Convergence Analysis of a Domain Decomposition FEM Approximation of the Isentropic Euler Equation

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Abstract

We analyze an iteration-by-subdomains algorithm of Dirichlet/Dirichlet type for the isentropic Euler equation, focusing on subsonic flows, which are the ones showing the most interesting features in a domain decomposition framework. The main attention is paid to the spatial decomposition, and the problem is advanced in time by means of a semi-implicit Euler scheme. We enforce the continuity on the interface of the inviscid flux, and, in the one-dimensional case, we prove convergence of the algorithm in characteristic variables for both the semi-discrete problem and the fully discrete one, where the equation is discretized in space via Streamline Diffusion Finite Elements. In both cases, the interface mapping is showed to be a contraction: in the semi-discrete case, for any choice of the time step Δt, with constant of order $e^{-C/\Delta t}$ ($C > 0$), in the fully discrete case, provided the entries of the stabilizing matrix are sufficiently small. Finally, some error estimates of energy type are given.

Keywords: Compressible Gas Dynamics, Domain decomposition, Streamline Diffusion Finite Element Methods

1 Introduction

The motion of an inviscid compressible gas is governed by the Euler equations, which express the conservation of mass, momentum and energy of the fluid. In several situations of practical interest, for instance when considering adiabatic flows, or when dealing with small time scales, or when the fluid is confined in a bounded region, the variations of entropy of the system can be neglected and the fluid can be assumed to be isentropic. In thermodynamics, an isentropic fluid is characterized by the existence of a function w called "*enthalpy*" (the heat function per unit mass) such that

$$\nabla w = \frac{1}{\rho}\,\nabla p,$$

p and ρ being the pressure and the density of the fluid, respectively. The other basic thermodynamical quantities, each of them a function of space $\mathbf{x}$ and time t depending on the given flow, are the temperature ϑ, and the internal energy per unit mass $\epsilon = w - p/\rho$.
When dealing with ideal gas dynamics, a fluid is isentropic when the pressure is a function of the sole density, and this assumption reads

$$p = K\rho^{\gamma},$$

where $K > 0$ and $\gamma \geq 1$ are a suitable constant and the ratio of specific heats, respectively. The enthalpy w and the internal energy ϵ are thus given by (see [CM93])

$$w = \frac{\gamma K \rho^{\gamma-1}}{\gamma - 1}, \qquad \epsilon = \frac{K \rho^{\gamma-1}}{\gamma - 1},$$

and the Euler equations for isentropic flows, in a given domain Ω, are

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \\ \rho \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla w + \rho \mathbf{b}, \end{cases}$$

with suitable boundary conditions on $\partial\Omega$, and where $\mathbf{b}(\mathbf{x}, t)$ is a given body force per unit mass. In general, these equations lead to a well-posed initial value problem only if $p'(\rho) > 0$: this agrees with the common experience that the increase of the surrounding pressure on a volume of fluid causes a decrease in the occupied volume and thus an increase in density.

In the recent years, several scientists faced the numerical approximation of the Euler equation in a domain decomposition framework, proposing and implemented algorithms for both sonic and transonic flows (among them, we recall the works by X.-C. Cai and his colleagues [CP99, CPS99]), but no convergence analysis for such algorithm has been provided. To our knowledge, only the work by V. Dolean *et al.* [DLN00] provides a theoretical convergence analysis in both two and three dimensions for a non-overlapping Schwarz algorithm: the main tool for the analysis in there are the Fourier transform, and a *frozen coefficients* technique, which consists in a linearization of the flux in the neighborhood of a constant state. In the following sections, we carry out a convergence analysis for an iteration-by-subdomains algorithm for one-dimensional flows, without freezeing the coefficients: the system is advanced in time by a semi-implicit method, which induces at each time step a linearization in the neighborhood of the previous state.

The paper is organized as follows. In Section 2, the characteristic formulation of the isentropic one dimensional Euler equation is derived. In Section 3 an iteration-by-subdomains algorithm in characteristic form is proposed, where the matching condition at the interface is the continuity of the characteristic variables. Focusing mainly on the spatial decomposition, the semi-discrete iterative algorithm is proved to converge. In Section 4, the problem is discretized in space by Streamline Diffusion Finite Elements, and some inflow-outflow type estimates for the single domain problem are given. In Section 5, we introduce a fully discrete version of the iterative algorithm, and we prove that the interface mapping is a contraction, provided the entries of the stabilizing matrix are sufficiently small, but independentely of h. In this sense, the result is optimal. Finally, in Section 6, coming back to the single domain problem, we firstly give some standard error estimates for the Streamline Diffusion Method, then we give some energy-type error estimates for the approximate solution in characteristic form.

2 The 1-D Isentropic Euler Equation

We consider here an inviscid isentropic compressible fluid in one space dimension: the vector of conserved variables is $\mathbf{W} = (\rho, \rho u)$, the flux vector $\mathbf{F}$ is given by $\mathbf{F}(\mathbf{W}) = \big(\rho u, \rho u^2 + p\big)$, and the conservative form of the equation reads

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = 0, \quad \text{in } Q_T := \Omega \times (0, T) \tag{2.1}$$

where $\Omega = (a, b) \subset \mathbf{R}$ is an interval.

In a region of smooth flow, using the Jacobian of the flux $\mathbf{F}(\mathbf{W})$, system (2.1) can be written in form of quasi-linear hyperbolic system

$$\frac{\partial \mathbf{U}}{\partial t} + A(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0 \quad \text{in } Q_T := \Omega \times (0,T). \tag{2.2}$$

Here $\mathbf{U} = (\rho, u)$ is the vector of physical unknowns, also called "*primitive variables*", and

$$A(\mathbf{U}) := \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix},$$

where $c = \sqrt{\partial p/\partial \rho} = \sqrt{K\gamma\rho^{\gamma-1}}$ is the *speed of sound*, K and γ being defined in the previous section. The quasi-linear system 2.2 is strictly hyperbolic, as the matrix A is diagonalizable with distinct real eigenvalues, namely $A = L\Lambda L^{-1}$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2)$, with

$$\lambda_1 = u + c, \quad \lambda_2 = u - c,$$

while L is the matrix of left eigenvectors, given by

$$L := \begin{pmatrix} c/\rho & 1 \\ -c/\rho & 1 \end{pmatrix}.$$

From a mathematical point of view, equation (2.2) has to be considered together with an initial condition $\mathbf{U}_0(x) = \mathbf{U}(x,0)$ and with suitable boundary conditions in order to have a well-posed initial-boundary value problem. Without entering the details of well-posedeness, we simply recall that it is not admissible to assign values on the outgoing components, since they could contradict the effect of the initial condition making it impossible for a solution to exist (for an extensive discussion on boundary conditions for hyperbolic problems, see for instance [Kre70] and [OS78]). Among the various set of boundary conditions that render this problem well posed, we consider the following ones

$$\begin{cases} \rho(a,t) = g_1(t) & t \in (0,T) \\ \rho(b,t) = g_2(t) & t \in (0,T), \end{cases} \tag{2.3}$$

namely, we assign the value of the density, or, equivalently, the value of the speed of sound. The same result we are going to present in the following could be obtained also with different choices of boundary conditions, for instance assigning the velocity on the left endpoint of the interval, $u(a,t) = b_1(t)$, and the density on the right end one $\rho(b,t) = b_2(t)$.

We require the initial value $\mathbf{U}_0(x)$ to be a continuous vector function, with first component attaining the values $g_1(0)$ and $g_2(0)$ at the endpoints of the interval, hence the solution of our problem is continuous for the whole time of smooth flow. In that order, we recall that the Euler system develops shocks in a finite time. So far, equation (2.2) fails, and one must use a weak formulation based on the conservative form of the equation (2.1). Moreover, we assume the solution $\mathbf{U}(x,t)$ to be bounded for the whole time of smooth flow.

With an iteration by subdomain approach in sight, we finally assume that the flow is subsonic, and directed from the left to the right, i.e. $0 < u < c$, so that $\lambda_1 > 0$ and $\lambda_2 < 0$ for each $(x,t) \in Q_T$, which amounts to have information traveling from each subdomain to the other one. In the domain deomposition framework, this is the most interesting case: in fact, if the flow is supersonic, both eigenvalues are positive, the whole information is a traveling wave from Ω_1 to Ω_2, and the domain decomposition approach is trivially reduced to the sequential solution firstly in Ω_1 and then in Ω_2.

