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Entanglement sharing and decoherence in the spin-bath

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The monogamous nature of entanglement has been illustrated by the derivation of entanglement-sharing inequalities—bounds on the amount of entanglement that can be shared among the various parts of a multipartite system. Motivated by recent studies of decoherence, we demonstrate an interesting manifestation of this phenomena that arises in system-environment models where there exists interactions between the modes or subsystems of the environment. We investigate this phenomenon in the spin-bath environment, constructing an entanglement-sharing inequality bounding the entanglement between a central spin and the environment in terms of the pairwise entanglement between individual bath spins. The relation of this result to decoherence will be illustrated using simplified system-bath models of decoherence.

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While entanglement is argued to be the distinguishing feature of quantum computers, responsible for their power [1], it is also the source of one of the major obstacles in their construction. *Decoherence*, the process by which a quantum superposition state decays into a classical, statistical mixture of states, is caused by entangling interactions between the system and its environment [2]. Somewhat paradoxically, the quantum entanglement between a system and its environment induces classicality in the system. While it is still a contentious topic as to whether quantum computation will be possible in the face of decoherence, Zurek [3] has demonstrated that decoherence is necessary to facilitate the measurement of a quantum system. Understanding decoherence lies at the heart of measurement, quantum information processing, and, more fundamentally, the transition from the quantum to the classical world.

The road to studying decoherence by explicitly modeling system-environment interactions has led to simple models of the quantum environment. Environments can be modeled as either baths of harmonic oscillators [4] or spins (with spin $\frac{1}{2}$) argued to represent distinct types of environmental modes [5]. The simplest system-environment models consist of a central spin (or qubit) coupled to the environment—i.e., the spin-boson model [4]—which has applications to the decoherence of qubits for quantum information processing.

Decoherence of a spin-$\frac{1}{2}$ particle at low temperatures may be conveniently modeled by the "central spin" model, which couples a central spin-$\frac{1}{2}$ particle $\mathcal{S}$ to a spin bath $\mathcal{B}$ of N spin-$\frac{1}{2}$ particles. A typical Hamiltonian for this model may be written in the form

$$H = H_{\mathcal{S}} + H_{\mathcal{B}} + H_{\mathcal{SB}}, \tag{1}$$

where $H_{\mathcal{S}}$ and $H_{\mathcal{B}}$ are the internal Hamiltonians of the central spin and spin bath, respectively, and $H_{\mathcal{SB}}$ is the coupling term. Denote the state of the system-environment at time t by $\rho_{\mathcal{SB}}(t)$. Initially at $t=0$ we take the central spin $\mathcal{S}$ to be in a pure state, uncorrelated with the bath. That is,

$$\rho_{\mathcal{SB}}(0) = |\psi\rangle_{\mathcal{S}}\langle\psi| \otimes \rho_{\mathcal{B}}(0) \tag{2}$$

for some initial state of the bath $\rho_{\mathcal{B}}(0)$. Typically $\rho_{\mathcal{B}}(0)$ is taken to be a thermal state of the Hamiltonian $H_{\mathcal{B}}$ or, at low temperatures, the ground state.

As the system evolves under H the central spin becomes coupled to the bath, and its reduced density matrix $\rho_{\mathcal{S}}(t)$ at later times is no longer pure. The central spin is said to have decohered, and the amount of decoherence is typically quantified by the von Neumann entropy of its reduced density matrix $S(\rho_{\mathcal{S}}(t))$.

More recently interactions between modes within the bath itself have been considered [6–8], which allow for appreciable correlations, such as entanglement, to arise between the modes of the bath.

In [6], Tessieri and Wilkie introduced coupling terms between spins in the bath Hamiltonian $H_{\mathcal{B}}$ and, taking the initial state of the bath as a thermal state of $H_{\mathcal{B}}$, found that this resulted in a suppression of the decoherence $S(\rho_{\mathcal{S}}(t))$. The amount of suppression increased as the effective energy scale of $H_{\mathcal{B}}$ increased relative to that of $H_{\mathcal{SB}}$, ultimately to the point where decoherence was negligible even after long times. This is somewhat surprising, as even small couplings $H_{\mathcal{B}}$ would usually be expected to eventually result in complete decoherence of the central spin. In this article we aim to demonstrate that this suppression effect may be understood to be a consequence of *entanglement sharing* and that it will be common to any central spin whose environment maintains appreciable internal entanglement while evolving in time.

