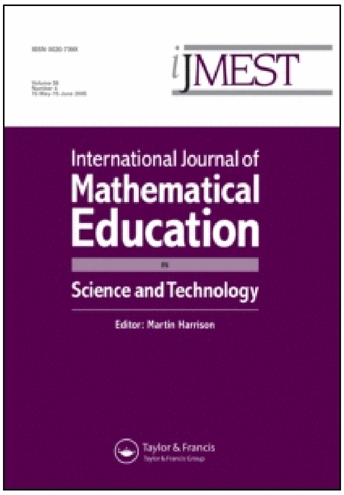

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Student interpretations of the terms in first-order ordinary differential equations in modelling contexts

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Student interpretations of the terms in first-order ordinary differential equations in modelling contexts

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A study of first-year undergraduate students' interpretational difficulties with first-order ordinary differential equations (ODEs) in modelling contexts was conducted using a diagnostic quiz, exam questions and follow-up interviews. These investigations indicate that when thinking about such ODEs, many students muddle thinking about the function that gives the quantity to be determined and the equation for the quantity's rate of change, and at least some seem unaware of the need for unit consistency in the terms of an ODE. It appears that shifting from amount-type thinking to rates-of-change-type thinking is difficult for many students. Suggestions for pedagogical change based on our results are made.

1. Introduction

Reform in the teaching of differential equations at the tertiary level has been driven in part by the same sorts of disappointing observations on student learning outcomes that have driven calculus reform [1, 2]. In addition, the ready availability of programmable graphics calculators and symbolic algebra packages such as *Mathematica* and *Maple* have raised the question of what it is important for students to be able to do themselves and what can be left for the technology to calculate. Furthermore, these packages have opened up opportunities for more sophisticated analyses and investigations of ordinary differential equations (ODEs) than was previously possible. Consequently, as Boyce [2] describes, in reform efforts there has been a move away from 'mere manipulative skills' teaching, to a greater emphasis on conceptual understanding, exploration and higher-level problem solving. Some of this reform work has been documented in a special issue of *The College Mathematics Journal* (Issue 5, 1994) and examples of the uses of graphics calculators and *Mathematica* are given in [3] and [4] respectively.

Aside from courses in differential equations, some authors as part of their reform calculus efforts have also introduced, early on in the curriculum, differential

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equations via physical models [3, 5]. The reasons given for doing so are partly motivational (the physical models illustrate the applicability and usefulness of the mathematics; the solution of differential equations provide a natural motivation for the learning of techniques of integration; also, introducing differential equations makes the subject fresh and therefore more interesting for students who have already studied calculus at the secondary level), though on the pedagogical side, it is believed that the physical applications will help deepen student understandings of the mathematics they are learning [3].

Despite these considerable reform efforts involving ODEs, not much research into students' understandings of ODEs appears to have been done, however. Rasmussen [6] has investigated student understandings of various aspects of solutions to ODEs, including graphical and numerical solutions. Two things of particular note for this research come from Rasmussen's work. First, Rasmussen posited that the switch from conceptualizing solutions as *numbers* (as is the case when solving algebraic equations) to conceptualizing solutions as *functions* (as is the case when solving ODEs) is akin to a paradigm shift and is non-trivial for students. Secondly, Rasmussen noted that some of the difficulties students had with graphical approaches stemmed from either thinking with an inappropriate quantity and/or losing focus of the intended underlying quantity. (This observation may be related to the height–slope confusion previously identified in the calculus literature [7, 8]).

Another piece of research in this field is by Habre [9], who explored students' strategies for solving ODEs in a reform setting. Of interest from this research is that despite an emphasis on qualitative (graphical) solution methods in the course, the majority of students interviewed still favoured algebraic approaches over graphical approaches at the end of the course, possibly reflecting the heavy algebraic focus of previous mathematical experiences. The research also suggested that students find it difficult to think in different modes (i.e. algebraic and graphical) simultaneously, which might also help explain why students typically don't use multiple modes to tackle problems. (As reported in [10], typically only 20% of engineering students use diagrams to aid their physics problem-solving in exams, and it has been found that even top students used graphs in only one quarter of their solution attempts in a test with non-routine calculus problems [11]).

