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Comment on "Grover search with pairs of trapped ions"

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In this Comment on Feng's paper [Phys. Rev. A **63**, 052308 (2001)], we show that Grover's algorithm may be performed exactly using the gate set given, provided that small changes are made to the gate sequence. An analytic expression for the probability of success of Grover's algorithm for any unitary operator U instead of Hadamard gate is presented.

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In Ref. [1], Feng attempts to describe how Grover's algorithm may be performed using trapped ions. In contrast to earlier proposals, Feng proposes using pairs of trapped ions. The advantage of this scheme is that it would eliminate one source of phase error known as superpositional wave-function oscillations [2]. However, we think that the incorrect gate sequence for Grover's algorithm is used in Ref. [1], and therefore the results obtained are not ideal. By making a small change to the gate sequence given, we show that Grover's algorithm may be performed exactly using only the operations introduced by Feng.

Two quantum operations are introduced by Feng [1]. The first operation is an X rotation, denoted by U,

$$U(\theta) = R_x(2\theta) \tag{1}$$

$$= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

A derivation for this operation may be found in Ref. [3]. To create a rotation by $\pi/2$ the authors suggest rotating by $\theta = 7\pi/4$, although this rotation may be obtained more easily by rotating by $\theta = 3\pi/4$. For clarity we define

$$W_n = R_x\left(-\frac{\pi}{2}\right)^{\otimes n}, \tag{3}$$

as in Eq. (6) of Ref. [1], where n is the number of qubits.

The second gate operation introduced is the $\Lambda_1 Y$ (controlled Y) operation. This operation is denoted by M in Feng's original paper:

$$M_1^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{4}$$

This operation may be used to create entanglement. It is not explicitly stated how this operation or the multiqubit $\Lambda_n Y$ operation is performed. However, using this operation, it is clear that it is possible to change the sign of one state, as is desirable in Grover's algorithm. This is described by the operation $P_m^{(n)}$ where n is the number of qubits and m is the number of the state to change sign. Using this gate set [$U(\theta)$ and M] Feng attempts to perform Grover's algorithm, which as he correctly realizes should be possible.

The major issue in Feng's [1] paper, we think, is to incorrectly implement Grover's algorithm. This is masked by the fact that the graphs given in Feng's paper (Figs. 1–3) are labeled incorrectly, and show amplitude, not probability. The maximum probability of success obtained in the search, using the method given in Ref. [1], for the $|111\rangle$ state was approximately ≈38%, and not the 62% shown on the graph. The actual probabilities for his implementation can be found by squaring amplitudes given, making all of the probabilities of success considerably less than those quoted.

A (correct) prescription for performing such a search is given by Grover [4] which may be implemented using X rotations instead of Hadamard gates. To correctly perform Grover's algorithm requires rotations of both $R_x(\pi/2)$ and $R_x(-\pi/2)$. These are not implemented in Feng's paper. Specifically, Eq. (4) in Ref. [1] should be specified by

$$D_2 = W_2 P_1^{(2)} W_2^{\dagger}, \tag{5}$$

and not

$$D_2 = W_2 P_1^{(2)} W_2. \tag{6}$$

Figure 1 shows the corrected circuit diagram for Grover's algorithm acting on two qubits. For larger systems of more than two qubits, similar circuits can be drawn.

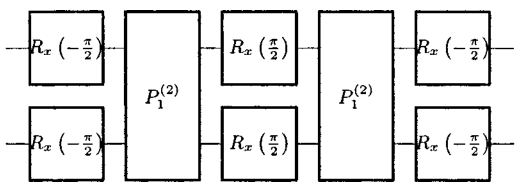

FIG. 1. Gate sequence of Grover search on two qubits.

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Although our change makes no difference to probability of success of two qubit states, the corresponding change for states of three or more qubits makes a significant difference to the success of the algorithm. The probability of finding a marked state may be found analytically for the modified gate sequences, as shown later in this paper. The probabilities exhibit the same periodic behavior, and have the same maximum probabilities expected from Grover's algorithm. With this small change to the gate sequence, Grover's algorithm may be performed exactly using the operations introduced by Feng.