A simple example of such a system is a single spin in a bath of spins with antiferromagnetic interactions between them. In the absence of the spin the ground state of the N bath spins would be something like a spin singlet which is highly entangled. If the single spin interacts antiferromagnetically with the bath spins, all it can do is flip individual spins in the bath. The total spin has to be conserved and hence will have a value of order $1/2$. If the bath is initialized

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in such a state, it will remain highly entangled throughout its interaction with the system spin.

Entanglement sharing refers to a striking difference between classical and quantum correlations—the latter may not be shared arbitrarily among several observables. The connection with decoherence is readily seen in a system of three spin-$\frac{1}{2}$ particles, labeled $\mathcal{S}$, B_1, and B_2, respectively. It has been shown [9] that entanglement between B_1 and B_2 limits the individual and collective entanglement they may have with $\mathcal{S}$. If a state of the system $\rho(t)$ is evolving under a Hamiltonian such as Eq. (1) and, moreover, if the "bath" B_1B_2 maintains appreciable entanglement, then it follows that there is a restriction on the entanglement between the central spin $\mathcal{S}$ and B_1B_2. For pure states this equivalent to a restriction on the amount that $\mathcal{S}$ may decohere. For mixed states we must also bound the classical correlations between $\mathcal{S}$ and B_1B_2 which may be done using a recent result of Koashi and Winter [10]. Entanglement between B_1 and B_2 thus suppresses all correlations between the central spin and the bath.

The situation becomes far more complicated for spin baths of N particles. The main difficulty is the plethora of different types of entanglement which exist in these baths and the absence of good entanglement measures for them. To overcome this difficulty we will assume that there is some symmetry in the Hamiltonians $H_{\mathcal{S}}$ and $H_{\mathcal{SB}}$. If the initial bath state $\rho_{\mathcal{B}}(t)$ is taken to be a thermal or eigenstate of $H_{\mathcal{B}}$, then the reduced state of the bath $\rho_{\mathcal{B}}(t)$ at later times will also obey this symmetry. For example, the simplest case is that considered by Tessieri and Wilkie where $H_{\mathcal{SB}}$ and $H_{\mathcal{B}}$ are completely symmetric. Here the pairwise entanglement between any two bath spins is the same, allowing us to quantify the bath entanglement by a single parameter.

In this paper we will obtain an entanglement-sharing inequality relating the entanglement between a central spin and a completely symmetric spin bath to the pairwise entanglement in the bath. This inequality is applicable to both pure and mixed states, and is sufficient to restrict decoherence where $\rho_{\mathcal{SB}}(t)$ is pure. We will then illustrate this damping effect in a simple model of decoherence originally proposed by Zurek [3] and the Tessieri-Wilkie model [6]. To conclude we will discuss possible extensions of this result to the bounding of classical correlations between the central spin and the bath.

To begin, let $\mathcal{S}$ be a central spin-$\frac{1}{2}$ particle and $\mathcal{B} = B_1B_2\cdots B_N$ a completely symmetric spin bath. As indicated above, the symmetry implies that the entanglement between any pair of bath spins B_i, B_j is the same, allowing us to use a single parameter as a measure of bath entanglement. This entanglement will be called the *intrabath* entanglement, while the entanglement between the central spin and the bath will be called the *system-bath* entanglement. To quantify these we will make use of a measure known as the *tangle* [9] whose definition we now briefly recall. For the reduced density matrix $\rho_{B_iB_j}$ of a pair of bath spins B_i, B_j define the spin-flipped density matrix

$$\tilde{\rho}_{B_iB_j} = (\sigma_y \otimes \sigma_y)\rho^*_{B_iB_j}(\sigma_y \otimes \sigma_y). \tag{3}$$

The asterisk denotes complex conjugation in the standard basis and σ_y is the Pauli Y matrix. The matrix $\rho_{B_iB_j}\tilde{\rho}_{B_iB_j}$ can be shown to have real non-negative eigenvalues, and we write their square roots in decreasing order as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The tangle between B_i and B_j is then defined as

$$\tau_{B_i|B_j} = (\max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\})^2. \tag{4}$$

This expression is for two spin-$\frac{1}{2}$ particles; however, the tangle between the central spin $\mathcal{S}$ and the bath $\mathcal{B}$ is also well defined for pure states of the combined system. The key point is that, because $\mathcal{S}$ is a spin-$\frac{1}{2}$ particle, only two dimensions of the bath-state space are required to expand the pure state in its Schmidt decomposition. The bath may therefore be imagined as a single spin-$\frac{1}{2}$ particle, with the tangle defined as before. Equation (4) can be further simplified for pure states so that the system-bath tangle is

$$\tau_{\mathcal{S}|\mathcal{B}} = 4 \det \rho_{\mathcal{S}}. \tag{5}$$

For further properties of the tangle, in particular its validity as an entanglement measure, we refer the reader to [9,11].