An aspect of conceptual understanding not addressed by the above research is students' ability in modelling contexts to both interpret in physical terms the various terms of an ODE and to translate from a physical description into a mathematical description. These two abilities are the focus of the present research and are of course flip sides of the same coin. These abilities are important as they are needed if students are to reason appropriately about solutions and ultimately if they are to develop skills in modelling themselves.

Although these aspects of student understanding of ODEs do not seem to have been previously investigated, similar things have been investigated in the contexts of algebraic word problems and various aspects of calculus problems. Thus, for example, it has been found that in algebraic word problem translations, common problems were word order matching/syntactic translation and static comparison methods [12]. Similarly, student difficulties with correctly distinguishing between constants and variables, and between dependent and independent variables in rates of change contexts has also been identified [13–15]. In addition, research on student understanding of kinematics graphs [7] and velocity and acceleration [16, 17], reveals that many do not clearly distinguish between distance, velocity and acceleration. It is also known that prior to their development of the concept of speed as an ordered ratio, children typically progress through a stage where they think of speed as a distance (the distance travelled in a unit of time) [18].

The goal of the research reported in this paper, which is to investigate student difficulties with translating between words (physical descriptions) and mathematics, is motivated by the observation that many student 'errors' in science and mathematics are not just careless slips, but are in fact systematic and furthermore common to significant numbers of students across a wide range of contexts. Furthermore, these systematic 'errors' have been found to be resistant to change by traditional instruction, and there is a general consensus that teaching needs to be cognizant of these systematic 'errors' if they are to be effectively addressed [19–22].

Why precisely the above should be the case is still a matter of debate and research [19], although there seems to be general agreement that fundamentally it is because students are not blank slates for their teachers to 'write upon', but they come to class with ideas and conceptions which may both aid or hinder further learning and also because students do not unproblematically absorb new teachings, but what they learn (or fail to learn) is affected by both their beliefs about learning and by how they attempt to make sense of what they are taught. (This is the 'constructivist principle' [23, 24].)

Beyond this general agreement though, there are a range of different ways of conceiving of student 'errors' (and hence the scare quotes around 'errors', as sometime these are conceived of as primitive understandings that are to some extent 'correct', and need to be developed and refined more than 'corrected'), some of which seem to be competing conceptions, while others appear to be describing different phenomena. Confrey [19] and Hammer [25] provide reviews of the various conceptions of student 'errors' and a discussion of their implications for pedagogy are provided in these reviews, while the ongoing nature of research in this field is illustrated in [23, 26].

One conception that seems particularly relevant to the current study, however, is that of a 'paradigm shift' or 'knowledge in transition'. As posited by Rasmussen [6], moving from the context of algebraic equations where solutions are constants, to that of ODEs where solutions are functions, represents a paradigm shift for students. Consequently, as with the scientific paradigm shifts as discussed by Kuhn, it can be expected that some students will have difficulties moving from the old ways of thinking to the new. Related to this view is the phenomenon of 'binary reversion' [20] where older and more familiar knowledge is inappropriately triggered by the context because it is more readily cognitively accessible than the newer, less familiar but relevant knowledge. These ideas are relevant because we will argue later that the 'rates of change' contexts of first-order ODEs can trigger both thinking about the function that describes a quantity rather than the function that describes the quantity's rate of change and constant rate of change concepts, which are inappropriate in a variable rate of change context.

To help understand why the above-identified problems should be so common (and why standard instruction almost encourages things to go wrong), and to help predict the sorts of problems one might expect in any area of learning, we use Perkins' theory [27] of default modes of thinking as a cognitive framework. According to this theory, the pattern-driven nature of human cognition leads to four default modes of thinking which, while they serve us well most of the time, can cause problems in novel situations or familiar situations that have been subtly changed (i.e. the typical sort of situations any student faces). These default modes are (giving only the negative side of the mode):

Fuzzy thinking: exemplified by a failure to clearly discriminate between closely related terms [20, 21, 28–30]; and overgeneralizing or having deficient applicability conditions [20, 26, 29].

Hasty thinking: exemplified by too rapidly deciding on a solution strategy or solution on the basis of a superficial examination of the most obvious features of a problem, rather than on deep processing (i.e. trying to pattern match the problem to one seen before) [30–32]. This thus represents a weakness in problem-solving approach, that is, it is a metacognitive weakness.

