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Generation of macroscopic superposition states with small nonlinearity

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We suggest a scheme to generate a macroscopic superposition state ("Schrödinger cat state") of a free-propagating optical field using a beam splitter, homodyne measurement, and a very small Kerr nonlinear effect. Our scheme makes it possible to reduce considerably the required nonlinear effect to generate an optical cat state using simple and efficient optical elements.

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Introduction. In the well-known cat state paradox, Schrödinger tried to demonstrate the possibility of generating a quantum superposition of a macroscopic system [1]. A superposition of two coherent states with a π phase difference and a large amplitude is considered a realization of such a macroscopic superposition and is sometimes called the "Schrödinger cat state." Recently, it has been found that the cat state of a propagating optical field is useful not only for the study of fundamental quantum physics but also for various applications to quantum information processing [2–9]. Once the optical cat state is generated, quantum teleportation [2–4], quantum nonlocality test [10], generation and purification [3,8] of entangled coherent states, quantum metrology [9], and quantum computation [5–7] will become closer to experimental realization using current technology.

It has been theoretically known that the cat state can be generated from a coherent state by a nonlinear interaction in a Kerr medium [11]. However, the Kerr nonlinearity of currently available media is too small to generate the cat state. It was pointed out that one needs an optical fiber of about 1500 km for an optical frequency of $\omega \approx 5 \times 10^{14}$ rad/sec to generate a coherent superposition state with currently available Kerr nonlinearity [12,13]. Even though it is possible in principle to make such a long nonlinear optical fiber, the effects of decoherence and phase fluctuations during the propagation will become too large.

Some alternative methods have been studied to generate a superposition of macroscopically distinguishable states using conditional measurements [15,16]. One drawback of these schemes is that a highly efficient photon-counting measurement is required to obtain a coherent superposition state, which is difficult using current technology. Cavity quantum electrodynamics has been studied to enhance nonlinearity [17]. Even though there have been experimental demonstrations of generating cat states in a cavity and in a trap [18,19], all the suggested schemes for quantum information processing with coherent states [2–9] require *free propagating* optical cat states.

Electromagnetically induced transparency (EIT) has been studied as a method to obtain a giant Kerr nonlinearity [20]. There has been an inspiring suggestion to generate cat states with it [21] but this developing technology of EIT has not been exactly at hand yet to generate a state in a quantum regime. Recently, Lund *et al.* proposed a simpler optical scheme to generate a propagating cat state of $|\alpha| \approx 2$ [22], which does not require Kerr-type nonlinearity nor photon-counting measurements.

In this paper, we study a probabilistic scheme to generate cat states with a small Kerr effect. We are particularly interested in generating a cat state of $|\alpha| \geqslant 10$, i.e., the average photon number over 100. Cat states with large amplitudes are preferred for quantum information processing. For example, higher precision is obtained for quantum metrology when large cat states are supplied [9]. Our scheme significantly reduces the required nonlinear effect to generate cat states using a beam splitter and homodyne measurement which are basic and efficient tools in quantum optics laboratories.

Generating a cat state with Kerr nonlinearity and its limitation. A cat state is defined as

$$|S_{(cat)\alpha,\varphi}\rangle = \mathcal{N}(\alpha,\varphi)(|\alpha\rangle + e^{i\varphi}|-\alpha\rangle), \tag{1}$$

where $\mathcal{N}(\alpha,\varphi)$ is a normalization factor, $|\alpha\rangle$ is a coherent state of amplitude α, and φ is a real local phase factor. Note that the relative phase φ can be approximately controlled by the displacement operation for a given cat state with $\alpha \gg 1$ [5,23]. The Hamiltonian of a single-mode Kerr nonlinear medium is $\mathcal{H}_{NL} = \omega a^\dagger a + \lambda (a^\dagger a)^2$, where a and $a^\dagger$ are annihilation and creation operators, ω is the energy level splitting for the harmonic-oscillator part of the Hamiltonian, and λ is the strength of the Kerr nonlinearity [11]. Under the influence of the nonlinear interaction the initial coherent state with the coherent amplitude α evolves to the following state at time τ:

$$|\psi(\tau)\rangle = e^{-|\alpha^2|/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-i\phi_n}}{\sqrt{n!}} |n\rangle, \tag{2}$$

where $\phi_n = \lambda \tau n^2$. When the interaction time $\lambda\tau$ in the medium is π/N with a positive integer N, the initial coherent state $|\alpha\rangle$ evolves to [14]

$$|\psi_N\rangle = \sum_{n=1}^{N} C_{n,N} |-\alpha e^{2in\pi/N}\rangle, \tag{3}$$

where $C_{n,N} = e^{i\zeta_n}/\sqrt{N}$. Comparing Eqs. (2) and (3) for an arbitrary N, we find an equation for the arguments ζ_n's of the coefficients of the coherent components, i.e.,