Since all the pairwise intrabath tangles are the same, we write $\tau_B \equiv \tau_{B_i|B_j}$ for all i, j. Our aim is to show how this τ_B constrains the system-bath tangle $\tau_{\mathcal{S}|\mathcal{B}}$. We will first consider the simplest case of pure states for an $N=2$ bath, since much is known about states of three spin-$\frac{1}{2}$ particles. Intuition built in this case will enable us to derive a related inequality for pure states of arbitrary sized baths.

For the two-spin bath, it was shown in [9] and [12] that there are two distinct types of entanglement between $\mathcal{S}$ and B_1B_2. $\mathcal{S}$ can be entangled with the spins B_1 and B_2 individually or with the bath B_1B_2 as a whole. The latter type is quantified by the *three-tangle* which we denote by $\tau_{\mathcal{S}|B_1|B_2}$. The total entanglement between $\mathcal{S}$ and $\mathcal{B}$ can now be written as

$$\tau_{\mathcal{S}|\mathcal{B}} = \tau_{\mathcal{S}|B_1} + \tau_{\mathcal{S}|B_2} + \tau_{\mathcal{S}|B_1|B_2}. \tag{6}$$

The three-tangle is invariant under permutations of the three spins and may be written alternatively as

$$\tau_{\mathcal{S}|B_1|B_2} = \tau_{\mathcal{S}|B_1B_2} - \tau_{\mathcal{S}|B_1} - \tau_{\mathcal{S}|B_2}, \tag{7}$$

$$\tau_{\mathcal{S}|B_1|B_2} = \tau_{B_1|\mathcal{S}B_2} - \tau_B - \tau_{B_1|\mathcal{S}}. \tag{8}$$

A simple consequence of this, together with the fact that the tangle is a positive quantity less than or equal to 1, is

$$\tau_B + \tau_{\mathcal{S}|B_1|B_2} \leq 1. \tag{9}$$

This inequality says that the intrabath entanglement plus the three-tangle part of the system-bath entanglement is always less than 1. On the other hand, the sum of $\tau_B + \tau_{\mathcal{S}|B_1} + \tau_{\mathcal{S}|B_2}$ can be greater than 1—it can take any value up to and including $4/3$ [12]. This suggests that intrabath entanglement has a stronger damping effect on the three-tangle component of $\tau_{\mathcal{S}|\mathcal{B}}$ than it does on the pairwise tangle component. We will therefore assume that, for a fixed intrabath tangle, a maximum system-bath entanglement is obtained when $\tau_{\mathcal{S}|B_1|B_2}=0$—that is, when it is composed entirely of the pairwise components in Eq. (6).

States of the $\mathcal{S}B_1B_2$ system with $\tau_{\mathcal{S}|B_1|B_2}=0$ are equivalent under local unitary operations to so-called W-class states of the form

$$|\psi\rangle = a|\uparrow\rangle_{\mathcal{S}}|\uparrow\downarrow\rangle_B + b|\uparrow\rangle_{\mathcal{S}}|\downarrow\uparrow\rangle_B + c|\downarrow\rangle_{\mathcal{S}}|\uparrow\uparrow\rangle_B + d|\uparrow\rangle_{\mathcal{S}}|\uparrow\uparrow\rangle_B, \tag{10}$$

where a,b,c,d are real and non-negative [12,13] and $a^2+b^2+c^2+d^2=1$. The tensor factors in each term refer to the state of the central spin and of the two bath spins, respectively. It is a simple matter to calculate the relevant tangles from Eqs. (4) and (5):

$$\tau_B = 4a^2b^2, \tag{11}$$

$$\tau_{\mathcal{S}|\mathcal{B}} = 4(a^2+b^2)c^2. \tag{12}$$

We will solve the equivalent and, as it turns out, slightly easier problem of maximizing τ_B for fixed $\tau_{\mathcal{S}|\mathcal{B}}=T$. That is, we must maximize

