Ion Trap Simulations of Quantum Fields in an Expanding Universe

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We propose an experiment in which the phonon excitation of ion(s) in a trap, with a trap frequency exponentially modulated at rate κ, exhibits a thermal spectrum with an "Unruh" temperature given by $k_B T = \hbar\kappa$. We discuss the similarities of this experiment to the response of detectors in a de Sitter universe and the usual Unruh effect for uniformly accelerated detectors. We demonstrate a new Unruh effect for detectors that respond to antinormally ordered moments using the ion's first blue sideband transition.

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It is well known that a comoving detector for field quanta in de Sitter space, moving in the conformal vacuum, will become excited with a frequency response equivalent to that of an inertial detector in a thermally excited field with a temperature $T = \hbar\sqrt{3\Lambda}/k_B$ where Λ is the cosmological constant [1]. A similar response is expected for a detector with constant acceleration, a, in a Minkowski vacuum [1–3] with $k_B T = \hbar a/2\pi c$. Quite clearly, such an effect would be very difficult to see given current technologies.

In this Letter we suggest an analogous system, based on detecting phonons of the vibrational modes of cold trapped ions [4]. In many ways this parallels a theme, pioneered by Unruh, of sonic equivalents for quantum fields in curved spacetime [5]. One limit of our result corresponds formally to the case considered by Scully *et al.* [6] for the very different physical problem of an atom, with constant acceleration moving through an optical cavity.

Our analogy is based on an alternative view in terms of the time-dependent redshift seen by an accelerated observer [7]. By controlling the trapping potentials of trapped ions it is possible to modulate the normal-mode frequencies so that they have the same time-dependent phase as redshifted (or blueshifted) frequencies seen by a constantly accelerated observer, or a comoving detector in de Sitter space.

In [7] it was shown that an accelerated detector moving through the Minkowski vacuum experiences time-dependent Doppler frequency shifts of the form $\nu \to \nu(t) = \nu \exp(\pm\kappa t)$ with $\kappa = a/c$, arising from the Lorentz boosts into the instantaneous rest frame of the detector. Thus we expect that an exponential modulation of the trapping frequency will lead to an Unruh-Davies-like effect. The specific choice of an exponential modulation of the trapping frequency is also cosmologically significant and corresponds to a de Sitter model Friedmann (empty) universe with zero spatial curvature and a scale factor given by $R(t) = \exp[(3\Lambda)^{1/2}t]$, where Λ is the cosmological constant [8]. For a general Friedmann universe the frequency of a photon would evolve as $\nu(t) = \nu_0 R(t_0)/R(t)$ as measured relative to some fiducial time t_0 when the frequency has value ν_0. For a large class of models $R(t) \sim t^{2/3}$ for early times and asymptotically varies exponentially for late times. It is well known that such expanding or contracting universes experience small but nonzero particle production [1]. For the de Sitter case of pure exponential expansion, the particle spectrum is thermal.

Trapped ions can be prepared (using laser cooling) in the ground state of the normal modes of vibration in the trap. The internal electronic state of each ion can be harnessed as a phonon detector using an external laser to couple the vibrational motion of a trapped ion to an electronic transition between states which we denote $g \leftrightarrow e$ [9]. The internal state can then be readout with high efficiency using a cycling transition. This scenario defines a *phonon detector* that may be turned on and off at will. We can implement various kinds of phonon detectors by carefully tuning the laser frequency ω_L to one of the vibrational sidebands of the ion. This enables one to realize rather unconventional phonon detectors that respond to antinormally ordered moments (blue sideband) as well as the more conventional normally ordered moments (red sideband) as we explain in more detail below.

For our purposes, the salient feature of a linear ion trap is that the quantized motion about each ion's equilibrium position $\hat{q}_m(t)$ ($m \in \{1, 2, \ldots, N\}$) can be regarded as the discretized version of a scalar field $\hat{\phi}(x = ml)$ where l is the mean position between each ion. Using the internal states of the ion as a phonon detector we, in effect, have a built-in field detector at each point of space. The detectors at each spatial point $x_m = ml$ can be turned on or off at will by utilizing an external laser addressing each ion. This is much harder to achieve in other quantum optical systems, such as atoms in small electromagnetic cavities or atoms in an expanding Bose-Einstein condensation (BEC).