The nonlinearity of the problem does not allow to define directly the characteristic variables $\mathbf{V}$: we therefore introduce them by means of the following differential form (see [Hir90], as well as [QV99])

$$d\mathbf{V} := Ld\mathbf{U} = \big(\frac{c}{\rho}d\rho + du, -\frac{c}{\rho} + du\big). \tag{2.4}$$

Hence, a direct integration provides

$$\begin{aligned} \mathbf{V}_1 &= u + \int \frac{c}{\rho}\, d\rho = u + \int \sqrt{K\gamma}\rho^{\gamma/2-3/2}\, d\rho \\ &= u + \frac{2}{\gamma-1} c + \text{const}, \end{aligned}$$

and similarly for $\mathbf{V}_2$. We thus have

$$\mathbf{V} = \left(u + \frac{2}{\gamma-1}\, c,\ u - \frac{2}{\gamma-1} c \right), \tag{2.5}$$

where the functions $\mathbf{V}_1$ and $\mathbf{V}_2$ (often denoted with R_- and R_+, respectively) are called the Riemann invariants of equation 2.2 and are constant along the *characteristic lines* $C_\pm = \{(x(t), t) \mid x'(t) = u \pm c\}$. Problem (2.2) can therefore be decoupled into its characteristic formulation

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} + \Lambda(\mathbf{V}) \dfrac{\partial \mathbf{V}}{\partial x} = 0 & \text{in } Q_T := \Omega \times (0, T) \\ \mathbf{V}_1(a, t) - \mathbf{V}_2(a, t) = \phi_1(t) & t \in (0, T) \\ \mathbf{V}_1(b, t) - \mathbf{V}_2(b, t) = \phi_2(t) & t \in (0, T) \end{cases} \tag{2.6}$$

where we have set

$$\phi_1(t) = \frac{4}{\gamma-1} \sqrt{K\ (g_1(t))^\gamma} \qquad \text{and} \qquad \phi_2(t) = \frac{4}{\gamma-1} \sqrt{K\ (g_2(t))^\gamma}.$$

Moreover, we can observe from (2.5) that the eigenvalues λ_1 and λ_2 can be expressed as linear combinations of the characteristic variables

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{4} \begin{pmatrix} 1+\gamma & 3-\gamma \\ 3-\gamma & 1+\gamma \end{pmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}. \tag{2.7}$$

Denoting with P the matrix in (2.7), system (2.6) can therefore be rewritten as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}_t + \begin{pmatrix} \sum_{j=1}^{2} P_{1j}\mathbf{V}_j & 0 \\ 0 & \sum_{j=1}^{2} P_{2j}\mathbf{V}_j \end{pmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.8}$$

Remark 2.1 From (2.7), we observe that, differently from the case of constant coefficients, system (2.8) is not constituted of two independent scalar equations coupled only through the boundary conditions, and this is a consequence of the nonlinearity of the original problem.

3 Domain Decomposition

Let $\alpha \in (a, b)$ and consider the decomposition of the domain Ω in the two non-overlapping subdomains $\Omega_1 := (a, \alpha)$, and $\Omega_2 := (\alpha, b)$, with interface $\Gamma = \{\alpha\}$. Let us denote with $\mathbf{W}_1$ and $\mathbf{W}_2$ the restrictions of the solution in Ω to the subdomains Ω_1 and Ω_2, respectively. Then, $\mathbf{W}_1$ and $\mathbf{W}_2$ must satisfy the Euler equations in Ω_1 and Ω_2 separately, where they inherit the boundary and initial conditions prescribed for

$\mathbf{W}$ on $\partial\Omega$ and at $t = 0$, and they have to be matched on Γ by means of suitable interface conditions. As a direct consequence of the fact that the variable $\mathbf{W}$ is a distributional solution of (2.1) in Ω, the natural requirement for a multidomain formulation of system 2.1 is the continuity on the interface of the inviscid flux, $\mathbf{F}(\mathbf{W}_1) = \mathbf{F}(\mathbf{W}_2)$, which splits in two conditions of Dirichlet type

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 n. \end{aligned} \tag{3.1}$$

In order to have a well-posed problem in each subdomain, we have to enforce on the interface as many boundary condition as the number of characteristic lines entering the subdomain. Thus, for subsonic flows, we have to enforce on the interface one boundary condition for each subdomain.

Remark 3.1 Since the velocity field is continuous across the interface, condition $(3.1)_1$ is equivalent to the requirement $\rho_1 = \rho_2$ for any $(\alpha, t) \in \Gamma \times (0, T)$ such that $u(\alpha, t) \neq 0$, which is in agreement with the physics of compressible fluid flows, allowing two kind of discontinuities: shock waves and contact discontinuities. We won't enter here the details of this topic, but we refer the interested reader to [GR94], [Jef76], [Smo83] or [CM93] for an exhaustive treatment of this subject. We only recall that in case of contact discontinuities the normal velocity is zero, the pressure is continuous, but density, tangential velocity and temperature may have non-zero jumps.

In the region of smooth flow, since the Riemann invariants are constant along the characteristics, we can enforce on the interface the continuity of the characteristic variables. Notice that the continuity of these latter variables guarantees the continuity of the physical ones. So far, we can consider the decomposed problem

$$\begin{cases} \dfrac{\partial \mathbf{U}^i}{\partial t} + A(\mathbf{U}^i)\dfrac{\partial \mathbf{U}^i}{\partial x} = 0 & \text{in } \Omega_i \times (0,T), \quad i = 1,2, \\ \mathbf{V}_1^1(\alpha,t) = \mathbf{V}_1^2(\alpha,t) & \forall t \in (0,T), \\ \mathbf{V}_2^1(\alpha,t) = \mathbf{V}_2^2(\alpha,t) & \forall t \in (0,T) \end{cases} \tag{3.2}$$

with the boundary conditions (2.3).

Remark 3.2 When considering discretisation, at the interface point α one has to enforce two additional conditions (besides the other two related to $\mathbf{V}$), in order to recover all the four interface variables, and this can be accomplished by imposing to the variables $\mathbf{U}$ to satisfy equation (2.2) at the interface point α for any outgoing component. If the flow is subsonic, we have to impose one additional conditions for both Ω_1 and Ω_2. An opportunity is to multiply equation (2.2) on the left by the matrix L and consider the components corresponding to the outgoing eigenvectors. For sake of simplicity, we can enforce the equations

$$\left[\mathbf{w}^r \cdot \left(\frac{\partial \mathbf{U}_1}{\partial t} + \lambda_r \frac{\partial \mathbf{U}_1}{\partial x}\right)\right](\alpha, t) = 0 \qquad \text{for } r = 1,2 \tag{3.3}$$

with $\mathbf{w}^1 := (c/\rho, 1)$, and $\mathbf{w}^2 := (-c/\rho, 1)$. Notice that, in the case of an hyperbolic system with constant coefficients, equations (3.3) above correspond to the natural choice of imposing the equation for the outgoing characteristic variable to be satisfied at the interface point α. Equations (3.3) can therefore be seen as a direct generalization of the constant coefficients case and are sometimes called the *compatibility* equations.

Problem 3.2 can be decoupled, owing to (2.4), into its characteristic form,

$$\begin{cases} \dfrac{\partial \mathbf{V}^i}{\partial t} + \Lambda(\mathbf{V}^i)\dfrac{\partial \mathbf{V}^i}{\partial x} = 0 & \text{in } \Omega_i \times (0,T), \quad i = 1,2, \\ \mathbf{V}_1^1(\alpha,t) = \mathbf{V}_1^2(\alpha,t) & \forall t \in (0,T), \\ \mathbf{V}_2^1(\alpha,t) = \mathbf{V}_2^2(\alpha,t) & \forall t \in (0,T) \end{cases} \tag{3.4}$$

with boundary conditions as in (2.6).

We are mainly interested in a spatial decomposition, thus, owing to (2.8), we advance in time the decomposed problem (3.4) by means of a semi-implicit method: at each time step, we linearize system (2.8) in the neighborhood of the previous one, leading to the following two systems of ordinary differential equations

In Ω_1 :

$$\begin{cases} \beta \mathbf{V}^{1,n+1} + \Lambda(\mathbf{V}^{1,n})\dfrac{d}{dx}\mathbf{V}^{1,n+1} = \beta \mathbf{V}^{1,n} \\ \mathbf{V}_1^{1,n+1}(a) - \mathbf{V}_2^{1,n+1}(a) = \phi_1(t^{n+1}) \end{cases}$$

In Ω_2 :

$$\begin{cases} \beta \mathbf{V}^{2,n+1} + \Lambda(\mathbf{V}^{2,n})\dfrac{d}{dx}\mathbf{V}^{2,n+1} = \beta \mathbf{V}^{2,n} \\ \mathbf{V}_1^{2,n+1}(b) - \mathbf{V}_2^{2,n+1}(b) = \phi_2(t^{n+1}) \end{cases}$$

where $\beta = 1/\Delta t$ is the inverse of the time step, which are coupled only through the interface conditions

$$\mathbf{V}_1^{2,n+1}(\alpha) = \mathbf{V}_1^{1,n+1}(\alpha) \qquad \text{and} \qquad \mathbf{V}_2^{2,n+1}(\alpha) = \mathbf{V}_2^{1,n+1}(\alpha).$$

3.1 An Iteration-by-subdomain algorithm for the time-discretized problem

At each time step an iterative procedure can be introduced to solve the coupled system. From now on, since we are not dealing with time, we drop any index referring to time discretisation, and we set $f^{(1)} := \beta \mathbf{V}^{1,n}$, $f^{(2)} := \beta \mathbf{V}^{2,n}$, $\phi_1 := \phi_1(t^{n+1})$, $\phi_2 := \phi_2(t^{n+1})$, as well as

$$\Lambda := \begin{cases} \Lambda(\mathbf{V}^{1,n}) & \text{in } \Omega_1 \\ \Lambda(\mathbf{V}^{2,n}) & \text{in } \Omega_2 \end{cases}$$

The iteration-by-subdomain procedure can therefore be written, for $k \geq 0$, as

In Ω_1 :

$$\begin{cases} \beta \mathbf{V}^{1,k+1} + \Lambda\dfrac{d}{dx}\mathbf{V}^{1,k+1} = f^{(1)} \\ \mathbf{V}_1^{1,k+1}(a) - \mathbf{V}_2^{1,k+1}(a) = \phi_1 \\ \mathbf{V}_2^{1,k+1}(\alpha) = \mathbf{V}_2^{2,k}(\alpha) \end{cases}$$

In Ω_2 :

$$\begin{cases} \beta \mathbf{V}^{2,k+1} + \Lambda\dfrac{d}{dx}\mathbf{V}^{2,k+1} = f^{(2)} \\ \mathbf{V}_1^{2,k+1}(\alpha) = \mathbf{V}_1^{1,k}(\alpha) \\ \mathbf{V}_1^{2,k+1}(b) - \mathbf{V}_2^{2,k+1}(b) = \phi_2, \end{cases} \tag{3.5}$$

having chosen any initial guess $\mathbf{V}_1^{1,0}(\alpha) \in \mathbf{R}$ and $\mathbf{V}_2^{2,0}(\alpha) \in \mathbf{R}$.