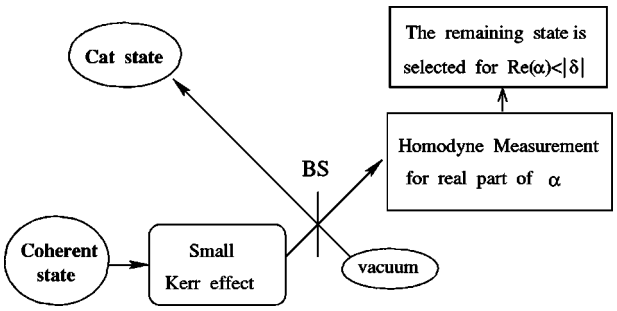

FIG. 1. Schematic of generating a cat state using small Kerr nonlinearity, a beam splitter, and a homodyne detection.

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{i\zeta_n}(-e^{2in\pi/N})^k = \exp(-i\pi k^2/N). \tag{4}$$

By solving the N coupled equations given by Eq. (4), the values ζ_n's are obtained as [24]

$$C_{n,N} = \frac{e^{i\zeta_n}}{\sqrt{N}} = \frac{1}{N}\sum_{k=0}^{N-1}(-1)^k \exp\left[-\frac{i\pi k}{N}(2n-k)\right]. \tag{5}$$

The process shown above can produce a large amount of entanglement in a short time [25]. The length L of the nonlinear cell corresponding to τ is $L=v\pi/2\lambda N$, where v is the velocity of light. For $N=2$, we obtain a desired cat state of the form (1) with $\varphi=\pi/2$ [11]. We pointed out that the nonlinear coupling λ is typically very small such that $N=2$ cannot be obtained in a length limit where the decoherence effect can be neglected.

Generating a cat state with smaller nonlinearity. If λ is not as large as required to generate the cat state, the state (3) with $N>2$ may be obtained by choosing an appropriate interaction time. From the state (3), it is required to remove all the other coherent component states except two coherent states of a π phase difference. First, we assume that the state (3) is incident on a 50-50 beam splitter with the vacuum onto the other input of the beam splitter, as shown in Fig. 1. The initial coherent amplitude α_i is taken to be real for simplicity. The state (3) with initial amplitude α_i after passing through the beam splitter becomes

$$|\psi_N\rangle = \sum_{n=1}^{N} C_{n,N}|-\alpha_i e^{2in\pi/N}/\sqrt{2}\rangle|-\alpha_i e^{2in\pi/N}/\sqrt{2}\rangle, \tag{6}$$

where all $|C_{n,N}|$'s have the same value. The real part of the coherent amplitude in the state (6) is then measured by homodyne detection in order to produce the cat state in the other path. By the measurement result, the state is reduced to

$$|\psi_N^{(1)}\rangle = \sum_{n=1}^{N} C_{n,N}^{(1)}(\alpha_i)|-\alpha_i e^{2in\pi/N}/\sqrt{2}\rangle, \tag{7}$$

where $C_{n,N}^{(1)}(\alpha_i)=\mathcal{N}_\psi\Sigma_{n=1}^{N}C_{n,N}\langle X|-\alpha_i e^{2in\pi/N}/\sqrt{2}\rangle$ with $\mathcal{N}_\psi$ the normalization factor and $|X\rangle$ the eigenstate of $\hat{X}=(a+a^\dagger)/\sqrt{2}$. After the homodyne measurement, the state is selected when the measurement result is in certain values. If coefficients $|C_{N/2,N}^{(1)}(\alpha_i)|$ and $|C_{N,N}^{(1)}(\alpha_i)|$ in Eq. (7) have the same nonzero value and all the other $|C_{n,N}^{(1)}(\alpha_i)|$'s are zero, then the state becomes a desired cat state. Suppose $N=4k$ where k is a positive integer number. If $X=0$ is measured in this case, the coefficients $|C_{n,N}^{(1)}(\alpha_i)|$'s will be the largest when $n=N/4$ and $n=3N/4$, and become smaller as n is far from these two points. The coefficients can be close to zero for all the other n's for an appropriately large α_i so that the resulting state may become a cat state of high fidelity.