The interaction Hamiltonian describing the coupling of the internal and vibrational degrees of freedom of the mth ion in a linear array of ions in a trap (in the interaction picture, and Lamb-Dicke limit) can be written as [9]

$$\hat{H}_I^{(m)} = \Omega_0 \eta \hat{q}_m(t) \sigma_x(t) \tag{1}$$

where $\hat{q}_m(t)$ is the local displacement of the ion, $\sigma_x(t)$ is

$$\sigma_x(t) = e^{-i\Delta t}\sigma_- + \text{H.c.}, \tag{2}$$

Ω_0 is the (scaled) Rabi frequency, $\Delta \equiv \omega_A - \omega_L$ is the detuning between atomic resonance and the laser, $k = \omega_L/c$ is the wave vector, and the Lamb-Dicke parameter is $\eta = \sqrt{\hbar k^2 \cos^2\theta/2M\nu}$, with M the mass of the ion and θ the angle between the laser and the trap axis. We have dropped the zeroth order term in η as in what follows we are assuming the resolved sideband limit and the laser is tuned within a transition linewidth of the red and blue sideband $|\Delta| = \nu$. The zeroth order term is well off resonance under this condition.

The detector can be turned on and off through the dependance on the external laser field in Ω_0, a somewhat unusual feature for field quanta detectors. Another unusual feature of this detector is that the transition frequency of the detector $\Delta = \omega_A - \omega_L$ can be varied by tuning the external laser. Conventional detectors would have a fixed transition frequency. This latter feature will enable us to define different kinds of phonon detectors.

The local oscillation amplitude can be expanded in terms of creation and annihilation operators for *global* normal modes (phonons) of the N-ion system by [9]

$$\hat{q}_m(t) = \imath \sum_{p=1}^{N} \frac{b_m^{(p)}}{\mu_p^{1/4}} (\hat{a}_p e^{-i\nu_p t} - \hat{a}_p^\dagger e^{i\nu_p t}) \tag{3}$$

where $b_m^{(p)}$ are coefficients determined by the transformation to normal modes. In the above, ν_p are the normal-mode trap frequencies given by $\nu_p = \sqrt{\mu_p}\nu$, all are proportional to the bare trap frequency ν, of which the eigenvalues of the normal-mode transformation are μ_p. For the center of mass mode $\nu_1 = \nu$ and for the breathing mode $\nu_2 = \sqrt{3}\nu$.

It is worth noting an important difference between this model and the usual treatment of a particle detector. In the case of a usual detector the frequency term Δ would be strictly positive, thus defining the positive and negative frequency components of the dipole $\sigma_\pm$. In Eq. (2), the parameter Δ can be positive or negative, so we cannot simply refer to positive or negative frequency components in absolute terms. However the operators $\sigma_\pm$ will retain their usual definition as raising and lowering operators.

In the case that $\Delta > 0$, the laser is detuned below the atomic transition, and we can resonantly excite so-called red sideband transitions when $\Delta \approx n\nu_p$. Near such a resonance ($n = 1$) we can make the rotating-wave approximation and describe the interaction by the usual Jaynes-Cummings model of a two-level system interacting with a bosonic degree of freedom. In physical terms it describes a Raman process in which one laser photon and one trap phonon are absorbed to excite the atom. A phonon detector defined this way would respond to the normally ordered moments of the phonon field amplitude. In the case that $\Delta < 0$, the laser is detuned above the atomic transition, and the resonant term for the first blue sideband is then a Raman process in which the atom is excited by the absorption of one laser photon and the *emission* of one phonon. Considered as a phonon counter this would correspond to a detector that responded to antinormally ordered moments of the phonon field amplitude.

If we exponentially chirp the trap frequency up or down such that $\nu \to \nu(t) = \nu e^{\pm\kappa t}$ with κ the modulation rate, the normal-mode creation and annihilation operators have the time dependance

$$\hat{a}_p(t) = \exp\left(\pm\imath \frac{\nu_p}{\kappa} e^{\kappa t_0}\right) \exp\left(\mp\imath \frac{\nu_p}{\kappa} e^{\kappa t}\right) \hat{a}_p(t_0). \tag{4}$$

We consider the chirp-up case first.

We now suppose that the coupling to the detector is turned on at the same time as the frequency modulation is turned on, and turned off at the same time as the frequency modulation is turned off. This is a rather different scenario from the usual discussion of the Unruh-Davies effect in which the detector is always coupled to the field and continuously accelerated; i.e., the redshift frequency modulation is always on.