3.1.1 Convergence Analysis

In order to prove the convergence of the iterative algorithm, following what is done by A. Quarteroni in [Qua90] for a spectral collocation method and by L. Gastaldi in [Gas92], both in the case of constant coefficients, we define, for each subdomain, the error vector (in characteristic form) as

$$\mathbf{E}_1^{i,k+1} := \mathbf{V}_1^{i,k+1} - \mathbf{V}_1^i, \qquad \mathbf{E}_2^{i,k+1} := \mathbf{V}_2^{i,k+1} - \mathbf{V}_2^i,$$

for $i = 1, 2$.
It can be easily viewed that the vector functions $\mathbf{E}^{i,k+1} := (\mathbf{E}_1^{i,k+1}, \mathbf{E}_2^{i,k+1})$, $i = 1, 2$ satisfy the following error equations

$$\text{In } \Omega_1: \quad a) \begin{cases} \beta \mathbf{E}_1^{1,k+1} + \lambda_1 \dfrac{d}{dx} \mathbf{E}_1^{1,k+1} = 0 \\ \beta \mathbf{E}_2^{1,k+1} + \lambda_2 \dfrac{d}{dx} \mathbf{E}_2^{1,k+1} = 0 \\ \mathbf{E}_1^{1,k+1}(a) = \mathbf{E}_2^{1,k+1}(a) \\ \mathbf{E}_2^{1,k+1}(\alpha) = \mathbf{E}_2^{2,k}(\alpha), \end{cases} \qquad \text{In } \Omega_2: \quad b) \begin{cases} \beta \mathbf{E}_1^{2,k+1} + \lambda_1 \dfrac{d}{dx} \mathbf{E}_1^{2,k+1} = 0 \\ \beta \mathbf{E}_2^{2,k+1} + \lambda_2 \dfrac{d}{dx} \mathbf{E}_2^{2,k+1} = 0 \\ \mathbf{E}_1^{2,k+1}(\alpha) = \mathbf{E}_1^{1,k}(\alpha) \\ \mathbf{E}_2^{2,k+1}(b) = \mathbf{E}_1^{2,k+1}(b). \end{cases} \tag{3.6}$$

We introduce the following sequence of *interface errors*:

$$\mathbf{E}_\alpha^k := \left[\mathbf{E}_1^{2,k}(\alpha)\right]^2 + \left[\mathbf{E}_2^{1,k}(\alpha)\right]^2 \quad \text{for } k \geq 1, \tag{3.7}$$

and we can prove the following results.

Lemma 3.1 *Assume that* $\frac{1}{\lambda_1} \in L^1(\Omega)$ *and* $\frac{1}{\lambda_2} \in L^1(\Omega)$. *Then, there exists a constant* $\sigma < 1$ *such that the interface error defined in* (3.7) *reduces at each iteration* $k \geq 1$ *according to the law*

$$\mathbf{E}_\alpha^{k+1} \leq \sigma \mathbf{E}_\alpha^k. \tag{3.8}$$

Proof. Since systems (3.6) are completely decoupled, we can have an explicit representation for the solutions of the error equations. Solving (3.6.a), firstly for $E_2^{1,k+1}$ and then for $E_1^{1,k+1}$, and proceeding in a similar way in (3.6.b), we get

$$\mathbf{E}_1^{1,k+1}(\alpha) = \exp\Big\{-\beta\Big(\Phi_1 + \Psi_1\Big)\Big\} \mathbf{E}_2^{2,k}(\alpha),$$
$$\mathbf{E}_2^{2,k+1}(\alpha) = \exp\Big\{-\beta\Big(\Phi_2 + \Psi_2\Big)\Big\} \mathbf{E}_1^{1,k}(\alpha),$$

with

$$\Phi_i := \int_{\Omega_i} \frac{dy}{\lambda_1(y)}, \qquad \Psi_i := -\int_{\Omega_i} \frac{dy}{\lambda_2(y)}, \quad i = 1, 2.$$

The subsonic assumption ($0 < u < c$) entails $\Phi_i > 0$ and $\Psi_i > 0$, and the thesis follows with

$$\sigma := \exp\Big(-\beta \ \min\{\Phi_1 + \Psi_1, \ \Phi_2 + \Psi_2\}\Big)$$

□

Theorem 3.1 *Under the assumptions of Lemma 3.1, the iteration-by-subdomain strategy in* (3.5) *converges as* $k \to \infty$, *uniformly with respect to the time step* Δt.

Proof. From the previous lemma we have $\lim_{k\to\infty} \mathbf{E}^k_\alpha = 0$, and, in order to complete the proof, we have to show that the convergence does not depend on the time step, and to prove that the error $\mathbf{E}^k(x)$ can be controlled, for each $x \in \Omega$, by the error on the interface $\mathbf{E}^k_\alpha$. It is not difficult to see that we have,

$$\left[\mathbf{E}^{1,k+1}(x)\right]^2 < 2\left[\mathbf{E}^{1,k}_2(\alpha)\right]^2 \qquad \text{and} \qquad \left[\mathbf{E}^{2,k+1}(x)\right]^2 < 2\left[\mathbf{E}^{2,k}_1(\alpha)\right]^2,$$

for any x in Ω_1 or Ω_2, respectively. Thus, for $i = 1, 2$,

$$\left[\mathbf{E}^{i,k+1}(x)\right]^2 < 2\left[\mathbf{E}^k_\alpha\right]^2,$$

and this concludes the proof. □

Each step in the iterative procedure (3.6) can be interpreted as an iterative mapping on the interface $\mathcal{M} : \mathbf{R}^2 \to \mathbf{R}^2$, which is defined, for each $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, as

$$\mathcal{M} : \begin{pmatrix} \xi_1 \\ \\ \xi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{V}^{1,k+1}_1(\alpha) \\ \\ \mathbf{V}^{2,k+1}_2(\alpha) \end{pmatrix} \tag{3.9}$$

where $\mathbf{V}^{1,k+1}_1(\alpha)$ and $\mathbf{V}^{2,k+1}_2(\alpha)$ are the solutions of systems (3.5) with incoming values on α given by $\mathbf{V}^{1,k+1}_2(\alpha) = \xi_2$ and $\mathbf{V}^{2,k+1}_1(\alpha) = \xi_1$. It is then immediate to prove the following result.

Lemma 3.2 *For any choice of the time step $\beta = 1/\Delta t$, the mapping $\mathcal{M}$ defined in* (3.9) *is a contraction. Moreover, there exists $C > 0$ such that the reduction factor K is given by*

$$K = e^{-\frac{C}{\Delta t}}.$$

Proof. Since all the problems involved are linear, it is enough to prove contractivity for the mapping $\mathcal{M}^0$, which is obtained from $\mathcal{M}$ when $f = \phi_1 = \phi_2 = 0$. Namely, it is enough to prove that there exists a constant $K < 1$ such that, for each $\xi \in \mathbf{R}^2$,

$$\left|\mathcal{M}^0\xi\right|^2 \leq K\,|\xi|^2 \tag{3.10}$$

A direct application of Lemma 3.1 concludes the proof with $C = \min\{\Phi_1 + \Psi_1,\ \Phi_2 + \Psi_2\}$. □

3.2 An Equivalent Algorithm

As a matter of fact, the procedure in (3.5) could be advanced (at least in principle) in parallel, but this is somehow redundant. In fact, the iteration by subdomain algorithm can be efficiently exploited in the following sequential way:

STEP 1. Given $\xi^0_1 \in \mathbf{R}$, solve in Ω_1, for each $k \geq 0$:

$$\begin{cases} \beta\mathbf{V}^{1,k+1}_2 + \lambda_2 \dfrac{d}{dx}\mathbf{V}^{1,k+1}_2 = f^{(1)}_2 \\ \\ \mathbf{V}^{1,k+1}_2(\alpha) = \xi^k_1, \end{cases} \quad \text{then} \quad \begin{cases} \beta\mathbf{V}^{1,k+1}_1 + \lambda_1 \dfrac{d}{dx}\mathbf{V}^{1,k+1}_1 = f^{(1)}_1 \\ \\ \mathbf{V}^{1,k+1}_1(a) = \mathbf{V}^{1,k+1}_2(a) + \phi_1. \end{cases}$$

STEP 2. Set $\xi^{k+1}_2 = \mathbf{V}^{1,k+1}_1(\alpha)$ and solve in Ω_2

$$\begin{cases} \beta\mathbf{V}^{2,k+1}_1 + \lambda_1 \dfrac{d}{dx}\mathbf{V}^{2,k+1}_1 = f^{(2)}_1 \\ \\ \mathbf{V}^{2,k+1}_1(\alpha) = \xi^{k+1}_2, \end{cases} \quad \text{then} \quad \begin{cases} \beta\mathbf{V}^{2,k+1}_2 + \lambda_2 \dfrac{d}{dx}\mathbf{V}^{2,k+1}_2 = f^{(2)}_2 \\ \\ \mathbf{V}^{2,k+1}_2(b) = \mathbf{V}^{2,k+1}_1(b) - \phi_2. \end{cases}$$

STEP 3. Set $\xi^{k+1}_1 = \mathbf{V}^{2,k+1}_2(\alpha)$, go to **STEP 1** and iterate.