$$g(a,b,c,d) = 4a^2b^2 \tag{13}$$

subject to the constraints

$$F_1(a,b,c,d) = 4(a^2+b^2)c^2 - T = 0, \tag{14}$$

$$F_2(a,b,c,d) = a^2+b^2+c^2+d^2-1 = 0. \tag{15}$$

This can be solved by the method of Lagrange multipliers, and we find that the maximum τ_B is given by

$$\tau_B = \frac{1}{4}(1+\sqrt{1-\tau_{\mathcal{S}|\mathcal{B}}})^2. \tag{16}$$

The corresponding entanglement-sharing inequality for the system-bath and intrabath tangles is then

$$\tau_{\mathcal{S}|\mathcal{B}} \leqslant \begin{cases} 1, & \tau_B \leqslant \frac{1}{4}, \\ 4(\sqrt{\tau_B}-\tau_B), & \tau_B \geqslant \frac{1}{4}. \end{cases} \tag{17}$$

For values of the intrabath tangle less than $1/4$ the system and the bath may be maximally entangled. As τ_B increases, however, we find that $\tau_{\mathcal{S}|\mathcal{B}}$ falls in an approximately linear fashion, and is 0 when the intrabath tangle is at a maximum. This confirms our expectation that strong quantum correlations in the environment limit decoherence effects, at least for pure states of the combined system.

We saw above that the three-tangle component of the system-bath entanglement was more strongly limited by the intrabath entanglement than the pairwise components $\tau_{\mathcal{S}|B_1},\tau_{\mathcal{S}|B_2}$. In the case of an N-spin bath it seems reasonable that we should expect the same, this time potentially for three-party and other higher-order quantum correlations between $\mathcal{S}$ and the bath. We will therefore assume that analogs of the W-class states are able to achieve maximum system-bath entanglement for a given intrabath entanglement. An inequality similar to Eq. (17) follows from this assumption and has been confirmed numerically for small values of N.

An analog of a W-class state should ideally be one where the system is only entangled with each of the bath spins individually. We will use a generalization of the states (10) given by

$$\begin{aligned} |W\rangle = {} & a_1|\uparrow\rangle_{\mathcal{S}}|\uparrow\uparrow\cdots\uparrow\uparrow\downarrow\rangle_B + a_2|\uparrow\rangle_{\mathcal{S}}|\uparrow\uparrow\cdots\uparrow\downarrow\uparrow\rangle_B + \cdots \\ & + a_N|\uparrow\rangle_{\mathcal{S}}|\downarrow\uparrow\cdots\uparrow\uparrow\uparrow\rangle_B + c|\downarrow\rangle_{\mathcal{S}}|\uparrow\uparrow\cdots\uparrow\uparrow\uparrow\rangle_B \\ & + d|\uparrow\rangle_{\mathcal{S}}|\uparrow\uparrow\cdots\uparrow\uparrow\uparrow\rangle_B \end{aligned} \tag{18}$$

for real a_i,c,d where $\sum_{i=1}^N a_i^2+c^2+d^2=1$. Here a_i is the coefficient of the state where the ith bath spin is down. From Eqs. (4) and (5) we find that the tangle between any pair of bath spins is given by

$$\tau_{B_i|B_j} = 4a_i^2a_j^2, \tag{19}$$

and the tangle between the central spin and the bath is given by

$$\tau_{\mathcal{S}|\mathcal{B}} = 4c^2\sum_{i=1}^{N} a_i^2. \tag{20}$$

The symmetry constraint implies that $a_i=a_j=a$ for all $i,j \leqslant N$, and it follows that

$$\tau_B = \tau_{B_i|B_j} = 4a^4, \tag{21}$$

$$\tau_{\mathcal{S}|\mathcal{B}} = 4Na^2c^2. \tag{22}$$

Fixing $\tau_{\mathcal{S}|\mathcal{B}}=D$ we can maximize τ_B as we did for the $N=2$ case and subsequently obtain a maximum τ_B at

$$\tau_B = \frac{1}{N^2}(1+\sqrt{1-\tau_{\mathcal{S}|\mathcal{B}}})^2, \tag{23}$$

with the corresponding entanglement-sharing inequality

$$\tau_{\mathcal{S}|\mathcal{B}} \leqslant \begin{cases} 1, & \tau_B \leqslant \frac{1}{N^2}, \\ N(2\sqrt{\tau_B}-N\tau_B), & \tau_B \geqslant \frac{1}{N^2}. \end{cases} \tag{24}$$

This inequality is identical to Eq. (17) up to a dimensional scaling. Note that the maximum possible pairwise tangle for a symmetric bath of N spin-$\frac{1}{2}$ particles has been shown to be $4/N^2$ [14] and that the system-bath tangle falls to 0 for this value of τ_B.