The probability $P_m(T, t_0)$ for the excitation of the ion from the ground state for a detector, turned on at $t = t_0$ and turned off at $t = T$, is [10]

$$P_m(T, t_0) = (\Omega_0 \eta)^2 \int_{t_0}^{T} dt' \int_{t_0}^{T} dt'' e^{i\Delta(t'-t'')} G(t', t'') \tag{5}$$

where the field correlation function is defined as

$$G(t', t'') = {}_{\text{vib}}\langle 0|\hat{q}_m(t')\hat{q}_m(t'')|0\rangle_{\text{vib}} \tag{6}$$

where we have assumed that the electronic and normal-mode vibrational degrees of freedom are all prepared in the ground state. Substituting Eq. (3) with the operators Eq. (4) into Eq. (6) yields

$$P_m(T, t_0) = (\Omega_0 \eta)^2 \sum_{p=1}^{N} \frac{|b_m^{(p)}|^2}{\sqrt{\mu_p}} |I_p(T, t_0)|^2 \tag{7}$$

where the integral is defined as

$$I_p(T, t_0) \equiv \int_{t_0}^{T} dt e^{i\Delta t} e^{[i(\nu_p/\kappa)e^{\kappa t}]}. \tag{8}$$

This formula was first obtained by Scully *et al.* [6] for the quite different physical setting of an atom accelerated through an optical cavity. With no loss of generality we may now set $t_0 = 0$ as the time for the initiation of the trap expansion. We first consider the case $\Delta > 0$, which is the red-sideband case. Our objective is to calculate the excitation probability for the two-level system near the red-sideband transition, which we label $P_m^R(T, t_0)$.

In an experiment one needs to vary Δ near the red or blue sideband, so we expect that Δ and ν_p are of the same order. We now consider the limit in which $\nu_p/\kappa \ll 1$ for all the normal modes. In other words we need to change the trap potential faster than the time scale of motion in the trap. This is the nonadiabatic limit. With this limit the frequency spectrum of an exponential modulation has a significant component at twice the trap frequency which will indeed drive two-phonon transitions required for squeezing. In other words there will be significant parametric heating of the ion. However, only in the case of exponential modulation do we get the Planckian spectrum. This is a alternative way to view Unruh excitation of a quantum counter. Furthermore we suppose the time over which the detector and modulation are on is such that $\kappa T \gg \nu_p/\kappa$. With these assumptions we can extend the upper limits of the integral from minus to plus infinity [11] to give

$$I_p = \frac{1}{\kappa b^{ia}} \Gamma(ia) e^{-(\pi/2)a}, \tag{9}$$

where $a = \Delta/\kappa$ and $b = \nu_p/\kappa$. If we now use the identity $|\Gamma(ix)|^2 = \pi/[x \sinh(\pi x)]$ for real x [12] we see that

$$|I_p|^2 = \frac{2\pi}{\kappa\Delta} \frac{1}{e^{2\pi\Delta/\kappa} - 1}. \tag{10}$$

Inserting this into Eq. (7) we can write P_m^R in the suggestive form

$$P_m^R = \frac{\Omega_0^2 \eta^2}{\kappa\Delta} \frac{2\pi}{e^{\hbar\Delta/k_B T} - 1} \sum_{p=1}^{N} \frac{|b_1^{(p)}|^2}{\sqrt{\mu_p}}. \tag{11}$$

This is Planck-like with Unruh temperature,

$$T \equiv \frac{\hbar\kappa}{k_B 2\pi}. \tag{12}$$

Equation (11) has a form analogous to a thermal spectrum at temperature T as seen by a uniformly accelerated observer moving through a particle-free inertial vacuum. The thermal form of the probability is independent of the phonon frequencies ν_p, since we have taken the upper limit t to infinity. The last summation expression is just a numerical factor, which can be computed [9], or dropped in the case of a single ion in the trap.

It is important to note that in order to obtain the Planck factor $(e^{2\pi\Delta/\kappa} - 1)^{-1}$ in Eq. (10), the signature of the Unruh effect, we made crucial use of a positive detuning $\Delta > 0$, corresponding to a red-sideband detuning. This resulted in the factor $(i)^{i\Delta/\kappa} = (e^{i\pi/2})^{i\Delta/\kappa} = e^{-\pi\Delta/2\kappa}$ that appears in I_p in Eq. (9). Dividing the square of this factor into the sinh function appearing in the denominator of $|I_p|^2$, resulting from the term $|\Gamma(i\Delta/\kappa)|^2$, produces the signature Bose-Einstein thermal distribution function.