Narrow thinking: related to hasty thinking, narrow thinking also represents a metacognitive weakness in that it is exemplified by a failure to consider alternative perspectives or solution strategies.

Sprawling thinking: may be useful when one is brainstorming, but is a problem when it leads one to lose track of what one is doing or 'to change horses midstream' [20]. This also represents a failure to develop effective problem-solving control and monitoring strategies.

2. Method

2.1. *Participants*

The participants were 59 first-year BSc students enrolled in a two-semester sequence of calculus and linear algebra. These students had all studied some calculus at secondary school, and had achieved reasonable results in these studies. During their semester of university study, the students had worked on a variety of physical systems which could be modelled by first-order ODEs, including unconstrained and logistic population growth, radioactive decay, the mixing of solutes in a tank, and Newton's law of cooling. Apart from solving the resultant ODE, during the course students were also expected to be able to interpret the physical meaning of the terms in an ODE given a description of the physical problem, and, given the description of a physical system covered in the course, to determine the governing ODE of that physical system.

2.2. *Procedure*

Probing student conceptual understanding is neither easy nor straightforward, and various methods for doing this each have their strengths and weaknesses as discussed below. Consequently, we used three different methods—a multiple choice diagnostic quiz, a short answer exam question, and one-on-one interviews—in order to triangulate our observations. We did not investigate students' ability to model a physical problem completely by themselves for two reasons. First, as part of the course aims, students were only expected to be able to model a problem that was a variation on one covered by the course. Thus their performance on such a question may simply reflect what they had memorized, not what they really understood. Second, it was envisaged that modelling a new problem would bring too many factors into play to allow easy interpretability of student answers. Consequently, all of the questions asked only focus on aspects of the whole problem of modelling.

2.2.1. Phase 1: Diagnostic quiz

In the first phase of the investigations, the participants were given a multiple choice diagnostic quiz in the last week of classes of their first semester of university study (48 out of the 59 were present at class on that day). The students were not given warning of the quiz, and so did not make any specific preparation for it (it was assumed that the quiz questions were either sufficiently basic and/or conceptual that this lack of preparation should not have been a problem for them). The timing of the quiz was such that the students had at that time spent some time studying first-order ODEs, methods for solving them, and various physical systems that can be modelled by these differential equations.

The diagnostic quiz included two questions chosen by the authors to be relatively simple models, similar to but not identical to models the students had seen either in class or on tutorial sheets (so that students could not simply 'remember' the correct answer), with the responses being chosen to represent hypothesized 'error types' (these hypotheses were based on a combination of teaching experience, results in the literature on related calculus questions, and an application of Perkins' [27] default thinking modes theory). (Normally, one would construct such a quiz after first doing some qualitative research on student thinking with a number of students. However, in this case, the choice of distracters was validated *post hoc* by the students' open responses to the exam question and by the one-on-one interviews.)

The two modelling questions on the quiz were:

Quiz Question 1. As light passes through a liquid or a solute it is absorbed (i.e. its intensity I, decreases) because it interacts with the molecules of the liquid or solute. For a solution of anthracene dissolved in dioxane, the absorption rate is proportional to its intensity I at that point (the proportionality constant is 0.0693 per mm). The intensity of light I, as a function of distance travelled x (in mm), into this solute is therefore described by the differential equation:

(a) $\dfrac{dI}{dx} = 0.0693$ (b) $\dfrac{dI}{dx} = -0.0693$ (c) $\dfrac{dI}{dx} = 0.0693I$ (d) $\dfrac{dI}{dx} = -0.0693I$

(e) $\dfrac{dI}{dx} = 0.0693x$ (f) $\dfrac{dI}{dx} = -0.0693x$ (g) $\dfrac{dI}{dx} = -0.0693Ix$ (h) $\dfrac{dI}{dx} = 0.0693Ix$

Quiz Question 2. The population P of fish in a pond at a fish farm as a function of time t will grow at a rate proportional to the population if left undisturbed and if there is plenty of food. In addition to this undisturbed growth rate, 5000 fish per year are also removed from the pond for sale. The differential equation which describes the growth of the fish population with time (in years) is given by:

(a) $dP/dt =$ term describing undisturbed growth $- 5000\,(dP/dt)$
(b) $dP/dt =$ term describing undisturbed growth $- 5000$
(c) $dP/dt =$ term describing undisturbed growth $- 5000t$
(d) $dP/dt =$ term describing undisturbed growth $- 5000P$
(e) $dP/dt =$ term describing undisturbed growth $- 5000Pt$

For question 1, the responses are in four pairs, with each pair differing only in sign to check for student awareness of sign issues. Responses (a) and (b) are of the simple form, $dI/dx =$ rate constant; (c) and (d) are of the form, $dI/dx =$ rate constant × dependent variable, a form familiar to students from unconstrained population growth, and with (d) being the correct answer; (e) and (f) are of the form, $\mathrm{d}I/\mathrm{d}x$ = rate constant × independent variable, a form which represents thinking in terms of 'amounts' rather than 'rates of change of amounts' in the familiar context of constant rate problems; and finally, (g) and (h) are as per (e) and (f) but with a factor of I included because the question states that rate of absorption is proportional to I. Apart from dropping the sign variation, the responses to question 2 were constructed along similar lines.

As can be seen, these two questions assessed students' ability to translate from a word problem to a mathematical equation. As translation is very difficult for most students and can involve a host of difficulties, to simplify the interpretation of the results it was decided to use the multiple choice format where students had to match the correct equation to the physical problem, a presumably much simpler task than developing the whole equation oneself.

The strengths of such quizzes, if the alternative responses are in fact distracters which match student thinking (see [33] for a discussion on the construction of useful multiple choice diagnostic questions), are that they indicate the prevalence of certain types of thinking [34] and if used as pre-tests and post-tests, can be used to assess how effective a course of instruction has been in helping students think in the accepted ways [35]. Their weakness, however, is that they don't necessarily reveal why students pick certain responses, and being forced response type questions, may force students to respond in different ways than they would if left to their own devices. Also, the results need to be interpreted with some caution, as it has been shown that students are not necessarily consistent in the way they answer questions which are presumably testing the same concepts [36, 37]. (This inconsistency presumably reflects the fact that the knowledge of many students is fragmented and that the way they think about problems can be context dependent [24, 29].) Despite these shortcomings however, we argue that they still provide useful information when correlated with other sources of information.

2.2.2. Phase 2: End-of-semester exam

The end-of-semester exam consisted of a variety of calculus questions, and included two questions on first-order ODEs. The part of the question which is relevant to this study is as follows:

Exam Question 3(c). A new type of drug is administered to a patient in a hospital by continuous intravenous drip. The differential equation describing the amount D (mg) of the drug in the patient at time t hours after it was first administered is $\mathrm{d}D/\mathrm{d}t = 100 - 0.01D^2$. Give the physical meaning of each of the three terms $\mathrm{d}D/\mathrm{d}t$, 100 and $-0.01D^2$ in the differential equation.

Thus, this question assesses students' ability to relate the terms in an ODE to processes in the physical model. Since this is a free response question, it overcomes some of the limitations of the forced response multiple choice questions in the diagnostic quiz. However, it too suffers from the limitation that because student responses were generally not justified, these responses only indirectly tell how students are thinking about the question. There is also the problem that some students may not choose their words very carefully and so what they write may not be precisely what they mean to say (this is another reason why multiple sources of information on student thinking are required). Nevertheless, in combination with other sources, this question can provide insights into student thinking. Furthermore, as argued in the Introduction in relation to 'fuzzy thinking', this lack of precision is undoubtedly one of the causes of errors in student thinking.

To aid the analysis of the results, the students' answers to this question were tabulated in order to ease comparisons both within and between students. After tabulation, each investigator read through the responses several times in order to get a feel for the sorts of categories answers could be sorted into. Each investigator then independently sorted responses into the agreed categories and compared results. There was high agreement between the investigators and the few disagreements were then discussed and a final categorization was decided upon. In addition to this, consistency of types of responses within this question and with the other questions was also determined.

2.2.3. Phase 3: Follow-up interviews

To check our interpretations of student responses to the first two phases of the study, during second semester, one-on-one follow-up interviews were conducted with eight students who had exhibited commonly occurring misconceptions on the diagnostic quiz. While the students targeted for these interviews were ones exhibiting common errors, participation was voluntary. The students interviewed scored from the 14th to the 71st percentile on the final exam, with five of the eight scoring near or above the class average. These facts give confidence that most if not all of the students interviewed had made a reasonably serious attempt at the course and that their (mis)conceptions are likely to be representative of the (mis)conceptions of the bulk of students who had problems with the questions investigated. The protocol for the interviews was as follows.

