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Analysis of the stochastic FitzHugh-Nagumo system

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In this paper we study a system of stochastic differential equations with dissipative nonlinearity which arise in certain neurobiology models. Besides proving existence, uniqueness and continuous dependence on the initial datum, we shall be mainly concerned with the asymptotic behaviour of the solution. We prove the existence of an invariant ergodic measure ν associated with the transition semigroup P_t; further, we identify its infinitesimal generator in the space $L^2(H;\nu)$.

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1 Introduction

Since the fundamental work of Hodgkin and Huxley [6], several equations were proposed to model the behavior of the signal propagation in a neural cell. The original model was formed by a system of four equations, the first one modeling how the electric impulses propagate along a long tube (the axon) that we model as a (normalized) segment $(0,1)$, while the remaining were concerned with various ions concentrations in the cell. Later, a more analytically tractable model were proposed by FitzHugh [5] and Nagumo [10]. In this paper we focus the interest on a stochastic version of the FitzHugh-Nagumo model. It consists of two variables, the first one, u, represents the *voltage variable* and the second one, w, is the *recovery variable*, associated with the concentration of potassium ions in the axon. For a thorough introduction to the biological motivations of this model we refer to Murray [8] or Keener and Sneyd [7].

Let us consider the equation

$$\begin{aligned} \partial_t u(t,\xi) &= \partial_\xi\left(c(\xi)\partial_\xi u(t,\xi)\right) - p(\xi)u(t,\xi) + f(u(t,\xi)) - w(t,\xi) \\ &\qquad + \partial_t \beta_1(t,\xi), \qquad t \ge 0, \quad \xi \in [0,1], \\ \partial_t w(t,\xi) &= -\alpha w(t,\xi) + \gamma u(t,\xi) + \partial_t \beta_2(t,\xi), \\ &\qquad t \ge 0, \quad \xi \in [0,1], \end{aligned} \tag{1.1}$$

where u represents the electrical potential and w is the recovery variable; α, γ, $c(\xi)$ and $p(\xi)$ are given phenomenological coefficients satisfying the conditions stated below; β_1, β_2 are independent Brownian motions; f is a nonrandom real-valued function with suitable smoothness properties: in the reduced FitzHugh-Nagumo system f is a polynomial of odd degree, precisely $f(u) = -u(u-1)(u-\xi_1)$, where $0 < \xi_1 < 1$ represents the voltage threshold. Problem (1.1) shall be endowed with boundary and initial conditions. The first one are necessary only for the potential $u(t,\xi)$ and we assume they are of Neumann type: $\partial_\xi u(t,0) = \partial_\xi u(t,1) = 0$; the initial condition are given, for simplicity, by continuous functions

$$u(0,\xi) = u_0(\xi), \qquad v(0,\xi) = v_0(\xi)$$

with $u_0, v_0 \in C([0,1])$.

We shall introduce the main assumptions on the coefficients of problem (1.1) that will be used without stating in the following. For this, it is necessary to introduce the operator A_0 on the space $L^2(0,1)$, defined on $D(A_0) = \{u \in H^2(0,1) \mid \partial_\xi u(\zeta) = 0,\ \zeta = 0,1\}$ by

$$A_0 u(\xi) = \partial_\xi \left(c(\xi) \partial_\xi u(\xi) \right), \qquad \xi \in [0,1],\ u \in D(A_0).$$

Hypothesis 1.1. *The constants α and γ are strictly positive real numbers; the functions $c(\xi)$ and $p(\xi)$ belong to $C^1([0,1])$, $c = \min_{[0,1]} c(\xi) > 0$ and $p = \min_{[0,1]} p(\xi) > 0$. Further, for ξ_1 from the definition of the FitzHugh-Nagumo nonlinearity, it holds*

$$3p - (\xi_1^2 - \xi_1 + 1) \geq 0. \tag{1.2}$$

There exists a complete orthonormal basis $\{e_k\}$ of $L^2(0,1)$ made of eigenvectors of A_0, such that the $\{e_k\}$ satisfy a uniform bound in the sup-norm, i.e., for some $M > 0$ it holds

$$|e_k(\xi)| \leq M, \qquad \xi \in [0,1],\ k \in \mathbb{N}. \tag{1.3}$$

Let β_1, β_2 be independent Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with continuous trajectories on $[0,T]$ for any $T > 0$; this means that

$$\beta_i \in C([0,T]; L^2(\Omega, L^2(0,1)))$$

with $\mathcal{L}(\beta_i(t,\cdot)) \cong \mathcal{N}(0, t\sqrt{Q_i})$ for suitable linear operators Q_i, $i = 1,2$ on $L^2(0,1)$.

With no loss of generality we can assume that the operators Q_i, $i = 1,2$ diagonalize on the same basis $\{e_k\}$. Therefore, there exist sequences λ_k^i, $i = 1,2$, $k \in \mathbb{N}$, of positive real numbers such that

$$Q_i e_k = \lambda_k^i e_k, \qquad i = 1,2, \quad k = 1,2,\dots.$$

Furthermore, we assume that

$$\sum_{i=1}^{2} \sum_{k=1}^{\infty} \lambda_k^i < \infty;$$

hence $\mathrm{Tr} Q_i < \infty$.

It is convenient to write (1.1) in an abstract form. To this end we set $H = L^2(0,1) \times L^2(0,1)$ endowed with the inner product

$$\langle (u_1, w_1), (u_2, w_2) \rangle_H = \gamma \langle u_1, u_2 \rangle_{L^2} + \langle w_1, w_2 \rangle_{L^2}$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the usual scalar product in $L^2(0,1)$ and γ is the constant from (1.1). The corresponding norm is denoted by $|\cdot|_H$. We also introduce the space $V = H^1(0,1) \times L^2(0,1)$ with the norm

$$\|x\|_V^2 = \gamma |x_1|_{H^1}^2 + |x_2|_{L^2}^2 .$$

On the space H we introduce the following operators:

$$\begin{aligned} &A : D(A) \subset H \to H, \quad D(A) = D(A_0) \times L^2(0,1) \\ &A \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} A_0 u & -w \\ \gamma u & -\alpha w \end{pmatrix} \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &F : D(F) := L^6(0,1) \times L^2(0,1) \to H \\ &F \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -u(u-\xi_1)(u-1) \\ 0 \end{pmatrix}. \end{aligned} \tag{1.5}$$

In the following, setting $X = \binom{u}{w}$, we rewrite equation (1.1) as

$$\begin{aligned} &\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t) \\ &X(0) = x \in H \end{aligned} \tag{1.6}$$

where $W(t) = (w_1(t), w_2(t))$ is a cylindrical Wiener process on H and Q is the operator matrix

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Our first result is an existence and uniqueness theorem for the solution of equation (1.6).

Theorem 1.1. *Let $x \in D(F)$ (resp. $x \in H$). Then, under the assumptions in Hypothesis 1.1, there exists a unique mild (resp. generalized) solution*

$$X \in L^2_W(\Omega; C([0,T]; H)) \cap L^2_W(\Omega; L^2([0,T]; V))$$

to equation (1.6) *which depends continuously on the initial condition.*

The proof will be given in Section 3. Starting from this result, we can introduce the transition semigroup $P_t : C_b(H) \to C_b(H)$ associated to the flow $X(t,\cdot)$ defined in equation (1.6), that is

$$P_t\phi(x) = \mathbb{E}\phi(X(t,x)), \quad \phi \in C_b(H), t \geq 0, x \in H. \tag{1.7}$$

In Theorem 4.1 we shall prove the existence of an invariant measure for P_t. After that, we shall prove that the associated Kolmogorov operator N_0 is dissipative in the space $L^p(H;\mu)$ and that its closure is m-dissipative.

