WATER RESOURCES RESEARCH, VOL. 40, W04601, doi:10.1029/2003WR002255, 2004

Drying front in a sloping aquifer: Nonlinear effects

Frank Stagnitti,[1] Ling Li,[2,3] J.-Yves Parlange,[4] Wilfried Brutsaert,[5] David A. Lockington,[2] Tammo S. Steenhuis,[4] Marc B. Parlange,[6] D. A. Barry,[7] and William L. Hogarth[8]

Received 10 April 2003; revised 18 December 2003; accepted 29 January 2004; published 9 April 2004.

[1] The profiles for the water table height $h(x, t)$ in a shallow sloping aquifer are reexamined with a solution of the nonlinear Boussinesq equation. We demonstrate that the previous anomaly first reported by *Brutsaert* [1994] that the point at which the water table h first becomes zero at $x = L$ at time $t = t_c$ remains fixed at this point for all times $t > t_c$ is actually a result of the linearization of the Boussinesq equation and not, as previously suggested [*Brutsaert*, 1994; *Verhoest and Troch*, 2000], a result of the Dupuit assumption. Rather, by examination of the nonlinear Boussinesq equation the drying front, i.e., the point x_f at which h is zero for times $t \geq t_c$, actually recedes downslope as physically expected. This points out that the linear Boussinesq equation should be used carefully when a zero depth is obtained as the concept of an "average" depth loses meaning at that time. *INDEX TERMS:* 1829 Hydrology: Groundwater hydrology; 1832 Hydrology: Groundwater transport; *KEYWORDS:* drying front, nonlinear effects, sloping aquifer

Citation: Stagnitti, F., L. Li, J.-Y. Parlange, W. Brutsaert, D. A. Lockington, T. S. Steenhuis, M. B. Parlange, D. A. Barry, and W. L. Hogarth (2004), Drying front in a sloping aquifer: Nonlinear effects, *Water Resour. Res.*, *40*, W04601, doi:10.1029/2003WR002255.

1. Introduction

[2] *Boussinesq* [1877] first formulated a general theory of unconfined groundwater flow in sloping aquifers. Since then, many analytical and numerical solutions of the Boussinesq equation have been developed and have been found to be particularly useful in modeling the hydrology of upland watersheds [e.g., *Brutsaert and Nieber*,1977; *Parlange et al.*, 1981; *Sanford et al.*, 1993; *Brutsaert*, 1994; *Lockington*, 1997; *Brutsaert and Lopez*, 1998; *Szilagyi and Parlange*, 1998; *Szilagyi et al.*, 1998; *Hogarth and Parlange*, 1999; *Verhoest and Troch*, 2000; *Parlange et al.*, 2001]. A common technique to developing analytical solutions for sloping aquifers is to linearize the diffusive term in the Boussinesq equation [e.g., *Brutsaert*,1994]. However, as indicated by *Brutsaert* [1994], an anomaly is observed after the drying front first appears. If $h(x, t)$ is the thickness of the water layer taken perpendicular to the bedrock, having a slope angle i and x being the spatial coordinate measured along the bedrock with $x = 0$ being the lower end and $x = L$ being the upper end of the bedrock, then "a difficulty arises as $h(x, t)$ becomes 0 at $x = L$. Physically, after this occurs, the point of $h = 0$ would be expected to slide down along the bottom of the aquifer from $x = L$ in the direction of $x = 0$" [*Brutsaert*, 1994, p. 2762]. *Verhoest and Troch* [2000, p. 796], using essentially Brutsaert's solution, elaborated that "as time increases, the solution prevents the point where the groundwater table reaches the bedrock from sliding down along the bottom of the aquifer, resulting in slowing down the decrease of the outflow rate". This suggestion that the Dupuit assumption is the cause of the problem [*Brutsaert*, 1994; *Verhoest and Troch*, 2000] was quite reasonable as it is well known in that the Dupuit assumption can lead to poor profiles even when the drainage rate is accurate.

[3] This note is concerned with elucidating the cause of this difficulty. Previous solutions used the linearized form of the Boussinesq equation. We shall demonstrate here that this difficulty does not arise with the nonlinear version of the Boussinesq equation and that it is not a Dupuit assumption but the linearization which is the cause of the problem.