Remark 3.3 Notice that the values of $\mathbf{V}_1^{1,k+1}(a)$ in **STEP 1** and $\mathbf{V}_2^{2,k+1}(b)$ in **STEP 2** are completely determined by the physical boundary conditions and by the values of $\mathbf{V}_2^{1,k+1}(a)$ and $\mathbf{V}_1^{2,k+1}(b)$, respectively.

Owing to Lemma 3.1 and Theorem 3.1, we can prove convergence for the above algorithm.

Theorem 3.2 *The iteration by subdomain in* **STEP 1** - **STEP 3** *is equivalent to* (3.5)*, and it converges as* $k \to \infty$*, independently of the choice of the time step* Δt*.*

Proof. Straightforward by observing that the sequences $\left\{\xi_1^k\right\}_k$ and $\left\{\xi_2^k\right\}_k$ are subsequences of $\left\{\mathbf{V}_1^{2,k}(\alpha)\right\}_k$ and $\left\{\mathbf{V}_2^{1,k}(\alpha)\right\}_k$, respectively, which stem from the iterative procedure in (3.5), and are convergent. □

4 Fully discrete finite element approximation for the single domain problem

In this section, following what is done in [Gas92] for linear hyperbolic systems with constant coefficients, we focus our attention on the finite dimensional approximation for the system stemming from a semi-implicit time discretisation of system (3.4).

4.1 The scalar case and its finite element approximation via the Streamline Diffusion Method

Since system (3.4) consists of two scalar transport equations coupled only through the boundary conditions, let us consider the following problem

$$\begin{cases} \dfrac{\beta}{\lambda(x)}\, u + u' = f(x) \qquad \text{in } \Omega = (a,\, b) \\ \\ u(a) = \xi \end{cases} \tag{4.1}$$

where $\beta > 0$, $\lambda(x) \geq \lambda_* > 0$ $\forall x \in \Omega$ and we have denoted with u' the space derivative of u with respect to x, *i.e.* $u' := \frac{du}{dx}$.
Problem (4.1) is well known to have a unique solution, which is given, for $x \in \Omega$, by

$$u(x) = \exp\left(-\beta \int_a^x \frac{dy}{\lambda(y)}\right) \times \left[\xi + \int_a^x f(t)\, \exp\left(\beta \int_a^t \frac{dy}{\lambda(y)}\right) dt\right]. \tag{4.2}$$

In order to approximate the solution of problem (4.1) with finite elements, let $\mathcal{T}_h$ be a subdivision of the interval Ω into a finite number of subintervals $[x_{j-1}, x_j]$ such that $|x_j - x_{j-1}| \leq h$ for $j = 1, \ldots, N$, where

$$a = x_0 < x_1 < \ldots < x_N = b.$$

We introduce the following finite element spaces:

$$V^h(\Omega) := \left\{v \in C^0(\overline{\Omega}) \;\middle|\; v_{|K} \in \mathbb{P}_k,\ \forall K \in \mathcal{T}_h\right\} \qquad k \geq 1, \tag{4.3}$$

$$V_a^h(\Omega) := \left\{v \in V^h(\Omega) \;\middle|\; v(a) = 0\right\}, \tag{4.4}$$

and the Streamline Diffusion approximation of problem (4.1) reads as follows:

$$\text{Find } u_h \in V^h : \quad u_h(a) = \xi_h \qquad \text{and} \qquad a_h(u_h, v_h) = L_h(v_h), \qquad \forall v_h \in V_a^h \tag{4.5}$$

where

$$\begin{aligned} a_h(w,v) &= \int_\Omega \left[\frac{\beta}{\lambda_h(x)}\, wv + w'v\right] + \sum_{K\in\mathcal{T}} \delta h \int_K \left[\frac{\beta}{\lambda_h(x)}\, w + w'\right] v' \\ &= \int_\Omega \left[\frac{\beta}{\lambda_h(x)}\, w + w'\right] \left[v + \delta h v'\right] \end{aligned}$$

and

$$L_h(v) = \int_\Omega f_h v + \sum_{K\in\mathcal{T}} \delta h \int_K f_h v' = \int_\Omega f_h \left(v + \delta h v'\right),$$

and, moreover, λ_h, f_h and ξ_h are suitable approximations of the data λ, f and ξ.
In order to have well-posedness for problem (4.5) we assume the bilinear form $a_h(.,.)$ to be positive, namely

$$\mu^* := \inf_{x\in\Omega} \left(\frac{\beta}{\lambda_h(x)} + \frac{1}{2}\frac{\beta\delta h \lambda_h'(x)}{\lambda_h^2(x)}\right) > 0, \tag{4.6}$$

which is fulfilled for each δ such that

$$0 < \delta < \delta^0 := 2\,\frac{\min_\Omega \lambda_h(x)}{\max\{-\min_\Omega \lambda_h'(x), 0\}}\, h^{-1} \tag{4.7}$$

The above condition has to be interpreted in the following way: if $\min_\Omega \lambda_h'(x) > 0$, then no upper bound is needed for δ. Implicitely, we have also assumed that $\min_\Omega \lambda_h' > -\infty$.

4.1.1 The finite element approximation of the vector case

In this section we go back to the complete system (3.4) and we introduce the finite element spaces

$$V_b^h(\Omega) := \left\{v \in V^h(\Omega) \;\middle|\; v(b) = 0\right\} \tag{4.8}$$

and

$$\mathcal{W}^h(\Omega) := \left[V^h(\Omega)\right]^2, \qquad \mathcal{W}_0^h(\Omega) := V_a^h(\Omega) \times V_b^h(\Omega). \tag{4.9}$$

At each time step n, neglecting any index referring to the time step, the stabilized fully discrete formulation for system (2.6) reads:

Find $\mathbf{V}_h \in \mathcal{W}^h(\Omega)$ *such that*

$$\begin{cases} \displaystyle\int_\Omega \left(\beta\Lambda_h^{-1}\mathbf{V}_h + \frac{d\mathbf{V}_h}{dx} - \mathbf{f}_h, \varphi + hD\frac{d\varphi}{dx}\right) dx = 0 \qquad \forall\, \varphi \in \mathcal{W}_0^h(\Omega) \\ \mathbf{V}_1(a) - \mathbf{V}_2(a) = \phi_1 \\ \mathbf{V}_1(b) - \mathbf{V}_2(b) = \phi_2 \end{cases} \tag{4.10}$$

where clearly $\Lambda_h = \Lambda(\mathbf{V}_h^{n-1})$ and $\mathbf{f}_h = \beta\Lambda_h^{-1}\mathbf{V}_h^{n-1}$, while $D = \mathrm{diag}(\delta_1, \delta_2)$ is a suitable diagonal matrix: δ_1 and δ_2 must satisfy (4.7), with λ_h replaced by $\lambda_1(\mathbf{V}_h)$ and $-\lambda_2(\mathbf{V}_h)$, respectively.

4.2 Inflow-outflow estimates for the finite element approximation

In this section, adapting to our problem the approach of [GG93] and [Gas92], we give some estimates of inflow-outflow type for the scalar problem (4.5) which will be used in the sequel. In that order, let us consider the following problems:

(P1) *Find* $u_h \in V^h(\Omega)$ *such that*

$$\begin{aligned} \int_\Omega \left\{ \frac{\beta}{\lambda_h(x)}\, u_h + u_h' - f_h \right\} \cdot (v + \delta h v') = &\quad 0 \qquad \forall v \in V_a^h(\Omega) \\ u_h(a) = &\quad \chi \end{aligned} \tag{4.11}$$

and

(P2) *Find* $u_h^m \in V^h(\Omega)$ *such that*

$$\begin{aligned} \int_\Omega \left\{ \frac{\beta}{\lambda_h(x)}\, u_h^m + (u_h^m)' - f_h \right\} \cdot (v + \delta h v') = &\quad 0 \qquad \forall v \in V_a^h(\Omega) \\ u_h^m(a) = &\quad \chi_m \end{aligned} \tag{4.12}$$

We are in the position to prove the following result.

