PHYSICAL REVIEW A **73**, 022311 (2006)

Entanglement under restricted operations: Analogy to mixed-state entanglement

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(Received 4 January 2005; revised manuscript received 15 September 2005; published 3 February 2006)

We show that the classification of bipartite pure entangled states when local quantum operations are restricted yields a structure that is analogous in many respects to that of mixed-state entanglement. Specifically, we develop this analogy by restricting operations through local superselection rules, and show that such exotic phenomena as bound entanglement and activation arise using pure states in this setting. This analogy aids in resolving several conceptual puzzles in the study of entanglement under restricted operations. In particular, we demonstrate that several types of quantum optical states that possess confusing entanglement properties are analogous to bound entangled states. Also, the classification of pure-state entanglement under restricted operations can be much simpler than for mixed-state entanglement. For instance, in the case of local Abelian superselection rules all questions concerning distillability can be resolved.

DOI: 10.1103/PhysRevA.73.022311 PACS number(s): 03.67.Mn, 03.65.Ud, 03.75.−b, 05.30.−d

I. INTRODUCTION

Entanglement of quantum systems is a potentially powerful resource for quantum information processing [1]. However, in the presence of noise, it is currently not known precisely *which* entangled states are useful, and a vast theory of mixed-state entanglement has developed to classify states according to their entanglement properties [2].

In this paper, we show that the theory of pure-state entanglement when quantum operations are restricted—described formally by a superselection rule (SSR)—precisely replicates the structure of mixed-state entanglement, including such exotic properties as bound entanglement and activation. This analogy is useful both for the theory of mixed-state entanglement, and for that of pure-state entanglement under restricted operations. After over a decade of debate on issues such as the nonlocality of a single photon [3–9] and the role of a phase reference in quantum teleportation [10–18], we resolve these conceptual issues by demonstrating that entanglement under restricted operations can be viewed as *bound* by the restriction. In addition, we demonstrate that the surprising results for entanglement under constraints [19–23] arise from the coexistence of two distinct operational notions of entanglement, and that distinguishing these notions realizes the entire structure of the preexisting mixed-state entanglement theory. Thus, we demonstrate that the specialized concepts of the field of mixed-state entanglement (such as activation and multicopy distillation) can be applied to a wide variety of practical situations. Moreover, unsolved questions for mixed-state entanglement have analogous questions in the context of pure-state entanglement under restrictions, and these can be answered in some cases. It is hoped that this formal analog of the complex and surprising structure of mixed-state entanglement in another situation—one that is conceptually straightforward to understand and interpret—will ultimately lead to new results in mixed-state entanglement theory.

II. CLASSIFYING MIXED-STATE ENTANGLEMENT

In this section, we present some known results for the classification of mixed-state entanglement, with our own bias and some new terminology. For an extensive review of mixed-state entanglement, see Ref. [2].

Central to the theory of entanglement is the classification of the states of a quantum system shared between two parties (Alice and Bob) who can perform only local quantum operations and classical communication (LOCC). This limitation on their operations means, on the one hand, that certain states cannot be prepared by the two parties starting from some uncorrelated fiducial state, and on the other hand that certain states shared by the two parties may serve as resources allowing them to perform tasks not possible with LOCC alone. In this paper, it will be important to distinguish between various sets of states characterized by either (i) the operations required to prepare them or (ii) the resource they provide for quantum information processing tasks. To emphasize the distinctions between these sets, we adopt a slightly unconventional terminology for mixed-state entanglement. First, we identify the class of bipartite states that are locally preparable, that is, preparable by LOCC (starting with some uncorrelated fiducial state). We denote this class LP. Second, we identify the class of states that are distillable [24], denoted D. States are distillable if n copies can be converted into nr pure maximally entangled states via LOCC for some $r>0$ in the limit $n\to\infty$.

A *pure* state is either locally preparable or distillable (either in LP or in D), depending on whether it is a product state or not (i.e., a state of the form $|\psi\rangle_A\otimes|\phi\rangle_B$ or not). For mixed states, the set LP is the set of states that possess a convex decomposition into product states (the separable states). Identifying the class of mixed states that are distillable is important for quantum information processing, but unfortunately it is not known how to determine if a general bipartite mixed state is distillable or not [2]. One property of the class D is certain, though: in contrast to the situation for pure states, there are mixed states that are neither locally preparable nor distillable, called bound entangled states [25].

Part of the difficulty in identifying the set of distillable mixed states arises from the asymptotic nature of the definition of distillability, as it is not known how to characterize all possible distillation protocols that act on a potentially infinite number of copies. In the following, we will make use of a related class with a simpler characterization: the class of states that are 1-distillable [26,27], denoted 1-D. We define and motivate this class as follows. First, we note that distillability is decidable on a 2×2-dimensional space, wherein all separable states are in LP, and all nonseparable states are distillable [28]. On an arbitrary bi-partite space, we define a state ρ to be 1-distillable if there exists an operation implementable with LOCC, represented by a completely positive map $\mathcal{E}$ [1], that maps ρ onto a 2×2-dimensional subspace of the bipartite system such that $\mathcal{E}(\rho)$ is nonseparable (and thus distillable). If a state is 1-distillable analogs then it is distillable. For pure states, 1-distillability is equivalent to distillability, and thus every pure state is either locally preparable or 1-distillable. This is not the case for mixed states. Due to the existence of bound entangled states (i.e., states that are neither locally preparable nor distillable), and the fact that $1\text{-D}\subset D$, there exist mixed states that are neither locally preparable nor 1-distillable. We shall refer to all such states as 1-bound.

