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# Eigenvalues of Casimir invariants for $U_q[\mathrm{osp}(m|n)]$

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For each quantum superalgebra $U_q[\mathrm{osp}(m|n)]$ with $m>2$, an infinite family of Casimir invariants is constructed. This is achieved by using an explicit form for the Lax operator. The eigenvalue of each Casimir invariant on an arbitrary irreducible highest weight module is also calculated. © *2005 American Institute of Physics.* [DOI: 10.1063/1.2137712]

## I. INTRODUCTION

Representations of quantum superalgebras are known to provide solutions to the Yang-Baxter equation and represent the symmetries that underly supersymmetric exactly solvable (or integrable) models. Many such examples have arisen in the context of modelling systems of strongly correlated electrons.[1–5] More recently, the properties of solvability and supersymmetry have been applied to other areas, such as the solution of the Kondo model,[6] integrable superconformal field theory[7] and disordered systems.[8] Developing the representation theory of the quantum superalgebras is a useful step towards the complete understanding of such models. However, in many respects the representation theory of quantum superalgebras is not a straightforward generalization of the quantum algebra case, principally because not all representations of quantum superalgebras are unitary.[9] This leads, for example, to the existence of indecomposable representations not arising in the quantum algebra case, which generally make the analysis of supersymmetric models problematic (e.g., see Ref. 8).

In this paper we construct the Casimir invariants (central elements) of quantized orthosymplectic superalgebras. Our method of construction follows from the general results of Ref. 10 and the explicit form of the Lax operator obtained in Ref. 11. A fundamental problem is to determine the eigenvalues of the Casimir invariants when acting on an arbitrary finite-dimensional irreducible module. To date, the eigenvalues have only been calculated for the type I quantum superalgebras,[12,13] while the results for $U_q[\mathrm{osp}(1|n)]$ follow from an isomorphism derived in Ref. 14. In this paper we perform the calculations for the remaining nonexceptional quantum superalgebras, namely $U_q[\mathrm{osp}(m|n)]$ for $m>2$. The procedure we use for calculating the eigenvalues of the Casimir invariants when acting on any irreducible module is based on the early work by Perelomov and Popov[15,16] and Nwachuku and Rashid.[17] In doing so we follow the method used in Refs. 18 and 19 for the classical general and orthosymplectic superalgebras, respectively, which was adapted in Ref. 13 to cover $U_q[\mathrm{gl}(m|n)]$. Although the concepts are much the same as in those cases, the combination of the $q$-deformation and the more complex root system of $U_q[\mathrm{osp}(m|n)]$ makes the calculations in this paper more technically challenging.

In the following section we introduce our notation for $U_q[\mathrm{osp}(m|n)]$ and state the Lax operator. In Sec. III we develop the formulas for the Casimir invariants of $U_q[\mathrm{osp}(m|n)]$. The bulk of the calculations are in Sec. IV where the eigenvalues of the Casimir invariants are derived in detail.

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## II. THE QUANTIZED ORTHOSYMPLECTIC SUPERALGEBRA $U_q[\mathrm{osp}(m|n)]$

The quantum superalgebra $U_q[\mathrm{osp}(m|n)]$ is a $q$-deformation of the classical orthosymplectic superalgebra. A brief explanation of $U_q[\mathrm{osp}(m|n)]$ is given below, with a more thorough introduction to $\mathrm{osp}(m|n)$ and the $q$-deformation to be found in Ref. 11.

First we need to define the notation. The grading of $a$ is denoted by $[a]$, where

$$[a] = \begin{cases} 0, & a = i, \quad 1 \leqslant i \leqslant m, \\ 1, & a = \mu, \quad 1 \leqslant \mu \leqslant n. \end{cases} \tag{1}$$

Throughout this paper we use greek letters $\mu,\nu$, etc., to denote odd indices and italic letters $i,j$, etc., for even indices. If the grading is unknown, the usual $a,b,c$, etc., are used. Which convention applies will be clear from the context. Throughout the paper we also use the symbols $\bar{a}$ and $\xi_a$, which are given by

$$\bar{a} = \begin{cases} m+1-a, & [a]=0, \\ n+1-a, & [a]=1, \end{cases}$$

and

$$\xi_a = \begin{cases} 1, & [a]=0, \\ (-1)^a, & [a]=1. \end{cases}$$

As a weight system for $U_q[\mathrm{osp}(m|n)]$ we take the set $\{\varepsilon_i, 1\leqslant i \leqslant m\} \cup \{\delta_\mu, 1 \leqslant \mu \leqslant n\}$, where $\varepsilon_{\bar{i}}=-\varepsilon_i$ and $\delta_{\bar{\mu}}=-\delta_\mu$. Conveniently, when $m=2l+1$ this implies $\varepsilon_{l+1}=-\varepsilon_{l+1}=0$. Acting on these weights, we have the invariant bilinear form defined by

$$(\varepsilon_i,\varepsilon_j) = \delta^i_j, \quad (\delta_\mu,\delta_\nu) = -\delta^\mu_\nu, \quad (\varepsilon_i,\delta_\mu) = 0, \quad 1 \leqslant i,j \leqslant l, 1 \leqslant \mu,\nu \leqslant k.$$

When describing an object with unknown grading indexed by $a$ the weight will be described generically as $\varepsilon_a$. This should not be assumed to be an even weight.

The even positive roots of $U_q[\mathrm{osp}(m|n)]$ are composed entirely of the usual positive roots of $o(m)$ together with those of $\mathrm{sp}(n)$, namely,

$$\varepsilon_i \pm \varepsilon_j, \quad 1 \leqslant i < j \leqslant l,$$

$$\varepsilon_i, \quad 1 \leqslant i \leqslant l \quad \text{when } m = 2l+1,$$

$$\delta_\mu + \delta_\nu, \quad 1 \leqslant \mu,\nu \leqslant k,$$

$$\delta_\mu - \delta_\nu, \quad 1 \leqslant \mu < \nu \leqslant k.$$

The root system also contains a set of odd positive roots, which are

$$\delta_\mu + \varepsilon_i, \quad 1 \leqslant \mu \leqslant k, \quad 1 \leqslant i \leqslant m.$$

Throughout this paper we choose to use the following set of simple roots:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leqslant i < l,$$

$$\alpha_l = \begin{cases} \varepsilon_l + \varepsilon_{l-1}, & m = 2l, \\ \varepsilon_l, & m = 2l+1, \end{cases}$$

$$\alpha_\mu = \delta_\mu - \delta_{\mu+1}, \quad 1 \leqslant \mu < k,$$

$$\alpha_s = \delta_k - \varepsilon_1.$$

Note this choice is only valid for $m > 2$.

In $U_q[\mathrm{osp}(m|n)]$ the graded commutator is realized by

$$[A,B] = AB - (-1)^{[A][B]}BA$$

and tensor product multiplication is given by

$$(A \otimes B)(C \otimes D) = (-1)^{[B][C]}(AC \otimes BD).$$

Using these conventions, we have the following:

*Definition 2.1: The quantum superalgebra* $U_q[\mathrm{osp}(m|n)]$ *is generated by simple generators* $e_a, f_a, h_a$ *subject to the relations*

$$[h_a, e_b] = (\alpha_a, \alpha_b)e_b,$$

$$[h_a, f_b] = -(\alpha_a, \alpha_b)f_b,$$

$$[h_a, h_b] = 0,$$

$$[e_a, f_b] = \delta_b^a \frac{(q^{h_a} - q^{-h_a})}{(q - q^{-1})},$$

$$[e_a, e_a] = [f_a, f_a] = 0 \quad \text{for } (\alpha_a, \alpha_a) = 0,$$

*We remark that* $U_q[\mathrm{osp}(m|n)]$ *has the structure of a quasitriangular Hopf superalgebra. In particular, there is a linear mapping known as the coproduct,* $\Delta: U_q[\mathrm{osp}(m|n)] \to U_q[\mathrm{osp}(m|n)]^{\otimes 2}$, *which is defined on the simple generators by*

$$\Delta(e_a) = q^{1/2h_a} \otimes e_a + e_a \otimes q^{-1/2h_a},$$

$$\Delta(f_a) = q^{1/2h_a} \otimes f_a + f_a \otimes q^{-1/2h_a},$$

$$\Delta(q^{\pm 1/2h_a}) = q^{\pm 1/2h_a} \otimes q^{\pm 1/2h_a},$$

*and extends to arbitrary elements according to the homomorphism property, namely*,

$$\Delta(AB) = \Delta(A)\Delta(B).$$

There are further defining relations such as the $q$-Serre relations, but they are not needed in this paper.

The quasitriangular property guarantees the existence of a universal $R$-matrix, which provides a solution to the Yang-Baxter equation. Before elaborating, we need to introduce the graded twist map.