2. Analysis

[4] Using the following dimensionless variables,

$$x^* = x/L \tag{1}$$

$$h^* = h/D \tag{2}$$

$$t^* = tDk \cos i/fL^2 \tag{3}$$

where D is the initial water depth, f is the drainable porosity and k is the saturated hydraulic conductivity, then using these variables, Boussinesq's equation can be written as [*Brutsaert*, 1994]:

$$\frac{\partial h^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left(h^* \frac{\partial h^*}{\partial x^*}\right) + \varepsilon \frac{\partial h^*}{\partial x^*} \tag{4}$$

[1]School of Ecology and Environment, Deakin University, Warrnambool, Australia.
[2]Department of Environmental Engineering, University of Queensland, Brisbane, Queensland, Australia.
[3]Also at Centre for Eco-Environmental Modelling, Hohai University, Nanjing, China.
[4]Department of Biological and Environmental Engineering, Cornell University, Ithaca, New York, USA.
[5]Department of Civil and Environmental Engineering, Cornell University, Ithaca, New York, USA.
[6]Department of Geography and Environmental Engineering, Johns Hopkins University, Baltimore, Maryland, USA.
[7]Institute for Infrastructure and Environment, School of Engineering and Electronics, University of Edinburgh, Edinburgh, UK.
[8]Science and Information Technology, University of Newcastle, Callaghan, New South Wales, Australia.

0043-1397/04/2003WR002255

where

$$\varepsilon = L \tan i / D, \tag{5}$$

h^* satisfies the following initial and boundary conditions,

$$h^* = 1, 0 < x^* < 1; \text{ at } t^* = 0 \tag{6}$$

$$h^* = 0, \text{ at } x^* = 0 \text{ for } t^* > 0 \tag{7}$$

$$\frac{\partial h^*}{\partial x^*} = -\varepsilon \text{ at } x^* = 1 \text{ for } 0 < t^* < t_c^* \tag{8}$$

Equation (8) is the condition of zero flux at $x^* = 1$ and t_c^* is the time when

$$h^*(1, t_c^*) = 0 \tag{9}$$

Of course, for $t^* > t_c^*$,

$$h^*\left(x_f^*, t^*\right) = 0 \tag{10}$$

Equation (10) is not a boundary condition but is used to define x_f^* where x_f^* is the point along the bedrock at which for $t^* > t_c^*$, $h^* = 0$. When there is no drying front, as is the case for the linearized solution of the Boussinesq equation, then $x_f^* = 1$ for all times, $t > t_c^*$. Note that at the position on the bedrock where $h^* = 0$, the flux, q^*, given by

$$q^* = -h^* \frac{\partial h^*}{\partial x^*} - \varepsilon h^*, \tag{11}$$

is automatically zero for $t > t_c$ as it must be for an impervious surface, as long as $\partial h^*/\partial x$ is not infinite. Thus the numerical solution of equation (4) will tell us directly whether x_f^*, when $h^* = 0$, remains equal to one as in the linear case or whether $x_f^* < 1$ for $t^* > t_c$.

3. Results and Discussion

[5] An explicit finite difference scheme was developed for the solution of equation (4) subject to the initial and boundary conditions stated in equations (5)–(8)

$$h_{i,n+1}^* = h_{i,n}^*(1 + a) - a h_{i-1,n}^* + b\left(h^{*2}_{i+l,n} - 2h^{*2}_{i,n} + h^{*2}_{i-1,n}\right),$$
$$i = 2, 3 \ldots M - 1, \tag{12a}$$

$$h_{1,n+1}^* = 0 \tag{12b}$$

$$h_{M,n+1}^* = h_{M-1,n+1}^* - \varepsilon \Delta x^*, \tag{12c}$$

where $a = \frac{\varepsilon \Delta t^*}{\Delta x^*}$, $b = \frac{\Delta t^*}{2\left(\Delta x^*\right)^2}$, Δx^* and Δt^* are the grid size and time step, respectively, i and n are the indices that result from the discretization, and $i = 1$ and M are the two boundary nodes (for $x^* = 0$ and 1, respectively). The stability condition is $\Delta t^* \leq \min\left(\frac{1}{2\left(\Delta x^*\right)^2}, \frac{\varepsilon}{\Delta x^*}\right)$. No special treatment was needed for determination of the moving front x_f obtained for $h_{i,n+1}^*$ effectively equal to zero in agreement with the definition of equation (10) (the first, very small, negative value is in fact chosen because of the discretization). The accuracy of this solution method was also checked using PDE2D [*Sewell*, 1993] software. Both solutions were found to be in excellent agreement.