Remarkably, by appropriately extending the set of operations that Alice and Bob can perform beyond LOCC, all states become either locally preparable or 1-distillable. We describe this extension of operations as supplementing LOCC with an additional resource. Clearly, additional power will affect the boundaries of what Alice and Bob can prepare or distill; we are interested in a resource that precisely removes the proper gap between LP and 1-D. Consider extending LOCC to allow all operations that preserve the positivity of the partial transpose of states [29]. With this additional resource, all states with positive partial transpose (PPT) can be prepared locally. All states that are not PPT are 1-distillable with this additional power in the sense that they can be mapped by a PPT-preserving operation $\mathcal{E}$ onto a 2×2-dimensional space such that $\mathcal{E}(\rho)$ is nonseparable [30]. States that are not locally preparable with LOCC, but locally preparable given LOCC plus the additional resource, can be said to become locally preparable given the resource, denoted BLP. Similarly, the class of states that are not 1-distillable but become 1-distillable given the resource we denote as B1-D. For mixed bipartite states under PPT-preserving operations, the class BLP contains all PPT bound entangled states and the class B1-D contains all non-PPT states that are not 1-distillable; both classes are nonempty [25–27]. See Fig. 1.

The categories BLP and B1-D are related in an interesting way. Through an isomorphism between bipartite quantum states and quantum operations, any PPT-preserving operation can be implemented probabilistically using LOCC and a specific state in BLP (i.e., a specific PPT bound entangled state) [31]. Recall that any B1-D state becomes 1-distillable if Alice and Bob are given the additional resource of all PPT-preserving operations. Thus, for every $\rho\in\text{B1-D}$ there exists a state $\sigma\in\text{BLP}$ such that $\sigma\otimes\rho$ is 1-distillable. We say that the state $\sigma\in\text{BLP}$ *activates* the entanglement of the state $\rho\in\text{B1-D}$ usig only LOCC operations [32]; see Fig. 2.

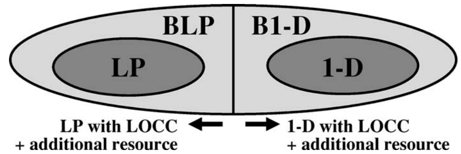

FIG. 1. Illustration of the division of all bipartite mixed states into four classes. When restricted to LOCC, there is a proper gap between what is locally preparable and what is 1-D. This gap contains 1-bound states. If an additional resource is supplied, allowing for all PPT-preserving operations, then all 1-bound states either become locally preparable (BLP) or become 1-distillable (B1-D).

Another remarkable feature of mixed-state entanglement is that, although states in B1-D are not 1-distillable, they may nevertheless be distillable [33]. We define a state ρ to be n-distillable (in n-D) if there exists an LOCC operation $\mathcal{E}_n$ onto a (2×2)-dimensional space such that $\mathcal{E}_n(\rho^{\otimes n})$ is nonseparable. In other words, the joint state $\rho^{\otimes n}$ is 1-distillable. If a state is n-distillable for some n then it is distillable. (In fact, it has been shown [33] that n-D is a proper subset of D for all finite n.) Thus, mixed-state entanglement exhibits multicopy distillability, meaning that there exist states in B1-D that are not 1-distillable but that are n-distillable for some $n\geqslant 2$.

There remain, however, many open questions regarding the general structure of mixed-state entanglement. Perhaps the most important question from the point of view of quantum information processing is: Are *all* states in B1-D distillable?

III. AN ANALOGY IN QUANTUM OPTICS

To introduce the concepts and results developed later in this paper, we first begin by providing a simple example of how the phenomena arising in the context of mixed-state entanglement have precise analogs in the structure of pure-state entanglement under a restriction on operations. Specifically, we consider some well-studied states from quantum optics and the restriction of a local photon-number SSR.

A. Local photon-number superselection rule

The restriction of a local photon-number SSR implies that a party cannot prepare coherent superpositions of states of different local photon number (starting with states without coherence), nor measure such coherences, nor implement a

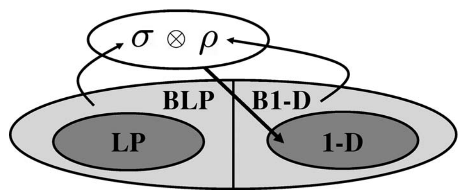

FIG. 2. Illustration of activation of bound entanglement. A BLP state σ can be used to "activate" the entanglement of a B1-D state ρ, i.e., the combined state $\rho\otimes\sigma$ is 1-D.

transformation that creates such coherence. For instance, if Alice is restricted by a local photon-number superselection rule, she cannot prepare a state of the form $(1/\sqrt{2})(|0\rangle_A + |1\rangle_A)$, where $|n\rangle_A$ denotes an n-photon eigenstate of a mode in Alice's possession. However, she can prepare the state $(1/\sqrt{2})(|01\rangle_A + |10\rangle_A)$ on a pair of modes in her possession (where $|01\rangle_A = |0\rangle_{A_1} \otimes |1\rangle_{A_2}$, etc.), because this state is an eigenstate of total local photon number.

A local photon-number SSR applies to multiparty quantum optics experiments when the parties do not share a common phase reference [16,19]. This connection between SSRs and reference frames can be seen as follows. In optical experiments, states of an optical mode are always referred to some phase reference. Consider several optical modes distributed between two parties, Alice and Bob. Suppose there is a third party, Charlie, who has a local phase reference—for example, a high intensity laser—to which the quantum states of Alice and Bob's optical modes can be referred. Suppose further that Alice and Bob do not share this phase reference, i.e., their lasers are not phase-locked with Charlie's. The relative phase between their phase references and Charlie's is therefore completely unknown.

We now demonstrate that this unknown phase relation leads to a local photon-number SSR. Let Alice prepare a quantum state of her local optical modes, which she represents by a density operator ρ_A relative to her phase reference. If the relative phase between Alice and Charlie's phase references was known to be ϕ, then this same state would be represented by the density operator $e^{-i\phi\hat{N}_A}\rho_A e^{i\phi\hat{N}_A}$ relative to Charlie's phase reference, where $\hat{N}_A$ is Alice's local photon number operator. Given that ϕ is completely unknown, one must average over its possible values to obtain the state relative to Charlie. This state is

$$\mathcal{U}_A[\rho_A] \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i\phi\hat{N}_A}\rho_A e^{i\phi\hat{N}_A}, \tag{1}$$

which is equivalent to

$$\mathcal{U}_A[\rho_A] = \sum_n \Pi_n^A \rho_A \Pi_n^A, \tag{2}$$

where Π_n^A is the projector onto the nth eigenspace of $\hat{N}_A$. The map $\mathcal{U}_A$ removes all coherence between states of differing total photon number on Alice's systems. Similarly, any operations Alice implements relative to her phase reference are redescribed relative to Charlie's phase reference as operations that commute with $\mathcal{U}_A$. Thus, relative to Charlie, Alice experiences a restriction on operations that is described by a superselection rule for local photon number $\hat{N}_A$ as defined in Ref. [21]. A similar argument applies to states and operations of Bob relative to Charlie, characterized by a map $\mathcal{U}_B$. Thus, the situation where Alice and Bob lack Charlie's phase reference is a restriction formally equivalent to a superselection rule for local photon number.

