Better Bell-inequality violation by collective measurements

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(Received 6 April 2006; published 30 May 2006)

The standard Bell-inequality experiments test for violation of local realism by repeatedly making local measurements on individual copies of an entangled quantum state. Here we investigate the possibility of increasing the violation of a Bell inequality by making collective measurements. We show that the nonlocality of bipartite pure entangled states, quantified by their maximal violation of the Bell-Clauser-Horne inequality, can always be enhanced by collective measurements, even without communication between the parties. For mixed states we also show that collective measurements can increase the violation of Bell inequalities, although numerical evidence suggests that the phenomenon is not common as it is for pure states.

DOI: 10.1103/PhysRevA.73.052116 PACS number(s): 03.65.Ud, 03.67.Mn

I. INTRODUCTION

It is one of the most remarkable features of quantum physics that measurements on separated systems cannot always be described by local realistic theories [1–6]. Typically, this phenomenon is revealed by the violation of a Bell inequality, which are constraints that have to be satisfied by any local realistic description. Bell-inequality violations have been observed experimentally in various physical systems, such as entangled photon pairs, as reviewed in Ref. [7] and entangled $^{9}Be^{+}$ ions [8]. For a background on Bell inequalities readers are referred to Ref. [9], and references therein.

Usually, experiments to test Bell inequalities involve making many measurements on individual copies of the quantum system with the system being prepared in the same way for each measurement. In this paper, we consider a somewhat different scenario and ask if quantum nonlocality can be enhanced by making joint local measurements on multiple copies of the entangled state. We will use the maximal Bell-inequality violation of a quantum state ρ as our measure of nonlocality. Our interest is to determine if $\rho^{\otimes N}$, when compared with ρ, can give rise to a higher Bell-inequality violation for some $N>1$.

A very similar problem was introduced by Peres [10] who considered Bell-inequality violations under collective measurements but allowed the experimenters to make an auxiliary measurement on their systems and postselect on both getting a specific outcome of their measurement. Numerically, Peres showed that with collective measurements and postselection [11], a large class of two-qubit states give rise to better Bell-inequality violation. Moreover, explicit examples were given to illustrate that collective measurements with postselection can be used to detect the nonlocality of a larger set of entangled states.

That postselection can be used to reveal such "hidden nonlocality," was already shown in 1994 by Popescu [12] using sequential measurements. After that, Gisin [13] also demonstrated that (without collective measurements) postselection itself in the form of local filtering operations can be used to detect a larger set of two-qubit entangled states. It is worth noting that an experimental demonstration of hidden nonlocality has been reported in Ref. [14].

In this paper, we will show that postselection is not necessarily to improve Bell-inequality violation. In order to find such examples for mixed states we have resorted to various numerical approaches that are described in Ref. [15] and provide upper and lower bounds on the optimal violation of a given Bell inequality by a given quantum state. The two algorithms described in Ref. [15] make use of convex optimization techniques, specifically semidefinite programs [16,17]. The first, henceforth referred as the LB algorithm, is an algorithm that can be used to determine, for a given quantum state ρ, a lower bound of its maximal violation of a given Bell inequality. This can be seen as an extension of the See-Saw iteration developed by Werner and Wolf [9] to Bell inequalities with more than two outcomes. As with many other numerical optimization techniques, the LB algorithm converges to a local maximum of the global optimization problem, and hence, feeding the algorithm with various random initial guesses is essential. Unless otherwise stated, Bell-inequality violations presented hereafter refer to the best violation that we could find either analytically, or numerically using this LB algorithm.

Complementarily, the other algorithm, which we shall refer as the UB algorithm, is one that can be used to determine an upper bound on the maximal violation of ρ for a given Bell inequality. The technique involves relaxing the complicated optimization over measurements in the Bell experiment to a sequence of semidefinite programs using techniques that have been developed in the general context of nonlinear optimization theory [18,19] and applied in quantum information theory in other contexts [20,21]. These methods provide global upper bounds on the Bell-inequality violation that can be accurately and efficiently computed. The upper bounds obtained via this algorithm are often not tight, but are sometimes nontrivial [15]. For ease of reference, these upper bounds are marked where they appear with †. In the event that a violation presented is known to be maximal (such as those computable using the Horodecki's criterion [22]), an * will be attached.