The *graded twist map* $T: U_q[\mathrm{osp}(m|n)]^{\otimes 2} \to U_q[\mathrm{osp}(m|n)]^{\otimes 2}$ is given by

$$T(a \otimes b) = (-1)^{[a][b]}(b \otimes a).$$

For convenience, $T \circ \Delta$, the twist map composed with the coproduct, is denoted $\Delta^T$. Then a *universal R-matrix*, $\mathcal{R}$, is an even, nonsingular element of $U_q[\mathrm{osp}(m|n)]^{\otimes 2}$ satisfying the following properties:

TABLE I. The action of the vector representation $\pi$ on the simple generators of $U_q[\text{osp}(m|n)]$.

| $\alpha_a$ | $\pi(e_a)$ | $\pi(f_a)$ | $\pi(h_a)$ |
|---|---|---|---|
| $\alpha_i, 1 \leqslant i < l$ | $E^i_{i+1} - E^{\overline{i+1}}_{\bar{i}}$ | $E^{i+1}_i - E^{\bar{i}}_{\overline{i+1}}$ | $E^i_i - E^{\bar{i}}_{\bar{i}} - E^{i+1}_{i+1} + E^{\overline{i+1}}_{\overline{i+1}}$ |
| $\alpha_l, m=2l$ | $E^{l-1}_{\bar{l}} - E^{l}_{\overline{l-1}}$ | $E^{\bar{l}}_{l-1} - E^{\overline{l-1}}_{l}$ | $E^{l-1}_{l-1} + E^l_l - E^{\overline{l-1}}_{\overline{l-1}} - E^{\bar{l}}_{\bar{l}}$ |
| $\alpha_l, m=2l+1$ | $E^l_{l+1} - E^{l+1}_{\bar{l}}$ | $E^{l+1}_l - E^{\bar{l}}_{l+1}$ | $E^l_l - E^{\bar{l}}_{\bar{l}}$ |
| $\alpha_\mu, 1 \leqslant \mu < k$ | $E^\mu_{\mu+1} + E^{\overline{\mu+1}}_{\bar{\mu}}$ | $E^{\mu+1}_\mu + E^{\bar{\mu}}_{\overline{\mu+1}}$ | $E^{\mu+1}_{\mu+1} - E^{\overline{\mu+1}}_{\overline{\mu+1}} - E^\mu_\mu + E^{\bar{\mu}}_{\bar{\mu}}$ |
| $\alpha_s$ | $E^{\mu=k}_{i=1} + (-1)^k E^{\overline{i=1}}_{\overline{\mu=k}}$ | $-E^{i=1}_{\mu=k} + (-1)^k E^{\overline{\mu=k}}_{\overline{i=1}}$ | $-E^{i=1}_{i=1} + E^{\bar{i}=\bar{1}}_{\bar{i}=\bar{1}} - E^{\mu=k}_{\mu=k} + E^{\bar{\mu}=\bar{k}}_{\bar{\mu}=\bar{k}}$ |

$$\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}, \quad \forall\, a \in U_q[\text{osp}(m|n)],$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}. \tag{2}$$

Here $\mathcal{R}_{ab}$ represents a copy of $\mathcal{R}$ acting on the $a$ and $b$ components, respectively, of $U_1 \otimes U_2 \otimes U_3$, where each $U$ is a copy of the quantum superalgebra $U_q[\text{osp}(m|n)]$. When $a > b$ the usual grading term from the twist map is included, so, for example, $\mathcal{R}_{21} = [\mathcal{R}^T]_{12}$, where $\mathcal{R}^T = T(\mathcal{R})$ is the *opposite universal R-matrix*.

The $R$-matrix is significant because it is a solution to the Yang-Baxter equation, which is prominent in the study of integrable systems,[20]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

A superalgebra may contain many different universal $R$-matrices, but there is always a unique one belonging to $U_q[\text{osp}(m|n)]^- \otimes U_q[\text{osp}(m|n)]^+$, with its opposite $R$-matrix in $U_q[\text{osp}(m|n)]^+ \otimes U_q[\text{osp}(m|n)]^-$. Here $U_q[\text{osp}(m|n)]^-$ is the Hopf subsuperalgebra generated by the lowering generators $\{f_a\}$ and Cartan elements $\{h_a\}$, while $U_q[\text{osp}(m|n)]^+$ is generated by the raising generators $\{e_a\}$ and the Cartan elements. These particular $R$-matrices arise out of the $\mathbb{Z}_2$-graded version of Drinfeld's double construction.[21] In this paper we consider the universal $R$-matrix belonging to $U_q[\text{osp}(m|n)]^- \otimes U_q[\text{osp}(m|n)]^+$.

We also need to define the vector representation for $U_q[\text{osp}(m|n)]$. Let End $V$ be the space of endomorphisms of $V$, an $(m+n)$-dimensional vector space. Then the irreducible *vector representation* $\pi: U_q[\text{osp}(m|n)] \to \text{End } V$ acts on the $U_q[\text{osp}(m|n)]$ generators as given in Table I, where $E^a_b$ is the elementary matrix with a 1 in the $(a,b)$ position and zeroes elsewhere.

One quantity that repeatedly arises in calculations for both classical and quantum Lie superalgebras is $\rho$, the *graded half-sum of positive roots*. In the case of $U_q[\text{osp}(m|n)]$ it is given by

$$\rho = \frac{1}{2}\sum_{i=1}^{l}(m-2i)\varepsilon_i + \frac{1}{2}\sum_{\mu=1}^{k}(n-m+2-2\mu)\delta_\mu.$$

This satisfies the property $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha)$ for all simple roots $\alpha$.

*The Lax operator for* $U_q[\text{osp}(m|n)]$: Let $\mathcal{R}$ be the universal $R$-matrix of $U_q[\text{osp}(m|n)]$ and $\pi$ the vector representation. The Lax operator associated with $\mathcal{R}$ is given by

$$R = (\pi \otimes \text{id})\mathcal{R} \in (\text{End } V) \otimes U_q[\text{osp}(m|n)].$$

It has been shown in Ref. 11 that the Lax operator is given by

$$R=\sum_a E_a^a\otimes q^{h_{\varepsilon_a}}+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[b]}E_b^a\otimes q^{h_{\varepsilon_a}}\hat{\sigma}_{ba},$$

where the simple operators $\hat{\sigma}_{ba}$ are given by

$$\hat{\sigma}_{ii+1}=-\hat{\sigma}_{\overline{i+1}\bar{i}}=q^{1/2}e_iq^{1/2h_i},\quad 1\leqslant i<l,$$

$$\hat{\sigma}_{l-1\bar{l}}=-\hat{\sigma}_{l\overline{l-1}}=q^{1/2}e_lq^{1/2h_l},\quad m=2l,$$

$$\hat{\sigma}_{l\bar{l}}=0,\quad m=2l,$$

$$\hat{\sigma}_{ll+1}=-q^{-1/2}\hat{\sigma}_{l+1\bar{l}}=e_lq^{1/2h_l},\quad m=2l+1,$$

$$\hat{\sigma}_{\mu\mu+1}=\hat{\sigma}_{\overline{\mu+1}\bar{\mu}}=q^{-1/2}e_\mu q^{1/2h_\mu},\quad 1\leqslant\mu<k,$$

$$\hat{\sigma}_{\mu=ki=1}=(-1)^kq\hat{\sigma}_{i=\bar{1}\bar{\mu}=\bar{k}}=q^{1/2}e_sq^{1/2h_s};$$

and the remaining operators can be calculated using the following:

(i) the $q$-commutation relations,

$$q^{(\alpha_c,\varepsilon_b)}\hat{\sigma}_{ba}e_cq^{1/2h_c}-(-1)^{([a]+[b])[c]}q^{-(\alpha_c,\varepsilon_a)}e_cq^{1/2h_c}\hat{\sigma}_{ba}=0,\quad \varepsilon_b>\varepsilon_a,$$

where neither $\varepsilon_a-\alpha_c$ nor $\varepsilon_b+\alpha_c$ equals any $\varepsilon_x$; and

(ii) the induction relations

$$\hat{\sigma}_{ba}=q^{-(\varepsilon_b,\varepsilon_a)}\hat{\sigma}_{bc}\hat{\sigma}_{ca}-q^{-(\varepsilon_c,\varepsilon_c)}(-1)^{([b]+[c])([a]+[c])}\hat{\sigma}_{ca}\hat{\sigma}_{bc},\quad \varepsilon_b>\varepsilon_c>\varepsilon_a,$$

where $c\neq\bar{b}$ or $\bar{a}$.

To define the opposite Lax operator $R^T=(\pi\otimes\mathrm{id})\mathcal{R}$ we require the graded conjugation action †, which is defined on the simple generators by (see Ref. 11)

$$e_a^\dagger=f_a,\quad f_a^\dagger=(-1)^{[a]}e_a,\quad h_a^\dagger=h_a.$$

It is consistent with the coproduct and extends naturally to all remaining elements of $U_q[\mathrm{osp}(m|n)]$, satisfying the following properties:

$$(\hat{\sigma}_{ab})^\dagger=(-1)^{[a]([a]+[b])}\hat{\sigma}_{ba},$$

$$(ab)^\dagger=(-1)^{[a][b]}b^\dagger a^\dagger,$$

$$(a\otimes b)^\dagger=a^\dagger\otimes b^\dagger,$$

$$\Delta(a)^\dagger=\Delta(a^\dagger).$$

Then the opposite $R$-matrix is given by

$$R^T=\sum_a E_a^a\otimes q^{h_{\varepsilon_a}}+(q-q^{-1})\sum_{\varepsilon_b>\varepsilon_a}(-1)^{[a]}E_a^b\otimes\hat{\sigma}_{ab}q^{h_{\varepsilon_a}},$$

where

$$\hat{\sigma}_{ab}=(-1)^{[b]([a]+[b])}\hat{\sigma}_{ba}^\dagger,\quad \varepsilon_b>\varepsilon_a.$$

## III. CASIMIR INVARIANTS OF $U_q[\mathrm{osp}(m|n)]$

We now use the Lax operator to construct a family of Casimir invariants and then to calculate their eigenvalues when acting on an irreducible highest weight module. Before constructing the Casimir invariants, however, we need to define a new object. Let $h_\rho$ be the unique element of the Cartan subalgebra $H$ satisfying

$$\alpha_i(h_\rho) = (\rho, \alpha_i), \quad \forall\ \alpha_i \in H^*.$$

Then from Ref. 10 we have the following theorem.