Of course, we have only demonstrated this inequality for the W-class states, Eq. (18). To verify the inequality numerically for small values of N we calculated $\tau_{\mathcal{S}|\mathcal{B}}$ and τ_B for random states having the appropriate bath symmetry. A sample size of 1×10^7 was used, and to reduce the sample space we used the generalized Schmidt decomposition [13]. No violations of Eq. (24) were found for $N \leqslant 5$.

The extension of Eq. (24) to mixed states ρ, where the formula (5) is no longer valid is straightforward. Given a pure-state decomposition $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ we may define the average system-bath tangle by

$$\overline{\tau_{\mathcal{S}|\mathcal{B}}}(\rho) = \sum_i p_i \tau_{\mathcal{S}|\mathcal{B}}(|\psi_i\rangle). \tag{25}$$

The minimum $\overline{\tau_{\mathcal{S}|\mathcal{B}}}(\rho)$ over all pure- state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ can then be used to quantify the quantum correlations between the system and the bath.

The concavity of Eq. (23) allows us to write

$$\frac{1}{N^2}[1 + \sqrt{1 - \tau_{\mathcal{S}|\mathcal{B}}^{\min}(\rho)}]^2 \geqslant \sum_i p_i \tau_B(|\psi_i\rangle). \tag{26}$$

On the other hand, the tangle is convex, so we have $\Sigma_i p_i \tau_B(|\psi_i\rangle) \geqslant \tau_B(\rho)$ and thus obtain the inequality

$$\frac{1}{N^2}[1 + \sqrt{1 - \tau_{\mathcal{S}|\mathcal{B}}^{\min}(\rho)}]^2 \geqslant \tau_B(\rho), \tag{27}$$

which we can be inverted to obtain the entanglement-sharing inequality for mixed states.

One simple model of decoherence where the inequality (24) is immediately applicable is an exactly solvable model introduced by Zurek [3] and recently used to investigate the structure of the decoherence induced by spin environments [15]. The system is always in a pure state, so there are no classical correlations and a bound on the system-bath entanglement is a bound on the decoherence.

The Hamiltonian of this model, after applying the complete symmetry constraint, is written

$$H_{\mathcal{SB}} = \frac{1}{2} g \sum_{k=1}^{N} \sigma_z^{(s)} \sigma_z^{(B_k)}. \tag{28}$$

It is possible to analytically solve this model to give a good illustration of how the decoherence of the central spin—as quantified by the decay of the off-diagonal elements of the reduced density operator of the system [15]—is suppressed by the presence of entanglement between the bath spins. Starting with a separable system-bath ($\mathcal{SB}$) state

$$|\Psi_{\mathcal{SB}}\rangle = (\chi|\downarrow\rangle_{\mathcal{S}} + \gamma|\uparrow\rangle_{\mathcal{S}}) \otimes |\mathcal{B}(0)\rangle, \tag{29}$$

the state of $\mathcal{SB}$ at an arbitrary time t is

$$|\Psi_{\mathcal{SB}}(t)\rangle = \chi|\downarrow\rangle_{\mathcal{S}}|\mathcal{B}_\downarrow(t)\rangle + \gamma|\uparrow\rangle_{\mathcal{S}}|\mathcal{B}_\uparrow(t)\rangle, \tag{30}$$

where

$$|\mathcal{B}_\downarrow(t)\rangle = |\mathcal{B}_\uparrow(-t)\rangle = \exp\left(igt\sum_{k=1}^{N} \sigma_z^{b_k}/2\right)|\mathcal{B}(0)\rangle. \tag{31}$$