This paper is organized as follows. In Sec. II, we present a measurement scheme which we will use to determine the Bell-Clauser-Horne inequality violation for any bipartite

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pure state. These measurements led to the largest violation that we were able to find and may even be maximal. Then, in Sec. III, we show that for bipartite pure entangled states, collective measurements can lead to a greater violation of the Bell-CH inequality. The corresponding scenario for mixed entangled states is analyzed in Sec. IV. We then conclude with a summary of results and some future avenues of research.

II. BELL-CH VIOLATION FOR PURE TWO QUDITS

In this section, we present a measurement scheme which gives rise to the largest Bell-Clauser-Horne (henceforth abbreviated as Bell-CH) inequality [3] violation that we have found for arbitrary pure two-*qudit* states, i.e., quantum states describing a composite of two d-dimensional quantum subsystems. We find using this inequality for probabilities rather than correlations to be convenient for our purposes and the equivalence between the Bell-CH inequality and the Bell-Clauser-Horne-Shimony-Holt (henceforth abbreviated as Bell-CHSH) inequality [2] in the ideal limit, implies that if the conjectured measurement scheme is optimal for the Bell-CH inequality, it will also give rise to the maximal Bell-CHSH inequality violation for any pure two-qudit state.

The Bell-CH inequality is meant for an experimental setup involving two observers, Alice (A) and Bob (B). Each of these observers can perform two alternative measurements, and each measurement gives rise to two possible outcomes which we shall label by ±. The Bell-CH inequality is as follows [3]:

$$\mathcal{S}_{\rm lhv} = p_{AB}^{+-}(1,1) + p_{AB}^{+-}(1,2) + p_{AB}^{+-}(2,1) - p_{AB}^{+-}(2,2) - p_A^{+}(1) - p_B^{-}(1) \leq 0, \tag{1}$$

where $p_{AB}^{+-}(k,l)$ refers to the joint probability that experimental outcome + and − are observed at the site of A and B, respectively, given that Alice performs the kth and Bob performs the lth measurement; the marginal probabilities $p_A^{+}(k)$ and $p_B^{-}(l)$ are similarly defined. In quantum mechanics, these probabilities are calculated according to

$$p_{AB}^{+-}(k,l) = \mathrm{tr}(\rho A_k^{+} \otimes B_l^{-}),$$

$$p_A^{+}(k) = \mathrm{tr}(\rho A_k^{+} \otimes \mathbf{1}_B), \quad p_B^{-}(l) = \mathrm{tr}(\rho \mathbf{1}_A \otimes B_l^{-}), \tag{2}$$

where we have denoted by A_+^k the positive-operator-valued measure (POVM) element associated with the "+" outcome of Alice's kth measurement and B_-^l the POVM element associated with the "−" outcome of Bob's lth measurement.

The maximal Bell-inequality violation for a quantum state is invariant under a local unitary transformation. As such, the maximal Bell-inequality violation for any bipartite pure quantum state is identical to its maximal violation when written in the Schmidt basis [23,24]. In this basis, an arbitrary bipartite pure state in d dimension, $|\Psi_d\rangle$ takes the form $|\Psi_d\rangle = \Sigma_{i=1}^{d} c_i |\varphi_i\rangle_A |\varphi_i\rangle_B$, where $\{|\varphi_i\rangle_A\}$ and $\{|\varphi_i\rangle_B\}$ are local orthonormal bases of subsystem possessed by observer A and B, respectively, and $\{c_i\}_{i=1}^{d}$ are the Schmidt coefficients of $|\Psi_d\rangle$. Without loss of generality, we may also assume that $c_1 \geq c_2 \geq \cdots \geq c_d \geq 0$. Then $|\Psi\rangle_d$ is entangled if and only if $d > 1$. Now, let us consider the following measurement settings for Alice, which were first adopted in Ref. [5]:

$$A_1^{\pm} = \tfrac{1}{2}[\mathbf{1}_d \pm Z], \quad A_2^{\pm} = \tfrac{1}{2}[\mathbf{1}_d \pm X],$$

$$Z \equiv \oplus_{i=1}^{\lfloor d/2 \rfloor} \sigma_z + \Pi, \quad X \equiv \oplus_{i=1}^{\lfloor d/2 \rfloor} \sigma_x + \Pi,$$

$$[\Pi]_{ij} = 0 \ \ \forall\, i,j \neq d, \quad [\Pi]_{dd} = d \bmod 2, \tag{3}$$

where σ_x and σ_z are, respectively, the Pauli x and z matrices.