2 Preliminary results

Before we proceed with the analysis of the abstract stochastic equation (1.6) it is necessary to study the properties of the operators A and F.

Lemma 2.1. *Set $\eta = \frac{1}{3}(\xi_1^2 - \xi_1 + 1)$; then*

$$F_\eta \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f(u) - \eta u \\ 0 \end{pmatrix}$$

is m-dissipative, that is, it is dissipative and $I - F_\eta$ maps $D(F)$ onto H, i.e., $\mathrm{Rg}(I - F_\eta) = H$.

Proof. Let $x = \binom{u}{w}$, $y = \binom{v}{z} \in H$. By definition,

$$\begin{aligned}
&\langle F(x) - F(y) - \eta(x-y), x-y\rangle_H \\
&\qquad = \gamma \,\langle f(u) - f(v) - \eta(u-v), u-v\rangle_{L^2} \le \gamma \left(\sup_{r\in\mathbb{R}} f'(r) - \eta \right) |u-v|^2_{L^2}\,.
\end{aligned}$$

We note that $\sup_{r\in\mathbb{R}} f'(r) = \frac{1}{3}(\xi_1^2 - \xi_1 + 1)$; thus the last term in the previous inequality vanishes and F_η is dissipative.

Let us show that $I - F_\eta$ is surjective. In fact, observe that its first component $-f(u) + (\eta+1)u$ a polynomial of degree 3 with positive derivative. Hence it is invertible. Its second component is the identity and, obviously, invertible. This concludes the proof. □

Remark 2.1. *We denote f_η the first component of F_η. Setting $\xi_0 = (1+\xi_1)/3$, $f_\eta(u) = f(u) - \eta u$ can be rewritten as*

$$f_\eta(u) = -(u - \xi_0)^3 - \xi_0^3.$$

Let us introduce the notation $A_\eta = A + \eta \begin{pmatrix} I \\ 0 \end{pmatrix}$ and F_η as above; then we may rewrite equation (1.6) as

$$\begin{aligned}
\mathrm{d}X(t) &= (A_\eta X(t) + F_\eta(X(t)))\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t) \\
X(0) &= x \in H
\end{aligned} \tag{2.1}$$

Lemma 2.2. *A_η is m-dissipative and in particular, there exist $\omega_1, \omega_2 > 0$ such that*

$$\langle A_\eta x, x\rangle \le -\omega_1 |x|^2_H \tag{2.2}$$

$$\langle A_\eta x, x\rangle \le -\omega_2 \|x\|^2_V. \tag{2.3}$$

Proof. First of all, we observe that the operator A_0 satisfies the inequality:

$$\langle \partial_\xi (c\,\partial_\xi u), u\rangle_{L^2} \le 0.$$

In fact, with $c = \min_{[0,1]} c(\xi)$, we have:

$$\int_0^1 \partial_\xi (c(\xi)\, \partial_\xi u(\xi)) u(\xi)\, \mathrm{d}\xi$$
$$= c(\xi) u(\xi) \partial_\xi u(\xi)\Big|_0^1 - \int_0^1 c(\xi) (\partial_\xi u(\xi))^2\, \mathrm{d}\xi \le -c\, |Du|_{L^2}^2 \le 0.$$

Now set $p = \min\limits_{[0,1]} p(\xi) > 0$ and $\omega_1 = \min\{p - \eta, \alpha\}$. For $x = \binom{u}{v}$ we have

$$\begin{aligned}\langle Ax, x\rangle &\le \gamma \langle \partial_\xi (c\, \partial_\xi u), u\rangle_{L^2} - \gamma(p-\eta)\, |u|_{L^2}^2 - \gamma\langle u, v\rangle + \gamma \langle u, v\rangle - \alpha |v|_{L^2}^2 \\ &\le -\gamma(p-\eta) |u|_{L^2}^2 - \alpha |v|_{L^2}^2 \le -\omega_1\, |x|_H^2 .\end{aligned}$$

This proves (2.2).

As (2.3) is concerned, we have

$$\langle Ax, x\rangle \le -c\gamma\, |Du|_{L^2}^2 - \gamma(p-\eta)\, |u|_{L^2}^2 - \alpha\, |v|_{L^2}^2 \le -\omega_2 (\gamma\, |u|_{H^1} + |v|^2) = -\omega_2\, \|x\|_V^2$$

for $\omega_2 = \min\{c, p - \eta, \alpha\}$.

Now let us show the m-dissipativity.We need to prove that $I - A_\eta$ is surjective. Fix $x_0 = (u_0, v_0) \in H$ and let we consider the following equation

$$\begin{cases} u - A_0 u + v = u_0 \\ v - \gamma u + \alpha v = v_0. \end{cases}$$

Note that the second equality can be rewritten as

$$v = \frac{1}{1+\alpha} v_0 + \frac{\gamma}{1+\alpha} u; \tag{2.4}$$

then, substituting v with the right member of (2.4) we obtain

$$\left[\left(1 - \frac{\gamma}{1+\gamma}\right) I - A_0\right] u = u_0 - \frac{1}{1+\alpha} v_0.$$

Using the m-dissipativity of A_0 and (see for instance [11]) we obtain that previous equation admits a solution $u \in L^2(0,1)$. We can then compute v by means of (2.4). It follows that for every x_0 there exists $x = (u, v)$ such that $(I - A_\eta)x = x_0$, that is A_η is m-dissipative. $\square$

From the above result it follows that A_η is the infinitesimal generator of a C_0 semigroup of contractions. Further, the following holds.

Proposition 2.3. *A_η generates an analytic C_0-semigroup of contractions e^{tA_η} on H and it is of negative type.*

Proof. Note that A_0 and $-\alpha I$ generate analytic semigroups on $L^2(0,1)$ while γI is a bounded linear operator on the same space. Thus, the proof easily follows by applying the results in [9, Section 4]. Moreover, the dissipativity condition (1.2) implies that $\left\|e^{tA_\eta}\right\| \le e^{-\omega t}$, that is, A_η is of negative type. $\square$

For the moment, we notice that from the above lemmata we obtain the dissipativity of the sum $A_\eta + F_\eta$.

Lemma 2.4. *Recall assumption (1.2), that we can write as*

$$p - \eta \geq 0 \tag{2.5}$$

where $p = \min_{[0,1)} |p(\xi)|$. *Then* $A_\eta + F_\eta = A + F$ *is dissipative.*

Proof. Observe that

$$\begin{aligned}\langle (A+F)x, x\rangle = \langle (A_\eta + F_\eta)x, x\rangle_H &\leq -\gamma p |u|^2_{L^2} - \alpha |v|^2_{L^2} + \gamma\eta |u|^2_{L^2} \\ &\leq - \min\{p-\eta, \alpha\}\, |x|^2_H;\end{aligned}$$

thus the dissipativity condition is satisfied if $p \geq \eta$. □

Setting $\omega = \min\{p-\eta, \alpha\}$, the statement of Lemma 2.4 can be rewritten as

$$\langle (A+F)x, x\rangle_H \leq -\omega |x|^2_H.$$

2.1 An approximating problem

In this section we show an existence and uniqueness result for a family of approximating problems of system (2.1) with a Lipschitz continuous nonlinearity. Consider, for any $\varepsilon > 0$, the following approximation of F_η, $F_{\eta,\varepsilon}$, given as

$$F_{\eta,\varepsilon}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_{\eta,\varepsilon}(u) \\ 0 \end{pmatrix}, \qquad f_{\eta,\varepsilon}(u) = \frac{f(u) - \eta u}{1 + \varepsilon(1 - \xi_0(u-\xi_0) + (u-\xi_0)^2)}.$$

It is easily seen that $F_{\eta,\varepsilon}$ is Lipschitz continuous and

$$|F_{\eta,\varepsilon}(x) - F_\eta(x)|_H \to 0, \quad x \in L^6(0,1) \times L^2(0,1)$$

when $\varepsilon \to 0$. Moreover it easy to see that

$$|F_\eta(x)|_H \leq C(1 + |x|^3_H), \quad x \in D(F), \tag{2.6}$$

for suitable $C > 0$.