[6] Figures 1–3 present the numerical solutions for three representative dimensionless hillslope aquifers given by ε = 10, 1.0, and 0.1, respectively. In Figures 1–3 the top graph presents the shape of the dimensionless water table h^* (x^*, t^*) at various times t^*, prior to and following t_c^*. Notice that after $t^* = t_c^*$ i.e., after the first time when h^* (1, t^*) = 0, the point x_f^* "slides" down the bedrock and does not remain at $x^* = 1$ as in the case of the linearized solution [*Brutsaert*, 1994]. The behavior of x_f^* as a function of t^* for $t^* > t_c^*$ is presented by the middle graph in Figures 1–3.

[7] It is now possible to understand why for $t^* > t_c^*$ there is a receding drying front for the nonlinear case, but not for the linearized approximation. Reverting to variables with dimensions, to compare with the linear case, the flux can be written as

$$q/h = -k\left[\frac{\partial h}{\partial x}\cos i + \sin i\right] \tag{13}$$

for the nonlinear case and

$$q/h = -k\left[h^{-1}\overline{D}\frac{\partial h}{\partial x}\cos i + \sin i\right] \tag{14}$$

in the linear case, where $\overline{D}$ is some judiciously chosen "average", i.e., fixed, depth. q/fh is the velocity of the fluid in the x-direction and in particular for $h = 0$ is equal to dx_f/dt, which must be zero for $t = t_c$, and the question is why a zero velocity, i.e., $x_f = L$ can or cannot be maintained for $t > t_c$.

[8] Using again dimensionless variables equation (13) then becomes

$$-\frac{dx_f{}^*}{dt^*} = \varepsilon + \frac{\partial h^*}{\partial x^*} \tag{15}$$

and from equation (14)

$$-h^* \frac{dx_f{}^*}{dt^*} = h^*\varepsilon + \frac{\partial h^*}{\partial x^*}\frac{\overline{D}}{D}. \tag{16}$$

At $t^* = t_c^*$, equation (15) gives $\partial h^*/\partial x^* = -\varepsilon$ in agreement with equation (8). For $t^* > t_c^*$ the aquifer continues drying and even if there were no receding drying front $|\partial h^*/\partial x^*|$ would have to decrease, and eventually be zero. As soon as $|\partial h^*/\partial x^*|$ is smaller than ε, equation (15) shows that dx_f^*/dt cannot be zero, thus a receding drying front must start, as observed numerically. The numerical solution did not require the use of equation (15), i.e., the latter is not a "boundary condition". Rather equation (15) should be automatically satisfied once the numerical solution is obtained and indeed we have done so. The bottom graph in Figures 1–3 is a plot of both dx_f^*/dt^* and $\varepsilon + \partial h^*/dx^*$ as generated from the numerical solutions. The complete agreement indeed confirms that equation (15) tracks the position of x_f^*.