Each student was first reminded of their answer for questions 1 and 2 of the diagnostic quiz and was then asked to explain why they chose that answer. If their response seemed incomplete or unclear, they were prompted to elaborate (sometimes this involved asking the student to explain why they didn't choose some of the other answers).

Following this, each student was asked to more fully explain their answers to Exam Question 3(c). To flesh out their thinking more, among other things they were also asked what they thought the units of each term in the ODE were.

To investigate whether student misinterpretations of the constant '100' term in exam question 3(c) was due to unfamiliarity with the operation of a continuous intravenous drip (though this was discussed in the course) or whether it represented something more general, the following more familiar physical situation was presented to the interviewed students.

Interview Question. A car initially travelling at 100 km/h suddenly loses engine power and so starts to decelerate at a rate proportional to its velocity squared (the proportionality constant is k) due to wind resistance. The car's velocity v as a function of time t since the loss of power is thus described by the differential equation:

(a) $\mathrm{d}v/\mathrm{d}t = 100 - kv^2$
(b) $\mathrm{d}v/\mathrm{d}t = -kv^2$

During each interview, the interviewer made a written record of student responses and at the end of the interview, also made a few additional notes regarding how the student had responded (such as recording that a particular line of questioning had been curtailed as the student had appeared to be getting quite frustrated by their inability to sort things out in their mind). After the interviews, student responses to the various parts of the follow-up interviews were tabulated to allow for easy comparison of student responses in order to identify similarities and differences in responses. This table also allowed for easy comparison of these students' interview responses with their answers to other phases of the investigation.

3. Results

Although not a part of this study, we'd like to note that the students investigated developed reasonable manipulative ability with ODEs, as evidenced by the fact that on the final exam question 2(a), which asked students to 'find the explicit general solution to $dy/dx = y\ln(x)$, $x > 0$', the average mark was 5.25/7, with 42% of the class getting full marks for the question. (The main reasons for losing marks were algebraic errors, or errors in performing the integration.) This relatively good performance on a 'traditional question' in comparison to the relatively poor performance on the conceptual questions, as will be shown below, reinforces previous observations in the literature that performance on 'traditional (manipulative or algorithmic) questions' does not necessarily give an instructor a clear idea of how much students have learned conceptually.

In presenting the results of our investigation, it is most informative to start with Phase 2, student interpretations of the terms in the ODE of Exam Question 3. As shown in table 1, and in contrast to their performance on Exam Question 2(a), many students performed rather poorly on this question.

Apart from this general observation, a detailed analysis of student responses revealed various patterns of responses. Firstly, as one would expect, the majority of students could interpret the dD/dt term adequately, but even here almost a quarter of students made incorrect interpretations. Some of these students mixed amount and rate terminology in their answers, for example, '... is the amount of drug (D) in the body with relation to time, or the rate of change of concentration of the drug', while some others just talked in terms of amounts, for example, '... represents how much the amount of drug changes due to time'. One might think, particularly in relation to the first answer, that these students are just being imprecise in their language use, and that it is likely they have correct conceptions. This interpretation cannot be ruled out for some students, as two students also interpreted the constant '100' in terms of 'amounts in/administered', and yet got

	Term		
Student Interpretation	dD/dt	100	$-0.01D^2$
Correct	42	10	21
No attempt	3	6	6
Other incorrect	14	43	32

Table 1. Student results on Exam Question 3(c).

both the diagnostic questions correct, suggesting that they had some level of understanding. However, as will be shown below, many student answers clearly indicate that the issue is not merely one of imprecise language usage, but of a confusion between related 'amounts' and 'rates of change of amounts'.

Consider now student interpretations of the constant '100' term. These answers could be categorized as follows: there were six 'no attempts', ten answers were deemed 'correct' and five were categorized as miscellaneous. Seven answers were in terms of 'amounts', and as mentioned above, some of these may have been just imprecise language use. Examples include: '... represents the amount of drug going into the patient's body', and, '... is the amount of drug D going in'.