Note, however, that the $\{B_l^{\pm}\}_{l=1}^{2}$ given in Ref. [5] is not optimal. In fact, given the measurements for Alice in Eq. (3), the optimization of Bob's measurement settings can be carried out explicitly [25]. Using the resulting analytic expression for Bob's optimal POVM [15], the optimal expectation value of the Bell-CH operator [27] for $|\Psi_d\rangle$ can be computed and we find

$$\langle \mathcal{B}_{CH} \rangle_{|\Psi_d\rangle} = \frac{1}{2} \sum_{n=1}^{\lfloor d/2 \rfloor} \sqrt{(c_{2n-1}^2 + c_{2n}^2)^2 + 4 c_{2n}^2 c_{2n-1}^2} + \frac{\gamma}{2} c_d^2 - \frac{1}{2}, \tag{4}$$

where $\gamma \equiv d \bmod 2$ [28].

Effectively, this measurement scheme corresponds to first ordering each party's local basis vectors $\{|\varphi_i\rangle\}_{i=1}^{d}$ according to their Schmidt coefficients, and grouping them pairwise in descending order from the Schmidt vector with the largest Schmidt coefficient. Physically, this can be achieved by Alice and Bob each performing an appropriate local unitary transformation. Each of their Hilbert space can then be represented as a direct sum of two-dimensional subspaces, which can be regarded as a one-qubit space, plus a one-dimensional subspace if d is odd. The final step of the measurement consists of performing the optimal measurement [22] in each of these two-qubit spaces as if the other spaces did not exist.

From here, it is easy to see that if we have a maximally entangled state, i.e., $|\Psi_d\rangle_{\rm ME} = 1/\sqrt{d} \Sigma_{i=1}^{d} |\varphi_i\rangle_A |\varphi_i\rangle_B$, then (4) gives

$$\langle \mathcal{B}_{CH} \rangle_{|\Psi_d\rangle_{\rm ME}} = \begin{cases} \dfrac{1}{\sqrt{2}} - \dfrac{1}{2}^{*}: d \text{ even} \\[2ex] \dfrac{\sqrt{2}(d-1)+1}{2d} - \dfrac{1}{2}: d \text{ odd} \end{cases}. \tag{5}$$

Under this measurement scheme, the Bell-CH inequality violation for a maximally entangled state with even d is thus the maximum allowed by Cirelson's bound [29] whereas that of maximally entangled state with odd d is not.

How good is the measurement scheme (3)? It is constructed so that for two qubits, i.e., when $d=2$, (4) gives the same violation found in Refs. [4,5], and is the maximal violation determined by Horodecki *et al.* [22]. The measurement given by (3) is hence optimal for two-qubit states. For higher dimensional quantum systems, we have looked at randomly generated pure two-qudit states ($d=3, \ldots, 10$) with their (unnormalized) Schmidt coefficients uniformly chosen at random from the interval $(0,1)$. For all of the 20 000 states generated for each d, we found that with (3) as the initial

TABLE I. Best known Bell-CH inequality violation for some bipartite pure entangled states, obtained from (3) with and without collective measurements. Also included below is the upper bound of $\langle \mathcal{B}_{CH}\rangle_{|\Psi\rangle}$ obtained from the UB algorithm. The first column of the table gives the number of copies N involved in the measurements. Each quantum state is labeled by their nonzero Schmidt coefficients, which are separated by : in the subscripts attached to the ket vectors; e.g., $|\Psi\rangle_{1:2:3:3}$ is the state with unnormalized Schmidt coefficients $\{c_i\}_{i=1}^4=\{1,2,3,3\}$. For each quantum state there is a bold entry corresponding to the smallest N such that the lower bound of $\langle \mathcal{B}_{CH}\rangle_{|\Psi\rangle}$ on the maximal violation exceeds the single-copy upper bound coming from the UB algorithm. * represents maximal violation and † represents upper bounds on maximal violation.