The fidelity between the state (7) obtained by our process and a "perfect" cat state of the form (1) with appropriate amplitude is

$$\begin{aligned} f(\alpha_i,N,X) &= \max_{\varphi}[|\langle S_{(cat)\alpha_i/\sqrt{2},\varphi}|\psi_N^{(1)}\rangle|^2] \\ &= \max_{\varphi}\left(\mathcal{N}(\alpha_i,\varphi)^2\mathcal{N}_\psi^2\left|\sum_{n=1}^{N} C_{n,N}^{(1)}\right.\right. \\ &\quad\times\exp\left[-\frac{\alpha_i^2}{2}(1+e^{2in\pi/N})\right] + e^{i\varphi}\sum_{n=1}^{N} C_{n,N}^{(1)} \\ &\quad\left.\left.\times\exp\left[-\frac{\alpha_i^2}{2}(1-e^{2in\pi/N})\right]\right|^2\right). \end{aligned} \tag{8}$$

The success probability to get a cat state is

$$\begin{aligned} \mathcal{P}(\alpha_i,N,\delta) &= \int_\delta dX \mathrm{Tr}[\rho_1|X\rangle\langle X|] \\ &= \int_\delta dP \sum_{nm}^{N}\langle -\alpha_i e^{2in\pi/N}/\sqrt{2}|X\rangle\langle X| - \alpha_i e^{2im\pi/N}/\sqrt{2}\rangle \\ &\quad\times\exp\left[-\frac{\alpha_i^2}{2}(1-e^{2i(m-n)\pi/N})\right] \end{aligned} \tag{9}$$

where $\rho_1=\mathrm{Tr}_2[|\psi_N\rangle_{1212}\langle\psi_N|]$ and δ is the range in which the high fidelity is obtained. Note that the initial coherent amplitude α_i needs to be larger as N increases for better fidelity.

We first examine an example of $\alpha_i=20$ and $\lambda\tau=\pi/20$, i.e., the interaction time (or the nonlinear strength) is an order of magnitude shorter (weaker) than the required value. After passing through the nonlinear medium, the fidelity between the generated state and an ideal cat state is $F\approx 0.1$. The probability distribution of P, which is the conjugate variable of X, is shown in Fig. 2(a). After beam splitting and the homodyne measurement are applied, the state is drastically reduced to a cat state with amplitude $|\alpha|=\alpha_i/\sqrt{2}\approx 14.1$. The maximum fidelity of this cat state is when the measurement result is $X=0$ for $\varphi=\pi$. Fig. 2(b) shows two well separated peaks of the cat state produced for the case of $X=0$. A high fidelity is obtained for a certain range δ of the measurement outcome as shown in Fig. 3. The total success probability can be calculated by integrating Eq. (9) over δ. The success probability for $F>0.99999$ is numerically obtained as $\approx 10\%$, which means that only 10 trials are required on average to obtain a cat state of $F>0.99999$. The success probability for $F>0.99$ is about 4% if $\lambda\tau=\pi/40$. If $\lambda\tau=\pi/60$, the maximum fidelity is 0.975 and the success probability for $F>0.9$ is 2%. Our scheme generates well separated peaks even for $\lambda\tau=\pi/200$ as shown in Fig. 4. Even though the