The most common response, though, was in terms of 'an initial condition or arbitrary constant' (16 students), for example: '... is the initial amount of the drug in the body', '... is a constant that represents the initial level of drug administered', and '... arbitrary constant'. Note that these answers are clearly in terms of 'amounts' and cannot simply be imprecise language use.

One possible reason as to why so many students interpreted the '100' term in this way is given by one of the interview responses where the student said that the '100 is a constant' and so must be 'the initial amount administered'. Furthermore, this student argued that 'you need a variable for it to be a rate of change'. This is reminiscent of Elby's [24] what-you-see-is-what-you-get (WYSIWYG) conjecture, in that because '100' was seen as 'constant', it was not seen as a 'rate of change', and the 'obvious' link to make (when making a 'fuzzy' overgeneralization) is that constants in ODE problems are 'initial conditions'.

Another, though very similar possibility revealed in the interviews, came from four students' answers to the decelerating car question. These students all argued that the ODE modelling the situation was '$\mathrm{d}v/\mathrm{d}t = 100 - kv^2$' because the car started at 100 km/h and the $-kv^2$ showed how it slowed down. For example, various students argued:

'... car traveling at 100 km/h – rate at which slowing';
'... initial velocity = 100 km/h – something because you will slow down';
'... 100 is initial velocity and velocity decreases from something' and
'... initially traveling at 100 and can't decelerate from zero'.

Clearly these students are all thinking about an equation for velocity rather than for its rate of change as required, and because of this, they have put the initial condition into the ODE. (This is reminiscent of the student descriptions of $\mathrm{d}D/\mathrm{d}t$ as being for 'the amount of drug D in the body in relation to time'.)

Moving back to the exam question, the other common interpretation (15 students) of the '100' term was that it represented an 'equilibrium amount' or a 'maximum amount', for example: '... is the equilibrium constant' and '... is the limiting/max amount. It is the equilibrium solution'. Again it is unlikely that these students are simply being imprecise in their use of language, but have again started thinking in terms of 'amounts' rather than in terms of 'rates of change of amounts'.

This last set of responses clearly shows the power of context on student thinking, as the most likely explanation for why they interpreted the '100' term as an 'equilibrium or maximum amount' is because Question 3(a) asked the students to 'find the equilibrium solution of the differential equation', and for this ODE, this happens to numerically evaluate to 100. Contextual influences were also evident in student interpretations of the $-0.01D^2$ term as well, as eight students who had interpreted the '100' term as either an 'initial condition/amount' or as a 'maximum or equilibrium amount', then went on to interpret the $-0.01D^2$ term as a 'rate or amount administered' in order to account for their knowledge that a drip continuously delivers drug to a patient. For example: '... is the rate at which the drug is being administered' and '... is the amount of drug that the continuous intravenous drip gives the patient'.

Another prominent category of interpretation for the $-0.01D^2$ term was in terms of amounts, with 13 (22%) having answers like, '... is the amount of drug being used up by the patient's body'. In comparison, these students' interpretations of the '100' term were fairly evenly spread between 'amount in', 'amount initially administered' and 'equilibrium or maximum amount' type answers. It is possible that those with the combination 'amount in/amount used up' for the '100, $-0.01D^2$' terms may have just been sloppy with their use of words, but the others are clearly on the wrong track and are thinking in terms of 'amounts' rather than in terms of 'rates of change of amounts'.

Further evidence of the above interpretation that many students are thinking in terms of amounts rather than rates of change of amounts comes from the interviews where five of the eight students gave the units of $\mathrm{d}D/\mathrm{d}t$ as being 'mg/h' while giving the units of the '100' term as being 'mg'. Two of these five students realized that this was a problem (thus indicating that they hadn't thought to check for unit consistency originally) but could not resolve it, while another two explicitly stated their belief that the units did not have to be consistent! Either way, the idea that one can use unit consistency to check one's thinking was not a part of any of these students' problem-solving strategies.

Moving now to the diagnostic quiz, the above results indicate that we were correct in our anticipation that amount type thinking would be the most likely error in student thinking. Table 3 in the appendix gives the full breakdown of student answers to the two diagnostic quiz questions, while table 2 combines responses into broader categories to ease interpretation.