The state of the system is then described by the reduced density operator

$$\rho_{\mathcal{S}} = |\chi|^2|\downarrow\rangle_{\mathcal{S}}\langle\downarrow| + \chi\gamma^* r(t)|\downarrow\rangle_{\mathcal{S}}\langle\uparrow| + \chi^*\gamma r^*(t)|\uparrow\rangle_{\mathcal{S}}\langle\downarrow| + |\gamma|^2|\uparrow\rangle_{\mathcal{S}}\langle\uparrow|, \tag{32}$$

where the *decoherence* factor [15] $r(t) = \langle\mathcal{B}_\uparrow(t)|\mathcal{B}_\downarrow(t)\rangle$ can be easily calculated. The absolute value of this factor is bounded by $0 \leqslant |r(t)|^2 \leqslant 1$, corresponding to complete decoherence to a statistical mixture (0) and no loss of coherence (1), respectively. The $\mathcal{SB}$ tangle $\tau_{\mathcal{S}|\mathcal{B}}(t)$ can be written in terms of this factor by

$$\tau_{\mathcal{S}|\mathcal{B}}(t) = 4|\chi|^2|\gamma|^2[1 - |r(t)|^2]. \tag{33}$$

We first consider an initial bath state of the form

$$|\mathcal{B}(0)\rangle = \bigotimes_{k=1}^{N} (\alpha|\downarrow\rangle_{B_k} + \beta|\uparrow\rangle_{B_k}), \tag{34}$$

which is completely separable, with each individual bath spin in an identical state (preserving the symmetry). It is a relatively simple exercise to calculate the decoherence factor,

$$|r(t)|^2 = [|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2\cos(2gt)]^N. \tag{35}$$

As argued in Zurek *et al.* [15], as $N \to \infty$, the average value $\langle|r(t)|^2\rangle \to 0$, implying complete decoherence of the initial state. This is the average over time, since for large N, $|r(t)|^2$ is predominantly 0 (over time) but will revive to 1 periodically. However, as $N \to \infty$, these revival approach δ functions in time. With no intrabath entanglement ($\tau_B = 0$), there is no bound on $\tau_{\mathcal{S}|\mathcal{B}}$, resulting in maximal possible entanglement between system and bath. Unentangled baths of this form were the topic of Ref. [15].

We now consider an initial entangled environment state. Following from the previous construction of the entanglement-sharing constraint, we choose an initial state of the form

$$|\mathcal{B}(0)\rangle = \frac{a}{\sqrt{N}}(|\downarrow\downarrow\cdots\downarrow\downarrow\uparrow\rangle_{\mathcal{B}} + |\downarrow\downarrow\cdots\downarrow\uparrow\downarrow\rangle_{\mathcal{B}} + \cdots + |\uparrow\downarrow\cdots\downarrow\downarrow\downarrow\rangle_{\mathcal{B}}) + d|\downarrow\downarrow\cdots\downarrow\downarrow\rangle_{\mathcal{B}}, \tag{36}$$

where $a^2 + d^2 = 1$, such that the entanglement between any two bath spins is $\tau_B = 4a^4$. Since the system-bath interaction does not flip spins, for such initial states the intrabath entanglement is invariant over the evolution. In other words, the bath spins maintain their entanglement. From this initial bath state, the decoherence factor is

$$|r(t)|^2 = |a|^4 + |d|^4 + 2|a|^2|d|^2\cos(2gt), \tag{37}$$

which, first, does not average to zero in the limit of large N and, in fact, will not be zero at anytime for given values of a and d (see Fig. 1). This can be interpreted as a suppression of decoherence, since at no time will the system ever be a complete statistical mixture of states.

The inequality only places a nontrivial upper bound on the system-bath entanglement when $\tau_B \geqslant 1/N^2$. For the states considered here, this corresponds to the parameter range $1/\sqrt{2} \leqslant a \leqslant 1$, to which we will now restrict ourselves. The system-bath tangle is given by

$$\tau_{\mathcal{S}|\mathcal{E}} = 2|a|^2(1 - |a|^2)[1 - \cos(2gt)]. \tag{38}$$

From the intrabath tangle $\tau_B = 4a^4$, the entanglement-sharing inequality (24) gives an upper bound on the system-bath tangle of

$$\tau_{\mathcal{S}|\mathcal{E}}^{\max} = 4|a|^2(1 - |a|^2) \tag{39}$$

and it is simple to show that $\tau_{\mathcal{S}|\mathcal{E}} \leqslant \tau_{\mathcal{S}|\mathcal{E}}^{\max}$. In turn, this constrains the lower bound on the decoherence factor. This simple example demonstrates that entanglement in the environment can constrain entanglement between the system and