If, on the other hand, we had instead chosen $\Delta = -|\Delta| < 0$ corresponding to a negative detuning to the blue sideband, the previous factor would become $(e^{i\pi/2})^{-i|\Delta|/\kappa} = e^{\pi|\Delta|/2\kappa}$. This choice then produces alternatively the probability for excitation on the blue sideband

$$P_m^B = \frac{\Omega_0^2 \eta^2}{\kappa|\Delta|} \frac{2\pi}{1 - e^{-\hbar|\Delta|/k_B T}} \sum_{p=1}^{N} \frac{|b_1^{(p)}|^2}{\sqrt{\mu_p}}, \tag{13}$$

with the same definition of the Unruh temperature as in Eq. (12).

We will call such a distribution an *antinormally ordered* Unruh effect since the vibrational excitation from the ground to the excited state takes place by an absorption of a photon and an emission of a phonon to the electronic state $|e\rangle$, i.e., by a term such as $\hat{a}_p^\dagger \sigma_+$. As discussed, such a detector responds to the antinormally ordered moments of the phonon field amplitude. Note that as $\kappa \to 0$ ($T \to 0$) in Eq. (13) we get a *finite* contribution to the probability for excitation P_m. In the case of red-sideband detuning, the usual Unruh effect analogy in Eq. (11), $P_m \to 0$ as $\kappa \to 0$. This limit corresponds to a fixed trap frequency ν for which we get no excitation as described above (analogous to an inertial observer who would detect no Unruh particle excitations).

For a finite chirp between times t_0 and T we can develop a general expression for $P_m(T, t_0)$ in terms of the incomplete gamma functions $\gamma(\mu, x) = \int_0^x dt t^{\mu-1} e^{-t}$ and $\Gamma(\mu, x) = \int_x^\infty dt t^{\mu-1} e^{-t}$ such that $\gamma(\mu, x) + \Gamma(\mu, x) = \Gamma(\mu)$. Let us write $\int_{y_0}^{y_T} dy = \int_0^\infty dy - \int_0^{y_0} dy - \int_{y_T}^\infty dy$ for general finite duration limits in Eq. (8) with the definition $y_t = e^{\kappa t}$. By scaling the integration variable to $y = y_{t_0} x$ in the second integral on the right hand side and $y = y_T x$ in the third integral on the right hand side one can easily show that

$$I_p(T, t_0) = \frac{1}{\kappa} \frac{\Gamma(ia) e^{-(\pi/2)a}}{b^{ia}} [1 - \gamma'(ia, -iby_{t_0}) - \Gamma'(ia, -iby_T)], \tag{14}$$

where we have defined the normalized gamma functions $\gamma'(\mu, x) = \gamma(\mu, x)/\Gamma(\mu)$ and $\Gamma'(\mu, x) = \Gamma(\mu, x)/\Gamma(\mu)$ such that $\gamma'(\mu, x) + \Gamma'(\mu, x) = 1$. Thus we obtain

$$P_m(T, t_0) = \frac{\Omega_0^2 \eta^2}{\kappa\Delta} \frac{2\pi}{e^{\hbar\Delta/k_B T} - 1} \sum_{p=1}^{N} \frac{|b_1^{(p)}|^2}{\sqrt{\mu_p}} \left| 1 - \gamma'\left(i\frac{\Delta}{\kappa}, -i\frac{\nu_p}{\kappa} e^{\kappa t_0}\right) - \Gamma'\left(i\frac{\Delta}{\kappa}, -i\frac{\nu_p}{\kappa} e^{\kappa T}\right) \right|^2. \tag{15}$$

The previous expression for the total excitation probability in Eq. (11) corresponds to $P_m(\infty, -\infty)$ using Eq. (15) above, which formally corresponds to sweeping the trap frequency from an initial zero value to an infinite final value. Considering a more realistic situation applicable to experiments, let us consider $t_0 = 0$ and a finite trap expansion time T, which corresponds to sweeping the trap frequency from $\nu \to \nu e^{\kappa T}$. Considering Eq. (15) as a function of the detuning Δ with parameters ν_p/κ and κT we can recover Eq. (11) under the following conditions

$$\frac{\nu_p}{\kappa} \ll 1, \qquad \frac{\nu_p}{\kappa} e^{\kappa T} \gg 1 \quad \forall\ p = (1, 2, \ldots, N), \quad (16)$$

which makes the incomplete gamma functions small compared to unity. As an example, taking $\nu_p/\kappa = 0.01$ and $(\nu_p/\kappa)\exp(\kappa T) = 100$, which requires that $\kappa T > 9.2$, approximates the full Unruh case $P_m(\infty, -\infty)$ quite well.