Hence, for $\varepsilon > 0$, we are concerned with the family of equations

$$\begin{aligned} \mathrm{d}X(t) &= (A_\eta X(t) + F_{\eta,\varepsilon}(X(t)))\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t) \\ X(0) &= x \in H \end{aligned} \tag{2.7}$$

which can be seen as an approximating problem of (1.6).

There exists a well established theory on stochastic evolution equations in Hilbert spaces, see Da Prato and Zabcyck [2], that we shall apply in order to show that for any $\varepsilon > 0$ Equation (2.7) admits a unique solution $X_\varepsilon(t)$. Let us recall from Proposition 2.3 that A_η is the infinitesimal generator of a strongly continuous semigroup e^{tA_η}, $t \geq 0$, on H; we also claim that the following inequality hold:

$$\int_0^t \mathrm{Tr}[e^{sA_\eta} Q e^{sA^*_\eta}]\,\mathrm{d}s < \infty, \quad \forall t \geq 0. \tag{2.8}$$

(see below). If these properties are satisfied, then the so-called stochastic convolution process

$$W_{A_\eta}(t) = \int_0^t e^{(t-s)A_\eta} \sqrt{Q}\, \mathrm{d}W(t)$$

is a well-defined mean square continuous, $\mathcal{F}_t$-adapted Gaussian process (see [2, Theorem 5.2]) and we can give the following

Definition 2.5. *Given a $\mathcal{F}_t$-adapted cylindrical Wiener process on probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a process $X(t)$, $t \geq 0$, is a mild solution of (2.7) if it satisfies $\mathbb{P}$-a.s. the following integral equation*

$$X(t) = e^{tA_\eta}x + \int_0^t e^{(t-s)A_\eta} F_{\eta,\varepsilon}(X(s))\, \mathrm{d}s + \int_0^t e^{(t-s)A_\eta} \sqrt{Q}\, \mathrm{d}W(t). \tag{2.9}$$

Let us check that in our assumptions, condition (2.8) holds.

Proposition 2.6. *A_η and Q satisfy the following inequality:*

$$\sup_{t\geq 0} \int_0^t \mathrm{Tr}[e^{sA_\eta} Q e^{sA_\eta^*}]\, \mathrm{d}s < \infty.$$

Proof. Recall that if S, T are linear operators defined on an Hilbert space H such that $S \in \mathcal{L}(H)$ and T is of trace class, then

$$\mathrm{Tr}(ST) = \mathrm{Tr}(TS) \leq \|S\|_{\mathcal{L}(H)} \mathrm{Tr}(T). \tag{2.10}$$

Taking into account the self-adjointness of A_η and the above remark we obtain

$$\mathrm{Tr}[e^{tA_\eta} Q e^{tA_\eta^*}] \leq \mathrm{Tr}(Q) \left\| e^{tA_\eta} \right\|^2_{\mathcal{L}(H)} \leq \mathrm{Tr}(Q) e^{-2\omega t},$$

hence

$$\int_0^\infty \mathrm{Tr}[e^{sA_\eta} Q e^{sA_\eta^*}]\, \mathrm{d}s \leq \int_0^\infty \mathrm{Tr}(Q)\, e^{-2\omega s}\, \mathrm{d}s < \infty.$$

□

Proposition 2.7. *The stochastic convolution is $\mathbb{P}$-almost surely continuous on $[0, \infty)$ and it verifies the following estimate*

$$\mathbb{E} \sup_{t\geq 0} \left| W_{A_\eta}(t) \right|^{2m}_H \leq C \tag{2.11}$$

for some positive constant C.

Proof. Note that for any $\alpha \in (0, 1)$ it holds

$$\int_0^\infty s^{-\alpha} \mathrm{Tr}[e^{sA_\eta} Q e^{sA_\eta^*}]\, \mathrm{d}s < \infty.$$

In fact,

$$\begin{aligned}\int_0^\infty s^{-\alpha} \mathrm{Tr}[e^{sA_\eta} Q e^{sA_\eta^*}]\, \mathrm{d}s &\leq \mathrm{Tr}(Q) \int_0^\infty s^{-\alpha} \left\| e^{sA_\eta} \right\|^2_{\mathcal{L}(H)} \mathrm{d}s \\ &\leq \mathrm{Tr}(Q) \int_0^\infty s^{-\alpha} e^{-2\omega_1 s}\, \mathrm{d}s < \infty.\end{aligned}$$

Now the thesis follows by ([3, Theorem 5.2.6]. □

Definition 2.8. *Let $L^2_W(\Omega; C([0,T]; H))$ denote the Banach space of all $\mathcal{F}_t$-measurable, pathwise continuous processes, taking values in H, endowed with the norm*

$$\|X\|_{L^2_W(\Omega;C([0,T];H))} = \left(\mathbb{E} \sup_{t\in[0,T]} |X(t)|_H^2 \right)^{1/2}$$

while $L^2_W(\Omega; L^2([0,T]; V))$ denotes the Banach space of all mappings $X : [0,T] \to V$ such that $X(t)$ is $\mathcal{F}_t$-measurable, endowed with the norm

$$\|X\|_{L^2_W(\Omega;L^2([0,T];V))} = \left(\mathbb{E} \int_0^T \|X(t)\|_V^2 \,\mathrm{d}t \right)^{1/2}.$$

With the above notation, Proposition 2.7 implies that $W_A(t) \in L^2(\Omega; C([0,T]; H))$ for arbitrary $T > 0$. Also, from Propositions 2.3 and 2.6 it follows that for $\varepsilon > 0$ the approximating problems admit a unique solution.

Proposition 2.9. *Let $x \in H$. Then, for any $\varepsilon > 0$ there exist a unique mild solution $X_\varepsilon(t,x)$ to equation (2.7) such that*

$$X_\varepsilon \in L^2_W(\Omega; C([0,T]; H)) \cap L^2_W(\Omega; L^2([0,T]; V)).$$

Proof. From [2, Theorem 7.4] we have that for any $x \in H$ problem (2.7) has a unique mild solution $X_\varepsilon(t,x)$ such that

$$\mathbb{E} \sup_{t\in[0,T]} |X_\varepsilon(t,x)|_H^p < C(1 + |x|^p), \qquad p > 2,$$

which further admits a continuous modification; this proves that $X_\varepsilon \in L^2_W(\Omega; C([0,T]; H))$. Now, we apply Ito's formula to the function $\phi(x) = |x|^2$ (although this is only formal, the following computations can be justified via a truncation argument) and we find that

$$\begin{aligned} |X_\varepsilon(t,x)|^2 = |x|^2 + 2\int_0^t \langle A_\eta X_\varepsilon(s,x) + F_{\eta,\varepsilon}(X(s,x)), X_\varepsilon(s,x)\rangle \,\mathrm{d}s \\ + 2\int_0^t \left\langle X_\varepsilon(s,x), \sqrt{Q}\,\mathrm{d}W(s) \right\rangle + \mathrm{Tr}(Q)\, t, \end{aligned} \tag{2.12}$$

where

$$\int_0^t \langle X_\varepsilon(s,x), \sqrt{Q}\,\mathrm{d}W(s)\rangle$$

is a square integrable martingale such that, by [2, Theorems 3.14 and 4.12],

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \left\langle X_\varepsilon(s,x), \sqrt{Q}\,\mathrm{d}W(s) \right\rangle \right| \le 3\mathrm{Tr}(Q)\, \mathbb{E} \left(\int_0^T |X_\varepsilon(s,x)|_H^2 \,\mathrm{d}s \right).$$