**Theorem 3.1:** *Let $\mathcal{V}$ be the representation space of $\tau$, an arbitrary finite-dimensional representation of $U_q[\mathrm{osp}(m|n)]$. If $\Gamma \in (\mathrm{End}\ \mathcal{V}) \otimes U_q[\mathrm{osp}(m|n)]$ satisfies*

$$\partial(a)\Gamma = \Gamma\partial(a), \quad \forall\ a \in U_q[\mathrm{osp}(m|n)], \tag{3}$$

*where $\partial \equiv (\pi \otimes \mathrm{id})\Delta$, then*

$$C = (\mathrm{str} \otimes \mathrm{id})(\tau(q^{2h_\rho}) \otimes I)\Gamma,$$

*belongs to the center of $U_q[\mathrm{osp}(m|n)]$. Above str denotes the supertrace.*

Now choose $\tau$ to be the vector representation $\pi$. Recalling that the universal $R$-matrix satisfies

$$\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}, \quad \forall\ a \in U_q[\mathrm{osp}(m|n)],$$

it is clear that

$$\partial(a)R^TR = R^TR\partial(a), \quad \forall\ a \in U_q[\mathrm{osp}(m|n)].$$

Hence if we set $A \in (\mathrm{End}\ V) \otimes U_q[\mathrm{osp}(m|n)]$ to be

$$A = \frac{(R^TR - I \otimes I)}{(q - q^{-1})},$$

the operators $A^l$ will satisfy condition (3) for all non-negative integers $l$. Thus the operators $C_l$ defined as

$$C_l = (\mathrm{str} \otimes \mathrm{id})(\pi(q^{2h_\rho}) \otimes I)A^l, \quad l \in \mathbb{Z}^+,$$

form a family of Casimir invariants. Here $A$ coincides with the matrix of Jarvis and Green[22] in the classical limit $q \to 1$, as do the invariants $C_l$.

Now write the Lax operator $R$ and its opposite $R^T$ in the form

$$R = I \otimes I + (q - q^{-1}) \sum_{\varepsilon_b \geqslant \varepsilon_a} E^a_b \otimes X^b_a,$$

$$R^T = I \otimes I + (q - q^{-1}) \sum_{\varepsilon_b \leqslant \varepsilon_a} E^a_b \otimes X^b_a.$$

In terms of the operators $\hat{\sigma}_{ba}$, this implies

$$X^b_a = \begin{cases} \dfrac{q^{h_{\varepsilon_a}} - I}{q - q^{-1}}, & a = b, \\ (-1)^{[b]} q^{h_{\varepsilon_a}} \hat{\sigma}_{ba}, & \varepsilon_a < \varepsilon_b, \\ (-1)^{[b]} \hat{\sigma}_{ba} q^{h_{\varepsilon_b}}, & \varepsilon_a > \varepsilon_b. \end{cases}$$

Writing $A$ as

$$A=\sum_{a,b}E^a_b\otimes A^b_a,$$

we obtain

$$A^b_a=(1+\delta^a_b)X^b_a+(q-q^{-1})\sum_{\varepsilon_c\leqslant\varepsilon_a,\varepsilon_b}(-1)^{([a]+[c])([b]+[c])}X^c_aX^b_c.$$

This produces a family of Casimir invariants

$$C_l=\sum_a(-1)^{[a]}q^{(2\rho,\varepsilon_a)}A^{(l)a}{}_a,$$

where the operators $A^{(l)b}{}_a$ are recursively defined as

$$A^{(l)b}{}_a=\sum_c(-1)^{([a]+[c])([b]+[c])}A^{(l-1)c}{}_aA^b_c. \tag{4}$$

Note that $A$ corresponds to the matrix $A$ given for the nongraded case in Ref. 23. Following a line of reasoning similar to that in Ref. 24 it can be shown that when acting on an irreducible module $V(\Lambda)$, $A$ satisfies the following polynomial identity:

$$\prod_{a=1}^{m+n}(A-\alpha_a(\Lambda)I)=0,$$

where

$$\alpha_a(\Lambda)=\frac{q^{(\varepsilon_a,\varepsilon_a+2\Lambda+2\rho)-C(\Lambda_0)}-1}{q-q^{-1}}$$

and $C(\Lambda_0)=(\delta_1,\delta_1+2\rho)=m-n-1$. In the limit $q\to1$ this reduces to the identity given in Ref. 24.

## IV. EIGENVALUES OF THE CASIMIR INVARIANTS

Now that we have found a family of Casimir invariants, we wish to calculate their eigenvalues on a general irreducible finite-dimensional module. Let $V(\Lambda)$ be an arbitrary irreducible finite-dimensional module with highest weight $\Lambda$ and highest weight state $|\Lambda\rangle$. Define $t^{(l)}_a$ to be the eigenvalue of $A^{(l)a}{}_a$ on this state, so

$$A^{(l)a}{}_a|\Lambda\rangle=t^{(l)}_a|\Lambda\rangle.$$

Once we have calculated $t^{(l)}_a$ we will use this result to find the eigenvalues of the Casimir invariants $C_l$.

To evaluate $t^{(l)}_a$, note that if $\varepsilon_b>\varepsilon_a$ then $A^{(l)b}{}_a$ is a raising operator, implying $A^{(l)b}{}_a|\Lambda\rangle=0$. Thus from Eq. (4) we deduce

$$\begin{aligned}t^{(l)}_a|\Lambda\rangle&=t^{(l-1)}_at^{(1)}_a|\Lambda\rangle+\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[a]+[b]}A^{(l-1)b}{}_aA^a_b|\Lambda\rangle\\&=t^{(l-1)}_at^{(1)}_a|\Lambda\rangle+\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[a]+[b]}A^{(l-1)b}{}_a[X^a_b+(q-q^{-1})X^a_bX^a_a]|\Lambda\rangle\\&=t^{(l-1)}_at^{(1)}_a|\Lambda\rangle+\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[a]+[b]}q^{(\Lambda,\varepsilon_a)}A^{(l-1)b}{}_aX^a_b|\Lambda\rangle.\end{aligned}$$

Now we know that

$$A^l \partial(X_b^a) = \partial(X_b^a) A^l. \tag{5}$$

This can be used to calculate $A^{(l)b}{}_a X_b^a|\Lambda\rangle$ for $\varepsilon_a < \varepsilon_b$. First we need an expression for $\Delta(X_b^a)$. The $R$-matrix properties give

$$(\Delta \otimes I)R = R_{13}R_{23} \Rightarrow (I \otimes \Delta)R^T = R_{12}^T R_{13}^T.$$

In terms of $X_b^a$, this implies

$$\begin{aligned}
&I \otimes I \otimes I + (q-q^{-1}) \sum_{\varepsilon_a \leqslant \varepsilon_b} E_a^b \otimes \Delta(X_b^a) \\
&\quad = \left(I \otimes I \otimes I + (q-q^{-1}) \sum_{\varepsilon_a \leqslant \varepsilon_b} E_a^b \otimes X_b^a \otimes I\right)\left(I \otimes I \otimes I + (q-q^{-1}) \sum_{\varepsilon_a \leqslant \varepsilon_b} E_a^b \otimes I \otimes X_b^a\right) \\
&\quad = I \otimes I \otimes I + (q-q^{-1}) \sum_{\varepsilon_a \leqslant \varepsilon_b} E_a^b \otimes (X_b^a \otimes I + I \otimes X_b^a) \\
&\quad + (q-q^{-1})^2 \sum_{\varepsilon_a \leqslant \varepsilon_c \leqslant \varepsilon_b} (-1)^{([a]+[c])([b]+[c])} E_a^b \otimes X_b^c \otimes X_c^a.
\end{aligned}$$

Hence for all $\varepsilon_a < \varepsilon_b$,

$$\Delta(X_b^a) = X_b^a \otimes I + I \otimes X_b^a + (q-q^{-1}) \sum_{\varepsilon_a \leqslant \varepsilon_c \leqslant \varepsilon_b} (-1)^{([a]+[c])([b]+[c])} X_b^c \otimes X_c^a.$$

We also need an expression for $\pi(X_b^a)$ for $\varepsilon_a \leqslant \varepsilon_b$. In Ref. 11 we found the generators for $R^T$ in the vector representation are given by

$$\hat{\sigma}_{ab} q^{h_{\varepsilon_a}} = E_b^a - (-1)^{[a]([a]+[b])} \xi_a \xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)} E_{\bar{a}}^{\bar{b}}, \quad \varepsilon_a < \varepsilon_b.$$

From this we deduce that

$$\pi(X_b^a) = (-1)^{[a]} E_b^a - (-1)^{[a][b]} \xi_a \xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)} E_{\bar{a}}^{\bar{b}}, \quad \varepsilon_a < \varepsilon_b.$$

Also, we know

$$\pi(X_a^a) = (q-q^{-1})^{-1} \pi(q^{h_{\varepsilon_a}} - I) = (q-q^{-1})^{-1}\left(q^{(\varepsilon_a,\varepsilon_a)(E_a^a - E_{\bar{a}}^{\bar{a}})} - I\right).$$

Applying these, we find that if $\varepsilon_a < \varepsilon_b$ then

$$\begin{aligned}
\partial(X_b^a) &= (\pi \otimes I)\Delta(X_b^a) = \pi(X_b^a) \otimes (I + (q-q^{-1})X_a^a) + (I + (q-q^{-1})\pi(X_b^b)) \otimes X_b^a \\
&\quad + (q-q^{-1}) \sum_{\varepsilon_a < \varepsilon_c < \varepsilon_b} (-1)^{([a]+[c])([b]+[c])} \pi(X_b^c) \otimes X_c^a \\
&= ((-1)^{[a]} E_b^a - (-1)^{[a][b]} \xi_a \xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)} E_{\bar{a}}^{\bar{b}}) \otimes q^{h_{\varepsilon_a}} + q^{(\varepsilon_b,\varepsilon_b)(E_b^b - E_{\bar{b}}^{\bar{b}})} \otimes X_b^a \\
&\quad + (q-q^{-1}) \sum_{\varepsilon_a < \varepsilon_c < \varepsilon_b} (-1)^{([a]+[c])([b]+[c])} \\
&\quad \times ((-1)^{[c]} E_b^c - (-1)^{[b][c]} \xi_b \xi_c q^{(\rho,\varepsilon_c-\varepsilon_b)} E_{\bar{c}}^{\bar{b}}) \otimes X_c^a.
\end{aligned}$$