Lemma 4.1 *Assume* $\lambda_h(x) \geq M_1 > 0$ *for all* $x \in \Omega$, $\lambda_h' \in L^\infty(\Omega)$ *and that* (4.7) *is satisfied. Let* u_h *and* u_h^m *be the solutions to problems* (4.11) *and* (4.12) *above. Then there exists a constant* $H_\Omega < 1$ *such that*

$$\left(u_h - u_h^m\right)^2(b) \leq H_\Omega\, \left(\chi - \chi_m\right)^2, \tag{4.13}$$

provided δ *is sufficiently small.*

Proof. The difference $e_m := u_h - u_h^m$ satisfies the following error equation

$$\begin{aligned} \int_\Omega \left\{ \frac{\beta}{\lambda_h(x)}\, e_m + e_m' \right\} \cdot (v + \delta h v') = 0 \\ e_m(a) = \chi - \chi_m \end{aligned} \tag{4.14}$$

If $\lambda_h(b) < \lambda_h(a)$, we take in (4.14) $v = e_m$ and we get

$$\begin{aligned} 0 &= \int_\Omega \frac{\beta}{\lambda_h}\, e_m^2 + \int_\Omega e_m e_m' + \delta h \int_\Omega \frac{\beta}{\lambda_h} e_m e_m' + \delta h \int_\Omega (e_m')^2 \\ &= \int_\Omega \left(\frac{\beta}{\lambda_h} + \frac{\beta \delta h \lambda_h'}{2\lambda_h^2} \right) e_m^2 + \frac{1}{2} \left[\left(1 + \frac{\beta\delta h}{\lambda_h} \right) e_m^2 \right]_a^b + \delta h \int_\Omega (e_m')^2. \end{aligned} \tag{4.15}$$

The third term on the right hand side is positive, while assumption (4.6) provides positivity also for the first one, so that we have

$$\left(1 + \frac{\beta\delta h}{\lambda_h(b)} \right) e_m^2(b) \;\leq\; \left(1 + \frac{\beta\delta h}{\lambda_h(a)} \right) e_m^2(a).$$

Inequality (4.13) follows with

$$H_\Omega \;:=\; \frac{1 + \frac{\beta\delta h}{\lambda_h(a)}}{1 + \frac{\beta\delta h}{\lambda_h(b)}} \;<\; 1.$$

If $\lambda_h(b) \geq \lambda_h(a)$, let $\varphi \in W^{1,\infty}(\Omega)$ be the linear function such that $\varphi(a) = 0$ and $\varphi(b) = 1$, *i.e.* $\varphi(x) = \frac{x-a}{b-a}$. We take $v = (1+\eta\varphi)\, e_m$ in (4.14), with $\eta > 0$, and we get

$$
\begin{aligned}
0 &= \int_\Omega \left\{ \frac{\beta}{\lambda_h}\, e_m + e_m' \right\} \cdot \left\{ (1+\eta\varphi)e_m + \delta h\, [(1+\eta\varphi)e_m]' \right\} \\
&= \int_\Omega \frac{\beta}{\lambda_h}\, e_m^2 (1+\eta\varphi) + \underbrace{\int_\Omega (1+\eta\varphi)e_m e_m'}_{(1)} + \underbrace{\delta h \int_\Omega \frac{\beta}{\lambda_h}(1+\eta\varphi)e_m e_m'}_{(2)} + \delta h \int_\Omega \frac{\beta}{\lambda_h}\eta\varphi' e_m^2 \\
&\quad + \delta h \int_\Omega (1+\eta\varphi)(e_m')^2 + \underbrace{\delta h \int_\Omega \eta\varphi' e_m e_m'}_{(3)}
\end{aligned}
\tag{4.16}
$$

Since φ is linear,

$$
\begin{aligned}
(1) &= \frac{1}{2}\left[(1+\eta\varphi)e_m^2\right]_a^b - \frac{1}{2}\int_\Omega \eta\varphi' e_m^2 \\
(2) &= \frac{\delta h}{2}\left[\frac{\beta}{\lambda_h}(1+\eta\varphi)e_m^2\right]_a^b + \frac{1}{2}\int_\Omega \frac{\beta\delta h\lambda_h'}{\lambda_h^2}(1+\eta\varphi)e_m^2 - \frac{1}{2}\int_\Omega \frac{\beta\delta h}{\lambda_h}\eta\varphi' e_m^2 \\
(3) &= \frac{\delta h}{2}\left[\eta\varphi' e_m^2\right]_a^b - \frac{\delta h}{2}\int_\Omega \eta\varphi'' e_m^2 \;=\; \frac{\delta h}{2}\left[\eta\varphi' e_m^2\right]_a^b,
\end{aligned}
$$

and this entails

$$
\begin{aligned}
0 = &\int_\Omega \left\{ \left(\frac{\beta}{\lambda_h} + \frac{\beta\delta h\lambda_h'}{2\lambda_h^2}\right)(1+\eta\varphi) + \frac{1}{2}\left(\frac{\beta\delta h}{\lambda_h} - 1\right)\eta\varphi' \right\} e_m^2 \\
&+ \frac{1}{2}\left[\left\{ \left(1 + \frac{\beta\delta h}{\lambda_h}\right)(1+\eta\varphi) + \delta h\eta\varphi' \right\} e_m^2\right]_a^b + \delta h\int_\Omega (1+\eta\varphi)(e_m')^2
\end{aligned}
\tag{4.17}
$$

The third term of the sum is positive independently of η. Concerning the first one we have two opportunities: if $\beta\delta h - \lambda_h > 0$ in Ω, this term is positive without any further restriction on η, while if $\beta\delta h - \lambda_h < 0$ for some $x \in \Omega$, taking into account the definition of φ, its positivity is guaranteed if, for instance

$$
\eta \;\leq\; \eta^* \;:=\; 2\,(b-a)\,\mu^*. \tag{4.18}
$$

where μ^* is the one defined in (4.6). We obtain from (4.17)

$$
\left[\frac{\delta\eta h}{b-a} + \left(1 + \frac{\beta\delta h}{\lambda_h(b)}\right)(1+\eta)\right] e_m^2(b) \;\leq\; \left[\frac{\delta\eta h}{b-a} + \left(1 + \frac{\beta\delta h}{\lambda_h(a)}\right)\right] e_m^2(a). \tag{4.19}
$$

Let

$$
\delta_0 \;:=\; \sup\left\{ \delta > 0 \;\middle|\; \eta_* := \beta\delta h \frac{\lambda_h(b) - \lambda_h(a)}{\lambda_h(a)\,[\lambda_h(b) + \beta\delta h]} \;<\; \eta^* \right\} \tag{4.20}
$$

and define

$$
\delta^* := \min\left\{\delta^0 \;,\; \delta_0\right\}, \tag{4.21}
$$

where δ^0 is the one introduced in (4.7). Thus, for any $\eta \in]\eta_*, \eta^*]$, inequality (4.13) follows with

$$H_\Omega \;:=\; \frac{\frac{\delta\eta h}{b-a} + \left(1 + \frac{\beta\delta h}{\lambda_h(a)}\right)}{\frac{\delta\eta h}{b-a} + \left(1 + \frac{\beta\delta h}{\lambda_h(b)}\right)\,(1+\eta)} \;<\; 1, \tag{4.22}$$

provided $0 < \delta \,<\, \delta^*$. □

5 Fully Discrete Multidomain Formulation and the Iterative Algorithm

In this section we go back to the multidomain formulation of Section 3 and we prove convergence for an iteration-by-subdomains procedure in the fully discrete case.
To this aim, for sake of simplicity, we assume that the interface α coincides with a node of the mesh, and consider the finite element spaces which are the restrictions to $\Omega_1 = (a, \alpha)$ and $\Omega_2 = (\alpha, b)$ of the spaces $\mathcal{W}^h(\Omega)$ and $\mathcal{W}_0^h(\Omega)$, namely

$$\mathcal{W}^h(\Omega_j) := \left[V^h(\Omega_j)\right]^2, \qquad j = 1, 2, \tag{5.1}$$

and

$$\mathcal{W}_0^h(\Omega_1) := V_a^h(\Omega_1) \times V_\alpha^h(\Omega_1) \qquad \text{and} \qquad \mathcal{W}_0^h(\Omega_2) := V_\alpha^h(\Omega_2) \times V_b^h(\Omega_2), \tag{5.2}$$

and consider the discretized version of the multidomain formulation (3.4), where, as usual, superindeces denote subdomains, whereas subindeces denote components,

$$\int_{\Omega_i} \left(\beta\Lambda_h^{-1}\mathbf{V}_h^i + \frac{d}{dx}\mathbf{V}_h^i - \mathbf{f}_h^{(i)}, \varphi + hD\frac{d\varphi}{dx}\right) dx = 0 \quad \forall\, \varphi \,\in\, \mathcal{W}_0^h(\Omega_i), \quad i = 1, 2 \tag{5.3}$$

$$\mathbf{V}_{h,1}^1(a) = \mathbf{V}_{h,2}^1(a) + \phi_1 \tag{5.4}$$

$$\mathbf{V}_{h,2}^2(b) = \mathbf{V}_{h,1}^2(b) - \phi_2 \tag{5.5}$$

$$\mathbf{V}_{h,2}^1(\alpha) = \mathbf{V}_{h,2}^2(\alpha) \tag{5.6}$$

$$\mathbf{V}_{h,1}^2(\alpha) = \mathbf{V}_{h,1}^1(\alpha) \tag{5.7}$$

where, as usual, $\mathbf{f}^{(i)}$ denotes the restriction of $\mathbf{f}$ to Ω_i, $(i = 1, 2)$.
We introduce, as in the continuous case, an iterative procedure to solve system (5.3)-(5.7) above. At the $(m+1)$-th iteration, it reads as follows

$$\int_{\Omega_1} \left(\beta\Lambda_h^{-1}\mathbf{V}_h^{1,m+1} + \frac{d}{dx}\mathbf{V}_h^{1,m+1} - \mathbf{f}_h^{(1)}, \varphi + hD\frac{d\varphi}{dx}\right) dx = 0 \quad \forall\, \varphi \,\in\, \mathcal{W}_0^h(\Omega_1) \tag{5.8}$$

$$\int_{\Omega_2} \left(\beta\Lambda_h^{-1}\mathbf{V}_h^{2,m+1} + \frac{d}{dx}\mathbf{V}_h^{2,m+1} - \mathbf{f}_h^{(2)}, \psi + hD\frac{d\psi}{dx}\right) dx = 0 \quad \forall\, \psi \,\in\, \mathcal{W}_0^h(\Omega_2) \tag{5.9}$$