We note that although the term "superselection rule" was initially introduced to describe an *in principle* restriction on quantum states and operations [34], it has been emphasized by Aharonov and Susskind [35] that whether or not coherent superpositions of a particular observable are possible is a practical matter, depending on the availability of a suitable reference system. Modern arguments in favor of this view may be found in Refs. [21,36,37], and we follow the practice of using the term "superselection rule" to describe both in principle and practical restrictions on operations.

B. Bound entanglement in pure-state quantum optics

In such situations, there has been considerable debate over the entanglement properties of certain types of states, such as the two-mode single-photon state [3–7]

$$\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B). \tag{3}$$

There is a temptation to say that this state is entangled simply because of its nonproduct form. However, it is far more useful to consider whether or not this state satisfies certain *operational* notions of entanglement. One such notion is whether a state can be used to violate a Bell inequality. Another is whether it is useful as a resource for quantum information processing, for instance, to teleport qubits or implement a dense coding protocol. In the context of a local photon-number SSR, this two-mode single-photon state fails to satisfy either of these notions of entanglement, because all such tasks would require Alice and Bob to violate the local photon-number SSR. A different but equally common notion of entanglement is that a state is entangled if it cannot be prepared by LOCC. The two-mode single-photon state certainly *does* fit *this* notion because the pure nonproduct states cannot be prepared by LOCC. Thus we see that operational notions of entanglement that coincided for pure states under unrestricted LOCC, namely being not locally preparable and being useful as a resource for tasks such as teleportation or violating a Bell inequality, do not coincide under a local photon-number SSR, and the state in question is judged entangled by one notion and not the other.[1]

This already has the flavor of the phenomenon of bound entanglement. However, strictly speaking, the two notions of entanglement that were explored in Sec. II were local preparability and 1-distillability. As one might expect from the above comments, these two notions do not coincide either, as we now show.

Consider a state of the form

$$\frac{1}{\sqrt{2}}(|01\rangle_A|10\rangle_B + |10\rangle_A|01\rangle_B). \tag{4}$$

This state is certainly not locally preparable. In addition, it *can* be used to violate a Bell inequality, implement dense coding, and so on, despite the SSR. This is because Alice and Bob can still implement any measurements they please in the two-dimensional subspaces spanned by $|01\rangle$ and $|10\rangle$. Thus, a useful notion of distillability for a bipartite pure state in the

[1]Of course, if there is no local photon-number SSR, this state would satisfy all of these notions of entanglement, as emphasized by van Enk [7]. In particular, no such SSR would apply if all parties share a common phase reference, as discussed in Sec. III C.

context of a local photon number superselection rule is whether n copies of the state can be converted into nr copies of $(1/\sqrt{2})(|01\rangle_A|10\rangle_B+|10\rangle_A|01\rangle_B)$ for some $r>0$. A useful notion of 1-distillability for a bipartite pure state in the context of a local photon number superselection rule is whether it can be projected to a nonproduct state in some 2×2 subspace, where the two-dimensional local spaces are eigenspaces of local photon number.

By this definition, the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ is clearly not 1-distillable under the local photon number SSR because the subspace spanned by $\{|0\rangle, |1\rangle\}$ cannot be mapped to the subspace spanned by $\{|01\rangle, |10\rangle\}$ under the restricted operations. This establishes the existence of pure states that are neither locally preparable nor 1-distillable under the local photon number SSR. Thus, they are analogous to the 1-bound states introduced for mixed-state entanglement.

Another class of states whose entanglement properties have been discussed recently in the quantum optics literature are those that are separable but not locally preparable under a local photon-number superselection rule [10,19]. Examples of such states[2] are

$$|+\rangle_A|+\rangle_B, \quad |-\rangle_A|-\rangle_B, \tag{5}$$

where $|\pm\rangle=(1/\sqrt{2})(|0\rangle\pm|1\rangle)$. Because of the SSR, these states cannot be prepared locally. But they are not 1-distillable either because they are product states. Thus, they also lie in the gap between what is locally preparable and what is 1-distillable. Verstraete and Cirac [19] identified such states as a "new type of nonlocal resource," and van Enk [38] identified states of the form of Eq. (5) as a standard unit of this nonlocal resource, which he called a "refbit." We can identify these states also as analogues of the 1-bound states of mixed-state entanglement.

C. A resource to "lift" the superselection rule

In the context of a SSR, there is also a resource that precisely removes the gap between what is locally preparable and 1-distillable (as occurred in mixed-state entanglement by extending LOCC to all PPT-preserving operations). Recall, as described above, that a local photon-number SSR applies if Alice and Bob are uncorrelated with the phase reference of Charlie, who is preparing the bipartite quantum states. Clearly, if Alice and Bob are given phase references that are precisely correlated with Charlie's, then they no longer face any restrictions beyond that of LOCC. Thus, for Alice and Bob to possess a shared phase reference is for them to possess a resource that "lifts" the SSR. Given this resource, states such as $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ become 1-distillable, while states such as $|+\rangle_A|+\rangle_B$ become locally preparable.

D. Activation and distillation in pure-state quantum optics

Finally, we demonstrate that there exist analogous processes of activation and multicopy distillation in this scenario. Both of these processes have been discussed (albeit using different terminology) by van Enk [38] for the specific quantum optical state examples we present here.