Substituting this expression into Eq. (5) and equating the $(a,b)$ entries, we find

$$(-1)^{[a]}A^{(l)a}{}_{a}q^{h_{\varepsilon_a}}-\delta^{a}_{\bar{b}}(-1)^{[a][b]}\xi_a\xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)}A^{(l)a}{}_{a}q^{h_{\varepsilon_a}}+q^{(\varepsilon_b,\varepsilon_b)}A^{(l)b}{}_{a}X^{a}_{b}$$
$$+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_c<\varepsilon_b}((-1)^{[c]}A^{(l)c}{}_{a}X^{a}_{c}-\delta^{b}_{\bar{c}}(-1)^{[b][c]}\xi_b\xi_c q^{(\rho,\varepsilon_c-\varepsilon_b)}A^{(l)\bar{b}}{}_{a}X^{a}_{c})$$
$$=(-1)^{[a]}q^{h_{\varepsilon_a}}A^{(l)b}{}_{b}-\delta^{a}_{\bar{b}}(-1)^{[a][b]}\xi_a\xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)}q^{h_{\varepsilon_a}}A^{(l)b}{}_{b}+(-1)^{[a]+[b]}q^{(\varepsilon_a,\varepsilon_b)}X^{a}_{b}A^{(l)b}{}_{a}$$
$$-(q-q)^{-1}\delta^{a}_{\bar{b}}\sum_{\varepsilon_a<\varepsilon_c<\varepsilon_b}(-1)^{[b][c]}\xi_b\xi_c q^{(\rho,\varepsilon_c-\varepsilon_b)}X^{a}_{c}A^{(l)b}{}_{\bar{c}}.$$

Simplifying gives

$$(-1)^{[a]+[b]}q^{(\varepsilon_a,\varepsilon_b)}X^{a}_{b}A^{(l)b}{}_{a}-q^{(\varepsilon_b,\varepsilon_b)}A^{(l)b}{}_{a}X^{a}_{b}$$
$$=((-1)^{[a]}-\delta^{a}_{\bar{b}}q^{(\rho,\varepsilon_a-\varepsilon_b)})q^{h_{\varepsilon_a}}(A^{(l)a}{}_{a}-A^{(l)b}{}_{b})$$
$$+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_c<\varepsilon_b}((-1)^{[c]}-\delta^{b}_{\bar{c}}q^{(\rho,\varepsilon_c-\varepsilon_b)})A^{(l)c}{}_{a}X^{a}_{c}$$
$$+(q-q^{-1})\delta^{a}_{\bar{b}}\sum_{\varepsilon_a<\varepsilon_c<\varepsilon_b}(-1)^{[b][c]}\xi_b\xi_c q^{(\rho,\varepsilon_c-\varepsilon_b)}X^{a}_{c}A^{(l)\bar{a}}{}_{\bar{c}}.$$

Remembering that $\varepsilon_a<\varepsilon_b$, we apply this to the highest weight state $|\Lambda\rangle$ to obtain

$$-q^{(\varepsilon_b,\varepsilon_b)}A^{(l)b}{}_{a}X^{a}_{b}|\Lambda\rangle=q^{(\Lambda,\varepsilon_a)}((-1)^{[a]}-\delta^{a}_{\bar{b}}q^{2(\rho,\varepsilon_a)})(t^{(l)}_{a}-t^{(l)}_{b})|\Lambda\rangle$$
$$+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_c<\varepsilon_b}((-1)^{[c]}-\delta^{b}_{\bar{c}}q^{2(\rho,\varepsilon_c)})A^{(l)c}{}_{a}X^{a}_{c}|\Lambda\rangle. \tag{6}$$

The next step is to calculate $A^{(l)b}{}_{a}X^{a}_{b}|\Lambda\rangle$ for $\varepsilon_a<\varepsilon_b$. It is first convenient to order the indices according to $b>c\Leftrightarrow\varepsilon_b<\varepsilon_c$. With this ordering we say an element $a>0$ if $\varepsilon_a<0$, $a=0$ if $\varepsilon_a=0$, and $a<0$ if $\varepsilon_a>0$. Using this convention, it is apparent the solution to (6) will be of the form

$$A^{(l)b}{}_{a}X^{a}_{b}|\Lambda\rangle=q^{(\Lambda,\varepsilon_a)}(-1)^{[a]}\sum_{a>c\geq b}\alpha^{a}_{bc}(t^{(l)}_{a}-t^{(l)}_{c})|\Lambda\rangle, \tag{7}$$

where $\alpha^{a}_{bc}$ is a function of $a$, $b$, and $c$. Now from Eq. (6) we have

$$(q-q^{-1})\sum_{a>c>b}(-1)^{[c]}A^{(l)c}{}_{a}X^{a}_{c}|\Lambda\rangle$$
$$=-q^{(\varepsilon_b,\varepsilon_b)}A^{(l)b}{}_{a}X^{a}_{b}|\Lambda\rangle+(q-q^{-1})\sum_{a>c>b}\delta^{b}_{\bar{c}}q^{-2(\rho,\varepsilon_b)}A^{(l)c}{}_{a}X^{a}_{c}|\Lambda\rangle$$
$$-(-1)^{[a]}q^{(\Lambda,\varepsilon_a)}(1-\delta^{a}_{\bar{b}}(-1)^{[a]}q^{2(\rho,\varepsilon_a)})(t^{(l)}_{a}-t^{(l)}_{b})|\Lambda\rangle$$
$$=-q^{(\varepsilon_{b+1},\varepsilon_{b+1})}A^{(l)b+1}{}_{a}X^{a}_{b+1}|\Lambda\rangle$$
$$+(q-q^{-1})\sum_{a>c>b+1}\delta^{b+1}_{\bar{c}}q^{-2(\rho,\varepsilon_{b+1})}A^{(l)c}{}_{a}X^{a}_{c}|\Lambda\rangle$$
$$-(-1)^{[a]}q^{(\Lambda,\varepsilon_a)}(1-\delta^{b+1}_{\bar{a}}(-1)^{[a]}q^{2(\rho,\varepsilon_a)})(t^{(l)}_{a}-t^{(l)}_{b+1})|\Lambda\rangle$$
$$+(q-q^{-1})(-1)^{[b+1]}A^{(l)b+1}{}_{a}X^{a}_{b+1}|\Lambda\rangle.$$

Substituting in the form of the solution given in Eq. (7) produces

$$q^{(\varepsilon_b,\varepsilon_b)}\sum_{a>d\geq b}\alpha^a_{bd}(t^{(l)}_a-t^{(l)}_d)|\Lambda\rangle$$
$$=(q^{(\varepsilon_{b+1},\varepsilon_{b+1})}-(q-q^{-1})(-1)^{[b+1]})\sum_{a>d\geq b+1}\alpha^a_{(b+1)d}(t^{(l)}_a-t^{(l)}_d)|\Lambda\rangle$$
$$-(1-\delta^a_{\bar{b}}(-1)^{[a]}q^{2(\rho,\varepsilon_a)})(t^{(l)}_a-t^{(l)}_b)|\Lambda\rangle+(1-\delta^a_{\overline{b+1}}(-1)^{[a]}q^{2(\rho,\varepsilon_a)})(t^{(l)}_a-t^{(l)}_{b+1})|\Lambda\rangle$$
$$+(q-q^{-1})\sum_{a>c>b}\delta^b_{\bar{c}}q^{-2(\rho,\varepsilon_b)}\sum_{a>d\geq c}\alpha^a_{\bar{b}d}(t^{(l)}_a-t^{(l)}_d)|\Lambda\rangle$$
$$-(q-q^{-1})\sum_{a>c>b+1}\delta^{b+1}_{\bar{c}}q^{-2(\rho,\varepsilon_{b+1})}\sum_{a>d\geq c}\alpha^a_{(\overline{b+1})d}(t^{(l)}_a-t^{(l)}_d)|\Lambda\rangle. \tag{8}$$

Set

$$\alpha^a_{bd}=\bar{\alpha}_{bd}(1-\delta^a_{\bar{d}}(-1)^{[a]}q^{2(\rho,\varepsilon_a)}).$$

Then from Eq. (8) we obtain

$$\bar{\alpha}_{bb}=-q^{-(\varepsilon_b,\varepsilon_b)}$$

and

$$\bar{\alpha}_{b(b+1)}=q^{-(\varepsilon_b,\varepsilon_b)}[(q^{(\varepsilon_{b+1},\varepsilon_{b+1})}-(q-q^{-1})(-1)^{[b+1]})\bar{\alpha}_{(b+1)(b+1)}+1+(q-q^{-1})\delta^b_{\overline{b+1}}q^{-2(\rho,\varepsilon_b)}\bar{\alpha}_{\bar{b}(b+1)}]$$
$$=q^{-(\varepsilon_b,\varepsilon_b)-(\varepsilon_{b+1},\varepsilon_{b+1})}(q-q^{-1})((-1)^{[b+1]}-\delta^b_{\overline{b+1}}q^{-2(\rho,\varepsilon_b)}).$$

To simplify this expression note that $q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}=q^{-(\varepsilon_b,\varepsilon_b)-(\varepsilon_{b+1},\varepsilon_{b+1})}$ in all cases except for $[b]=0$, $b=l$, $m=2l$, in which case $q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}=q^2q^{-(\varepsilon_b,\varepsilon_b)-(\varepsilon_{b+1},\varepsilon_{b+1})}$. However $[b]=0$, $b=l$, $m=2l$ if and only if $\delta^b_{\overline{b+1}}=1$, and in that case we find $\bar{\alpha}_{b(b+1)}=0$. Hence for all values of $b$ we can write

$$\bar{\alpha}_{b(b+1)}=(q-q^{-1})q^{-2(\rho,\varepsilon_b)}((-1)^{[b+1]}q^{2(\rho,\varepsilon_{b+1})}-\delta^b_{\overline{b+1}}).$$