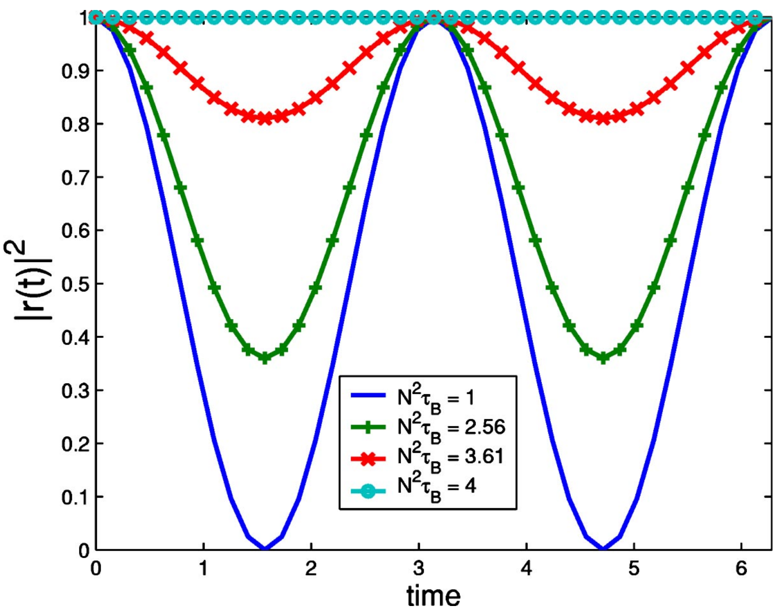

FIG. 1. (Color online) Plot of the temporal evolution of the decoherence factor $|r(t)|^2$ with an initial entangled environmental state of the form of Eq. (36) for different values of intrabath tangle. We see that the entanglement in the bath acts to suppress the oscillation of $|r(t)|^2$, meaning the state of the system remains coherent.

environment, and hence limit the effect of decoherence. Of course, in this example we have not considered any intrinsic central spin or bath dynamics.

It is also possible to calculate the intrabath entanglement for the Tessieri-Wilkie model [6], where the initial state of the bath is a thermal state and thus the overall state at time t is mixed. In the Tessieri-Wilkie model, the system is described by

$$H_{\mathcal{S}} = \frac{\omega_0}{2}\sigma_z^{(0)} + \beta\sigma_x^{(0)}, \tag{40}$$

the bath,

$$H_{\mathcal{B}} = \sum_{i=1}^{N} \frac{\omega_i}{2}\sigma_z^{(i)} + \beta\sum_{i=1}^{N} \sigma_x^{(i)} + \lambda\sum_{i=1}^{N-1}\sum_{j=1}^{N} \sigma_x^{(i)}\sigma_x^{(j)}, \tag{41}$$

and the interaction,

$$H_{\mathcal{SB}} = \lambda_0\sum_{i=1}^{N} \sigma_x^{(i)}\sigma_x^{(0)}. \tag{42}$$

Following Ref. [6], $\beta=0.01$, $\lambda_0=1$, and $\omega_0=0.8288$; however, we set $\omega_i=1$ such that all baths spins are identical. The bath starts in the thermal state $\rho_{\mathcal{B}}(0)=\exp(-H_{\mathcal{B}}/kT)/\{\mathrm{Tr}[\exp(-H_{\mathcal{B}}/kT)]\}$, such that varying the intrabath coupling strength λ varies the initial entanglement between the bath spins. To see the effects of decoherence, the central spin is initialized in the state $|\psi_{\mathcal{S}}(0)\rangle=(|\uparrow\rangle+|\downarrow\rangle)/\sqrt{2}$. In the absence of the bath, the central spin will simply precess, exhibiting Rabi oscillations. Interactions with the bath that decohere the spin will prevent such coherent oscillations.

Figure 2 shows how an entangled bath can suppress the decohering effects of the bath, allowing coherent oscillations of the central spin. Since the bath begins in thermal equilibrium, its state does not vary significantly over its evolution (especially if N is large). Hence, if the initial state is en-