Combining $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ (a state which, by itself, is *not* 1-distillable under a local photon-number SSR) with $|+\rangle_A|+\rangle_B$ one obtains a state that is 1-distillable. The state $|+\rangle_A|+\rangle_B$ is said to *activate* the entanglement of $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$. This is seen as follows. Let Alice and Bob both perform a quantum non-demolition measurement of local photon number, and post-select the case where they both find a local photon number of 1. The resulting state is

$$\Pi_1^A \otimes \Pi_1^B\left[\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)|+\rangle_A|+\rangle_B\right] \propto \frac{1}{\sqrt{2}}(|01\rangle_A|10\rangle_B+|10\rangle_A|01\rangle_B). \tag{6}$$

We note that the controversy over the use of the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ to demonstrate quantum nonlocality [3–6,8,9] can be resolved by recognizing the role of activation. As we have shown, this state is not 1-distillable when Alice and Bob do not share a correlated local phase reference (i.e., when a local photon-number SSR applies). However, violations of a Bell inequality have recently been demonstrated experimentally using this state [8,9]. One can take two different perspectives on such an experiment. It is illustrative to consider them both.

In Ref. [8], in addition to the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$, a correlated pair of coherent states $|\alpha\rangle_A|\alpha\rangle_B$, where $|\alpha\rangle\equiv\Sigma_n(e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!})|n\rangle$, are assumed to be shared between Alice and Bob. These modes are used as the local oscillators in the homodyne detections at each wing. Noting that neither $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ nor $|\alpha\rangle_A|\alpha\rangle_B$ are 1-distillable under the superselection rule, it is unclear how it is possible to violate the Bell inequality using such resources. The resolution of the puzzle is that the product of coherent states $|\alpha\rangle_A|\alpha\rangle_B$ (similar to the state $|+\rangle_A|+\rangle_B$) *activates* the entanglement of the two-mode single photon state. To see this, we note that the same measurement of local photon number as described above projects the state onto a nonproduct state of random but definite local photon number, allowing for a demonstration of nonlocality within the constraints of the SSR. (Such a measurement is, in fact, implemented using an ideal homodyne detection. Loosely speaking, each observer's homodyne detection apparatus couples the two local modes at a beam splitter and then measures the number of photons in each of the two output ports. This incorporates a measurement of the total local photon number because the latter quantity can be obtained as the sum of the number in each output port. The difference of these two photocounts, which is typically the quantity of interest in homodyne detection, yields the information necessary to demonstrate the Bell inequality violation.)

An experimental demonstration of nonlocality using the two-mode single photon state can also be described as follows [9]. Rather than treating the local oscillators as coherent states, they are treated as correlated classical phase refer-

[2]References [10,19] considered states such as the equal mixture of $|+\rangle_A|+\rangle_B$ and $|-\rangle_A|-\rangle_B$. For simplicity, we restrict our attention to pure states.

ences. In this case, they constitute an additional resource that "lifts" the restriction of the local photon-number SSR, and the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ becomes 1-distillable. These two alternative descriptions are equally valid [37].

The existence of such activation processes also resolves a controversy concerning the source of entanglement in the experimental realization of continuous-variable quantum teleportation [39]. Again, we consider two different perspectives on the experiment.

The first perspective is a variant of the one presented by Rudolph and Sanders [10]. In our language, it can be synopsized as follows. Alice and Bob are presumed to be restricted in the operations they can perform by a local photon-number superselection rule. They share a two-mode squeezed state $|\gamma\rangle=\sqrt{1-\gamma^2}\Sigma_{n=0}^{\infty}\gamma^n|n,n\rangle$, where $0\leqslant\gamma\leqslant 1$. In addition, they share two other modes prepared in a product of coherent states $|\alpha\rangle|\alpha\rangle$.[3] The former is the purported entanglement resource in the teleportation protocol, while the latter is a quantum version of a shared phase reference. These states are analogous to $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ and $|+\rangle_A|+\rangle_B$, respectively—neither is 1-distillable when considered on its own. So the question arises as to how teleportation could possibly have been achieved when neither the purported entanglement resource nor the quantum shared phase reference are 1-distillable. The resolution to this puzzle is that although individually, neither is 1-distillable, together they are: the quantum shared phase reference *activates* the entanglement in the two-mode squeezed state.

The second perspective is one wherein the shared phase reference is treated classically [39]. As described above, this acts as a resource which lifts the SSR, and causes the two-mode squeezed state to become 1-distillable.

An analog of multicopy distillation can also be demonstrated in our quantum optical example. For instance, the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ is two-distillable. The protocol, introduced in Ref. [20] and discussed in greater detail in Ref. [40], is as follows. As in the activation example above, Alice and Bob both perform a quantum nondemolition measurement of local photon number (on both local modes) and post-select the case where they both find a local photon number of 1. The resulting state is

$$\Pi_1^A\otimes\Pi_1^B\left[\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)\right]^{\otimes 2}$$
$$\propto\frac{1}{\sqrt{2}}(|01\rangle_A|10\rangle_B+|10\rangle_A|01\rangle_B),\qquad(7)$$

where $|\psi\rangle^{\otimes 2}=|\psi\rangle|\psi\rangle$. A process very similar to this two-copy distillation has been demonstrated in quantum optics experiments (see Ref. [41]), where correlated but unentangled photon pairs from parametric downconversion were made incident on the two input modes of a beamsplitter, so each photon transforms to a state of the form $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$. Subsequently, measurements on the two output modes are postselected for one photon detection at each output mode. The fact that their postselected results are consistent with a description of an entangled state demonstrates that the entanglement of the state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ has been distilled by making use of two copies.

We see that the remarkable (and often confusing) entanglement properties of states when local operations are restricted can be understood by recognizing that different operational notions of entanglement do not coincide in this case, leaving a structure akin to that of mixed-state entanglement.

[3]The state assigned to this pair of resources in Ref. [10] is simply a mixed version of the one we consider here.