Now that we have found $\bar{\alpha}_{bb}$ and $\bar{\alpha}_{b(b+1)}$, they can be used to calculate the remaining $\bar{\alpha}_{bd}$. From Eq. (8) we observe that if $d>b+1$ then

$$\bar{\alpha}_{bd}=q^{-(\varepsilon_b,\varepsilon_b)}(q^{(\varepsilon_{b+1},\varepsilon_{b+1})}-(q-q^{-1})(-1)^{[b+1]})\bar{\alpha}_{(b+1)d}+(q-q^{-1})q^{-(\varepsilon_b,\varepsilon_b)}\sum_{d\geq c>b}\delta^b_{\bar{c}}q^{-2(\rho,\varepsilon_b)}\bar{\alpha}_{\bar{b}d}$$
$$-(q-q^{-1})q^{-(\varepsilon_b,\varepsilon_b)}\sum_{d\geq c>b+1}\delta^{b+1}_{\bar{c}}q^{-2(\rho,\varepsilon_{b+1})}\bar{\alpha}_{(\overline{b+1})d}. \tag{9}$$

Now define $\theta_{xy}$ by

$$\theta_{xy}=\begin{cases}1, & x<y,\\ 0, & x\geq y.\end{cases}$$

Then Eq. (9) can be rewritten as

$$\bar{\alpha}_{bd}=q^{-(\varepsilon_b,\varepsilon_b)}(q^{(\varepsilon_{b+1},\varepsilon_{b+1})}-(q-q^{-1})(-1)^{[b+1]})\bar{\alpha}_{(b+1)d}$$
$$+(q-q^{-1})q^{-(\varepsilon_b,\varepsilon_b)}q^{2(\rho,\varepsilon_c)}(\theta_{bc}\theta_{c(d+1)}\delta^b_{\bar{c}}-\theta_{(b+1)c}\theta_{c(d+1)}\delta^{b+1}_{\bar{c}})\bar{\alpha}_{cd},\quad d>a+1. \tag{10}$$

Consider $\bar{\alpha}_{bd}$ for any $b>l$. Both $\theta_{b\bar{b}}$ and $\theta_{(b+1)(\overline{b+1})}$ will equal 0, so

$$\bar{\alpha}_{bd} = q^{-(\varepsilon_b,\varepsilon_b)}(q^{(\varepsilon_{b+1},\varepsilon_{b+1})} - (q-q^{-1})(-1)^{[b+1]})\bar{\alpha}^{a}_{(b+1)d}$$
$$= q^{-(\varepsilon_b,\varepsilon_b)}q^{-(\varepsilon_{b+1},\varepsilon_{b+1})}\bar{\alpha}_{(b+1)d} = q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}\bar{\alpha}_{(b+1)d}.$$

Since

$$\bar{\alpha}_{(d-1)d} = (-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d-\varepsilon_{d-1})},$$

we obtain

$$\bar{\alpha}_{bd} = (-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d-\varepsilon_b)}, \quad d > b > l.$$

Substituting this together with our expression for $\bar{\alpha}_{bb}$ into Eq. (10), we find

$$\begin{aligned}\bar{\alpha}_{bd} &= q^{-(\varepsilon_b,\varepsilon_b)}(q^{-(\varepsilon_{b+1},\varepsilon_{b+1})} - \delta^{b+1}_{\overline{b+1}}(q-q^{-1}))\bar{\alpha}_{(b+1)d} \\ &\quad + (q-q^{-1})^2 q^{-(\varepsilon_b,\varepsilon_b)}(-1)^{[d]}q^{2(\rho,\varepsilon_d)}(\theta_{b\bar{b}}\theta_{\bar{b}d} - \theta_{(b+1)(\overline{b+1})}\theta_{(\overline{b+1})d}) \\ &\quad - (q-q^{-1})q^{-(\varepsilon_b,\varepsilon_b)}q^{-(\varepsilon_d,\varepsilon_d)}q^{2(\rho,\varepsilon_d)}(\delta^{\bar{b}}_d - \delta^{\overline{b+1}}_d), \quad d > b+1. \end{aligned} \tag{11}$$

But for $d > b+1$,

$$\begin{aligned}\theta_{b\bar{b}}\theta_{\bar{b}d} - \theta_{(b+1)(\overline{b+1})}\theta_{(\overline{b+1})d} &= \delta^b_l\theta_{\bar{l}d} - \delta^{\bar{b}}_d\theta_{bl} \\ &= \delta^b_l(1-\delta^d_{\bar{l}}) - \delta^{\bar{b}}_d(1-\delta^b_l) = \delta^b_l - \delta^{\bar{b}}_d.\end{aligned}$$

Also, $-[(-1)^{[d]}(q-q^{-1}) + q^{-(\varepsilon_d,\varepsilon_d)}]\delta^{\bar{b}}_d = -q^{(\varepsilon_d,\varepsilon_d)}\delta^{\bar{b}}_d$, so Eq. (11) reduces to

$$\begin{aligned}\bar{\alpha}_{bd} &= (q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}q^{-2\delta^b_{\overline{b+1}}} - \delta^{b+1}_{\overline{b+1}}q^{-1}(q-q^{-1}))\bar{\alpha}_{(b+1)d} + \delta^b_l q^{-1}(q-q^{-1})^2(-1)^{[d]}q^{2(\rho,\varepsilon_d)} \\ &\quad - \delta^{\bar{b}}_d(q-q^{-1})q^{2(\rho,\varepsilon_d)} + \delta^{\overline{b+1}}_d(q-q^{-1})q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}q^{-2\delta^b_{\overline{b+1}}}q^{2(\rho,\varepsilon_d)} \\ &= (q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}q^{-2\delta^b_{\overline{b+1}}} - \delta^{b+1}_{\overline{b+1}}q^{-1}(q-q^{-1}))\bar{\alpha}_{(b+1)d} + \delta^b_l q^{-1}(q-q^{-1})^2(-1)^{[d]}q^{2(\rho,\varepsilon_d)} \\ &\quad + (q-q^{-1})q^{-2(\rho,\varepsilon_b)}(\delta^{\overline{b+1}}_d - \delta^{\bar{b}}_d), \quad d > b+1.\end{aligned}$$

Recall that for $b > l$ we have

$$\bar{\alpha}_{bd} = (-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d-\varepsilon_b)}, \quad d > b.$$

Then when $b = l$ we find

$$\begin{aligned}\bar{\alpha}_{bd} &= (q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}q^{-2\delta^b_{\overline{b+1}}} - \delta^{b+1}_{\overline{b+1}}q^{-1}(q-q^{-1}))(-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d-\varepsilon_{b+1})} \\ &\quad + q^{-1}(q-q^{-1})^2(-1)^{[d]}q^{2(\rho,\varepsilon_d)} - (q-q^{-1})q^{-2(\rho,\varepsilon_b)}\delta^{\bar{l}}_d \\ &= (-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d-\varepsilon_b)}[\delta^{b+1}_{\overline{b+1}}(1-(q-q^{-1}) + (q-q^{-1})) \\ &\quad + \delta^b_{\overline{b+1}}(q^{-2} + q^{-1}(q-q^{-1})\ ] - (q-q^{-1})q^{-2(\rho,\varepsilon_b)}\delta^{\bar{l}}_d \\ &= (q-q^{-1})q^{-2(\rho,\varepsilon_b)}((-1)^{[d]}q^{2(\rho,\varepsilon_d)} - \delta^b_{\bar{d}})\end{aligned}$$

for all $d > b+1$. Comparing this with our earlier results for $d = b+1$ and $b > l$, we have

$$\bar{\alpha}_{bd} = (q-q^{-1})q^{-2(\rho,\varepsilon_b)}((-1)^{[d]}q^{2(\rho,\varepsilon_d)} - \delta^b_{\bar{d}}), \quad \forall\, b \geqslant l, \quad d > b.$$

But for $b < l$ we know

$$\bar{\alpha}_{bd} = q^{2(\rho,\varepsilon_{b+1}-\varepsilon_b)}\bar{\alpha}_{(b+1)d} + (q-q^{-1})q^{-2(\rho,\varepsilon_b)}(\delta_d^{\overline{b+1}} - \delta_d^{\bar{b}}), \quad d > b+1.$$

Hence for all $b$ we obtain

$$\bar{\alpha}_{bd} = (q-q^{-1})q^{-2(\rho,\varepsilon_b)}\left((-1)^{[d]}q^{2(\rho,\varepsilon_d)} - \sum_{c=b}^{d-1}\delta_d^{\bar{c}} + \sum_{c=b}^{d-2}\delta_d^{\overline{c+1}}\right)$$
$$= (q-q^{-1})q^{-2(\rho,\varepsilon_b)}((-1)^{[d]}q^{2(\rho,\varepsilon_d)} - \delta_d^{\bar{b}}), \quad d > b.$$

Thus for all $a > b$

$$A^{(l)b}{}_a X_b^a|\Lambda\rangle = q^{(\Lambda,\varepsilon_a)}(-1)^{[a]}\sum_{a>c\geq b}\alpha_{bc}^a(t_a^{(l)} - t_c^{(l)})|\Lambda\rangle, \tag{12}$$

where $\alpha_{bc}^a$ is given by

$$\alpha_{bc}^a = \begin{cases} -q^{-(\varepsilon_b,\varepsilon_b)}(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)}), & c=b, \\ (q-q^{-1})q^{-2(\rho,\varepsilon_b)}((-1)^{[c]}q^{2(\rho,\varepsilon_c)} - \delta_c^{\bar{b}})(1-\delta_{\bar{c}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)}), & c > b. \end{cases}$$

### A. Constructing the Perelomov-Popov matrix equation

The expression (12) can now be substituted into the equation

$$t_a^{(l)}|\Lambda\rangle = t_a^{(l-1)}t_a^{(1)}|\Lambda\rangle + \sum_{\varepsilon_a<\varepsilon_b}(-1)^{[a]+[b]}q^{(\Lambda,\varepsilon_a)}A^{(l-1)b}{}_a X_b^a|\Lambda\rangle$$

to find a matrix equation for the various $t_a^{(l)}$. The matrix factor is an analogue of the Perelomov-Popov matrix introduced in Refs. 15 and 16, which was used to calculate the eigenvalues of the Casimir invariants of various classical Lie algebras.