N	$\lvert\Psi_{2:1}\rangle$	$\lvert\Psi_{1:1:1}\rangle$	$\lvert\Psi_{1:2:3}\rangle$	$\lvert\Psi_{1:2:3:4}\rangle$	$\lvert\Psi_{1:2:3:3}\rangle$	$\lvert\Psi_{1:1:1:1:1}\rangle$
	Lower bound					
1	0.14031*	0.13807	0.16756	0.18431	0.19259	0.16569
2	0.14031	0.18409	0.18307	0.19624	0.20516	0.19882
3	**0.16169**	**0.19944**	0.19451	0.20275	0.20685	0.20545
4	0.16169	0.20455	**0.19642**	0.20388	0.20706	**0.20678**
5	0.17964	0.20625	0.20254	0.20596	0.20710	0.20704
10	0.19590	0.20710	0.20643	0.20704	0.20711	0.20711
	Upper bound					
1	0.14031*	0.18409†	0.19624†	0.20711†	0.20711†	0.20569†

measurement setting, the (iterative) LB algorithm never gives a $\langle \mathcal{B}_{CH}\rangle_{|\Psi_d\rangle}$ that is different from (4) by more than 10^{-15}, thus indicating that (4) is, at least, a local maximum of the optimization problem.

Furthermore, for another 8000 randomly generated pure two-qudit states, 1000 each for $d=3,\ldots,10$, an extensive numerical search using more than 4.6×10^6 random initial measurement guesses has not led to a single instance where $\langle \mathcal{B}_{CH}\rangle_{|\Psi_d\rangle}$ is higher than that given in (4) [30]. These numerical results suggest that the measurement scheme given by (3) may be the optimal measurement that maximizes the Bell-CH inequality violation for arbitrary pure two-qudit states.

III. MULTIPLE COPIES OF PURE STATES

Let us now look into the problem of whether nonlocal correlations can be enhanced by performing collective measurements on $N>1$ copies of an entangled quantum state [31]. As our first example of nonlocality enhancement, consider again those maximally entangled states residing in Hilbert space with odd d. It is well known their maximal Bell-CH/Bell-CHSH inequality violation cannot saturate Cirelson's bound [32]. In fact, their best known Bell-CH inequality violation [5] is that given in (5). By combining N copies of these quantum states, it is readily seen that we effectively end up with another maximally entangled state of d^N dimension. It then follows from (5) that their Bell-CH violation under collective measurements increases monotonically with the number of copies N (see also Table I, columns 3 and 7). In fact, it can be easily shown that this violation approaches asymptotically the Cirelson's bound [29] in the limit of large N. Therefore, if the maximal violation of these quantum states is given by (5), collective measurements can already give better Bell-CH violation with $N=2$. Even if the maximal violation is not given by (5), it can be seen, by comparing the upper bound of the single-copy violation from the UB algorithm and the lower bound of the N-copy violation, from Table I that for $d=3$ and $d=5$, a Bell-CH violation better than the maximal single-copy violation can always be obtained when N is sufficiently large.

Such an enhancement is even more pronounced in the case of nonmaximally entangled states. In particular, for N copies of a (nonmaximally entangled) two-qubit state written in the Schmidt basis,

$$|\Psi_2\rangle^{\otimes N} = (\cos\phi|00\rangle + \sin\phi|11\rangle)^{\otimes N}, \tag{6}$$

where $0<\phi\leq\pi/4$ [33]. The Bell-CH violation given by (4) is

$$\langle \mathcal{B}_{CH}\rangle_{|\Psi_2\rangle} = \frac{p}{\sqrt{2}} + \frac{1-p}{2}\sqrt{1+\sin^2 2\phi} - \tfrac{1}{2}, \tag{7}$$

where

$$p = 1 - \tfrac{1}{2}\cos^{2(N-1)}\phi \sum_{m=0}^{N-1} \tan^{2m}\phi [1-(-1)^{(N-1)!/m!(N-1-m)!}],$$

is the total probability of finding $|\Psi_2\rangle^{\otimes N}$ in one of the *perfectly correlated two-dimensional subspaces* (i.e., a subspace with $c_{2n-1}=c_{2n}$) upon reordering of the Schmidt coefficients in descending order.