IV. PURE-STATE ENTANGLEMENT UNDER GENERAL RESTRICTIONS

We now develop the analogy between mixed-state entanglement and pure-state entanglement when the allowed local quantum operations are restricted by a *general* (not necessarily Abelian) SSR. We continue to consider only pure states, because, although one could characterize mixed-state entanglement under such restrictions, the classification of such states would be at least as difficult as unrestricted mixed-state entanglement.

A. Restricting operations through general superselection rules

We formulate a restriction on operations generally in the form of a SSR associated with a finite or compact Lie group G [21,36]. (A different concept of entanglement under restrictions on operations is discussed in Ref. [42].)

The SSR we describe can be defined operationally as follows. Suppose Alice and Bob share a pair of systems, described by a Hilbert space $\mathcal{H}^A\otimes\mathcal{H}^B$, the states on which were prepared and described by a third party, Charlie. Suppose further that the local reference frames of Alice, Bob, and Charlie, which transform via a group G, are uncorrelated: that is, the element $g\in G$ relating Alice's and Charlie's local frames is completely unknown, as is the element $g'\in G$ relating Bob and Charlie's local frames. It follows that a preparation represented by a density matrix ρ on $\mathcal{H}^A$ relative to Alice's frame is represented by the density matrix $\mathcal{G}_A[\rho]$ relative to Charlie's frame, where

$$\mathcal{G}_A[\rho]\equiv\int_G \mathrm{d}v(g)T^A(g)\rho T^{A\dagger}(g),\qquad(8)$$

with $T^A(g)$ a unitary representation of g on $\mathcal{H}^A$, and $\mathrm{d}v$ the group-invariant (Haar) measure. The operations that Alice can implement relative to Charlie's frame are represented by completely positive maps $\mathcal{O}_A$ that commute with $\mathcal{G}_A$. A similar result holds for the operations that Bob can implement. The joint LOCC operations that Alice and Bob can implement relative to Charlie's frame are those represented by maps $\mathcal{O}_{AB}$ that commute with $\mathcal{G}_A\otimes\mathcal{G}_B$. These are said to be locally G-invariant [21]. This restriction on operations is referred to as a local SSR for G.

A local superselection rule for G induces the following structure in the local Hilbert spaces (we consider $\mathcal{H}^A$):

$$\mathcal{H}^A = \bigoplus_n \mathcal{H}_n^A, \tag{9}$$

i.e., each local Hilbert space is split into "charge sectors" labeled by n and each carrying inequivalent representations T_n^A of G. Each sector can be further decomposed into a tensor product

$$\mathcal{H}_n^A = \mathcal{M}_n^A \otimes \mathcal{N}_n^A, \tag{10}$$

of a subsystem $\mathcal{M}_n^A$ carrying an irreducible representation T_n^A and a subsystem $\mathcal{N}_n^A$ carrying a trivial representation of G. For an Abelian SSR, such as the photon-number SSR discussed in Sec. III, the subsystems $\mathcal{M}_n^N$ are one-dimensional, and so the additional tensor product structure within the irreps is not required; for a general SSR, they can be nontrivial. The subsystems $\mathcal{N}_n^A$ are G-invariant noiseless subsystems relative to the decoherence map $\mathcal{G}_A$ [43]. The action of $\mathcal{G}_A$ on a density operator ρ in terms of this decomposition is

$$\mathcal{G}_A[\rho] = \sum_n \mathcal{D}_{An} \otimes \mathcal{I}_{An}(\Pi_n^A \rho \Pi_n^A), \tag{11}$$

where Π_n^A is the projection onto the charge sector n, $\mathcal{D}_{An}$ is the completely-positive trace-preserving map that takes every operator on $\mathcal{M}_n^A$ to a constant times the identity operator on that space, and $\mathcal{I}_{An}$ is the identity map over operators in the space $\mathcal{N}_n^A$. The effect of the local SSR, then, is to remove the ability to prepare states or measure operators that have coherence between different local charge sectors or that are not completely mixed over the subsystems $\mathcal{M}_n^A$. The same structure arises for $\mathcal{H}^B$ and provides an analogous decomposition of $\mathcal{G}_B$. For further details, see Refs. [21,36].

To address the issue of distillability of a state, we now demonstrate how to treat multiple systems under a local SSR. If the system that Alice exchanges with Charlie is made up of several systems, $\mathcal{H}^A = \otimes_i \mathcal{H}^{A_i}$, which are all defined relative to Alice's frame, the uncertainty in the element $g \in G$ relating Alice's frame to Charlie's is represented by Eq. (8) using the tensor representation $T^A = \otimes_i T^{A_i}$.

B. The analogy: general results

We now present our main results which demonstrate that the structure of mixed-state entanglement is analogous in many respects to the structure of pure-state entanglement with a general restriction on local operations. The set of LOCC operations that are locally G-invariant will be denoted by G-LOCC. The set of pure bipartite states that are locally preparable under a SSR for G, that is, preparable by G-LOCC, will be denoted by $\mathrm{LP}_{G\text{-SSR}}$. A pure bipartite state is in $\mathrm{LP}_{G\text{-SSR}}$ iff (i) the state is a product state and (ii) it is locally G-invariant. (Thus, not all pure product states are in $\mathrm{LP}_{G\text{-SSR}}$.) A state $|\psi\rangle$ is 1-D with G-LOCC, denoted $1\text{-}\mathrm{D}_{G\text{-SSR}}$, if there exists an operation $\mathcal{E}$ in G-LOCC mapping $|\psi\rangle$ onto a 2×2-dimensional space such that $\mathcal{E}[|\psi\rangle\langle\psi|]$ is locally G-invariant and nonseparable. It follows from the main theorem of Bartlett and Wiseman [21] that $|\psi\rangle$ is in $1\text{-}\mathrm{D}_{G\text{-SSR}}$ iff $\mathcal{G}_A \otimes \mathcal{G}_B[|\psi\rangle\langle\psi|]$ is 1-distillable with unrestricted LOCC. Both $\mathrm{LP}_{G\text{-SSR}}$ and $1\text{-}\mathrm{D}_{G\text{-SSR}}$ are nonempty; explicit examples of each can be constructed as product/nonproduct states within 2×2 subspaces or subsystems that are invariant relative to $\mathcal{G}_A \otimes \mathcal{G}_B$.