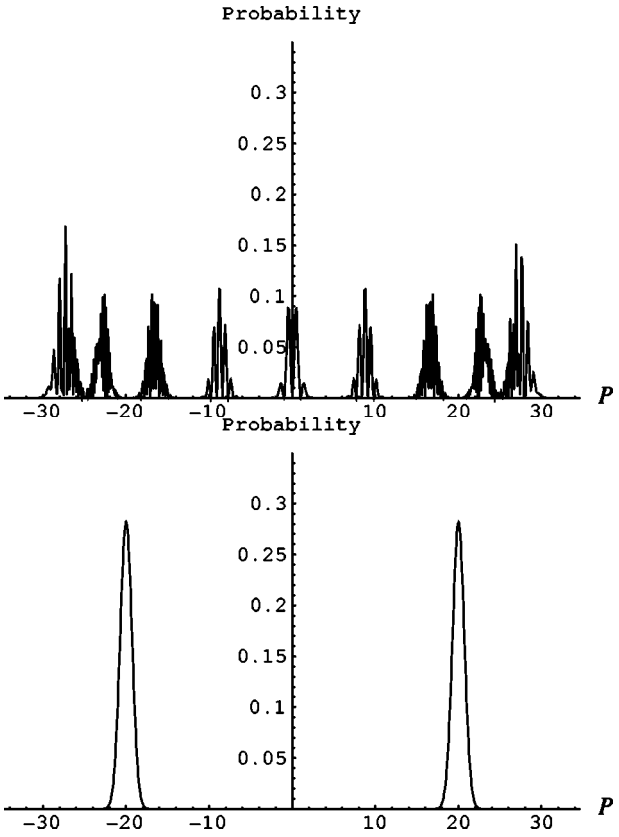

FIG. 2. (a) The probability distribution of the quadrature variable P for the state after passing the small nonlinear medium but *before* the beam splitter in Fig. 1. The interaction time $\lambda\tau=\pi/20$ and the initial amplitude $\alpha_i=20$. (b) The probability distribution for the state *after* the beam splitter and homodyne measurement of result $X=0$. It is obvious from the figures that a cat state with well separated peaks is obtained after the process.

resulting state in this case is somewhat different from the ideal cat state in Eq. (1), this shows that one may observe a conspicuous signature of Schrödinger cat state (macroscopic superposition) even with a $1/100$ times weaker nonlinearity compared with the currently required level.

Error sources. In our proposal, a coherent state can be well approximated by a laser field. Beam splitting of a propagating field with a vacuum can also be efficiently performed using current technology [26]. However, decoherence and random phase fluctuations in a nonlinear medium would be main sources of errors in our scheme. We have neglected these effects because the initial state would pass through a relatively short length of medium. We now roughly assess the effects of decoherence and random phase fluctuations as follows.

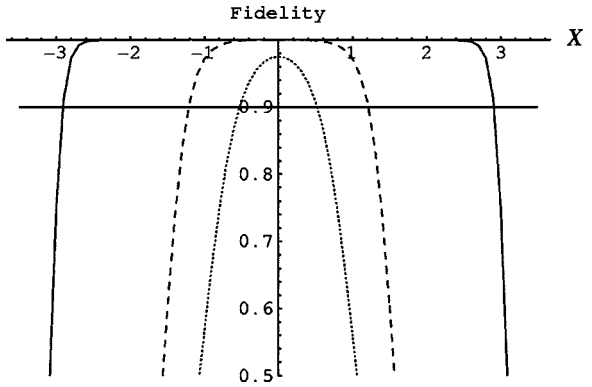

FIG. 3. Fidelity F of the generated cat state against the measurement outcome X where $N=20$ (solid line), $N=40$ (dashed line), $N=60$ (dotted line). The initial amplitude is $\alpha_i=20$.

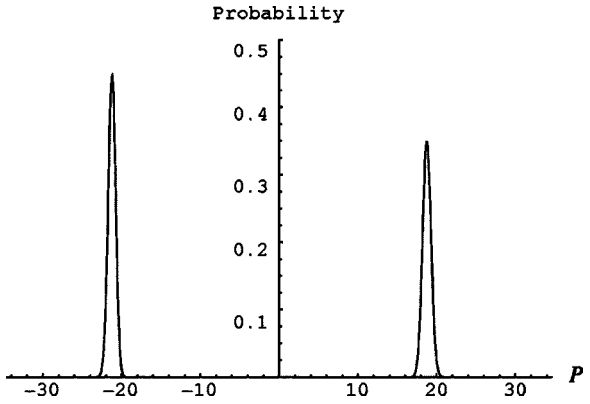

FIG. 4. The probability distribution of P for the state conditioned on the homodyne measurement of result $X=0$ after the beam splitter when the interaction time is $\lambda=\pi/200$ and the initial amplitude is $\alpha_i=20$.

Besides the gradual reduction of amplitude by the loss of the average energy, photon losses will cause the loss of the phase information and make the final state mixed. An analysis of Eqs. (3)–(6) shows that photon losses only at the later stage in the nonlinear medium will significantly affect the phase of the final state. We assume that photon losses only at the final stage change the phase. In this case, if an odd number of photons are lost the phase of the final state is flipped by π while it does not change if an even number of photons are lost. The final state is then represented by

$$\rho_{env}^{(1)} = (1-\mathcal{P}_f)|\Psi_N^{(1)}\rangle\langle\Psi_N^{(1)}| + \mathcal{P}_f|\Phi_N^{(1)}\rangle\langle\Phi_N^{(1)}|, \qquad (10)$$

where $\mathcal{P}_f=\Sigma_{n=0}^{\infty}p_{2n+1}$, p_n is the probability of losing n photons, $|\Psi_N^{(1)}\rangle=|\psi^{(1)}_{N_{\alpha_i e^{-\gamma\tau/2},\varphi}}\rangle$, $|\Phi_N^{(1)}\rangle=|\psi^{(1)}_{N_{\alpha_i e^{-\gamma\tau/2},\varphi+\pi}}\rangle$, and γ is the energy decay rate. The probability p_n is given by a Poisson distribution. If the probability of losing photons at the final stage is 10%, the maximum fidelity will be $F\approx 0.88$. If the probability increases to 30% (60%), the maximum fidelity will decrease to $F\approx 0.71$ ($F\approx 0.55$).