Moreover we have

$$\int_0^t \langle A_\eta X_\varepsilon(s,x), X(s,x)\rangle \,\mathrm{d}s \le -\omega_2 \int_0^t \|X_\varepsilon(s,x)\|_V^2 \,\mathrm{d}s$$

and

$$\int_0^t \langle F_{\eta,\varepsilon}(X_\varepsilon(s,x)), X_\eta(s,x)\rangle \,\mathrm{d}s \le 0.$$

Hence, taking the expectation of both members in (2.12) we obtain

$$\mathbb{E}\sup_{t\in[0,T]} |X(t,x)|^2 + \omega_2 \mathbb{E}\int_0^T \|X_\varepsilon(s,x)\|_V^2 \,\mathrm{d}s \le |x|^2 + (6+\eta)\int_0^T \mathbb{E}\sup_{s\in[0,t]} |X_\varepsilon(s,x)|_H^2 \,\mathrm{d}t.$$

By Gronwall's lemma this yields

$$\mathbb{E}\sup_{t\in[0,T]} |X_\varepsilon(t,x)|_H^2 + \omega_2 \mathbb{E}\int_0^T \|X_\varepsilon(s,x)\|_V^2 \,\mathrm{d}s \le C(|x|^2+1). \tag{2.13}$$

We conclude that $X_\varepsilon \in L^2(\Omega; L^2([0,T];V))$. □

3 Existence and uniqueness result

Here we make use of the results given in the last section to show that problem (2.1) admits a unique solution. Our main result can be stated as follows.

Theorem 3.1. *For every $x \in D(F)$ (resp. $x \in H$), there exists a unique mild (resp. generalized) solution $X \in L^2_W(\Omega; C([0,T];H)) \cap L^2_W(\Omega; L^2([0,T];V))$ to equation* (1.6) *which satisfies*

$$\mathbb{E}\,|X(t,x) - X(t,\bar{x})|_H^2 \le C\,|x-\bar{x}|_H^2\,. \tag{3.1}$$

Proof. As shown in the proof of Proposition 2.9, $\{X_\varepsilon\}_{\varepsilon\ge 0}$ satisfies

$$\mathbb{E}\sup_{t\in[0,T]} |X_\varepsilon(t,x)|_H^2 + \omega_1 \mathbb{E}\int_0^t \|X_\varepsilon(s,x)\|_V^2 \,\mathrm{d}s \le C(|x|^2+1), \qquad t\ge 0.$$

therefore it is bounded in $L^2_W(\Omega; C([0,T];H)) \cap L^2_W(\Omega; L^2([0,T];V))$.

We are going to show the following estimates

$$\mathbb{E}\int_0^T |f_{\eta,\varepsilon}(X_\varepsilon(t,x))|_H^2 \,\mathrm{d}t \le C, \tag{3.2}$$

$$\mathbb{E}\sup_{t\in[0,T]} |X_\varepsilon(t,x) - X_\lambda(t,x)|_H^2 \le C(\lambda+\varepsilon), \tag{3.3}$$

where we use the same symbol C to denote several positive constants independent of ε. Using the above results, we conclude that $\{X_\varepsilon\}_\varepsilon$ is a Cauchy sequence on

$$L^2(\Omega; C([0,T];H)) \cap L^2(\Omega; L^2([0,T];V))$$

and, consequently, it converges uniformly on $[0,T]$ to a process $X(t,x)$.

Step 1. We begin with the continuous dependence on the initial condition. Let us consider the difference $X_\varepsilon(t,x) - X_\varepsilon(t,\bar{x})$, for $x,\bar{x}\in H$.

Note that

$$\begin{aligned} &\mathrm{d}X_\varepsilon(t,x) - \mathrm{d}X_\varepsilon(t,\bar x) \\ &\qquad = A_\eta\left[X_\varepsilon(t,x) - X_\varepsilon(t,\bar x)\right]\,\mathrm{d}t + \left[F_{\eta,\varepsilon}(X_\varepsilon(t,x)) - F_{\eta,\varepsilon}(X_\varepsilon(t,\bar x))\right]\,\mathrm{d}t \end{aligned}$$

hence

$$\begin{aligned} &|X_\varepsilon(t,x) - X_\varepsilon(t,\bar x)|_H^2 \\ &\qquad = |x-\bar x|^2 + 2\int_0^t \langle A_\eta(X_\varepsilon(s,x) - X_\varepsilon(s,\bar x)), X_\varepsilon(s,x) - X_\varepsilon(s,\bar x)\rangle\,\mathrm{d}s \\ &\qquad\quad + 2\int_0^t \langle F_{\eta,\varepsilon}(X_\varepsilon(s,x)) - F_{\eta,\varepsilon}(X_\varepsilon(s,\bar x)), X_\varepsilon(s,x) - X_\varepsilon(s,\bar x)\rangle\,\mathrm{d}s \end{aligned}$$

and therefore

$$\mathbb{E}\,|X_\varepsilon(t,x) - X_\varepsilon(t,\bar x)|_H^2 \le \mathbb{E}\,|x-\bar x|_H^2 - 2\omega\int_0^t \mathbb{E}\,|X_\varepsilon(s,x) - X_\varepsilon(s,\bar x)|_H^2\,\mathrm{d}s.$$

Applying Gronwall's lemma we obtain

$$\mathbb{E}\,|X_\varepsilon(t,x) - X_\varepsilon(t,\bar x)|_H^2 \le e^{-2\omega t}\,|x-\bar x|_H^2\,. \tag{3.4}$$

The continuity condition (3.1) easily implies uniqueness of the mild solution on $D(F)$ and of the generalized solution on H. Consequently, it only remains to prove existence.

Step 2. Next, let us consider estimate (3.2). We shall apply Ito's formula to the function

$$\phi(x) = \int_0^1 g_\varepsilon(u(\xi))\,\mathrm{d}\xi, \qquad x = (u(\xi), v(\xi)) \in H,$$

where

$$g_\varepsilon(r) = -\int_0^r f_{\eta,\varepsilon}(s)\,\mathrm{d}s, \qquad r \in \mathbb{R}^+,\ \varepsilon > 0.$$

It is not difficult to show that, for any $x \in D(F)$,

$$D\phi(x) = \begin{pmatrix} -f_{\eta,\varepsilon}(u) \\ 0 \end{pmatrix} \qquad \text{and} \qquad D^2\phi(x) = \begin{pmatrix} -f'_{\eta,\varepsilon}(u) & 0 \\ 0 & 0 \end{pmatrix},$$

thus

$$\begin{aligned} &\langle A_\eta X_\varepsilon + F_{\eta,\varepsilon}(X_\varepsilon), D\phi(X_\varepsilon)\rangle = -\gamma\,\langle \partial_\xi(c(\cdot)\partial_\xi U_\varepsilon), f_{\eta,\varepsilon}(U_\varepsilon)\rangle \\ &\qquad + \gamma\,\langle (p(\xi)-\eta)U_\varepsilon, f_{\eta,\varepsilon}(U_\varepsilon)\rangle + \gamma\,\langle V_\varepsilon, f_{\eta,\varepsilon}(U_\varepsilon)\rangle - \gamma\,|f_{\eta,\varepsilon}(U_\varepsilon)|^2\,. \end{aligned}$$