As we have shown, the experimental signature of the exponential modulation of the trap frequency is the Planck-like form for the excitation probability for the two-level electronic system in each ion. In such experiments it is the ratio of the excitation probability on the red ($\Delta = \nu$) and blue ($|\Delta| = \nu$) sidebands that is determined: $R_e = P_m^R/P_m^B$, as this number is independent of the Rabi frequency, the Lamb-Dicke parameter, and the time of interaction between the vibrational and electronic degrees of freedom [13,14]. Let us consider the case of a single ion with trap frequency ν. Using Eqs. (11) and (13) we see that

$$R_e = \frac{1 - e^{-2\pi\nu/\kappa}}{e^{2\pi\nu/\kappa} - 1}. \quad (17)$$

In a typical experiment one can detect R values as low as 0.05 with about 20% error. This implies that $\nu/\kappa \approx 0.5$. At secular frequencies of $\nu/2\pi = 0.1$–1 MHz, we need a modulation frequency of the order of a few hundred khz to MHz; a not particularly difficult requirement for fast electronics.

The key issue, however, is the absolute size of the excitation probabilities at the red and blue sideband. This is determined by the prefactor $(\Omega_0\eta)^2/(\kappa\nu)$. Defining $z = 2\pi\nu/\kappa$ the equations for the excitation probability at the red and blue sidebands are

$$P^R = \left(\frac{\Omega_0 \eta}{\nu}\right)^2 \frac{z}{e^z - 1} \quad (18)$$

$$P^B = \left(\frac{\Omega_0 \eta}{\nu}\right)^2 \frac{z}{1 - e^{-z}}. \quad (19)$$

As we expect z to be of the order of unity, we require that the secular frequency is within 1 order of magnitude of the effective Rabi frequency. This corresponds to a rather weakly bound ion, but should be achievable if stimulated Raman transitions are used to couple the two-level system. For example the relevant transition in $^9Be^+$ can have an effective Rabi frequency of the order of $\Omega_0/2\pi = 500$ kHz [15]. If we use the center of mass mode with secular frequency of $\nu/2\pi = 200$ kHz, and a Lamb-Dicke parameter of $\eta = 0.2$, the prefactor is 0.25. At a more conservative estimate of $\eta = 0.05$, well within the Lamb-Dicke regime, the prefactor drops to a value of 0.015. These numbers are encouraging enough to suggest the plausibility of observing an analogous Unruh-like effect in today's linear ion traps. Of course achieving a perfect exponential modulation will not be easy. Finally we note that the effect we describe can be seen even for a single trapped ion, although the quantum scalar field analogy is rather strange in that case as it corresponds to a universe with a single global mode: a very small universe indeed.

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[10] Equation (5) for the total excitation probability stems from the expression $P_m(T, t_0) = \sum_k |A_k^{(m)}|^2$ where the amplitude for a transition from the ground state $|0\rangle$ to any arbitrary excited state $|\phi_k\rangle$ is given by the first order perturbation theory $A_k^{(m)} = (-i/\hbar)\langle\phi_k| \int_{t_0}^T dt \hat{H}_I^{(m)}(t)|0\rangle$, and use has been made of the completeness relation $\sum_k |\phi_k\rangle\langle\phi_k| = 1$. The non-zero contribution in (5) arises from the term $\hat{a}_p^\dagger \sigma_+$ in (1).
[11] In deriving Eq. (9) from Eq. (8) we have made the change of variables $\tau = \kappa t - \alpha$ where $\nu_p/\kappa \equiv \exp(-\alpha)$. For the stated limits of $\kappa T \gg \nu_p/\kappa \gg 1$ we can extend the upper limits to $\pm\infty$ to obtain Eq. (9).
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