k) = 1/2\pi$ at zero temperature. On the other hand, in the weak coupling limit $c' \to 0$, the quasimomentum distributions are not uniform. The mutual statistics are governed by $\alpha(k, k') \approx -\frac{c'}{\pi(k-k')^2}$. Thus, if $c = 0$, one recovers Bose statistics.

To conclude, we have derived the low-energy properties and the Haldane exclusion statistics of the 1D integrable anyon gas. We have given analytic expressions for the ground state energy, dispersion relations, finite-size corrections, and the Haldane statistical parameter and have explicitly considered the strong and weak coupling regimes. We found that the anyonic statistical interaction and the dynamical interaction implement a continuous range of Haldane exclusion statistics, from Bose statistics to Fermi statistics and beyond.

The authors thank C. Lee, H.-Q. Zhou, and M. Bortz for helpful discussions and the Australian Research Council for support.

[1] J.M. Leinaas and J. Myrheim, Nuovo Cimento Soc. Ital. Fis. B **37**, 1 (1977); F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982); A. Khare, *Fractional Statistics and Quantum Theory* (World Scientific, Singapore, 2005).
[2] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[3] F.E. Camino, W. Zhou, and V.J. Goldman, Phys. Rev. B **72**, 075342 (2005); E.-A. Kim, M. Lawler, S. Vishveshwara, and E. Fradkin, Phys. Rev. Lett. **95**, 176402 (2005).
[4] S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005); N.E. Bonesteel, L. Hormozi, G. Zikos, and S.H. Simon, Phys. Rev. Lett. **95**, 140503 (2005).
[5] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); *Fractional Statistics and Anyon Superconductivity*, edited by F. Wilczek (World Scientific, Singapore, 1990).
[6] Z.N.C. Ha, Phys. Rev. Lett. **73**, 1574 (1994); M.V.N. Murthy and R. Shankar, Phys. Rev. Lett. **73**, 3331 (1994).
[7] Z.N.C. Ha, Nucl. Phys. B **435**, 604 (1995). The idea of the 1D anyon gas was suggested in this paper.
[8] Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994); D. Bernard and Y.-S. Wu, cond-mat/9404025.
[9] S.B. Isakov, Phys. Rev. Lett. **73**, 2150 (1994).
[10] A. Kundu, Phys. Rev. Lett. **83**, 1275 (1999).
[11] E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[12] B. Paredes, P. Fedichev, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **87**, 010402 (2001).
[13] J.X. Zhu and Z.D. Wang, Phys. Rev. A **53**, 600 (1996).
[14] B. Paredes *et al.*, Nature (London) **429**, 277 (2004); T. Kinoshita, T. Wenger, and B.S. Weiss, Science **305**, 1125 (2004).
[15] G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004); G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005).
[16] M.T. Batchelor, M. Bortz, X.-W. Guan, and N. Oelkers, J. Stat. Mech. (2005) L10001.
[17] C.N. Yang and C.P. Yang, J. Math. Phys. (N.Y.) **10**, 1115 (1969).
[18] C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994); D. Sen and R.K. Bhaduri, Phys. Rev. Lett. **74**, 3912 (1995).
[19] This relation provides a direct way to calculate the thermodynamics of the TG gas via exclusion statistics.
[20] B. Sutherland, Phys. Rev. B **56**, 4422 (1997).