We claim that

$$\begin{aligned} f'_{\eta,\varepsilon}(u) = -\varepsilon\,&\frac{(-1+2\,(u-\xi_0))\left(-(u-\xi_0)^3 - {\xi_0}^3\right)}{\left(1+\varepsilon\left(1-u+(u-\xi_0)^2+\xi_0\right)\right)^2} \\ &- \frac{3\,(u-\xi_0)^2}{1+\varepsilon\left(1-u+(u-\xi_0)^2+\xi_0\right)} \end{aligned} \tag{3.5}$$

is always negative; then it follows that

$$-\int_0^1 (\partial_\xi c(\xi)\partial_\xi u) f_{\eta,\varepsilon}(u)\,\mathrm{d}\xi = -c(\xi)\partial_\xi u(\xi)\Big|_0^1 - \int_0^1 c(\xi)(\partial_\xi u(\xi))^2 f'_\varepsilon(u(\xi))\,\mathrm{d}\xi \le 0$$

and, for any $\sigma > 0$

$$\langle v, f_{\eta,\varepsilon}(u)\rangle \le \sigma\,|v|^2 + \frac{1}{\sigma}\,|f_{\eta,\varepsilon}(u)|^2 .$$

From the above inequalities it follows that for σ sufficiently large

$$\begin{aligned}
&\langle A_\eta X_\varepsilon + F_{\eta,\varepsilon}(X_\varepsilon), D\phi(X_\varepsilon)\rangle \\
&\qquad \le \gamma\sigma\left(\eta + \|p\|^2_{L^\infty([0,1])}\right)|U_\varepsilon|^2 + \gamma\sigma\,|V_\varepsilon|^2 + \gamma\left(\frac{2}{\sigma} - 1\right)|f_{\eta,\varepsilon}(U_\varepsilon)|^2 \qquad (3.6)\\
&\qquad \le -C\,|f_{\eta,\varepsilon}(U_\varepsilon)|^2 + K\,|X_\varepsilon|^2
\end{aligned}$$

for suitable constants C, K. Further,

$$\mathrm{Tr}(QD^2\phi(X_\varepsilon)) = -\sum_{k=1}^\infty \left\langle Q_1 f'_{\eta,\varepsilon}(U_\varepsilon)e_k, e_k\right\rangle = -\sum_{k=1}^\infty \lambda_k \int_0^1 f'_{\eta,\varepsilon}(U_\varepsilon(\xi))e_k^2(\xi)\,\mathrm{d}\xi.$$

Now we observe that

$$\frac{\left|-3(u-\xi_0)^2 - \varepsilon(-\xi_0 + 2(u-\xi_0))(-(u-\xi_0)^3 - \xi_0^3)\right|}{1+\varepsilon - \xi_0\varepsilon(u-\xi_0) + \varepsilon(u-\xi_0)^2} \le 4\left(|u-\xi_0|^2 + \varepsilon\right),$$

so that for ε sufficiently small, taking into account (3.5) and the uniform bound condition on the e_k stated in assumption (1.3), we have

$$\begin{aligned}
\left|f'_\varepsilon(u(\xi))e_k^2\right| &\le 4\left(|u(\xi)-\xi_0|^2 + \varepsilon\right)\frac{\left|e_k^2(\xi)\right|}{1+\varepsilon - \xi_0\varepsilon(u(\xi)-\xi_0) + \varepsilon(u(\xi)-\xi_0)^2} \\
&\le C\left(|u(\xi)-\xi_0|^2 + \varepsilon\right) \le C\left(|u(\xi)|^2 + 1\right),
\end{aligned}$$

therefore

$$\begin{aligned}
\mathbb{E}\int_0^t \left|\mathrm{Tr}\left[QD^2\phi(X_\varepsilon(s))\right]\right|\,\mathrm{d}s &\le \mathbb{E}\int_0^t \mathrm{d}s\left(\int_0^1 \sum_{k=1}^\infty \lambda_k\left|f'_{\eta,\varepsilon}(U_\varepsilon(s)(\xi))e_k^2(\xi)\right|\,\mathrm{d}\xi\right) \\
&\le C\left(1 + \mathbb{E}\int_0^t |X_\varepsilon(s)|_H^2\,\mathrm{d}s\right). \qquad (3.7)
\end{aligned}$$

Estimates (3.6) and (3.7) yield

$$\begin{aligned}
&\mathbb{E}\phi(X_\varepsilon(t,x)) + \mathbb{E}\int_0^t |f_{\eta,\varepsilon}(X_\varepsilon(s,x))|^2\,\mathrm{d}s \\
&\qquad\qquad \le \phi(x) + C\left(1 + \mathbb{E}\int_0^t |X_\varepsilon(s,x)|^2\,\mathrm{d}s\right) \le C,
\end{aligned}$$

and therefore

$$\mathbb{E}\int_0^T |f_{\eta,\varepsilon}(X_\varepsilon(t,x))|^2 \,\mathrm{d}t \le C,$$

so that inequality (3.2) is proved.

Step 3. We proceed to estimate (3.3). We observe that

$$\mathrm{d}(X_\lambda(t,x)-X_\varepsilon(t,x)) = [A_\eta(X_\lambda(t,x)-X_\varepsilon(t,x)) + F_{\eta,\lambda}(X_\lambda(t,x)) - F_{\eta,\varepsilon}(X_\varepsilon(s,x))]\,\mathrm{d}t.$$

Hence, using Ito's formula as before we get

$$\begin{aligned}
&\mathbb{E}\sup_{[0,T]} |X_\lambda(t,x)-X_\varepsilon(t,x)|^2 \\
&\qquad =\mathbb{E}\int_0^T \langle A_\eta(X_\lambda(s,x)-X_\varepsilon(s,x)), X_\lambda(s,x)-X_\varepsilon(s,x)\rangle\,\mathrm{d}s \\
&\qquad\quad + \mathbb{E}\int_0^T \langle F_{\eta,\lambda}(X_\lambda(s,x) - F_{\eta,\varepsilon}(X_\varepsilon(s,x)), X_\lambda(s,x)-X_\varepsilon(s,x)\rangle\,\mathrm{d}s \\
&\qquad \le -\omega_2\mathbb{E}\int_0^T \|X_\lambda(s,x)-X_\varepsilon(s,x)\|^2\,\mathrm{d}s \\
&\qquad\quad + \mathbb{E}\int_0^T \langle f_{\eta,\lambda}(U_\lambda(s,x)) - f_{\eta,\varepsilon}(U_\varepsilon(s,x)), U_\lambda(s,x)-U_\varepsilon(s,x)\rangle\,\mathrm{d}s.
\end{aligned}$$

Now set

$$\begin{aligned}
h_\varepsilon(u) &= \frac{f_{\eta,\varepsilon}(u)}{1+\varepsilon-\varepsilon\xi_0(u-\xi_0)+\varepsilon(u-\xi_0)^2} + \frac{\xi_0^3}{1+\varepsilon(1-\xi_0(u-\xi_0)+(u-\xi_0)^2)} \\
&= \frac{-(u-\xi_0)^3}{1+\varepsilon(1-\xi_0(u-\xi_0)+(u-\xi_0)^2)}.
\end{aligned}$$