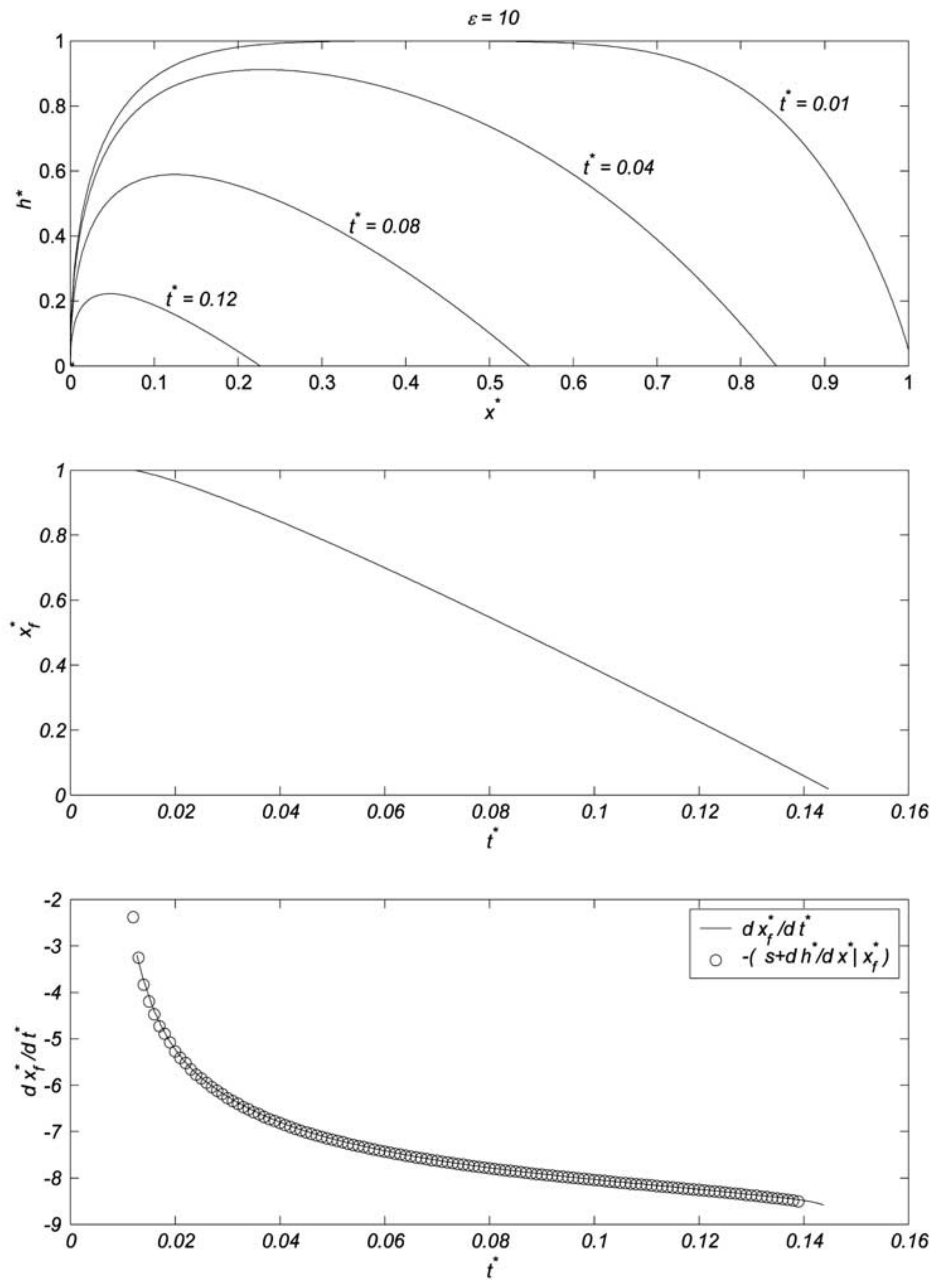

Figure 1. Hillslope type ε = 10. Note for all figures the asterisk in the figure captions has been dropped for convenience. (top) Water table height h^* as a function of dimensionless distance x^*, (middle) x_f^* as a function of time t^* for $t^* > t_c^*$, and (bottom) dx_f^*/dt^* as a function of $\varepsilon + dh^*/dx_f^*$ at x_f^*.