Result 1. With LOCC constrained by a local SSR for G, the classes of pure bipartite states that are locally preparable ($\mathrm{LP}_{G\text{-SSR}}$) or 1-distillable ($1\text{-}\mathrm{D}_{G\text{-SSR}}$) are both nonempty.

As with mixed-state entanglement, there is a proper gap between these two classes. The class of states in the gap contains both product and nonproduct pure states, and is analogous to the class of 1-bound states in mixed-state entanglement. An explicit example of such a state is a product state that is not locally G-invariant for one or both parties.

Result 2. With LOCC constrained by a local SSR for G, there exists a nonempty class of states that are neither locally preparable nor 1-distillable (neither in $\mathrm{LP}_{G\text{-SSR}}$ nor in $1\text{-}\mathrm{D}_{G\text{-SSR}}$).

Moreover, it is possible to extend G-LOCC in such a way that any pure state in this gap becomes either locally preparable or 1-distillable. One simply lifts the restriction of the local superselection by providing Alice and Bob with Charlie's local frame, so that the local frames of the three parties are correlated. With this additional resource, Alice and Bob can now implement any operation in LOCC. Extending G-LOCC to LOCC divides the proper gap between $\mathrm{LP}_{G\text{-SSR}}$ and $1\text{-}\mathrm{D}_{G\text{-SSR}}$ into two classes, both of which are nonempty. All product states that are not locally G-invariant (i.e., product states not in $\mathrm{LP}_{G\text{-SSR}}$) *become* locally preparable with G-LOCC given the shared reference frame for G. We denote this class $\mathrm{BLP}_{G\text{-SSR}}$. This result follows directly from the fact that all pure product states are locally preparable with unrestricted LOCC. All nonproduct states $|\psi\rangle$ for which $\mathcal{G}_A \otimes \mathcal{G}_B[|\psi\rangle\langle\psi|]$ is not 1-distillable (i.e., nonproduct states not in $1\text{-}\mathrm{D}_{G\text{-SSR}}$) *become* 1-distillable with G-LOCC given the shared reference frame for G. We denote this class $\mathrm{B1\text{-}D}_{G\text{-SSR}}$. This result follows directly from the fact that all pure nonproduct states are 1-distillable with unrestricted LOCC.

Result 3. With LOCC constrained by a local SSR for G and the additional resource of a shared local reference frame for G, the SSR is "lifted," and all states in the proper gap either become locally preparable ($\mathrm{BLP}_{G\text{-SSR}}$) or become 1-distillable ($\mathrm{B\text{-}1\text{-}D}_{G\text{-SSR}}$). Both classes $\mathrm{BLP}_{G\text{-SSR}}$ and $\mathrm{B1\text{-}D}_{G\text{-SSR}}$ are nonempty.

Thus, we have demonstrated that the structure of Fig. 1 for mixed-state entanglement is analogous to the structure of pure-state entanglement under the restriction of a SSR. Although it is likely that the processes of activation and multicopy distillation also exist for general SSR, we only consider this aspect of the analogy in depth in the context of Abelian SSR. We turn to this in the next section.

V. ACTIVATION AND DISTILLATION OF PURE STATES CONSTRAINED BY AN ABELIAN SUPERSELECTION RULE

Although it has proven difficult to fully characterize activation and distillation processes in the context of mixed-state entanglement, it is straightforward to do so in the context of pure states with an Abelian SSR, as we now demonstrate. In particular, we completely classify all pure bipartite states in terms of the number of copies needed for distillation.

An Abelian SSR is a SSR for the group H, all the elements of which commute. (In the following, H refers exclusively to an Abelian group.) Superselection rules for local charge or particle number are examples, with the relevant Abelian group being U(1). The SSR for photon number considered in Sec. III is another example, which can be seen from the fact that the phase degree of freedom in quantum optics transforms via the U(1) group, so that a shared phase reference is an example of a shared reference frame for U(1). In the following, we will refer to the superselected quantity for a Abelian SSR as a "charge," and we will refer to local charge eigenstates simply as eigenstates. We begin with a useful lemma. (Note that this lemma fails for the case of non-Abelian groups.)

Lemma. If $|\Psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$ is a nonproduct state, then Alice and Bob can, with H-LOCC, project $|\Psi\rangle$ onto a 2×2 subspace $\mathcal{S}^A \otimes \mathcal{S}^B$ with local projectors Π^A and Π^B, such that $(\Pi^A \otimes \Pi^B)|\Psi\rangle$ is a nonproduct state.

Proof. Express $|\Psi\rangle$ using an eigenstate basis $\{|n,\alpha\rangle_A\}$ for $\mathcal{H}^A$ as

$$|\Psi\rangle = \sum_{n,\alpha} |n,\alpha\rangle_A \otimes |\phi_{n,\alpha}\rangle_B, \tag{12}$$

where n labels the "charge" and α other quantum numbers. (The states $|\phi_{n,\alpha}\rangle_B$ are not necessarily orthogonal and are not normalized.) For any nonproduct state $|\Psi\rangle$ there must exist at least two noncolinear $|\phi_{n,\alpha}\rangle_B$ in this decomposition, and thus there exists a two-dimensional subspace wherein the projections of these two vectors are noncolinear. ■

From this lemma, it follows that a state $|\Psi\rangle$ is in 1-D$_{H\text{-SSR}}$ iff there exists a subspace $\mathcal{S}^A \otimes \mathcal{S}^B$ that is locally H-invariant such that $(\Pi^A \otimes \Pi^B)|\Psi\rangle \neq 0$. It follows that for a nonproduct state that is not in 1-D$_{H\text{-SSR}}$, that is, a state in B1-D$_{H\text{-SSR}}$, any subspace $\mathcal{S}^A \otimes \mathcal{S}^B$ such that $(\Pi^A \otimes \Pi^B)|\Psi\rangle$ is a nonproduct state must fail to be locally H-invariant.