First recall that

$$A_a^b = (1+\delta_b^a)X_a^b + (q-q^{-1})\sum_{c\geq a,b}(-1)^{([a]+[c])([b]+[c])}X_a^c X_c^b,$$

where

$$X_a^b = \begin{cases} \dfrac{q^{h_{\varepsilon_a}} - I}{q-q^{-1}}, & a=b, \\ (-1)^{[b]}q^{h_{\varepsilon_a}}\hat{\sigma}_{ba}, & \varepsilon_a < \varepsilon_b, \\ (-1)^{[b]}\hat{\sigma}_{ba}q^{h_{\varepsilon_b}}, & \varepsilon_a > \varepsilon_b. \end{cases}$$

Then

$$A_a^a|\Lambda\rangle = 2X_a^a|\Lambda\rangle + (q-q^{-1})X_a^a X_a^a|\Lambda\rangle = (q-q^{-1})^{-1}(2(q^{h_{\varepsilon_a}}-1) + (q^{h_{\varepsilon_a}}-1)^2)|\Lambda\rangle$$

$$t_a^{(1)} = \frac{q^{2(\Lambda,\varepsilon_a)} - 1}{q-q^{-1}}.$$

Hence we obtain

$$t_a^{(l)} = \frac{(q^{2(\Lambda,\varepsilon_a)}-1)}{(q-q^{-1})}t_a^{(l-1)} + \sum_{b<a}(-1)^{[a]+[b]}q^{(\Lambda,\varepsilon_a)}\left(q^{(\Lambda,\varepsilon_a)}(-1)^{[a]}\sum_{b\leqslant c<a}\alpha_{bc}^a(t_a^{(l-1)}-t_c^{(l-1)})\right)$$

$$= \frac{(q^{2(\Lambda,\varepsilon_a)}-1)}{(q-q^{-1})}t_a^{(l-1)} - q^{2(\Lambda,\varepsilon_a)}\sum_{b<a}(-1)^{[b]}q^{-(\varepsilon_b,\varepsilon_b)}(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)})(t_a^{(l-1)}-t_b^{(l-1)})$$

$$+ (q-q^{-1})q^{2(\Lambda,\varepsilon_a)}\sum_{c<b<a}(-1)^{[c]}q^{-2(\rho,\varepsilon_c)}(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)})$$

$$\times ((-1)^{[b]}q^{2(\rho,\varepsilon_b)}-\delta_{\bar{c}}^b)(t_a^{(l-1)}-t_b^{(l-1)}).$$

Now consider the function $\gamma_b$ defined by

$$\gamma_b = (-1)^{[b]}q^{-(\varepsilon_b,\varepsilon_b)} - (q-q^{-1})\sum_{c<b}(-1)^{[c]}q^{-2(\rho,\varepsilon_c)}((-1)^{[b]}q^{2(\rho,\varepsilon_b)} - \delta_{\bar{c}}^b)).$$

We evaluate this for all $b$, remembering that $C(\Lambda_0)=(\delta_1,\delta_1+2\rho)=m-n-1$ and

$$\rho = \frac{1}{2}\sum_{i=1}^{l}(m-2i)\varepsilon_i + \frac{1}{2}\sum_{\mu=1}^{k}(n-m+2-2\mu)\delta_\mu.$$

We find

$$\gamma_b = (-1)^{[b]}q^{2(\rho,\varepsilon_b)}q^{-C(\Lambda_0)}$$

for all values of $b$. We also consider the function

$$\beta_a = 1-(q-q^{-1})\sum_{b<a}\gamma_b(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)}),$$

so that

$$t_a^{(l)} = \frac{(q^{2(\Lambda,\varepsilon_a)}\beta_a-1)}{(q-q^{-1})}t_a^{(l-1)} + q^{2(\Lambda,\varepsilon_a)}\sum_{b<a}\gamma_b(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{2(\rho,\varepsilon_a)})t_b^{(l-1)}. \tag{13}$$

Again, by considering the various cases individually we find

$$\beta_a = q^{(\varepsilon_a,2\rho+\varepsilon_a)-C(\Lambda_0)}$$

for any $a$, regardless of whether $m$ is even or odd. Substituting this result together with that for $\gamma_b$ into Eq. (13) gives

$$t_a^{(l)} = \frac{(q^{(\varepsilon_a,2\Lambda+2\rho+\varepsilon_a)-C(\Lambda_0)}-1)}{(q-q^{-1})}t_a^{(l-1)} + q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)}\sum_{b<a}(-1)^{[b]}q^{(2\rho,\varepsilon_b)}(1-\delta_{\bar{b}}^a(-1)^{[a]}q^{(2\rho,\varepsilon_a)})t_b^{(l-1)}.$$

This can be written in the matrix form

$$\underline{t}^{(l)} = M\underline{t}^{(l-1)},$$

where $M$ is a lower triangular matrix with entries

$$M_{ab} = \begin{cases} 0, & a<b, \\ (q-q^{-1})^{-1}(q^{(\varepsilon_a,2\Lambda+2\rho+\varepsilon_a)-C(\Lambda_0)}-1), & a=b, \\ q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)}((-1)^{[b]}q^{(2\rho,\varepsilon_b)}-\delta_{\bar{b}}^a), & a>b. \end{cases}$$

Then we have

$$\underline{t}^{(l)} = M^l \underline{t}^{(0)}, \quad \text{with } t_a^{(0)} = 1 \quad \forall\, a,$$

where $M$ is an analogue of the Perelomov-Popov matrix.

### B. Solving the matrix equation

This matrix equation for $t_a^{(l)}$ can now be used to calculate the eigenvalues of $C_l$. Loosely speaking, the problem reduces to diagonalizing the matrix $M$. Recall

$$C_l = \sum_a (-1)^{[a]} q^{(2\rho,\varepsilon_a)} A^{(l)a}{}_a.$$

Denote the eigenvalue of $C_l$ on $V(\Lambda)$ as $\chi_\Lambda(C_l)$. Then we have

$$\chi_\Lambda(C_l) = \sum_a (-1)^{[a]} q^{(2\rho,\varepsilon_a)} t_a^{(l)} = \sum_{a,b} (-1)^{[a]} q^{(2\rho,\varepsilon_a)} (M^l)_{ab}.$$

To calculate this we wish to diagonalize $M$. We assume the eigenvalues of $M$,

$$\alpha_a^\Lambda = \frac{(q^{(\varepsilon_a,2\Lambda+2\rho+\varepsilon_a)-C(\Lambda_0)} - 1)}{(q - q^{-1})},$$

are distinct. Then we need a matrix $N$ satisfying

$$(N^{-1}MN)_{ab} = \delta_b^a \alpha_a^\Lambda,$$

which implies

$$\chi_\Lambda(C_l) = \sum_{a,b,c} (-1)^{[a]} q^{(2\rho,\varepsilon_a)} (\alpha_b^\Lambda)^l N_{ab} (N^{-1})_{bc}. \tag{14}$$

Now

$$(MN)_{ab} = \alpha_b^\Lambda N_{ab}.$$

Substituting in the values for $M_{ab}$ gives

$$\alpha_a^\Lambda N_{ab} + q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)} \sum_{c<a} ((-1)^{[c]} q^{(2\rho,\varepsilon_c)} - \delta_{\bar{c}}^a) N_{cb} = \alpha_b^\Lambda N_{ab}. \tag{15}$$

Since the eigenvalues $\alpha_a^\Lambda$ are distinct, this implies

$$N_{ab} = 0, \quad \forall\, a < b.$$

Set

$$P_{ab} = \sum_{c \leq a} (-1)^{[c]} q^{(2\rho,\varepsilon_c)} N_{cb}. \tag{16}$$

Then Eq. (15) becomes

$$\begin{aligned} 2(\alpha_b^\Lambda - \alpha_a^\Lambda) N_{ab} &= q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)} P_{(a-1)b} - \theta_{0a} q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)} N_{\bar{a}b} \\ &\Rightarrow (\alpha_b^\Lambda - \alpha_a^\Lambda)(-1)^{[a]} q^{(-2\rho,\varepsilon_a)} (P_{ab} - P_{(a-1)b}) \\ &= q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)} P_{a-1b} - \theta_{0a} q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)} N_{\bar{a}b}, \end{aligned}$$

which simplifies to

$$P_{ab} = \frac{(\alpha_b^\Lambda - \alpha_a^\Lambda + (-1)^{[a]} q^{2(\Lambda+\rho,\varepsilon_a)-C(\Lambda_0)})}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} P_{(a-1)b} - \frac{\theta_{0a}(-1)^{[a]} q^{2(\Lambda+\rho,\varepsilon_a)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} N_{\bar{a}b}.$$

Set

$$\psi_a^b = \alpha_b^\Lambda - \alpha_a^\Lambda + (-1)^{[a]} q^{2(\Lambda+\rho,\varepsilon_a)-C(\Lambda_0)},$$

so this becomes

$$P_{ab} = \frac{\psi_a^b}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} P_{(a-1)b} - \frac{\theta_{0a}(-1)^{[a]} q^{2(\Lambda+\rho,\varepsilon_a)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} N_{\bar{a}b}. \tag{17}$$

Without loss of generality we can choose $N_{aa}=1 \,\forall\, a$, so $P_{bb}=(-1)^{[b]} q^{2(\rho,\varepsilon_b)}$. Then in the cases $0 \geqslant a > b$ and $a > b \geqslant 0$ the last term in Eq. (17) vanishes, giving

$$P_{ab} = (-1)^{[b]} q^{2(\rho,\varepsilon_b)} \prod_{c=b+1}^{a} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)}.$$

Similarly, for $a > \bar{b} > 0$ we obtain

$$P_{ab} = P_{\bar{b}b} \prod_{c=\bar{b}+1}^{a} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)}. \tag{18}$$