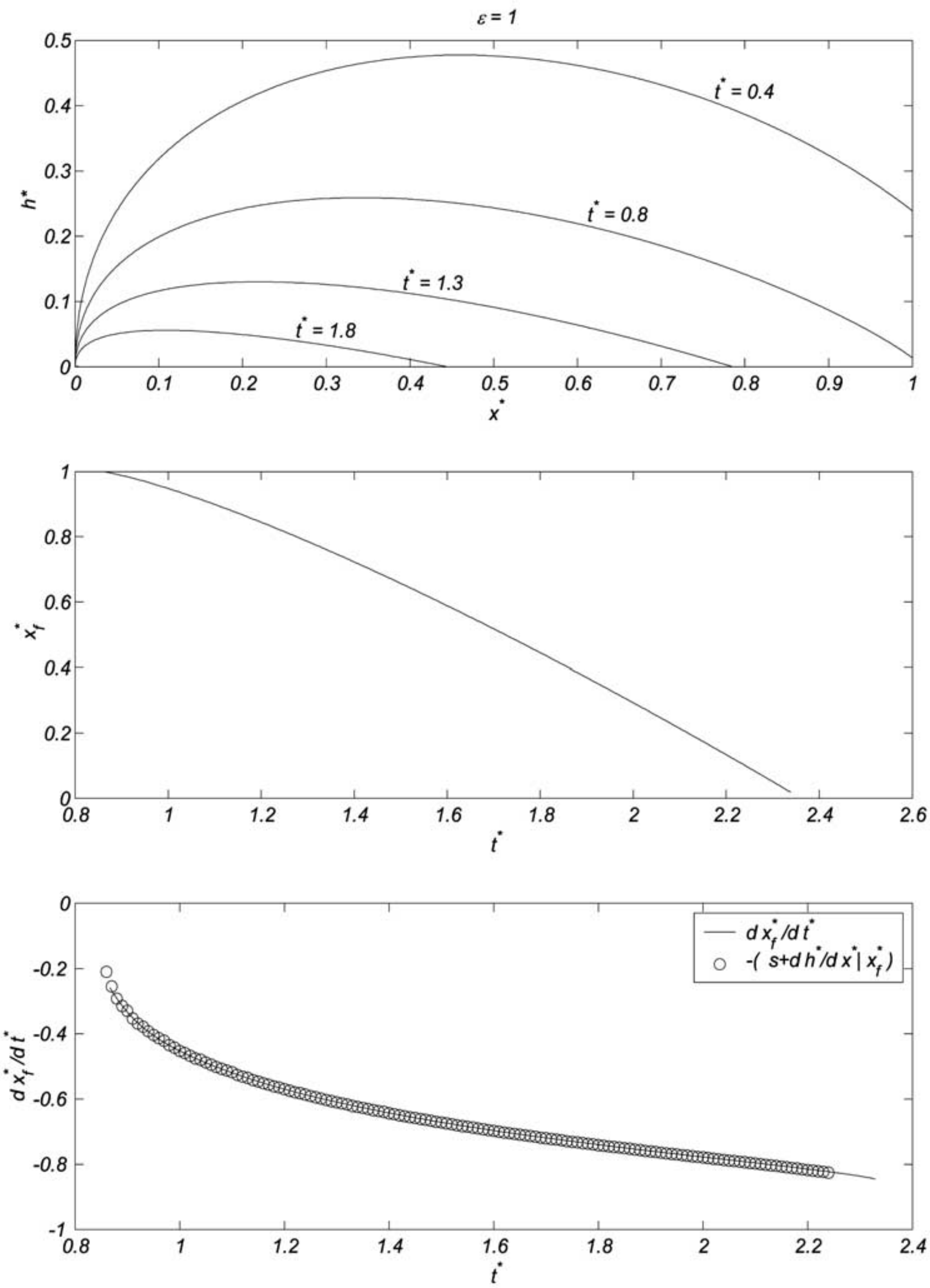

Figure 2. Hillslope type ε = 1.0. (top-bottom) As in Figure 1.