A. Activation under a local Abelian superselection rule

Theorem (activation). For all $|\Psi\rangle \in$ B1-D$_{H\text{-SSR}}$, there exists a $|\chi\rangle \in$ BLP$_{H\text{-SSR}}$ such that $|\Psi\rangle \otimes |\chi\rangle$ is in 1-D$_{H\text{-SSR}}$. We say that $|\chi\rangle$ has activated the entanglement in $|\Psi\rangle$.

Proof. Let

$$|\psi\rangle \equiv (\Pi^{A_1} \otimes \Pi^{B_1})|\Psi\rangle \tag{13}$$

be a nonproduct state on a 2×2 subspace $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$. Let $\{|\tilde{n}\rangle_{A_1}, |\tilde{n}'\rangle_{A_1}\}$ be a basis of eigenstates for $\mathcal{S}^{A_1}$, where $\tilde{n} \equiv (n,\alpha)$ and $\tilde{n}' \equiv (n',\alpha')$; note that it can occur that $n=n'$ due to the existence of other quantum numbers α. Similarly, let $\{|\tilde{m}\rangle_{B_1}, |\tilde{m}'\rangle_{B_1}\}$ form a basis for $\mathcal{S}^{B_1}$. Because $|\Psi\rangle$ is in B1-D$_{H\text{-SSR}}$, $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$ must fail to be H-invariant, and therefore either $n \neq n'$ or $m \neq m'$ or both. Let $\mathcal{S}^{A_2} \otimes \mathcal{S}^{B_2}$ be defined analogously to $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$, and define a state $|\chi\rangle$, confined to these subspaces, as follows:

$$|\chi\rangle \equiv (|\tilde{n}\rangle_{A_2} + |\tilde{n}'\rangle_{A_2}) \otimes (|\tilde{m}\rangle_{B_2} + |\tilde{m}'\rangle_{B_2}). \tag{14}$$

Because either $n \neq n'$ or $m \neq m'$ or both, this state is not locally H-invariant, and therefore is in BLP. Let $\mathcal{S}^A_{n+n'}$ be the H-invariant subspace of $\mathcal{S}^{A_1} \otimes \mathcal{S}^{A_2}$ spanned by

$$\{|\tilde{n}\rangle_{A_1} \otimes |\tilde{n}'\rangle_{A_2}, |\tilde{n}'\rangle_{A_1} \otimes |\tilde{n}\rangle_{A_2}\}. \tag{15}$$

Define $\mathcal{S}^B_{m+m'}$ similarly. Projecting $|\psi\rangle \otimes |\chi\rangle$ onto $\mathcal{S}^A_{n+n'} \otimes \mathcal{S}^B_{m+m'}$ can be performed probabilistically with H-LOCC, and can easily be shown to result in a nonproduct state that is locally H-invariant.

B. Distillation under a local Abelian superselection rule

We now present a complete characterization of the distillability properties of any pure state constrained by a local Abelian SSR. Specifically, we present two protocols for distillation of $|\Psi\rangle \in$ B1-D$_{H\text{-SSR}}$. Protocol A requires three copies of the state, and is based on activation. Protocol B requires only two copies of the state. In both protocols one chooses a local 2×2 subspace $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$ with certain properties, and $|\psi\rangle$ is defined in terms of $|\Psi\rangle$ as in Eq. (13).

Distillation protocol A. This protocol works if $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$ can be chosen such that ${}_{A_1}\langle \tilde{n}|\psi\rangle \in \mathcal{S}^{B_1}$ or ${}_{A_1}\langle \tilde{n}'|\psi\rangle \in \mathcal{S}^{B_1}$ is *not* locally H-invariant, and ${}_{B_1}\langle \tilde{m}|\psi\rangle \in \mathcal{S}^{A_1}$ or ${}_{B_1}\langle \tilde{m}'|\psi\rangle \in \mathcal{S}^{A_1}$ is not locally H-invariant (this requires $n \neq n'$ and $m \neq m'$). Projecting each copy k onto $\mathcal{S}^{A_k} \otimes \mathcal{S}^{B_k}$ yields, with some probability, three copies of $|\psi\rangle$. On the first copy, Alice measures $\{|\tilde{n}\rangle_{A_1}, |\tilde{n}'\rangle_{A_1}\}$, and on the second copy Bob measures $\{|\tilde{m}\rangle_{B_2}, |\tilde{m}'\rangle_{B_2}\}$ thereby collapsing A_2 and B_1, with some probability, to a product state that is not locally H-invariant, and thus in BLP$_{H\text{-SSR}}$. It can be shown, by following the proof of the activation theorem, that this product state is in fact sufficient to activate the entanglement in the third copy of $|\psi\rangle$. In this case, $|\Psi\rangle$ is in 3-D$_{H\text{-SSR}}$.

Distillation protocol B. Consider two copies of $|\Psi\rangle$. The protocol requires one to project the first copy onto $\mathcal{S}^{A_1} \otimes \mathcal{S}^{B_1}$ and the second copy onto $\mathcal{S}^{A_2} \otimes \mathcal{S}^{B_2}$, and then to project both copies onto $\mathcal{S}^A_{n+n'} \otimes \mathcal{S}^B_{m+m'}$ (this subspace is defined in the proof of the activation theorem). The resulting state has the form

$$\lambda_+ |S_+\rangle_A |S_+\rangle_B + \lambda_- |S_-\rangle_A |S_-\rangle_B, \tag{16}$$

where

$$|S_\pm\rangle_A = |\tilde{n}\rangle_{A_1} |\tilde{n}'\rangle_{A_2} \pm |\tilde{n}'\rangle_{A_1} |\tilde{n}\rangle_{A_2}, \tag{17}$$

and similarly for $|S_\pm\rangle_B$. It can be shown that λ_- is necessarily nonzero. Thus, if $\lambda_+ \neq 0$, then the resulting state is a nonproduct locally H-invariant state, and $|\Psi\rangle$ is in 2-D$_{H\text{-SSR}}$.

Theorem (distillation). All pure nonproduct states are distillable using H-LOCC, with at most three copies of the state required for distillation, that is, B1-D$_{H\text{-SSR}} \cup$ 1-D$_{H\text{-SSR}}$ = 3-D$_{H\text{-SSR}}$.