It is interesting to note that for these two-qubit states, their Bell-CH inequality violation for $N=2k-1$ copies, and $N=2k$ copies are identical [34] for all $k\geq 1$, as illustrated in the second column of Table I and in Fig. 1. This feature, however, does not seem to generalize to higher dimensions.

Like the odd-dimensional maximally entangled state, the violation of the Bell-CH inequality for *any* pure two-qubit entangled states, as given by (4), increases asymptotically towards the Cirelson bound [29] with the number of copies N, as can be seen in Fig. 1. A direct implication of this is that, with a sufficiently large number of copies, the nonlocality present in *any* weakly entangled pure two-qubit states is of no noticeable difference from that in a maximally entangled two-qubit state.

Similarly, if we consider N copies of pure two-qutrit entangled states written in the Schmidt form

$$|\Psi_3\rangle^{\otimes N} = (\cos\phi|00\rangle + \sin\phi\cos\theta|11\rangle + \sin\phi\sin\theta|22\rangle)^{\otimes N}, \tag{8}$$

where $0<\phi\leq\pi/4$, $0<\theta\leq\pi/4$, it can be verified that their Bell-CH inequality violation, as given by (4), also increases steadily with the number of copies. Thus, if (4) gives the maximal Bell-CH violation for pure two-qutrit states, a better Bell-inequality violation can also be attained by collective measurements using two copies of these quantum states. The explicit value of the violation can be found in columns 3 and 4 of Table I for two specific two-qutrit states. As above, even if the maximal Bell-CH violation is not given by (4), collec-

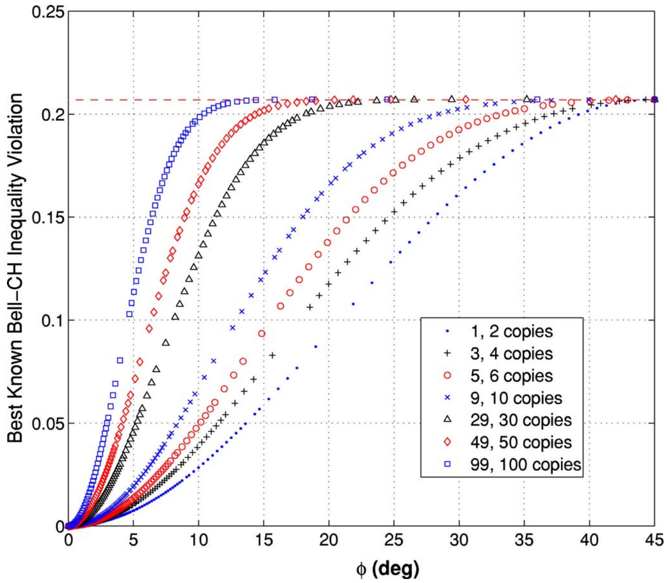

FIG. 1. (Color online) Best known Bell-CH inequality violation of pure two-qubit states obtained from (3), plotted as a function of ϕ, which gives a primitive measure of entanglement; $\phi=0$ for bipartite pure product state and $\phi=45°$ for bipartite maximally entangled state. The curves from right to left represent increasing numbers of copies. The dotted horizontal line at $1/\sqrt{2}-1/2$ is the maximal possible violation of Bell-CH inequality; correlations allowed by local realistic theories have values less than or equal to zero.

tive measurements with (3) can definitely give a violation that is better than the maximal-single-copy ones as a result of the bound coming from the UB algorithm for a single copy (see Table I). Corresponding examples for pure bipartite four-dimensional and five-dimensional quantum states can also be found in the table.

Some intuition for the way in which a better Bell-CH inequality violation may be obtained with collective measurements and the measurement scheme (3) is that the reordering of subspaces prior to the measurements (3) generally increases the total probability of finding two-dimensional subspaces with $c_{2n}=c_{2n-1}$, while ensuring that the remaining two-dimensional subspaces are at least as correlated as any of the corresponding single-copy two-dimensional subspaces. The measurement then effectively projects onto each of these subspaces (with Alice and Bob being guaranteed to obtain the same result) and then performs the optimal measurement on the resulting shared two-qubit state. Since the optimal measurements in each of these perfectly correlated two-dimensional subspaces give the maximal Bell-CH inequality violation, while the same measurements in the remaining two-dimensional subspaces give as much violation as the single-copy violation, the multiple-copy violation is thus generally greater than that of a single copy.