There are three striking things about table 2. First, only eight students got both questions correct. Secondly, consistent with observations made above, 20 students (45%) used amount-type thinking for their answers to both questions. Finally, some 14 students (32%) appeared to give inconsistent answers (i.e. answers which appear to involve different types of reasoning) for the two questions. As mentioned above, this apparent inconsistency might reflect the fact that many students' knowledge is highly fragmented and consequently is highly context-dependent [24, 29]. Note that while this inconsistency does not bode well for the rationale of this research, the fact that 45% nevertheless did seem to be consistent in the (not

	RC (2b)	× DV (2d)	× IV (2c, e)
RC (1a, b)	1	–	1*
× DV (1c, d)	8**	1	6*
× IV (1e–h)	7*	–	20

Table 2. Cross-tabulation of student responses to diagnostic quiz Questions 1 and 2. Here the row and column headers provide descriptors of the right-hand side of the ODEs, with RC = rate constant, DV = dependent variable (i.e. I or P), IV = independent variable (i.e. x or t), and * indicates apparently inconsistent answers, while ** indicates both quiz questions correct. Total number of participants = 44.

quite correct) ways they thought about these problems, supports the value of this kind of research to find out patterns in the errors students make.

The follow-up interviews confirmed that students were picking answers 1(e)–(h) and 2(c), (e) on the quiz because they were thinking in terms of amounts, and furthermore that there were two basic types of amount-type thinking going on. First, for the fish removal question, six of the eight students argued that the answer was 2(c) because $5000t$ gives the number of fish removed after t years. This result suggests that the issue is one of 'knowledge in transition' or of a 'paradigm shift' as the statement that '$5000t$ gives the number of fish removed after t years' is actually a correct statement (and presumably these students intuitively drew on their experience with constant rate problems to quickly come to this conclusion), it is just inappropriate for the current context.

Student reasoning on the light absorption question was somewhat different however, presumably because this question was sufficiently different from previously seen constant rate problems so as not to automatically trigger that schema. For this question, nearly all of the students interviewed basically argued that the ODE had to have an x on the right-hand side because I was a function of x. (Again note that these students are thinking in terms of $I(x)$ instead of its rate of change.) This suggests not only amount-type thinking (one student said that 'the x accounts for more light being absorbed with distance travelled'), but also a lack of insight into implicit functions (three students explicitly ruled out $\mathrm{d}I/\mathrm{d}x = -0.0693I$ as a solution because it was not explicitly a function of x as they expected). A further lack of understanding of implicit functions was revealed by one student's comment that the P in diagnostic quiz Question 2 was the maximum amount of fish, rather than $P(t)$ (an example of the constant/variable confusion found by other researchers [13–15]). It is unknown whether similar interpretations were made by other students of the I in diagnostic quiz Question 1, thus causing them to rule out solutions with an I in them.

4. Discussion

Although we found some context dependence in student answers, we also found that a significant proportion of students muddled 'amount' and 'rate of change of amount' type thinking when thinking about first-order ODEs used to model physical processes. In some instances, this may be simply imprecise language use by the students involved, but in many others it is clearly a lack of discrimination between these closely related terms. In fact, one student who was interviewed even went so far as to say, 'rate and amount of change mean the same to me'. This lack of discrimination between closely related concepts is often found in novices [20, 21, 28–30], and may simply reflect the human tendency for 'fuzzy' thinking [27], but its similarity with a stage that children typically go through in their thinking about speed (namely considering speed as a distance—the distance travelled per unit time [18]) suggests also that this may be an instance of 'knowledge in transition' or of difficulties making a 'paradigm shift' (cf. [6]) from thinking about the function which describes how a quantity varies to thinking about the equation which describes how a quantity's rate of change varies.

We found several consequences of this muddling of 'amounts' and 'rates of change of amounts'. First, a constant term in an ODE was interpreted by many either as an initial condition or as an equilibrium/maximum amount rather than as a constant rate of change. This may reflect Elby's [24] idea of WYSIWYG, namely that a 'constant' is a 'constant', that is something that is not changing and hence not a 'rate of change', with 'sprawling thinking' (i.e. losing track of what one is after [6, 20, 27]) also possibly contributing to the error. Similarly, several of the students interviewed put an initial condition into the ODE modelling a decelerating car. In this case, the students were clearly thinking in terms of the car's velocity rather than its rate of change. Interestingly, one student who from their knowledge of physics recognized dv/dt as representing acceleration, did not make this mistake as they realized that they were looking for an equation for acceleration rather than velocity. This raises the intriguing possibility that rephrasing questions so that they are about finding, for example, the equation for 'acceleration' rather than finding the equation for 'the rate of change of velocity', may help students stay focused on 'acceleration' instead of thinking about 'velocity' (cf. [38]). In all these cases, 'hasty' thinking [27], namely pattern matching based on superficial similarities with similar problems seen before, also undoubtedly contribute to the muddling.