It remains to find $P_{ab}$ for $\bar{b} \geqslant a > 0$. In this case, the last term in Eq. (17) contributes, giving

$$\begin{aligned} P_{ab} = {} & (-1)^{[b]} q^{2(\rho,\varepsilon_b)} \prod_{c=b+1}^{a} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)} - \frac{(-1)^{[a]} q^{2(\Lambda+\rho,\varepsilon_a)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} N_{\bar{a}b} \\ & - \sum_{d=\bar{l}}^{a-1} \frac{(-1)^{[d]} q^{2(\Lambda+\rho,\varepsilon_d)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_d^\Lambda)} N_{\bar{d}b} \prod_{c=d+1}^{a} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)}. \end{aligned} \tag{19}$$

Recall that if $b < a < 0$, then

$$\begin{aligned} N_{ab} &= \frac{q^{(2\Lambda,\varepsilon_a)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} P_{(a-1)b} \\ &= \frac{(-1)^{[b]} q^{2(\Lambda,\varepsilon_a)+2(\rho,\varepsilon_b)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_a^\Lambda)} \prod_{c=b+1}^{a-1} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)}. \end{aligned}$$

Substituting this into Eq. (19), we find

$$\begin{aligned} P_{\bar{b}b} = {} & (-1)^{[b]} q^{2(\rho,\varepsilon_b)} \prod_{c=b+1}^{\bar{b}} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)} - \frac{(-1)^{[b]} q^{-2(\Lambda+\rho,\varepsilon_b)-C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_{\bar{b}}^\Lambda)} \\ & - \sum_{d=\bar{l}}^{\bar{b}-1} \frac{(-1)^{[d]+[b]} q^{2(\rho,\varepsilon_d+\varepsilon_b)-2C(\Lambda_0)}}{(\alpha_b^\Lambda - \alpha_d^\Lambda)(\alpha_b^\Lambda - \alpha_{\bar{d}}^\Lambda)} \prod_{c=b+1}^{\bar{d}-1} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)} \prod_{c=d+1}^{\bar{b}} \frac{\psi_c^b}{(\alpha_b^\Lambda - \alpha_c^\Lambda)}, \end{aligned}$$

which can also be simplified to

$$P_{\bar{b}b}\prod_{c=b+1}^{\bar{b}}\frac{(\alpha_b^{\Lambda}-\alpha_c^{\Lambda})}{\psi_c^b}=(-1)^{[b]}q^{2(\rho,\varepsilon_b)}\left[1-\sum_{d=\bar{l}}^{\bar{b}}\frac{(-1)^{[d]}q^{2(\rho,\varepsilon_d)-2C(\Lambda_0)}}{\psi_d^b\psi_{\bar{d}}^b}\prod_{c=\bar{d}+1}^{d-1}\frac{(\alpha_b^{\Lambda}-\alpha_c^{\Lambda})}{\psi_c^b}\right]. \quad (20)$$

From this point we will consider the case $m=2l+1$. This is marginally more complicated than the case with even $m$. Define $\Phi_d^b$ to be

$$\Phi_d^b=\prod_{c=\bar{l}}^{d-1}\frac{(\alpha_b-\alpha_c)(\alpha_b-\alpha_{\bar{c}})}{\psi_c^b\psi_{\bar{c}}^b}$$
$$=\frac{(\alpha_b-\alpha_{d-1})(\alpha_b-\alpha_{\overline{d-1}})}{\psi_{d-1}^b\psi_{\overline{d-1}}^b}\Phi_{d-1}^b, \quad \Phi_{\bar{l}}^b=1.$$

Then $P_{\bar{b}b}$ can be written as

$$P_{\bar{b}b}=(-1)^{[b]}q^{2(\rho,\varepsilon_b)}\prod_{\substack{c=b+1\\c\neq 0}}^{\bar{b}}\frac{\psi_c^b}{(\alpha_b^{\Lambda}-\alpha_c^{\Lambda})}\left[\frac{\psi_0^b}{\alpha_b-\alpha_0}-\sum_{d=\bar{l}}^{\bar{b}}\frac{(-1)^{[d]}q^{2(\rho,\varepsilon_d)-2C(\Lambda_0)}}{\psi_d^b\psi_{\bar{d}}^b}\Phi_d^b\right].$$

Note that for $c\neq 0$,

$$\psi_c^b=\frac{q^{-C(\Lambda_0)}}{(q-q^{-1})}(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)}-q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)}+(q-q^{-1})(-1)^{[c]}q^{(\varepsilon_c,2\rho+2\Lambda)})$$
$$=\frac{q^{-C(\Lambda_0)}\tilde{\psi}_c^b}{(q-q^{-1})},$$

where

$$\tilde{\psi}_c^b=q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)}-q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)}.$$

So

$$\sum_{d=\bar{l}}^{\bar{b}}\frac{(-1)^{[d]}q^{2(\rho,\varepsilon_d)-2C(\Lambda_0)}}{\psi_d^b\psi_{\bar{d}}^b}\Phi_d^b=(q-q^{-1})\sum_{d=\bar{l}}^{\bar{b}}\frac{(-1)^{[d]}(q-q^{-1})q^{2(\rho,\varepsilon_d)}}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b}\Phi_d^b$$
$$=(q-q^{-1})\sum_{d=\bar{l}}^{\bar{b}}\frac{(q^{2(\varepsilon_d,\varepsilon_d)}-1)q^{2(\rho,\varepsilon_d)-(\varepsilon_d,\varepsilon_d)}}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b}\Phi_d^b \quad (21)$$

and

$$\Phi_{d+1}^b=\frac{(\alpha_b-\alpha_d)(\alpha_b-\alpha_{\bar{d}})}{\psi_d^b\psi_{\bar{d}}^b}\Phi_d^b$$
$$=\frac{(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{(\varepsilon_d,\varepsilon_d+2\rho+2\Lambda)})(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{(\varepsilon_d,\varepsilon_d-2\rho-2\Lambda)})}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b}\Phi_d^b$$

for $d\geqslant\bar{l}$. Now

$$(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{(\varepsilon_d,\varepsilon_d+2\rho+2\Lambda)})(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{(\varepsilon_d,\varepsilon_d-2\rho-2\Lambda)})$$
$$=q^{2(\varepsilon_d,\varepsilon_d)}(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{(\varepsilon_d,-\varepsilon_d+2\rho+2\Lambda)})(q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-q^{-(\varepsilon_d,\varepsilon_d+2\rho+2\Lambda)})$$
$$+q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}(1-q^{2(\varepsilon_d,\varepsilon_d)})+q^{2(\varepsilon_d,\varepsilon_d)}-1$$

$$= q^{2(\varepsilon_d,\varepsilon_d)}\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b - (q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)(q^{2(\varepsilon_d,\varepsilon_d)}-1).$$

Then, for $d\geqslant\bar{l}$,

$$\frac{\Phi_{d+1}^b}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)} = \left[\frac{q^{2(\varepsilon_d,\varepsilon_d)}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)} - \frac{(q^{2(\varepsilon_d,\varepsilon_d)}-1)}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b}\right]\Phi_d^b. \tag{22}$$

Now for $d=\bar{b}$,

$$\frac{(q^{2(\varepsilon_d,\varepsilon_d)}-1)q^{2(\rho,\varepsilon_d)-(\varepsilon_d,\varepsilon_d)}}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b} = \frac{(q^{2(\varepsilon_b,\varepsilon_b)}-1)q^{2(\rho,\varepsilon_{\bar{b}})-(\varepsilon_b,\varepsilon_b)}}{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)}-q^{-(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)})q^{(\varepsilon_b,2\rho+2\Lambda)}(q^{(\varepsilon_b,\varepsilon_b)}-q^{-(\varepsilon_b,\varepsilon_b)})}$$

$$= \frac{q^{2(\rho,\varepsilon_{\bar{b}})+(\varepsilon_b,\varepsilon_b)}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)},$$

which can be written as

$$\frac{(q^{2(\varepsilon_d,\varepsilon_d)}-1)q^{2(\rho,\varepsilon_d)-(\varepsilon_d,\varepsilon_d)}}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b} = \frac{q^{2(\rho,\varepsilon_{\bar{b}-1})-(\varepsilon_{\bar{b}-1},\varepsilon_{\bar{b}-1})}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}$$

when $b<l$. Hence Eq. (22) can be used to pairwise cancel the terms in the sum in Eq. (21). Adding the first two terms ($d=\bar{b},\bar{b}-1$), we find

$$q^{2(\rho,\varepsilon_{\bar{b}-1})-(\varepsilon_{\bar{b}-1},\varepsilon_{\bar{b}-1})}\left[\frac{\Phi_{\bar{b}}^b}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)} + \frac{(q^{2(\varepsilon_{\bar{b}-1},\varepsilon_{\bar{b}-1})}-1)}{\tilde{\psi}_{\bar{b}-1}^b\tilde{\psi}_{b+1}^b}\Phi_{\bar{b}-1}^b\right]$$

$$= q^{2(\rho,\varepsilon_{\bar{b}-1})-(\varepsilon_{\bar{b}-1},\varepsilon_{\bar{b}-1})}\frac{q^{2(\varepsilon_{\bar{b}-1},\varepsilon_{\bar{b}-1})}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}\Phi_{\bar{b}-1}^b$$

$$= \frac{q^{2(\rho,\varepsilon_{\bar{b}-2})-(\varepsilon_{\bar{b}-2},\varepsilon_{\bar{b}-2})}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}\Phi_{\bar{b}-1}^b.$$

Continuing to apply Eq. (22) in this manner gives

$$\sum_{d=\bar{l}}^{\bar{b}}\frac{(q^{2(\varepsilon_d,\varepsilon_d)}-1)q^{2(\rho,\varepsilon_d)-(\varepsilon_d,\varepsilon_d)}}{\tilde{\psi}_d^b\tilde{\psi}_{\bar{d}}^b}\Phi_d^b = \frac{q^{2(\rho,\varepsilon_{\bar{l}})+(\varepsilon_l,\varepsilon_l)}}{(q^{2(\varepsilon_b,\varepsilon_b+\rho+\Lambda)}-1)}\Phi_{\bar{l}}^b$$

$$= \frac{q^{2l+1-m}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}. \tag{23}$$