$$\mathbf{V}_{h,1}^{1,m+1}(a) = \mathbf{V}_{h,2}^{1,m+1}(a) + \phi_1 \tag{5.10}$$

$$\mathbf{V}_{h,2}^{2,m+1}(b) = \mathbf{V}_{h,1}^{2,m+1}(b) - \phi_2 \tag{5.11}$$

$$\mathbf{V}_{h,2}^{1,m+1}(\alpha) = \mathbf{V}_{h,2}^{2,m}(\alpha) \tag{5.12}$$

$$\mathbf{V}_{h,1}^{2,m+1}(\alpha) = \mathbf{V}_{h,1}^{1,m}(\alpha) \tag{5.13}$$

5.1 Convergence Analysis

Procedure (5.8)-(5.13) above can be interpreted as a discrete iterative mapping $\mathcal{M}_h : \mathbf{R}^2 \to \mathbf{R}^2$, acting in the following way:

$$\mathcal{M}_h \ : \ \begin{pmatrix} \mathbf{V}^{1,m}_{h,1}(\alpha) \\ \mathbf{V}^{2,m}_{h,2}(\alpha) \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{V}^{1,m+1}_{h,1}(\alpha) \\ \mathbf{V}^{2,m+1}_{h,2}(\alpha) \end{pmatrix} \tag{5.14}$$

where $\mathbf{V}^{1,m+1}_{h,1}(\alpha)$ and $\mathbf{V}^{2,m+1}_{h,2}(\alpha)$ are the restriction to the interface α of the solution of systems (5.8)-(5.10)-(5.12) and (5.9)-(5.11)-(5.13), respectively. The convergence properties of the mapping $\mathcal{M}_h$ are given in the following theorem.

Theorem 5.1 *Assume there exists a constant $M > 0$ such that the eigenvalues $\lambda_{1,h}$ and $\lambda_{2,h}$ of Λ_h satisfy $\lambda_{1,h}(x) \geq M$ and $-\lambda_{2,h}(x) \geq M$, for any $x \in \Omega$, respectively. Assume moreover that $\lambda'_{i,h} \in L^\infty(\Omega)$ (for $i = 1, 2$). Then, the discrete mapping $\mathcal{M}_h$ is a contraction on the interface, provided the entries of the diagonal matrix D are sufficiently small.*

Proof. The mapping $\mathcal{M}_h$ is linear, so it is enough to prove that it is contractive on the error, and to this aim it is immediate to see that the difference $\mathbf{E}^{i,m} := \mathbf{V}^i_h - \mathbf{V}^{i,m}_h$ $(i = 1, 2)$ satisfies the following error equations (as usual, subindices denote components)

$$\int_{\Omega_1} \left(\beta \Lambda_h^{-1} \mathbf{E}^{1,m+1}_h + \frac{d}{dx} \mathbf{E}^{1,m+1}_h, \varphi + hD \frac{d\varphi}{dx} \right) dx = 0 \quad \forall\, \varphi \in \mathcal{W}^h_0(\Omega_1) \tag{5.15}$$

$$\int_{\Omega_2} \left(\beta \Lambda_h^{-1} \mathbf{E}^{2,m+1}_h + \frac{d}{dx} \mathbf{E}^{2,m+1}_h, \psi + hD \frac{d\psi}{dx} \right) dx = 0 \quad \forall\, \psi \in \mathcal{W}^h_0(\Omega_2) \tag{5.16}$$

$$\mathbf{E}^{1,m+1}_{h,1}(a) = \mathbf{E}^{1,m+1}_{h,2}(a) \tag{5.17}$$

$$\mathbf{E}^{2,m+1}_{h,2}(b) = \mathbf{E}^{2,m+1}_{h,1}(b) \tag{5.18}$$

$$\mathbf{E}^{1,m+1}_{h,2}(\alpha) = \mathbf{E}^{2,m}_{h,2}(\alpha) \tag{5.19}$$

$$\mathbf{E}^{2,m+1}_{h,1}(\alpha) = \mathbf{E}^{1,m}_{h,1}(\alpha) \tag{5.20}$$

Equations (5.15) and (5.16) consist of two scalar equations coupled only through the boundary conditions. If the entries of D are small enough, we can apply Lemma (4.1) to both components of the error in Ω_1 and Ω_2.
Let us focus on Ω_1: from Lemma (4.1) there exists a constant $K_{\Omega_1} < 1$ such that

$$\left[\mathbf{E}^{1,m+1}_{h,2}\right]^2(a) \leq K_{\Omega_1} \left[\mathbf{E}^{1,m+1}_{h,2}\right]^2(\alpha),$$

and, owing to (5.17), there exists a constant $H_{\Omega_1} < 1$ such that

$$\left[\mathbf{E}^{1,m+1}_{h,1}\right]^2(\alpha) \leq H_{\Omega_1} \left[\mathbf{E}^{1,m+1}_{h,2}\right]^2(a) \leq H_{\Omega_1} K_{\Omega_1} \left[\mathbf{E}^{1,m+1}_{h,2}\right]^2(\alpha). \tag{5.21}$$

From a similar argument within Ω_2 there exist constants $H_{\Omega_2} < 1$ and $K_{\Omega_2} < 1$ such that

$$\left[\mathbf{E}^{2,m+1}_{h,2}\right]^2(\alpha) \leq H_{\Omega_2} \cdot K_{\Omega_2} \left[\mathbf{E}^{2,m+1}_{h,1}\right]^2(\alpha) \tag{5.22}$$

Gathering together (5.19), (5.20), (5.21) and (5.22) we have

$$|\mathcal{M}_h \mathbf{E}^m_h|^2 \leq \mathcal{K}\, |\mathbf{E}^m_h|^2$$

where

$$\mathcal{K} := \min \{H_{\Omega_1} K_{\Omega_1} \ , \ H_{\Omega_2} K_{\Omega_2}\} < 1,$$

and this concludes the proof. □

Remark 5.1 In our framework, the boundedness assumption on ρ and u entails boundedness also for the discrete solution and its derivative at each time step. Infact, since $\Lambda_h = \Lambda(\mathbf{V}_h^{n-1})$, where $\mathbf{V}_h^{n-1}$ (globally continuous and piecewise polynomial) is the discrete solution computed at the previous time step, we have $\mathbf{V}_h^n \in W^{1,\infty}(\Omega_j)$, and $\lambda_{j,h} \in W^{1,\infty}(\Omega_j)$ (for $j = 1, 2$). Finally, on one hand the hypotesis $-\lambda_{2,h}(x) \geq M$ is coherent with the assumption on the flow to be subsonic ($u < c$), whereas the hypotesis $\lambda_{1,h}(x) \geq M$ agrees with the assumption on the flow to be directed rightwards ($u > 0$).

6 Error estimates

In this section we go back to the single domain case and we study the approximation error we get from the characteristic approach to the Euler system. For that purpose, we firstly derive some standard approximation errors for the Streamline Diffusion Method.

6.1 Error Estimates for the Streamline Diffusion Method

We give here some standard error estimates for the Streamline Diffusion finite element discretisation of a transport problem.
In that order, we consider again problems 4.1 and 4.5: for $f(x) \in L^2(\Omega)$, and $1/\lambda(x) \in L^\infty(\Omega)$, the solution u of problem 4.1 belongs to $H^1(\Omega)$ and satisfies the following a priori estimate

$$\|u\|_{\mathbf{H}^1} \leq C \left(\|f\|_0 + |\xi|\right). \tag{6.1}$$

Under the coerciveness assumption (4.6), it is not difficult, by means of standard arguments, to prove the following error estimates.

Lemma 6.1 *Let u be the solution of problems* (4.1) *and let u_h be the solution of problem* (4.5) *with boundary condition $u_h(a) = \xi$, respectively. Assume that* (4.6) *is satisfied, that f, $f_h \in L^2(\Omega)$, λ, $\lambda_h \in L^\infty(\Omega)$ and $\lambda(x), \lambda_h(x) \geq \lambda_* > 0$ for all $x \in \Omega$. Then, the following error estimate holds*

$$\begin{aligned} \mu^* \|u - u_h\|_0^2 + \delta h \|u' - u_h'\|_0^2 \leq \\ \leq Ch\Big(\|f_h\|_0 + \xi^2\Big) + C\Big(\|f - f_h\|_0^2 + \|\lambda - \lambda_h\|_0^2\Big) \end{aligned} \tag{6.2}$$

*where μ^* is the constant in the coerciveness assumption (*4.6*), and C is a constant depending on β, Ω, λ_*, δ and k, but independent of h.*