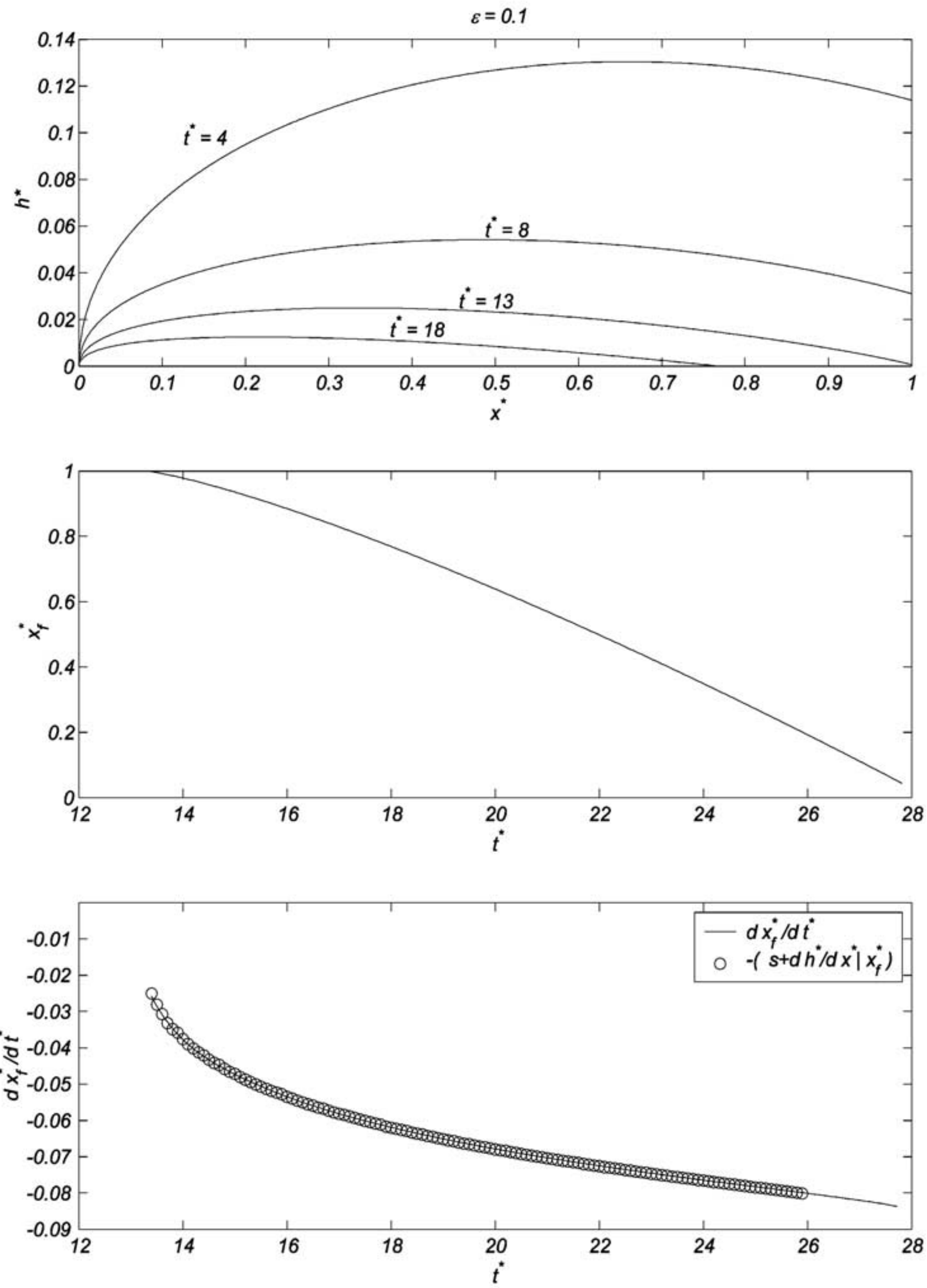

Figure 3. Hillslope type ε = 0.1. (top-bottom) As in Figure 1.

[9] The situation is very different for the linear case in equation (16). As time approaches t_c, $-(\overline{D}/D)(\partial h^*/\partial x^*)$ will approach zero. However since $h^* = 0$ for $t^* \geq t_c^*$ on the bedrock, so will $\partial h^*/\partial x^*$ remain equal to zero. Thus drying of the hillslope can proceed without any need for $|\partial h^*/\partial x^*|$ to become smaller, as in the nonlinear case. That is, there is no necessity in the linear case to have a drying front.

[10] We are very grateful to a reviewer for pointing out that our findings are consistent with a specific aspect of a known problem of the nonlinear diffusion equation, which is a parabolic equation for $h > 0$ but not for $h = 0$ making it "degenerate parabolic" [*Peletier*, 1971; *Aronson*, 1986]. Such equations need not have classical solutions and if at some instant the solution has compact support its support will remain compact for any later time. Of course, linearization makes the equation parabolic everywhere and thus eliminates the possibility of having compact support.

4. Conclusions

[11] We have demonstrated in this paper that the anomaly first observed by *Brutsaert* [1994] that the water table $h(x, t)$ remains at $x = L$ when $h = 0$ for all times $t > t_c$ where t_c is the first time when $h = 0$ at $x = L$ is indeed an artifact of the linearization of the Boussinesq equation and not a result of the application of the Dupuit assumption. In contrast to those observations we have demonstrated that the point x_f, i.e., the point along the bedrock at which h first becomes zero at a time $t > t_c$, indeed recedes from $x_f = 1$ at $t = t_c$ for all times greater than t_c. We have shown that the recession

of x_f is in exact agreement with equation (15) which results from the nonlinearity of the Boussinesq equation. This is an important reminder that one must be wary of possible nonphysical consequences when using a linearized Boussinesq equation when the aquifer hits bedrock.