The phase of the state in the medium can randomly fluctuate during the process. For example, if the phase fluctuates by $\Delta\varphi$ during the process, the final state will be $|\psi_N^{\Delta\varphi}\rangle \approx \Sigma_{n=1}^{N} C_{n,N}^{(1),\Delta\varphi}|-\alpha_i e^{i(2n\pi/N+\Delta\varphi)}/\sqrt{2}\rangle$, where $C_{n,N}^{(1),\Delta\varphi}=C_{n,N}\langle X|-\alpha_i e^{i(2n\pi/N+\Delta\varphi)}/\sqrt{2}\rangle$. We suppose the distribution of the phase fluctuation is Gaussian. The average fidelity between the phase-fluctuated state and the ideal cat state for the measurement result $X=0$ will then be $F=\int_{-\infty}^{\infty}d\Delta\varphi G(\Delta\varphi,\sigma) \times\langle S_{(cat)\alpha_i/\sqrt{2},\varphi_{\max}}|\psi_N^{\Delta\varphi}\rangle$, where $\varphi_{\max}$ is the fidelity-maximizing phase for $\Delta\varphi=0$ and $G(\Delta\varphi,\sigma)$ is the Gaussian distribution of $\Delta\varphi$ with standard deviation σ. The fidelity against the standard deviation σ is plotted in Fig. 5. It can be simply shown

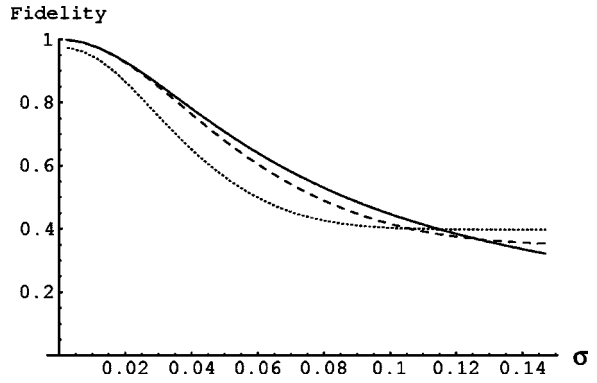

FIG. 5. The average fidelity between the ideal cat state and the phase-fluctuated state with standard deviation σ during the process in Fig. 1. The cases for $N=20$ (solid line), $N=40$ (dashed line), and $N=60$ (dotted line) have been plotted with the initial amplitude $\alpha_i=20$. The result of the homodyne measurement is considered $X=0$.

that the phase fluctuation in Fig. 5 is just the same order of the Gaussian weighted integration of $|\langle \alpha_i/\sqrt{2}|\alpha_i e^{i\Delta\varphi}/\sqrt{2}\rangle|^2$. This kind of phase fluctuation problem is typical in continuous-variable quantum optics experiments such as a squeezing experiment. One can significantly reduce this sensitivity by being less ambitious for making a large cat state, i.e., by reducing the amplitude of the initial coherent state.

Remarks. We have suggested an optical scheme using a beam splitter and homodyne detection to generate a cat state with relatively small nonlinearity. It has been found that the required nonlinear effect to generate a useful cat state with $|\alpha|>10$ and $F>0.9$ can be reduced to less than $1/30$. A signature of a Schrödinger cat state can be obtained even with a $1/100$ times weaker nonlinearity compared with the currently required level.

Our scheme is an effort to considerably reduce the required nonlinear effect to generate a cat state using a beam splitter and homodyne detection which are efficient tools in quantum optics laboratories. Experimental efforts are being made for optical fibers with loss as low as 0.01 db/km where a signal attenuates by half in about 300 km [27,28]. If one can reduce the required level of nonlinearity by, e.g., 30 times (or 100 times), such a level of nonlinear effect will be gained in an optical fiber of 50 km (or 15 km). Then there will be a significantly improved possibility of producing a cat state using the nonlinear fiber. Various nonlinear crystals may be considered instead of optical fibers. It might be possible to obtain even lower ratios of losses to nonlinearity by using whispering gallery modes of a microsphere constructed from a nonlinear material [29].

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