Proof. For every state $|\Psi\rangle$ in B1-D$_{H\text{-SSR}}$ for which distillation protocol A fails, protocol B necessarily succeeds. The proof is as follows. If protocol A fails, then for all choices of $\mathcal{S}^A \otimes \mathcal{S}^B$, the state $|\psi\rangle$ has a Schmidt basis [1] composed of local eigenstates, and protocol B can be shown to work whenever $|\psi\rangle$ is of this form. ■

Although related, this theorem is different from the one presented in Schuch *et al.* [23]. In Ref. [23], it was shown how states constrained by an Abelian SSR can be converted, using at most three copies, to the state $|\text{V-EPR}\rangle=|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B$. As we argued in Sec. III, this state can be considered to be bound entangled when local operations are constrained by a photon-number superselection rule. In addition, it is shown in Ref. [23] that this resource can be asymptotically converted to nonproduct locally H-invariant states. Our result is stronger in that we show it is possible to prepare (with some probability) an effective two-qubit entangled state that is locally H-invariant using only operations obeying the superselection rule and at most three copies of any entangled pure state. This effective two-qubit state may then be distilled by standard techniques to prepare locally H-invariant maximally entangled pure states at some asymptotic rate.

Finally, we note that we can completely characterize the distillability properties of any pure state constrained by a local Abelian SSR. We show that the class B1-D can be divided into three nonempty regions by establishing that 1-$\text{D}_{H\text{-SSR}}$ is a *proper* subset of 2-$\text{D}_{H\text{-SSR}}$ and 2-$\text{D}_{H\text{-SSR}}$ is a *proper* subset of 3-$\text{D}_{H\text{-SSR}}$ (i.e., protocol A sometimes fails while protocol B succeeds). States such as

$$|\psi'_{2\text{D}}\rangle=\frac{1}{\sqrt{2}}(|01\rangle_A|0\rangle_B+|10\rangle_A|1\rangle_B), \tag{18}$$

$$|\psi''_{2\text{D}}\rangle=\frac{1}{\sqrt{2}}(|01\rangle_A|+\rangle_B+|10\rangle_A|-\rangle_B), \tag{19}$$

$$|\psi'''_{2\text{D}}\rangle=\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B), \tag{20}$$

expressed in the Fock basis, are in 2-$\text{D}_{H\text{-SSR}}$ (using protocol B). None of them are locally H-invariant and therefore are not in 1-$\text{D}_{H\text{-SSR}}$. The state

$$|\psi_{3\text{D}}\rangle=\frac{1}{\sqrt{2}}(|0\rangle_A|+\rangle_B+|1\rangle_A|-\rangle_B), \tag{21}$$

is in 3-$\text{D}_{H\text{-SSR}}$ (as is any state in B1-D). However, because two copies of this state become separable when averaged (uniformly) over H locally, it is not in 2-$\text{D}_{H\text{-SSR}}$.

VI. DISCUSSION

In summary, we have shown how to reproduce the rich classification scheme of mixed-state entanglement by restricting local operations on the set of pure states so as to create a proper gap between what is locally preparable and what is 1-distillable. Debates over the entanglement properties of pure states under restricted operations, such as have appeared in the quantum optics literature, are resolved by recognizing novel categories of entanglement in this context. Our results suggest that the exotic structure of mixed-state entanglement is generic, and that developing entanglement theory under other sorts of restrictions is a promising direction for further research.

For example, recent interest in creating bipartite entangled states in condensed matter systems requires careful articulation of the operational meaning of entanglement, due to the various practical restrictions on operations on these systems. Local particle-number superselection rules often apply in practice, and as noted in Refs. [20,44,45], for example, the single-electron two-mode Fock state $(1/\sqrt{2})(|0\rangle_A|1\rangle_B+|1\rangle_A|0\rangle_B)$ has ambiguous entanglement properties under this restriction. Wiseman and Vaccaro [20] have introduced an operational measure, called entanglement of particles to quantify the distillable entanglement under a local particle-number SSR, and this two-mode single-electron Fock state has no entanglement of particles by this measure. For this reason, most proposals for creating bipartite entangled states make use of spin or orbital angular momentum degrees of freedom of multiple particles [46–48]. We note, however, that the two-mode single-electron Fock state is an entanglement resource akin to the two-mode single-photon state, which we have shown to be useful through activation or multi-copy distillation; also, a suitable shared U(1) reference frame could "lift" the restriction of the SSR, and the two-mode single-electron Fock state would be unambiguously entangled with such a resource. Moreover, entangled states between angular momentum degrees of freedom of different particles will yield no real advantage over the two-mode single-electron Fock state in situations wherein there is a local SU(2) SSR. Such a SSR will be in force, for instance, if the parties fail to share a Cartesian frame for spatial orientations [49]. As with quantum optical systems, we emphasize the need to be operational when classifying or quantifying entanglement.

The theory of entanglement for indistinguishable particles is another situation where our results may shed some light. Recent research has investigated the "quantum correlation between particles," [50,51] which relates to the correlations between indistinguishable particles inherent in the symmetry (or antisymmetry) of a many-particle wave function. Reference [20] argues that these quantum correlations are merely "fluffy bunny entanglement," [52] that is, operationally useless. Our work here supports this conclusion; we would say that the entanglement is bound by the restriction of the indistinguishability of particles. Nonetheless, in analogy with restrictions arising from SSRs, it may be worthwhile to consider the possibility of "lifting" this restriction through an appropriate shared resource.

The analogy we present here also suggests that it may be fruitful to think of standard LOCC as a restriction relative to the "more natural" PPT-preserving operations, and to consider whether a resource that lifts this restriction might be established with the same ease as a shared reference frame.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council and the State of Queensland. We thank John Vaccaro for many fruitful discussions and results on the entanglement properties of a single photon. We also thank Konrad Banaszek, Lucien Hardy, Michael Nielsen, Terry Rudolph, Steven van Enk, and Andrew White for helpful discussions.

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