Proof. Let us consider the following auxiliary problem

$$\begin{cases} \dfrac{\beta}{\lambda_h(x)}\ \hat{u} + \ \hat{u}' = f_h(x) \qquad \text{in } \Omega = (a,\, b) \\ \\ \hat{u}(a) = \xi \end{cases} \tag{6.3}$$

whose exact solution is given by

$$\hat{u}(x) = \exp\left(-\beta \int_a^x \frac{dy}{\lambda_h(y)}\right) \cdot \left[\xi + \int_a^x f_h(t)\, \exp\left(\beta \int_a^t \frac{dy}{\lambda_h(y)}\right) dt\right], \tag{6.4}$$

and, if $f_h \in L^2(\Omega)$, it satisfies an a priori estimate analogous to (6.1). We have

$$\mu^*\|u-u_h\|_0^2+\delta h\|u'-u_h'\|_0^2 \leq$$
$$\leq 2\left(\mu^*\|u-\hat{u}\|_0^2+\delta h\|u'-\hat{u}'\|_0^2+\mu^*\|\hat{u}-u_h\|_0^2+\delta h\|\hat{u}'-u_h'\|_0^2\right).$$

where $\hat{u}$ is the solution of problem (6.3). Let us focus on the first term: by standard manipulation and the use of Hölder and Jensen's inequalities, we get, up to a constant

$$\|u-\hat{u}\|_0^2 \leq |\Omega|^2 e^{\frac{2\beta|\Omega|}{\lambda_*}}\left(\|f-f_h\|_0^2+\|\lambda-\lambda_h\|_0^2\right) \tag{6.5}$$

(for more details see [Ger02]). Using then the fact that u and $\hat{u}$ are solutions of equations (4.1) and (6.3), we have for the second term, up to a multiplicative factor,

$$\|u'-\hat{u}'\|_0^2 \leq \|f-f_h\|_0^2+\frac{\beta^2}{\lambda_*^4}\left(\|\hat{u}\|_0^2\,\|\lambda-\lambda_h\|_0^2+\|\lambda_h\|_0^2\,\|u-\hat{u}\|_0^2\right), \tag{6.6}$$

where the L^2 boundedness of f_h entails L^2 boundedness also for $\hat{u}$.
Finally, let us focus on the last two terms in (6.5). The difference $(\hat{u}-u_h)$ satisfies the following equation

$$\begin{cases} \displaystyle\int_\Omega \left[\frac{\beta}{\lambda_h(x)}(\hat{u}-u_h)+(\hat{u}-u_h)'\right][\varphi+\delta h\varphi']=0 \qquad \forall\varphi\in V_0^h(\Omega) \\ (\hat{u}-u_h)(a)=0 \end{cases} \tag{6.7}$$

Let $\Pi_h^k\hat{u}$ be the interpolant of $\hat{u}$ in $V^h(\Omega)$ (notice that, since $\Omega\subset\mathbf{R}$, $H^1(\Omega)\subset C^0(\Omega)$, and the interpolant is well-defined for any $k\geq 1$); if we choose $\varphi=(\Pi_h^k\hat{u}-u_h)$, which belongs to $V_0^h(\Omega)$, we have

$$\begin{aligned}\int_\Omega &\left\{\frac{\beta}{\lambda_h(x)}(\Pi_h^k\hat{u}-u_h)^2 \;+\; \delta h\left[(\Pi_h^k\hat{u}-u_h)'\right]^2+\left(1+\frac{\beta\delta h}{\lambda_h(x)}\right)(\Pi_h^k\hat{u}-u_h)(\Pi_h^k\hat{u}-u_h)'\right\} \\ &= \int_\Omega\left[\frac{\beta}{\lambda_h(x)}(\Pi_h^k\hat{u}-\hat{u})+(\Pi_h^k\hat{u}-\hat{u})'\right]\left[(\Pi_h^k\hat{u}-u_h)+\delta h(\Pi_h^k\hat{u}-u_h)'\right].\end{aligned} \tag{6.8}$$

Let us focus on the left hand side in (6.8): an integration by parts of the third term, together with the fact that $(\Pi_h^k\hat{u}-u_h)(a)=0$, provides:

$$\begin{aligned}\int_\Omega &\left\{\frac{\beta}{\lambda_h(x)}(\Pi_h^k\hat{u}-u_h)^2 \;+\; \delta h\left[(\Pi_h^k\hat{u}-u_h)'\right]^2+\left(1+\frac{\beta\delta h}{\lambda_h(x)}\right)(\Pi_h^k\hat{u}-u_h)(\Pi_h^k\hat{u}-u_h)'\right\} \\ &\geq \mu^*\|\Pi_h^k\hat{u}-u_h\|_0^2+\delta h\|(\Pi_h^k\hat{u}-u_h)'\|_0^2+\frac{1}{2}(\Pi_h^k\hat{u}-u_h)^2(b).\end{aligned} \tag{6.9}$$

where the inequality stems from the coerciveness assumption (4.6) and the positiveness of β, δ, h, and λ_h. Now, let us consider the right hand side in (6.8): we have by standard arguments

$$\int_\Omega\left[\frac{\beta}{\lambda_h(x)}(\Pi_h^k\hat{u}-\hat{u})+(\Pi_h^k\hat{u}-\hat{u})'\right]\left[(\Pi_h^k\hat{u}-u_h)+\delta h(\Pi_h^k\hat{u}-u_h)'\right]\leq$$
$$\leq Ch^{-1}\|\Pi_h^k\hat{u}-\hat{u}\|_0^2+C\delta h\|(\Pi_h^k\hat{u}-\hat{u})'\|_0^2+\frac{\mu^*}{2}\|\Pi_h^k\hat{u}-u_h\|_0^2+\frac{\delta h}{2}\|(\Pi_h^k\hat{u}-u_h)'\|_0^2,$$

(again, for details see [Ger02]), and we thus have

$$\frac{\mu^*}{2}\|\Pi_h^k\hat{u}-u_h\|_0^2+\frac{\delta h}{2}\|(\Pi_h^k\hat{u}-u_h)'\|_0^2+\frac{1}{2}(\Pi_h^k\hat{u}-u_h)^2(b)\le$$

$$\le Ch^{-1}\|\Pi_h^k\hat{u}-\hat{u}\|_0^2+C\delta h\|(\Pi_h^k\hat{u}-\hat{u})'\|_0^2\le Ch\|\hat{u}\|_{H^1(\Omega)}^2,$$

where the last inequality follows from standard interpolation estimates for finite elements. Finally, using the fact that $\Pi_h^k\hat{u}(b)=\hat{u}(b)$, we can conclude

$$\begin{aligned}&\mu^*\|\hat{u}-u_h\|_0^2+\delta h\|(\hat{u}-u_h)'\|_0^2+(\hat{u}-u_h)^2(b)\le\\ &\le 2\left(\mu^*+Ch^{-1}\right)\|\hat{u}-\Pi_h^k\hat{u}\|_0^2+2C\delta h\|(\Pi_h^k\hat{u}-\hat{u})'\|_0^2 \qquad (6.10)\\ &\le Ch\|\hat{u}\|_{H^1(\Omega)}^2\le Ch\left(\|f_h\|_0^2+|\xi|^2\right),\end{aligned}$$

Gathering together estimates, (6.5), (6.5), (6.6), and (6.10), the thesis follows. □

6.2 Error estimates for the primitive variables

In this section we derive an energy estimate for the FEM approximation through the characteristic approach. Since our main attention focused on the spatial domain decomposition, we give in this section an estimate of the difference between the exact solution at time t^n, $\mathbf{U}(t^n,x)$, and the approximate one stemming from the characteristic approach. Since $\mathbf{V}_1=u+\frac{2}{\gamma-1}c$ and $\mathbf{V}_2=u-\frac{2}{\gamma-1}c$, the inverse change of variable is

$$\mathbf{U}(t,x)=\begin{cases}\rho(t,x)=F_1(\mathbf{V}(t,x))=\left\{\frac{1}{K\gamma}\left[\frac{\gamma-1}{4}(\mathbf{V}_1-\mathbf{V}_2)(t,x)\right]^2\right\}^{1/\gamma-1}\\ \\ u(t,x)=F_2(\mathbf{V}(t,x))=\frac{1}{2}(\mathbf{V}_1+\mathbf{V}_2)(t,x).\end{cases} \qquad (6.11)$$

Due to the nonlinearity of the change of variables, when we map the discretized (either in time or in both time and space) characteristic variables back to the primitive ones, we do not obtain the solution of a discretized version of the original problem in the primitive variables. However, we expect these resulting functions to be a good approximation of the primitive variables. Under these considerations, we denote with $\mathbf{V}^n$ the solution, at time step n, of the single domain problem discretized in time as in Section 3,

$$\beta\mathbf{V}^n+\Lambda^{n-1}\mathbf{V}_x^n=\beta\mathbf{V}^{n-1}, \qquad (6.12)$$

where $\Lambda^{n-1}=\mathrm{diag}(\lambda_1^{n-1},\lambda_2^{n-1})$, as defined therein, and we define, with a little abuse of notation,

$$\mathbf{U}^n(x):=\begin{cases}\rho^n(x)=F_1(\mathbf{V}^n(x))\\ \\ u^n(x)=F_2(\mathbf{V}^n(x))\end{cases}\qquad \mathbf{U}_h^n(x):=\begin{cases}\rho_h^n(x)=F_1(\mathbf{V}_h^n(x))\\ \\ u_h^n(x)=F_2(\mathbf{V}_h^n(x))\end{cases} \qquad (6.13)$$

where $\mathbf{V}_h^n(x)$ is the fully discrete approximation of $\mathbf{V}(t^n,x)$ via the Streamline Diffusion FEM. We are in position to prove the following result.