We note that, for any u, v,

$$(h_\lambda(u)-h_\varepsilon(v))(u-v) \le (h_\lambda(u)-h_\varepsilon(v))((u+h_\lambda^{1/3}(u)) - (v+h_\varepsilon^{1/3}(v))). \tag{3.8}$$

In fact,

$$\begin{aligned}
&(h_\lambda(u)-h_\varepsilon(v))(u-v) - (h_\lambda(u)-h_\varepsilon(v))((u+h_\lambda^{1/3}(u)) - (v+h_\varepsilon^{1/3}(v))) \\
&\qquad\qquad = -(h_\lambda^{1/3}(u)-h_\varepsilon^{1/3}(v))^2(h_\lambda^{2/3}(u)+h_\varepsilon^{1/3}(u)h_\lambda^{1/3}(u)+h_\lambda^{2/3}(u)) \le 0.
\end{aligned}$$

Moreover one can compute

$$\left|u-\xi_0+h_\varepsilon^{1/3}(u)\right| \le \varepsilon\,|h_\varepsilon(u)|\,,$$

therefore

$$\begin{aligned}
(h_\varepsilon(u)-h_\lambda(v))(u-v) &\le (|h_\varepsilon(u)|+|h_\lambda(v)|)\,(\varepsilon\,|h_\varepsilon(u)|+\lambda\,|h_\lambda(v)|) \\
&\le C(\varepsilon+\lambda)(|h_\varepsilon(u)|^2+|h_\lambda(v)|^2).
\end{aligned} \tag{3.9}$$

Furthermore, we observe that

$$\begin{aligned}
&\xi_0^3\left(-\frac{1}{1+\varepsilon-\varepsilon\xi_0(u-\xi_0)+\varepsilon(u-\xi_0)^2}+\frac{1}{1+\varepsilon-\varepsilon\xi_0(v-\xi_0)+\varepsilon(v-\xi_0)^2}\right)(u-v)\\
&=\xi_0^3\frac{-\varepsilon(u-\xi_0)^2+\lambda(v-\xi_0)^2}{(1+\varepsilon-\varepsilon\xi_0(u-\xi_0)+\varepsilon(u-\xi_0)^2)(1+\varepsilon-\varepsilon\xi_0(v-\xi_0)+\varepsilon(v-\xi_0)^2)}(u-v)\\
&\le\xi_0^3(\varepsilon+\lambda)\left[|u-v|+|u-\xi_0|^2+|v-\xi_0|^2+|u-\xi_0|^3+|v-\xi_0|^3\right] \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10) we get

$$\begin{aligned}
&(f_{\eta,\varepsilon}(u)-f_{\eta,\lambda}(v))(u-v)\le\\
&C(\varepsilon+\lambda)\left(|h_\varepsilon(u)|^2+|h_\lambda(v)|^2+|u-v|+|u-\xi_0|^2+|v-\xi_0|^2+|u-\xi_0|^3+|v-\xi_0|^3\right)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\mathbb{E}\sup_{[0,T]}|X_\lambda(t,x)-X_\varepsilon(t,x)|^2\\
&\qquad\le\mathbb{E}\int_0^T\int_0^1(f_{\eta,\lambda}(X_\lambda)-f_{\eta,\varepsilon}(X_\varepsilon))(X_\lambda-X_\lambda)\,\mathrm{d}\xi\,\mathrm{d}t\le C(\varepsilon+\lambda)
\end{aligned}$$

We conclude that there exists the limit $X=\lim_{\varepsilon\to0}X_\varepsilon$ in $L^2(\Omega;C([0,T];H))$ and, by (2.13), also that $X\in L^2(\Omega;L^2([0,T];V))$. Moreover, estimate (3.4) implies inequality (3.1). □

We conclude the section with another estimate which turns out to be useful when we will deal with the asymptotic behaviour of the solution.

Lemma 3.2. *The following estimate holds*

$$\mathbb{E}\,|X(t,x)|^{2m}\le C_m\left(1+e^{-m\omega_1 t}\,|x|^{2m}\right),\quad x\in H,t\ge0.$$

Proof. Let $Y(t)=X(t,x)-W_{A_\eta}(t)$. Then

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Y(t)=A_\eta Y(t)+F_\eta(Y(t)+W_{A_\eta}(t)),\qquad Y(0)=x.$$

Observe that

$$\begin{aligned}
\tfrac{1}{2m}\tfrac{\mathrm{d}}{\mathrm{d}t}|Y(t)|^{2m}&=|Y(t)|^{2m-2}\tfrac{\mathrm{d}}{\mathrm{d}t}|Y(t)|^2\\
&\le-\omega_1|Y(t)|^{2m}+\left\langle F_\eta(Y(t)+W_{A_\eta}(t)),Y(t)\right\rangle|Y(t)|^{2m-2}\\
&\le-\omega_1|Y(t)|^{2m}+\left\langle F_\eta(W_{A_\eta}(t)),Y(t)\right\rangle|Y(t)|^{2m-2}\\
&\le-\omega_1+\left|F_\eta(W_{A_\eta}(t))\right||Y(t)|^{2m-1}
\end{aligned}$$

Hence we conclude that

$$\tfrac{1}{2m}\tfrac{\mathrm{d}}{\mathrm{d}t}|Y(t)|^{2m}\le-\omega_1|Y(t)|^{2m}+C\left|F_\eta(W_{A_\eta}(t))\right|^{2m}.$$

for some $C > 0$. By Gronwall's lemma it follows that

$$|Y(t)|^{2m} \leq e^{-m\omega_1 t} |x|^{2m} + 2mC \int_0^t e^{-m\omega_1(t-s)} \left|F_\eta(W_{A_\eta}(s))\right|^{2m} \,\mathrm{d}s,$$

so that for some $C > 0$ (possibly different from the above):

$$\begin{aligned} &|X(t,x)|^{2m} \\ &\leq C \left(e^{-m\omega_1 t} |x|^{2m} + \int_0^t e^{-m\omega_1(t-s)} \left|F_\eta(W_{A_\eta}(s))\right|^{2m} \,\mathrm{d}s + \left|W_{A_\eta}(t)\right|^{2m} \right). \end{aligned} \tag{3.11}$$

Now recall that F_η has polynomial growth (see (2.6)); in particular we have that

$$\left|F_\eta(W_{A_\eta}(t))\right|^{2m} \leq C \left(1 + \left|W_{A_\eta}(t)\right|^3\right)^{2m} \leq C(1 + \left|W_{A_\eta}(t)\right|^{6m}).$$

Moreover, by (2.11), $\sup_{t \geq 0} \mathbb{E} \left|W_{A_\eta}(t)\right|^{2m} < C_m$, then

$$\begin{aligned} \int_0^t e^{-m\omega_1(t-s)} \left|F_\eta(W_{A_\eta}(t))\right|^{2m} \,\mathrm{d}s &\leq C \int_0^t e^{-m\omega_1(t-s)} \left(1 + \left|W_{A_\eta}(s)\right|^{6m}\right) \mathrm{d}s \\ &\leq C \int_0^t e^{-m\omega_1(t-s)} \left(1 + C_m^3\right) \mathrm{d}s \leq C_m'. \end{aligned}$$

Using the last estimate in (3.11) we conclude the proof.

□

4 Asymptotic behaviour of solutions

Let $P_t : C_b(H) \to C_b(H)$ be the transition semigroup associated to the flow $X(t, \cdot)$ defined in equation (1.6), that is

$$P_t\phi(x) = \mathbb{E}\phi(X(t,x)), \quad \phi \in C_b(H), t \geq 0, x \in H. \tag{4.1}$$

We are ready to prove the main result of the paper.