Hence in the case $m=2l+1$,

$$P_{\bar{b}b} = (-1)^{[b]}q^{2(\rho,\varepsilon_b)}\left[\frac{\psi_0^b}{\alpha_b-\alpha_0} - \frac{(q-q^{-1})}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}\right]\prod_{\substack{c=b+1\\c\neq 0}}^{\bar{b}}\frac{\psi_c^b}{(\alpha_b^\Lambda-\alpha_c^\Lambda)}.$$

By substituting in the formulas for $\psi_c^b$ and $\alpha_b$ and simplifying we obtain

$$P_{\bar{b}b} = (-1)^{[b]}q^{2(\rho,\varepsilon_b)}\left[1+(q-q^{-1})\frac{q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}-1)}\right]\prod_{c=b+1}^{\bar{b}}\frac{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)}-q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)})}{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)}-q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)})},$$

and thus for $a\geqslant\bar{b}>0$,

$$P_{ab} = (-1)^{[b]} q^{2(\rho,\varepsilon_b)} \left[ 1 + (q - q^{-1}) \frac{q^{(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)}}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)} - 1)} \right] \prod_{c=b+1}^{a} \frac{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)} - q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)})}{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)} - q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)})}.$$

Similarly, we find from Eqs. (18), (20), (21), and (23) that when $m$ is even then

$$P_{ab} = (-1)^{[b]} q^{2(\rho,\varepsilon_b)} \left[ 1 - \frac{q(q - q^{-1})}{(q^{2(\varepsilon_b,\varepsilon_b+2\rho+2\Lambda)} - 1)} \right] \prod_{c=b+1}^{a} \frac{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)} - q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)})}{(q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)} - q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)})}$$

for $a \geqslant \bar{b} > 0$. Hence we have found expressions for $P_{ab}$ for all $a,b$ satisfying $a \geqslant \bar{b} > 0$. At the end of the paper these, together with the earlier results for $P_{ab}$, will be used to calculate $\chi_\Lambda(C_l)$.

Now we return to the diagonalization of the matrix $N$. We know

$$(N^{-1}M)_{ab} = \alpha_a^\Lambda (N^{-1})_{ab}.$$

Substituting in the values for $M_{ab}$ gives

$$\alpha_b^\Lambda (N^{-1})_{ab} + (-1)^{[b]} q^{(2\rho,\varepsilon_b)-C(\Lambda_0)} \sum_{c>b} q^{(2\Lambda,\varepsilon_c)} (1 - \delta_{\bar{b}}^{c} (-1)^{[b]} q^{-2(\rho,\varepsilon_b)}) (N^{-1})_{ac} = \alpha_a^\Lambda (N^{-1})_{ab}. \tag{24}$$

Set

$$\hat{Q}_{ab} = \sum_{c \geqslant b} q^{2(\Lambda,\varepsilon_c)} (N^{-1})_{ac}.$$

We then solve for $\hat{Q}_{ab}$, with the calculations being very similar to those for $P_{ab}$. For $0 \leqslant b < a$ and $b < a \leqslant 0$ we find

$$\hat{Q}_{ab} = q^{2(\Lambda,\varepsilon_a)} \prod_{c=b}^{a-1} \frac{\psi_c^a}{(\alpha_a^\Lambda - \alpha_c^\Lambda)}.$$

For $m = 2l+1$ we obtain

$$\hat{Q}_{ab} = q^{2(\Lambda,\varepsilon_a)} \left[ 1 + (q - q^{-1}) \frac{q^{(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)}}{(q^{2(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)} - 1)} \right] \prod_{c=b}^{a-1} \frac{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)})}{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)})}$$

for $b \leqslant \bar{a} < 0$. Similarly, for even $m$ we find

$$\hat{Q}_{ab} = q^{2(\Lambda,\varepsilon_a)} \left[ 1 - \frac{q(q - q^{-1})}{(q^{2(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)} - 1)} \right] \prod_{c=b}^{a-1} \frac{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_c,2\rho+2\Lambda-\varepsilon_c)})}{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_c,2\rho+2\Lambda+\varepsilon_c)})}.$$

for $b \leqslant \bar{a} < 0$.

To use these results to calculate $\chi_\Lambda(C_l)$ we introduce a new function $Q_{ab}$, defined by

$$Q_{ab} = \sum_{c \geqslant b} (N^{-1})_{ac}.$$

Then from Eqs. (14) and (16) we deduce

$$\chi_\Lambda(C_l) = \sum_a (\alpha_a^\Lambda)^l P_{(\mu=\bar{1})a} Q_{a(\nu=1)}. \tag{25}$$

However we know

$$t_a^{(l)} = \frac{q^{2(\Lambda,\varepsilon_a)} - 1}{q - q^{-1}},$$

and

$$\sum_b (N^{-1})_{ab} t_b^{(1)} = \sum_{b,c} (N^{-1})_{ab} M_{bc} t_c^{(0)}$$

$$\Rightarrow \sum_b (N^{-1})_{ab} \frac{(q^{2(\Lambda,\varepsilon_b)} - 1)}{(q - q^{-1})} = \sum_b (N^{-1}M)_{ab}$$

$$= \sum_b \alpha_a^{\Lambda} (N^{-1})_{ab} = \sum_b (N^{-1})_{ab} \frac{(q^{(\varepsilon_a, 2\Lambda+2\rho+\varepsilon_a) - C(\Lambda_0)} - 1)}{(q - q^{-1})}.$$

Thus

$$Q_{a(\nu=1)} = q^{C(\Lambda_0) - (\varepsilon_a, 2\rho+2\Lambda+\varepsilon_a)} \hat{Q}_{a(\nu=1)}.$$

### C. Explicit formulas for the eigenvalues

Substituting our formulas for $P_{(\mu=\bar{1})a}$ and $Q_{a(\nu=1)}$ into Eq. (25), noting that for $a \neq 0$ exactly one of $a < 0$ or $a > 0$ is true, we find the eigenvalues of the Casimir invariants $C_l$ are given by

$$\chi_\Lambda(C_l) = \sum_a (-1)^{[a]} q^{C(\Lambda_0) - (\varepsilon_a, \varepsilon_a)} f(a) \left[ \frac{(q^{(\varepsilon_a, 2\rho+2\Lambda+\varepsilon_a) - C(\Lambda_0)} - 1)}{(q - q^{-1})} \right]^l \times \prod_{b \neq a} \frac{(q^{(\varepsilon_a, 2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_b, 2\rho+2\Lambda-\varepsilon_b)})}{(q^{(\varepsilon_a, 2\rho+2\Lambda+\varepsilon_a)} - q^{(\varepsilon_b, 2\rho+2\Lambda+\varepsilon_b)})},$$

where

$$f(a) = \begin{cases} 1 - (q - q^{-1}) \dfrac{q}{(q^{2(\varepsilon_a, \varepsilon_a + 2\rho + 2\Lambda)} - 1)}, & m = 2l, \\ 1 + (q - q^{-1}) \dfrac{q^{(\varepsilon_a, \varepsilon_a + 2\rho + 2\Lambda)}}{(q^{2(\varepsilon_a, \varepsilon_a + 2\rho + 2\Lambda)} - 1)}, & a \neq 0,\ m = 2l + 1, \\ 1, & a = 0,\ m = 2l + 1. \end{cases}$$

Throughout we assumed the eigenvalues were distinct. If they are not, the calculations are more complicated but the result is the same. Thus we have found the following.

**Theorem 4.1:** *The quantum superalgebra* $U_q[\mathrm{osp}(m|n)]$, *for* $m > 2$, *has an infinite family of Casimir invariants of the form*,

$$C_l = (\mathrm{str} \otimes I)(\pi(q^{2h_\rho}) \otimes I) A^l, \quad l \in \mathbb{Z}^+,$$

*where*

$$A = \frac{(R^T R - I \otimes I)}{(q - q^{-1})}.$$

*The eigenvalues of the invariants when acting on an arbitrary irreducible finite-dimensional module with highest weight* $\Lambda$ *are given by*

$$\chi_\Lambda(C_l) = \sum_a (-1)^{[a]} q^{C(\Lambda_0)-(\varepsilon_a,\varepsilon_a)} f(a) \left[ \frac{(q^{(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)-C(\Lambda_0)}-1)}{(q-q^{-1})} \right]^l \times \prod_{b\neq a} \frac{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)}-q^{(\varepsilon_b,2\rho+2\Lambda-\varepsilon_b)})}{(q^{(\varepsilon_a,2\rho+2\Lambda+\varepsilon_a)}-q^{(\varepsilon_b,2\rho+2\Lambda+\varepsilon_b)})},$$

*where*

$$f(a) = \begin{cases} 1-(q-q^{-1})\dfrac{q}{(q^{2(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)}-1)}, & m=2l, \\ 1+(q-q^{-1})\dfrac{q^{(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)}}{(q^{2(\varepsilon_a,\varepsilon_a+2\rho+2\Lambda)}-1)}, & a\neq 0,\ m=2l+1, \\ 1, & a=0,\ m=2l+1. \end{cases}$$

This completes the calculation of the eigenvalues of an infinite family of Casimir invariants of $U_q[\mathrm{osp}(m|n)]$ when acting on an arbitrary irreducible highest weight module, provided $m>2$. This had already been done for $U_q[\mathrm{osp}(2|n)]$, using a different method, in Ref. 12. Also every finite-dimensional representation of $U_q[\mathrm{osp}(1|n)]$ is isomorphic to a finite-dimensional representation of $U_{-q}[o(n+1)]$,[14] whose central elements are well understood. Hence the eigenvalues of a family of Casimir invariants, when acting on an arbitrary irreducible finite-dimensional highest weight module have now been calculated for all quantized orthosymplectic superalgebras. Together with the results for $U_q[\mathrm{gl}(m|n)]$,[13] this covers all nonexceptional quantized superalgebras.

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