As one may have noticed, our measurement protocol bears some resemblance with the entanglement concentration protocol developed by Bennett *et al.* [35]. In entanglement concentration Alice and Bob make slightly different projections onto subspaces that are spanned by all those ket vectors sharing the same Schmidt coefficients thus obtaining a maximally entangled state in a bipartite system of some dimension. One can also obtain improved Bell-inequality violations by adopting their protocol and first projecting Alice's Hilbert space into one of the perfectly correlated subspaces and performing the best known measurements for a Bell-inequality violation in each of these (not necessary two-dimensional) subspaces. We have compared the Bell-CH inequality violation of an arbitrary pure two-qubit state derived from each of these protocols and found that the violation obtained using our protocol always outperforms the other. The difference, nevertheless, diminishes as $N\rightarrow\infty$. This observation provides another consistency check of the optimality of (4).

IV. MULTIPLE COPIES OF MIXED STATES

The impressive enhancement in a pure state Bell-CH inequality violation naturally leads us to ask if the same conclusion can be drawn for mixed entangled states. The possibility of obtaining a better Bell-inequality violation with collective measurements, however, does not seem to generalize to all entangled states.

Our first counterexample comes from the two-dimensional Werner state [36], which can seen as a mixture of a singlet state and the maximally mixed state,

$$\rho_w = (1-p)\frac{\mathbf{1}_4}{3} + \frac{4p-1}{3}|\Psi^-\rangle\langle\Psi^-|, \tag{9}$$

where p is the probability of finding a singlet state in this mixture. This state is entangled for $p>1/2$ and violates the Bell-CH inequality if and only if [22] $p>p_w\equiv(3/\sqrt{2}+1)/4 \simeq 0.7803$. Using the LB algorithm [15], we have searched for the maximal violation of ρ_w with $p>p_w$ for $N\leq 4$ copies but no increase in the maximal violation of Bell-CH inequality has ever been observed (see Fig. 2). In fact, by using the UB algorithm [15], we find that for two copies of some Bell-CH violating Werner states, their maximal Bell-CH inequality violation is identical to the corresponding single-copy violation within numerical precision of 10^{-12}. This strongly suggests that for some Werner states the maximal Bell-CH inequality violation does not depend on the number of copies N.

There are, nevertheless, some two-qubit states whose maximal Bell-CH inequality violation for $N=3$ is higher than the corresponding single-copy violation. In contrast to the pure state scenario, the set of mixed two-qubit states seems to be dominated by those whose three-copy Bell-CH inequality violation is not enhanced. In fact, among 50 000 randomly generated Bell-CH violating two-qubit states [37], only about 0.38% of them were found to have their three-copy Bell-CH inequality violation greater than their maximal single-copy violation. Moreover, as can be seen in Fig. 3, they are all clustered at regions with relatively low linear entropy.

As with the pure state scenario, an enhancement of nonlocal correlations in the Bell-CH setting seems to be more prevalent in higher dimensional quantum systems. In particular, for all of the three-dimensional isotropic states [43]

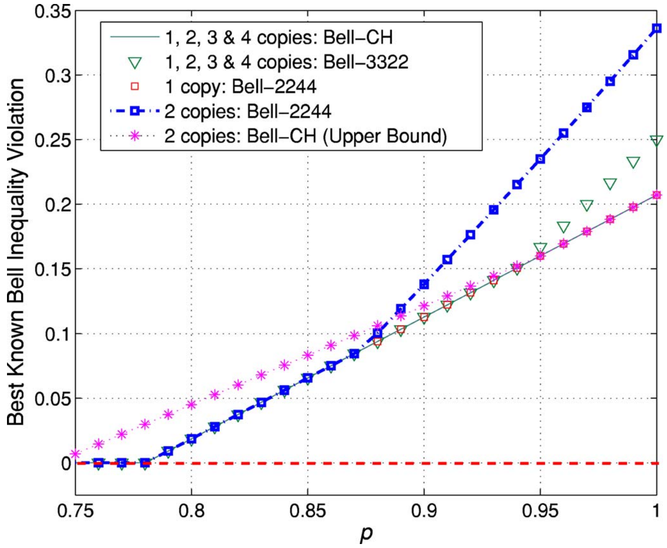

FIG. 2. (Color online) Best known expectation value of the Bell-CH, Bell-3322, and Bell-2244 operators with respect to the two-dimensional Werner states; p represents the overlap with a singlet state. Also included is the upper bound on the maximal $\langle \mathcal{B}_{CH} \rangle_{\rho_W^{\otimes 2}}$ obtained from the UB algorithm [15].