Theorem 4.1. *Under hypothesis 1.1 there exists a unique invariant measure μ for P_t.*

Proof. To discuss the existence of the invariant measure, it will be convenient to consider equation (1.6) on the whole real line. Therefore we extend the process $W(t)$ for $t < 0$ by choosing a process $\tilde{W}(t)$ with the same law as $W(t)$ but independent of it and setting

$$W(t) = \tilde{W}(-t), \quad t \leq 0.$$

Now, for any $\lambda > 0$, denote by $X_\lambda(t,x)$, $t \geq -\lambda$, the unique solution of

$$\begin{aligned} &\mathrm{d}X = [A_\eta X + F_\eta(X)]\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W(t) \\ &X(-\lambda) = x \in H. \end{aligned}$$

Then X_λ satisfies the following integral equation:

$$X_\lambda(t,x) = x + \int_{-\lambda}^{t} A_\eta X_\lambda(s,x) + F(X_\lambda(s,x))\,\mathrm{d}s + \int_{-\lambda}^{t} \sqrt{Q}\,\mathrm{d}W(s).$$

We note that

$$\begin{aligned} X(\lambda,x) &= x + \int_0^{\lambda} A_\eta X_\lambda(s,x) + F(X_\lambda(s,x))\,\mathrm{d}s + \int_0^{\lambda} \sqrt{Q}\,\mathrm{d}W(s) \\ &= x + \int_{-\lambda}^{0} A_\eta X_\lambda(s,x) + F(X_\lambda(s,x))\,\mathrm{d}s + \int_{-\lambda}^{0} \sqrt{Q}\,\mathrm{d}W(s) \\ &= X_\lambda(0,x). \end{aligned}$$

Thus, the theorem will be proved once we establish that

$$\lim_{\lambda\to\infty} \mathcal{L}(X_\lambda(0,x)) = \mu$$

weakly, for some $\mu \in M_1^+(H)$ and all $x \in H$. As in [2, Theorem 11.21] we will not prove only this, but we will show that there exists a random variable $Y \in L^2(\Omega; \mathcal{F}, \mathbb{P})$ such that

$$\lim_{\lambda\to\infty} \mathbb{E}\,|X_\lambda(t,x) - Y|^2 = 0, \quad x \in H, \tag{4.2}$$

and the law of Y is the required stationary distribution.

We first prove that (4.2) is true when $x = 0$. We put $X_\lambda(t,0) = X_\lambda(t)$. Proceeding as in Theorem 3.1 we obtain

$$\mathbb{E}\,|X_\lambda(t)|^2 \le -2\omega\mathbb{E}\int_{-\lambda}^{t} |X_\lambda(s)|_H^2\,\mathrm{d}s + 2\mathrm{Tr}[Q]t$$

Using Gronwall's lemma, we have

$$\mathbb{E}\,|X_\lambda(t)|^2 \le (2\mathrm{Tr}[Q](t+\lambda) + |x|^2)e^{-2\omega(t+\lambda)} \le C, \quad \forall\lambda > 0, \forall t \in [-\lambda,\infty]. \tag{4.3}$$

We can now prove (4.2). Let $\gamma < \lambda$; then

$$X_\lambda(t,0) = X_\gamma(t, X_\lambda(-\gamma,0)), \quad t \ge -\gamma$$

and, proceeding as in Theorem 3.1 we obtain an estimate similar to that in (3.4)

$$\begin{aligned} \mathbb{E}\,|X_\lambda(t,0) - X_\gamma(t,0)|^2 &= \mathbb{E}\,|X_\gamma(t, X_\lambda(-\gamma,0)) - X_\gamma(t,0)|^2 \\ &\le e^{-\omega_1(t+\gamma)}\mathbb{E}\,|X_\lambda(-\gamma)|_H^2 \le Ce^{-\omega_1(t+\gamma)}. \end{aligned} \tag{4.4}$$

Estimates (4.3) and (4.4) imply that $\{X_\lambda(0)\}_{\lambda\ge 0}$ is a bounded Cauchy sequence in $L_W^2(\Omega; H)$. Then there exists a random variable Y such that $\mathbb{E}\,|X_\lambda(0) - Y|_H^2 \to 0$, as $\lambda \to \infty$. Proceeding similarly we show that

$$\lim_{\lambda\to\infty} \mathbb{E}\,|X_\lambda(0,x) - X_\lambda(0)|^2 = 0, \quad \forall x \in H.$$

This ends the proof. □

Lemma 4.2. *For any $\phi \in C_b(H)$ and $x \in H$ there exists the limit*

$$\lim_{t\to\infty} P_t\phi = \int_H \phi(y)\mu(\mathrm{d}y).$$

Proof. Set $Y = \lim_{s\to-\infty} X(0,-s,x) \in L^2(\Omega; H)$, which exists in virtue of Theorem 4.1. Then

$$P_t\phi(x) = \mathbb{E}\left[\phi(X(t,0,x))\right] = \mathbb{E}\left[\phi(X(0,-t,x))\right].$$

By the dominated convergence theorem it follows that

$$\lim_{t\to\infty} P_t\phi(x) = \mathbb{E}\left[\phi(Y)\right] = \int_H \phi(y)\mu(\mathrm{d}y).$$

□

5 The infinitesimal generator of P_t

Lemma 5.1. *For any $p \geq 1$ P_t has a unique extension to a strongly continuous semigroup of contraction in $L^p(H,\mu)$ which we still denote by P_t.*

Proof. Let $\phi \in C_b(H)$ and μ_t be the law of $X(t,x)$. By Holder inequality we have that

$$|P_t\phi(x)|^p \leq P_t\, |\phi(x)|^p\,.$$

Integrating this identity with respect to μ over H and taking into account the invariance of μ, we obtain

$$\int_H |P_t\phi(x)|^p\, \mu(\mathrm{d}x) \leq \int_H P_t\, |\phi|^p\,(x)\mu(\mathrm{d}x) = \int_H |\phi(x)|^p\, \mu(\mathrm{d}x).$$

Since $C_b(H)$ is dense in $L^p(H,\mu)$, P_t can be uniquely extended to a contraction semigroup in $L^p(H,\mu)$. The strong continuity of P_t follows from the dominated convergence theorem. □

Taking into account the Hille-Yosida's theorem, from the previous lemma we deduce that the infinitesimal generator of P_t on $L^p(H,\mu)$ (which we denote by N) is closed, densely defined and it satisfies

$$|\lambda R(\lambda, A)| \leq 1.$$

We want to show that N is the closure of the differential operator N_0 defined by

$$N_0\phi = \frac{1}{2}\mathrm{Tr}[QD^2\phi(x)] + \langle x, AD\phi(x)\rangle + \langle F(x), D\phi(x)\rangle$$

on $\mathcal{E}_A(H) = $ linear span $\left\{\phi = e^{\langle x,h\rangle} \mid h \in D(A)\right\}$.

We recall that the operator

$$L\phi = \frac{1}{2}\mathrm{Tr}[QD^2\phi(x)] + \langle x, AD\phi(x)\rangle$$

is the Ornstein-Uhlenbeck operator and it verifies

$$|L\phi(x)| \le a + b\,|x|\,, \quad x \in H. \tag{5.1}$$

(see [1, Section 2.6]).

Then, in order to show that N_0 is well-defined as an operator with values in $L^p(H,\mu)$ we need that $F(x) \in L^p(H,\mu)$. This is provided by the following result.