[12] **Acknowledgments.** The first author acknowledges the support of the Australian Research Council International Linkage grant LX0211202 and Large Grant Scheme A10014154.

References

Aronson, D. G. (1986), The porous medium equation, *Lect. Notes Math.*, *1224*, 1–46.

Boussinesq, J. (1877), Essai sur la theorie des eaux courantes, *Mem. Acad. Sci. Inst. Fr.*, *23*(1), 252–260.

Brutsaert, W. (1994), The unit response of groundwater outflow from a hillslope, *Water Resour. Res.*, *30*, 2759–2764.

Brutsaert, W., and J. P. Lopez (1998), Basin-scale geohydrologic drought flow features of riparian aquifers in the Southern Great Plains, *Water Resour. Res.*, *34*, 233–240.

Brutsaert, W., and J. L. Nieber (1977), Regionalized drought flow hydrographs from a mature glaciated plateau, *Water Resour. Res.*, *13*, 637–643.

Hogarth, W. L., and J.-Y. Parlange (1999), Solving the Boussinesq equation using solutions of the Blasius equation, *Water Resour. Res.*, *35*, 885–888.

Lockington, D. A. (1997), Response of an unconfined aquifer to sudden change in boundary head, *J. Irrig. Drain. Eng.*, *123*, 24–27.

Parlange, J.-Y., R. D. Braddock, and G. Sander (1981), Analytical approximation to the solution of the Blasius equation, *Acta Mech.*, *38*, 119–125.

Parlange, J.-Y., F. Stagnitti, A. Heilig, J. Szilagyi, M. B. Parlange, T. S. Steenhuis, W. L. Hogarth, D. A. Barry, and L. Li (2001), Sudden drawdown and drainage of a horizontal aquifer, *Water Resour. Res.*, *37*, 2097–2101.

Peletier, L. A. (1971), Asymptotic behavior of solutions of the porous media equation, *SIAM J. Appl. Math.*, *21*, 542–551.

Sanford, W. E., J.-Y. Parlange, and T. S. Steenhuis (1993), Hillslope drainage with sudden drawdown: Closed form solution and laboratory experiments, *Water Resour. Res.*, *29*, 2313–2321.

Sewell, G. (1993), PDE2D: Easy to use software for general two-dimensional partial differential equations, *Adv. Eng. Software*, *17*, 105–112.

Szilagyi, J., and M. B. Parlange (1998), Base flow separation based on an analytical solution of the Boussinesq equation, *J. Hydrol.*, *204*, 251–260.

Szilagyi, J., M. B. Parlange, and J. D. Albertson (1998), Recession flow analysis for aquifer parameter determination, *Water Resour. Res.*, *34*, 1851–1858.

Verhoest, N. E. C., and P. A. Troch (2000), Some analytical solutions of the linearized Boussinesq equation with recharge for a sloping aquifer, *Water Resour. Res.*, *36*, 793–800.

D. A. Barry, Institute for Infrastructure and Environment, School of Engineering and Electronics, Crew Building, The King's Buildings, University of Edinburgh, Edinburgh EH9 3JN, UK.

W. Brutsaert, Department of Civil and Environmental Engineering, Cornell University, Ithaca, NY 14853, USA.

W. L. Hogarth, Science and Information Technology, University of Newcastle, Callaghan NSW 2308, Australia.

L. Li and D. A. Lockington, Department of Environmental Engineering, University of Queensland, Brisbane, 4072 Queensland, Australia.

J.-Y. Parlange and T. S. Steenhuis, Department of Biological and Environmental Engineering, Riley-Robb Hall, Cornell University, Ithaca, NY 14853, USA. (jp58@cornell.edu)

M. B. Parlange, Department of Geography and Environmental Engineering, Johns Hopkins University, Baltimore, MD 21218, USA.

F. Stagnitti, School of Ecology and Environment, Deakin University, Warrnambool, 3280 Victoria, Australia.