$$\rho_{I_3} = p|\Psi_3\rangle_{\mathrm{ME}}\langle\Psi_3| + (1-p)\frac{\mathbf{1}_9}{9} \tag{10}$$

that were found to violate the Bell-CH inequality, numerical results obtained from the LB algorithm suggest that the maximal violation increases steadily with the number of copies. Further results obtained using the UB algorithm show that with $N=3$, some of the Bell-CH violating ρ_{I_3} definitely give better Bell-CH violation with collective measurements. The results are summarized in Fig. 4.

Yet another question that one can ask is how much does the enhancement of nonlocal correlations depend on the choice of Bell inequality. To address this question, we have also studied the enhancement of nonlocal correlations with respect to other Bell inequalities for probabilities, in particular, the Bell-3322 inequality, the Bell-2233 inequality, and the Bell-2244 inequality [44,45]. For these Bell inequalities, we find that the possibility of enhancing nonlocal correlations does seem to depend on both the number of alternative settings and the number of possible outcomes involved in a Bell experiment. The dependence on the number of outcomes is particularly prominent in the case of Werner states, where a large range of Bell-2244-inequality-violating Werner states seem to achieve a higher two-copy violation, even though their maximal Bell-CH inequality violation apparently remains unchanged up to $N=4$ (Fig. 2).

The dependence on the number of alternative settings can be seen in the best known violation of ρ_{I_3} with respect to the Bell-CH inequality and the Bell-3322 inequality (Fig. 4). In particular, when the number of alternative settings is increased from 2 (in the case of Bell-CH inequality) to 3 (in the case of Bell-3322 inequality), the range of states whereby collective measurements were found to improve the Bell-inequality violation is drastically *reduced*.

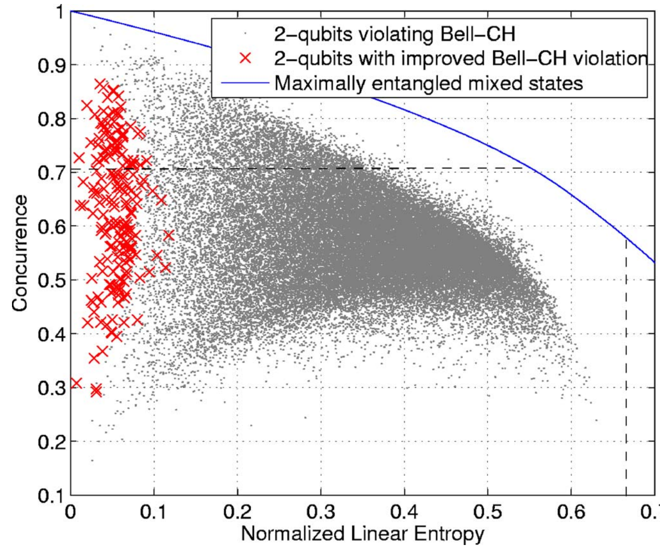

FIG. 3. (Color online) Distribution of two-qubit states sampled for improved Bell-CH violation by collective measurements. The maximally entangled mixed states, which demarcate the boundary of the set of density matrices on this concurrence-entropy plane [39,40], are represented by the solid line. Note that as a result of the chosen distribution over mixed states [37] this region is not well sampled. The region bounded by the solid line and the horizontal dashed line (with concurrence equal to $1/\sqrt{2}$) only contain two-qubit states that *violate* the Bell-CH inequality [42]; the region bounded by the solid line and the vertical dashed line (with normalized linear entropy equal to $2/3$) only contain states that *do not violate* the Bell-CH inequality [41,42]. Two-qubit states found to give a better three-copy Bell-CH violation are marked with red crosses.