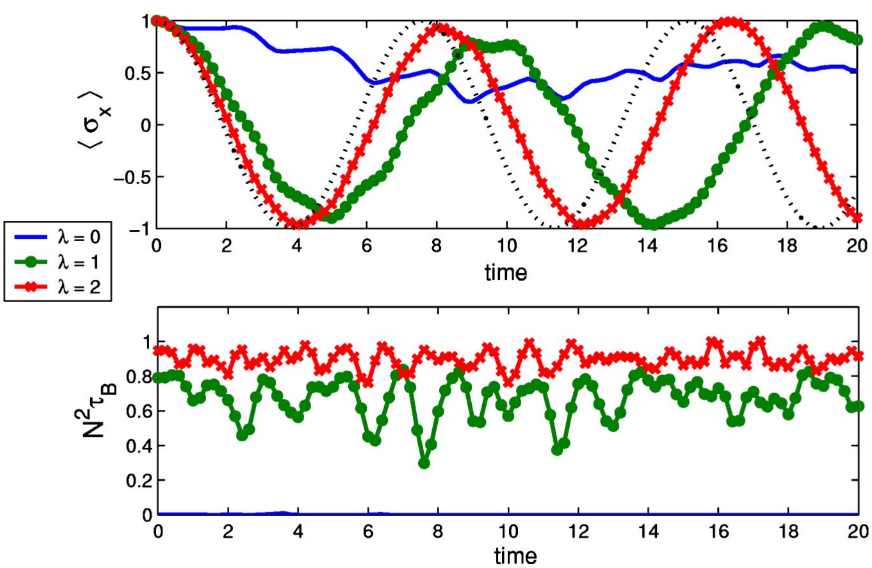

FIG. 2. (Color online) Rabi oscillations and the intrabath entanglement, quantified by the tangle between any two bath spins, for three different intrabath coupling strengths for the Tesseiri-Wilkie model, with $N=10$ bath spins. The dotted line in the $\langle\hat{\sigma}_x\rangle$ plot is the case of no system-bath interaction. As the intrabath coupling increases, so does the intrabath entanglement, and the Rabi oscillations approach the limit of no system-bath interaction.

tangled, this degree of entanglement is maintained throughout the evolution.

Since the initial state is mixed, classical correlations between system and bath will be a cause of decoherence. However, it is likely that the result of Koashi and Winter [10] may be extended to the central-spin model, thus showing that suppression of decoherence is a generic feature when spin-bath environments maintain a high degree of internal entanglement.

In order to gain insight into how intrabath entanglement can reduce decoherence we have considered two simple models in which all bath spins interact equally with one another. This represents a model for which the mean-field approximation for the interaction between spins is exact. More physical models will involve short-range interactions, yet we conjecture that they will exhibit essentially the same phenomena.

Recent studies of a central spin or qubit interacting with a reservoir of (identical) qubits has considered the process of *homogenization* [16], of which thermalization is a special case [17]. The system qubit is initially in some state ρ, with each bath spin in the identical state ξ. The aim of the process is to output all qubits in some arbitrarily small neighborhood of ξ. Thermalization is the case where ξ corresponds to the thermal state. This thermalization process is equivalent to the decoherence of the system qubit to a thermal state.

In this discrete time process, the system qubit interacts with a only single bath qubit at each time step and never the same qubit twice. It is shown that the partial swap operation uniquely determines a universal quantum homogenizer [16]. While there is no explicit interaction between bath qubits, their mutual interaction with the system qubit generates entanglement not only between the system and reservoir, but also intrabath entanglement. This entanglement is studied in [16] and the results agree with the entanglement-sharing arguments we have made here. Specifically, in the example considered, the entanglement between system and bath decreases in the long term, as more bath qubits become entangled with each other. Interestingly, it is shown that all entanglements are pairwise, with no multiparty entanglement present [18]. It would be interesting to extend the work in these articles by considering thermalization in the presence of a self-interacting bath. Of course, different methods would have to be employed, since the state of the bath qubits would change after each interaction.

Decoherence is the major stumbling block on the road to quantum computing. Here we have introduced a novel way of constraining the decoherence effects from a spin-bath environment. Such environmental models are of particular importance for predicting decoherence effects in solid-state qubits in the low-temperature regime [19,20].

We have used two simplified models as examples of how entanglement in the environmental bath may suppress decoherence. While we have only discussed spin baths, one could also envision similar effects for oscillator baths, where entangled spins may be replaced by multimode squeezed states. As well, we have focused upon two-party entanglement in the bath. The effects of m-party entangled states may be quite different.

The types of entangled states of the bath that may be created and maintained will depend explicitly upon the physical system in question. To discover if entanglement sharing can suppress decoherence in realistic situations requires calculations for specific quantum computer architectures. Only then will it be apparent if this unique property of entanglement can be used to our advantage in overcoming decoherence.

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