Lemma 5.2. *Under Hypothesis 1.1, there exists c_m depending only on A_η and F_η such that*

$$\int_H |x|_H^{2m}\ \mu(\mathrm{d}x) \le c_m. \tag{5.2}$$

Proof. Denote by $\mu_{t,x}$ the law of $X(t,x)$. Then by lemma 3.2 we have that for any $\beta > 0$

$$\begin{aligned}\int_H \frac{|y|^{2m}}{1+\beta\,|y|^{2m}}\mu_{t,x}(\mathrm{d}y) &\le \int_H |y|^{2m}\ \mu_{t,x}(\mathrm{d}y)\\ &= \mathbb{E}\,|X(t,x)|^{2m} \le C_m(1+e^{-m\omega_1 t}\,|x|^{2m}), \quad x \in H.\end{aligned}$$

Consequently, letting $t \to \infty$ we find, taking into account Lemma 4.2,

$$\begin{aligned}\int_H \frac{|y|^{2m}}{1+\beta\,|y|^{2m}}\mu(dy) = \lim_{t\to\infty} P_t\phi(x) &= \lim_{t\to\infty}\int_H \frac{|y|^{2m}}{1+\beta\,|y|^{2m}}\mu_{t,x}(\mathrm{d}y)\\ &\le \lim_{t\to\infty} C_m(1+e^{-m\omega_1 t}\,|x|^{2m}),\end{aligned}$$

which yields (5.2). □

Applying formula (2.6) we immediately obtain the following

Corollary 5.3. *We have*

$$\int_H |F_\eta(x)|^{2m}\ \mu(\mathrm{d}x) < \infty. \tag{5.3}$$

The corollary implies that $N_0\phi \in L^p(H,\mu)$ for all $\phi \in \mathcal{E}_A(H)$ as required. We can now show that $N_0\phi = N\phi$ for all $\phi \in \mathcal{E}_A(H)$.

Lemma 5.4. *For any $\phi \in \mathcal{E}_A(H)$ we have*

$$\mathbb{E}\left[\phi(X(t,x))\right] = \phi(x) + \mathbb{E}\left[\int_0^t N_0\phi(X(s,x))\,\mathrm{d}s\right], \quad t \ge 0,\ x \in H. \tag{5.4}$$

Moreover $\phi \in D(N)$ and $N_0\phi = N\phi$.

Proof. Equality (5.4) follows easily by applying Itô's formula. It remains to prove that $\mathcal{E}_A(H) \subset D(N)$ and $N_0\phi = N\phi$. Since it holds that

$$\lim_{h\to\infty} \frac{1}{h}\left(P_t\phi(x) - \phi(x)\right) = N_0\phi(x)$$

pointwise, it is enough to show that

$$\frac{1}{h}(P_h\phi - \phi), \quad h \in (0,1],$$

is equibounded in $L^p(H,\mu)$.

We note that, in view of (5.1) and (5.4), for any $x \in H$ we have

$$|P_h\phi(x) - \phi(x)| \le \int_0^h \mathbb{E}\left[a + b\,|X(s,x)| + |\phi|_0\,|F(X(s,x))|\right] \mathrm{d}s.$$

By Hölder's inequality we find that

$$\begin{aligned} |P_h\phi(x) - \phi(x)|^p &\le h^{p-1}\int_0^h \mathbb{E}\left[a + b\,|X(s,x)| + |\phi|_0\,|F(X(s,x))|\right]^p \mathrm{d}s \\ &\le c_p h^{p-1}\int_0^h \mathbb{E}\left[a + b\,|X(s,x)|\right]^p \mathrm{d}s + c_p h^p\,|\phi|_0^p \int_0^h \mathbb{E}\,|F(X(s,x))|^p \mathrm{d}s \\ &= c_p h^{p-1}\int_0^h P_s\left(a + b\,|\cdot|\right)^p \mathrm{d}s + c_p h^{p-1}\,|\phi|_0^p\int_0^h P_s\,|F(\cdot)|^p \mathrm{d}s. \end{aligned}$$

Integrating with respect to μ over H and taking into account the invariance of μ, the above formula yields

$$|P_h\phi - \phi|^p_{L^p(H,\mu)} \le h^p \int_H \left[(a + b\,|x|^p) + |\phi|^p\,|F(x)|^p\right]\mu(\mathrm{d}x) < \infty,$$

thanks to Corollary 5.3. Consequently $1/h(P_h\phi - \phi)$ is equibounded in $L^p(H,\mu)$ as claimed. □

Theorem 5.5. *Assume that Hypothesis 1.1 holds. Then N is the closure of N_0 in $L^p(H,\mu)$.*

Proof. By Lemma 5.4, N extends N_0. Since N is dissipative (it is the infinitesimal generator of a C_0 contraction semigroup), so it is N_0. Consequently N_0 is closable. Let us denote by $\bar{N}_0$ its closure. We have to show that $\bar{N}_0 = N$.

Let $\lambda > 0$ and $f \in \mathcal{E}_{A_\eta}(H)$. Consider the approximating equation

$$\lambda\phi_\varepsilon - L\phi_\varepsilon - \langle F_{\eta,\varepsilon}, D\phi_\varepsilon\rangle = f, \quad \varepsilon > 0 \tag{5.5}$$

By [1, Theorem 3.21] we have that Equation (5.5) has a unique solution $\phi_\varepsilon \in C_b^1(H)$ given by

$$\phi_\varepsilon(x) = \mathbb{E}\int_0^1 e^{-\lambda t} f(X_\varepsilon(t,x))\,\mathrm{d}t, \quad \forall x \in H.$$

Moreover, for all $h \in H$ we have

$$\langle D\phi_\varepsilon(x), h\rangle = \int_0^\infty e^{-\lambda t}\mathbb{E}\left[\langle Df(X_\varepsilon(t,x)), DX_\varepsilon(t,x)[h]\rangle\right] \mathrm{d}t.$$

and by [4, Proposition 11.2.13] we have

$$\|DX_\varepsilon(t,x)\|_{\mathcal{L}(H)} \le e^{\omega t};$$

consequently we obtain

$$|D\phi_\varepsilon(x)|_H \le \frac{1}{\lambda-\omega}\,\|Df\|_0 .$$

Arguing as in [4, Theorem 11.2.14] we can write (5.5) as

$$\lambda\phi_\varepsilon - \bar{N}_0\phi_\varepsilon = f + \langle F_{\eta,\varepsilon} - F, D\phi_\varepsilon\rangle .$$

We claim that

$$\lim_{\varepsilon\to 0}\langle F_{\eta,\varepsilon} - F, D\phi_\varepsilon\rangle = 0 \quad \text{in } L^p(H,\mu).$$

In fact, we have

$$\int_H |\langle F_{\eta,\varepsilon}(x) - F(x), D\phi_\varepsilon\rangle|^p\ \mu(\mathrm{d}x) \le \frac{1}{\lambda-\omega}\,\|Df\|_0^p \int_H |F_{\eta,\varepsilon}(x) - F(x)|^p\ \mu(\mathrm{d}x).$$

Clearly,

$$\lim_{\varepsilon\to 0}|F_{\eta,\varepsilon}(x) - F(x)|^p = 0, \quad \mu - a.e.$$

Moreover

$$|F_{\eta,\varepsilon}(x) - F(x)|^p \le 2\,|F(x)|^p\,, \quad x\in H.$$

Therefore, the claim follows from the dominated convergence theorem, since

$$\int_H |F(x)|^p\ \mu(\mathrm{d}x) < \infty$$

in virtue of Corollary 5.3. In conclusion we have proved that the closure of the range of $\lambda - \bar{N}_0$ includes $\mathcal{E}_A(H)$ which is dense in in $L^p(H,\mu)$. Now the theorem follows from Lumer-Phillips theorem. □

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