Lemma 6.2 *Let $\mathbf{U}(t^n,x)$ and $\mathbf{V}(t^n,x)$ be the solutions of problems* (2.2) *and* (2.6) *respectively, at time $t=t^n$, and $\mathbf{V}_h^n(x)$ be the solution of problem* (4.10) *at time step n. Assume that, $\mathbf{U}(t^n,x)$, $\mathbf{V}(t^n,x)\in$*

$L^2(\Omega)$, and that $\mathbf{V}^{n-1}(x) \in L^\infty(\Omega)$. Assume moreover that $\lambda^* \geq |\lambda_j^{n-1}(x)|$, $|\lambda_{hj}^{n-1}(x)| \geq \lambda_* > 0$ ($j = 1, 2$), for all $x \in \Omega$, that $\gamma < 3$, and that (4.6) is satisfied. Then, at time step n, the following error estimate holds

$$\left\|\mathbf{U}(t^n, x) - \mathbf{U}_h^n(x)\right\|_0^2 \leq C \left\|\mathbf{V}(t^n, x) - \mathbf{V}^n(x)\right\|_0^2$$
$$+C \left\|\mathbf{V}^{n-1}(x) - \mathbf{V}_h^{n-1}(x)\right\|_0^2 + Ch \left(\left\|\mathbf{V}_h^{n-1}(x)\right\|_0^2 + |g_1(t^n)|^2 + |g_2(t^n)|^2\right),$$

where $g_1(t^n)$ and $g_2(t^n)$ are the boundary conditions in (2.3) for $t = t^n$, and where the constant C may depend on β, δ, Ω, and λ_*, but is independent of h.

Remark 6.1 A few comments on the assumptions of Lemma 6.2 are in order. The assumption that the exact solution belongs to $L^2(\Omega)$, at time $t = t^n$, in both primitive ($\mathbf{U}$) and characteristic form ($\mathbf{V}$), is not restrictive in the region of smooth flow. The bounds on the modulus of the time discrete λ_j^{n-1}, and fully discrete $\lambda_{h,j}^{n-1}$, $j = 1, 2$, approximations of the eigenvalues $u + c$ and $u - c$ are justified by the assumption we made on the flow to be subsonic. Finally, since for ideal gases the ratio $\gamma \sim 5/3$, the assumption on γ is not restrictive either.

Proof. First of all, notice that, under our assumptions, the function $\mathbf{V}^n$, solution of problem (6.12), belongs to $L^2(\Omega)$. Then, since

$$\left\|\mathbf{U}(t^n, x) - \mathbf{U}_h^n(x)\right\|_0^2 = \left\|\mathbf{U}_1(t^n, x) - \mathbf{U}_{h,1}^n(x)\right\|_0^2 + \left\|\mathbf{U}_2(t^n, x) - \mathbf{U}_{h,2}^n(x)\right\|_0^2,$$

we have to analyze both terms in the above summation. Concerning the first one, we observe that, since $\gamma < 3$, the function $F_1(.)$ is Lipschitz continuous, with Lipschitz constant that we indicate with L_1. We set, for simplicity of notations, $\mathcal{C} := \left[(\gamma - 1)^2/(16K\gamma)\right]^{1/\gamma - 1}$, and we obtain

$$\int_\Omega \left|\mathbf{U}_1(t^n, x) - \mathbf{U}_{h,1}^n(x)\right|^2 = \mathcal{C}^2 \int_\Omega \left| [\mathbf{V}_1(t^n, x) - \mathbf{V}_2(t^n, x)]^{\frac{2}{\gamma-1}} - \left[\mathbf{V}_{h,1}^n(x) - \mathbf{V}_{h,2}^n(x)\right]^{\frac{2}{\gamma-1}} \right|^2$$
$$\leq \mathcal{C}^2 L_1^2 \int_\Omega \left|[\mathbf{V}_1(t^n, x) - \mathbf{V}_2(t^n, x)] - [\mathbf{V}_{h,1}^n(x) - \mathbf{V}_{h,2}^n(x)]\right|^2$$
$$\leq 4\,\mathcal{C}^2 L_1^2 \left(\left\|\mathbf{V}_1(t^n, x) - \mathbf{V}_1^n(x)\right\|_0^2 + \left\|\mathbf{V}_1^n(x) - \mathbf{V}_{h,1}^n(x)\right\|_0^2 + \left\|\mathbf{V}_2(t^n, x) - \mathbf{V}_2^n(x)\right\|_0^2 \right.$$
$$\left. + \left\|\mathbf{V}_2^n(x) - \mathbf{V}_{h,2}^n(x)\right\|_0^2\right),$$

The linearity of $F_2(.)$ allows a simpler treatment of the second term:

$$\int_\Omega \left|\mathbf{U}_2(t^n, x) - \mathbf{U}_{h,2}^n(x)\right|^2 \leq \frac{1}{2}\left(\left\|\mathbf{V}_1(t^n, x) - \mathbf{V}_1^n(x)\right\|_0^2 + \left\|\mathbf{V}_1^n(x) - \mathbf{V}_{h,1}^n(x)\right\|_0^2\right.$$
$$\left. + \left\|\mathbf{V}_2(t^n, x) - \mathbf{V}_2^n(x)\right\|_0^2 + \left\|\mathbf{V}_2^n(x) - \mathbf{V}_{h,2}^n(x)\right\|_0^2\right).$$

Therefore, there exists a constant $\mathcal{K} = \max\{1/2, 4\mathcal{C}^2 L^2\}$ such that

$$\left\|\mathbf{U}(t^n, x) - \mathbf{U}_h^n(x)\right\|_0^2 \leq \mathcal{K} \left(\left\|\mathbf{V}(t^n, x) - \mathbf{V}^n(x)\right\|_0^2 + \left\|\mathbf{V}^n(x) - \mathbf{V}_h^n(x)\right\|_0^2\right)$$

Under our assumptions, we are in the position to use the estimates of the previous section, with $\mathbf{f} = \beta\Lambda^{-1}\mathbf{V}^{n-1}$, and $\mathbf{f}_h$ as in (4.10), and we get from Lemma 6.1

$$\left\|\mathbf{U}(t^n,x) - \mathbf{U}_h^n(x)\right\|_0^2 \leq C \left\|\mathbf{V}(t^n,x) - \mathbf{V}^n(x)\right\|_0^2$$
$$+Ch\left(\left\|\mathbf{f}_h\right\|_0^2 + |g_1(t^n)|^2 + |g_2(t^n)|^2\right) + C\left(\left\|\mathbf{f} - \mathbf{f}_h\right\|_0^2 + \left\|\mathbf{l}^{n-1} - \mathbf{l}_h^{n-1}\right\|_0^2\right)$$

where we have set $\mathbf{l}^{n-1} = (\lambda_1^{n-1}, \lambda_2^{n-1})$, $\mathbf{l}_h^{n-1} = (\lambda_{h1}^{n-1}, \lambda_{h2}^{n-1})$. We easily have

$$\|\mathbf{f}_h\|_0^2 \leq C \left\|\mathbf{V}^{n-1}\right\|_0^2,$$

and, owing to 2.7,

$$\left\|\mathbf{l}^{n-1} - \mathbf{l}_h^{n-1}\right\|_0^2 \leq \ \|P\|_*^2 \left\|\mathbf{V}^{n-1}(x) - \mathbf{V}_h^{n-1}(x)\right\|_0^2,$$

P and $\|.\|_*$ being the matrix in 2.7, and any compatible matrix norm, respectively.
Finally, the boundedness assumption on $\mathbf{l}^{n-1}$ and $\mathbf{V}^{n-1}$ entails (for details, again, see [Ger02])

$$\left\|\mathbf{f} - \mathbf{f}_h\right\|_0^2 \leq C \ \left\|\mathbf{V}^{n-1} - \mathbf{V}_h^{n-1}\right\|_0^2,$$

which concludes the proof. □

Remark 6.2 From the above error estimate, we can easily conclude that, if $\mathbf{V}_h^{n-1}(x)$ converges in the L^2-norm to $\mathbf{V}^{n-1}(x)$ uniformly in h, then the approximation error between $\mathbf{U}(t^n,x)$ and $\mathbf{U}_h^n(x)$ depends, as $h \to 0$, only on the approximation error in the time marching scheme for the characteristic variables, *i.e.*

$$\lim_{h\to 0} \left\|\mathbf{U}(t^n,x) - \mathbf{U}_h^n(x)\right\|_0^2 \leq \ C \ \left\|\mathbf{V}(t^n,x) - \mathbf{V}^n(x)\right\|_0^2.$$

Remark 6.3 The above results are valid at time step n, and our attention was mainly paid to the convergence of the iteration-by-subdomain procedure. Further work needs however to be done in order to link the approximation error at the *n-th* time step to the initial condition $\mathbf{U}_0(x)$.

7 Conclusions

We proposed an iteration-by-subdomain algorithm with interface matching conditions of Dirichlet/Dirichlet type, and we proved its convergence, for both the time discrete and the fully discrete problem, in the case of one dimensional isentropic flows. Convergence is achieved for any choice of the time step Δt in the time marching scheme, and independently of the mesh parameter h. Anyway, a few more comments are in order.
Firstly, we considered isentropic flows, which is not such a retrictive assumption, since this is a good approximation of several phenomena occurring in nature. Then, the result has been obtained for the quasi-linear form of Euler system, thus the convergence of the iterative algorithm is ensured only in the region of smooth flow. Since it is well known that the Euler system develops shocks in a finite time, further work needs to be done to extend this result also in the presence of shocks or rarefaction waves. Finally, a convergence result for the quasi-linear system in higher dimensions without freezing the coefficients is not yet available, and it appears rather complicated.
In conclusion, the result obtained is clearly not optimal, nevertheless it is an attempt to give a theoretical convergence analysis for a domain decomposition approach to the Euler system, a task that, to our knowledge, hasn't been faced yet.

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