The second major example of muddling of 'amount' and 'rate of change amount' type thinking was the common tendency for students to use ideas from constant rate problems where a change in a quantity is given by the rate constant $\times$ the independent variable. Thus for example, if P is the number of fish in a fishery and 5000 are removed per year, then '$dP/dt = -5000t$'. A contributing factor to this confusion for at least some students was their thinking that the dependence on the independent variable had to be explicit, thus failing to recognize that a dependence on the dependent variable is also *implicitly* a dependence on the independent variable. Again there is evidence of superficial pattern-matching to familiar problems and losing track of what one is after.

Another area where a lack of sufficient discrimination was evident was the lack of awareness, even appreciation, by some students, that all the terms in a physical equation must have the same units. This may reflect the possibility that they have not observed or realized (or not been required to realize) that while each individual factor in a physical equation or ODE may have differing units, each term as a whole must in fact have the same units. (Additionally, some students did not realize that proportionality factors in ODEs often have units too.) Since this problem was only revealed in the interviews (though student answers to other questions showed that few, if any, took units into consideration when thinking about the ODEs), it is unknown how common this problem is and thus is an area for further research.

5. Suggestions for pedagogical improvement

As a broad claim (and one oft made in both the calculus reform and physics reform literature, and by [1] in relation to reform in the teaching of differential equations), the inclusion of more qualitative or conceptual type questions into the curriculum is likely to be of some benefit, as these will force students to move away from a purely manipulation focus to more of a focus on understanding. More specifically, based on the experience of physics reform efforts, an effective way of dealing with the student misconceptions brought to light by questions 1 and 2 of the diagnostic quiz is likely to be Mazur's [39] Peer Instruction method (see also http://mazur-www.harvard.edu/education/pi.html). In this method, students would first be given in class a question like 1 or 2 from the quiz to think about individually and to choose an answer. A vote on the answer would reveal a diversity of opinion and consequently provide the cognitive dissonance required to motivate the students to discuss the question in small groups. (Having to defend their choice and argue against others is likely to lead to more learning than if the instructor were simply to resolve the dispute.) Mazur found that these group discussions usually zone in on the correct answer (provided that at least 30% of the students have a correct conceptual understanding), and the instructor can check that this has in fact happened by a second vote. At this point, the instructor can also confirm that the students have discovered the important checking strategy of confirming that the units of both sides of the differential equation agree by asking if the students know how they could positively prove an answer is incorrect. (Proving that an answer is in fact correct is of course considerably more difficult.) A possible drawback of this method, though, is that students may need to experience it regularly for it to work well.

Another technique that has been used in the past is to give students questions of the sort: ‘A student has provided the following answer to this problem. What is the error in their reasoning?’ If the error has been shown to be common by research, then many students will consequently become aware of their error through questions of this sort, and, by having to figure out where they've gone wrong themselves, are likely to learn a lot.

The above two suggestions work on the idea that misconceptions need to be confronted if they are to be overcome. Another approach is to recognize that it is common for humans to only discriminate between terms as much as appears to be necessary [20], and that the most common error identified above is most likely due to a failure to clearly discriminate between amounts, changes in an amount and the rate of change of that amount. Consequently, questions which force students to clearly discriminate between these terms may help reduce the sorts of errors we have identified.

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Appendix: Student results on the diagnostic quiz questions

	2a	2b*	2c	2d	2e	blank	Total
1a							0
1b		1	1				2
1c			2		1		3
1d*		8	2	1	1		12
1e		1	5				6
1f	1	5	8		1		15
1g	1	1	6				8
1h							0
blank					1	1	2
Total	2	16	24	1	4	1	48

*The correct responses.

Table 3. Complete cross-tabulation of student responses to questions 1 and 2 of the diagnostic quiz.

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