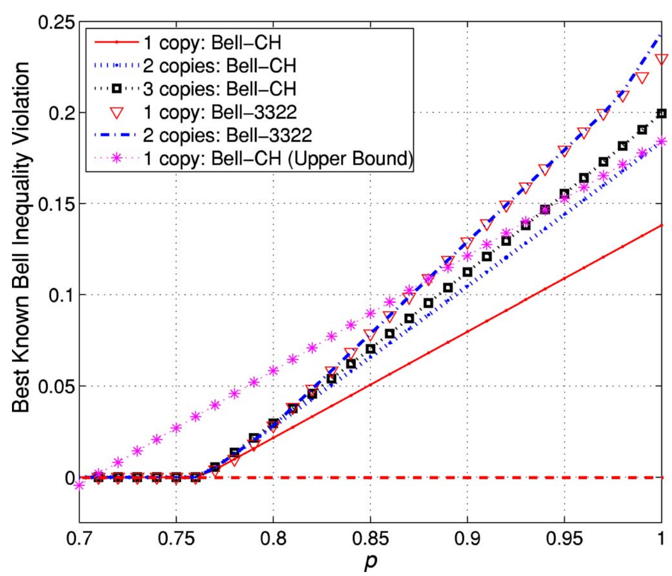

FIG. 4. (Color online) Best known expectation value of the Bell-CH and Bell-3322 operators with respect to the three-dimensional isotropic states; p is the fraction of maximally entangled two-qutrit state in the mixture. Also included is the upper bound on the maximal $\langle \mathcal{B}_{CH} \rangle_{\rho_{I_3}}$ obtained from the UB algorithm [15].

V. CONCLUSION

In this paper, we have focused on bipartite entangled systems and considered the enhancement of nonlocal correlations by collective measurements without postselection. This amounts to allowing an experiment in which a local unitary is applied to a number of copies of the state ρ prior to the Bell-inequality experiment.

We find that the Bell-CH inequality violation of all bipartite pure entangled states can be enhanced by allowing collective measurements even without postselection. For mixed entangled states, however, explicit examples (Werner states) have been presented to demonstrate that there may be entangled states whose nonlocal correlations cannot be enhanced in any Bell-CH experiments. In fact, the set of mixed two-qubit states whose Bell-CH violation can be increased with collective measurements seems to be relatively small.

We have also done some preliminary studies on how the usefulness of collective measurements depends on the choice of Bell inequality and on the dimension of the subsystem. Our data at the moment are consistent with the hypothesis that the usefulness of collective measurements in Bell-inequality experiments increases with the Hilbert space dimension and with the number of measurement outcomes allowed by Bell inequality. On the other hand, as the number of measurement settings allowed by the Bell inequality increases the advantage provided by collective measurements seems to diminish. However, note that we have not really performed the systematic study required to establish such trends, if they exist, due to the great numerical effort that would be required. Given these observations, it does seem that postselection is a lot more powerful than collective measurements on their own in increasing Bell-inequality violation.

An immediate question that follows from the present work is what is the class of quantum states whereby collective measurements can increase their Bell-inequality violation? One motivation for studying our problem is to understand better the set of quantum states that violate a Bell inequality and are thus inconsistent with local realism. It has been known for a long time that this set is a strict subset of the entangled states if projective [36] or even generalized measurements [46] on single copies of a system are permitted. One might wonder whether collective measurements without postselection allow us to violate Bell inequalities for a larger set of states. However, we do not know of examples where a state that does not violate a given Bell inequality becomes violating under collective measurements when no postselection is allowed. Moreover, for mixed states, the set of states whose violations increase when collective measurements are allowed appears to be rather restricted. This is consistent with the recent work by Masanes [47] which suggests that the set of states that violates a given Bell inequality under collective measurements without postselection is a subset of all distillable states.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council. We thank Ben Toner for stimulating discussions, and Henry Haselgrove and Eric Cavalcanti for interesting comments. In addition, Y.C.L. gratefully acknowledges the kind hospitality of the quantum information technology group of NUS, where part of this work was completed; helpful suggestions from Reinhard Werner, Jing Ling Chen, and Meng Khoon Tey are particularly appreciated.

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