In fact, a simple analytic expression can be derived for the probability of success of Grover's algorithm using an arbitrary rotation U in the place of the Hadamards gates. Consider one step of the algorithm, given by

$$Q = -P_\gamma U^\dagger P_\tau U. \tag{7}$$

In this equation, $|\gamma\rangle$ is the initially prepared state whose sign is also flipped by the Grover iteration. $|\tau\rangle$ is the marked state. The subspace spanned by $|\gamma\rangle$ and $U^\dagger|\tau\rangle$ is invariant under the operation Q. Using the identities

$$P_\gamma = I - 2|\gamma\rangle\langle\gamma|, \tag{8}$$

$$P_\tau = I - 2|\tau\rangle\langle\tau|, \tag{9}$$

it can be shown [4] that

$$Q\begin{bmatrix} |\gamma\rangle \\ U^\dagger|\tau\rangle \end{bmatrix} = \begin{bmatrix} (1-4|U_{\tau\gamma}|^2) & 2U_{\tau\gamma} \\ -2U^*_{\tau\gamma} & 1 \end{bmatrix}\begin{bmatrix} |\gamma\rangle \\ U^\dagger|\tau\rangle \end{bmatrix}, \tag{10}$$

where

$$U_{\tau\gamma} = \langle\tau|U|\gamma\rangle. \tag{11}$$

Notice that $U^\dagger|\gamma\rangle$ and $|\gamma\rangle$ are not orthogonal. We wish to represent the rotation Q in an orthonormal basis. To do this we introduce the vector

$$|\gamma'\rangle = \frac{|\gamma\rangle - U_{\tau\gamma}U^\dagger|\tau\rangle}{\sqrt{1-|U_{\tau\gamma}|^2}}. \tag{12}$$

$|\gamma'\rangle$ and $U^\dagger|\gamma\rangle$ form an effective spin orthonormal basis for the subspace on which Q acts; that is, $\langle\gamma'|\gamma'\rangle=1$ and $\langle\gamma'|U^\dagger|\tau\rangle=0$.

We now show that in this new basis, the Grover iteration simply represents a rotation in SU(2). An arbitrary rotation in SU(2) around a unit vector $\hat{\mathbf{n}}$ by an angle of 2ϕ may be represented by

$$R_{\hat{n}}(2\phi) = e^{-i\phi\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}} \tag{13}$$

$$= \cos(\phi)I - i\sin(\phi)\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}. \tag{14}$$

For a vector, $\hat{\mathbf{n}}=n_x\hat{\boldsymbol{i}}+n_y\hat{\boldsymbol{j}}$ in the XY plane, this rotation may be written as

$$R_{\hat{n}}(4\phi) = \begin{bmatrix} \cos(2\phi) & -\sin(2\phi)(n_y+in_x) \\ \sin(2\phi)(n_y-in_x) & \cos(2\phi) \end{bmatrix}. \tag{15}$$

Expressed as an operation in the pseudospin basis of $|0_L\rangle=|\gamma'\rangle$ and $|1_L\rangle=U^\dagger|\tau\rangle$, Eq. (7) yields

$$Q' = \begin{bmatrix} 1-2|U_{\tau\gamma}|^2 & -2U^*_{\tau\gamma}\sqrt{1-|U_{\tau\gamma}|^2} \\ 2U_{\tau\gamma}\sqrt{1-|U_{\tau\gamma}|^2} & 1-2|U_{\tau\gamma}|^2 \end{bmatrix}. \tag{16}$$

$$= \begin{bmatrix} \cos(2\theta) & -\dfrac{U^*_{\tau\gamma}}{|U_{\tau\gamma}|}\sin(2\theta) \\ \dfrac{U_{\tau\gamma}}{|U_{\tau\gamma}|}\sin(2\theta) & \cos(2\theta) \end{bmatrix}, \tag{17}$$

where we have defined the angle θ by

$$\sin\theta = |U_{\tau\gamma}|, \tag{18}$$

$$\cos\theta = \sqrt{1-|U_{\tau\gamma}|^2}. \tag{19}$$

Comparing the equation for an arbitrary rotation, Eq. (15), and the equation for Q', Eq. (17), we see that Q' is a rotation by an angle of 4θ around a unit vector defined by

$$\hat{\mathbf{n}} = -\mathrm{Im}\frac{U_{\tau\gamma}}{|U_{\tau\gamma}|}\hat{\imath} + \mathrm{Re}\frac{U_{\tau\gamma}}{|U_{\tau\gamma}|}\hat{\jmath}. \tag{20}$$

Clearly $\hat{\mathbf{n}}$ is a unit vector lying in the XY plane.

With appropriate definitions of θ [Eqs. (18) and (19)] and $\hat{\mathbf{n}}$ [Eq. (20)] we can express Q' as

$$Q' = e^{-i2\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}. \tag{21}$$

This shows that Q' is a rotation in SU(2). The angle rotated depends only on the magnitude of a single matrix element $U_{\tau\gamma}$. The direction of rotation is only dependent on the phase of the same matrix element $U_{\tau\gamma}$. This equation holds for arbitrary numbers of qubits and for arbitrary choice of U.

We now find the probability of success of Grover's algorithm. First, we consider the initial state $|\gamma\rangle$. In terms of a rotation we see from Eq. (12) that

$$|\gamma\rangle = \sqrt{1-|U_{\tau\gamma}|^2}|\gamma'\rangle + U_{\tau\gamma}U^\dagger|\tau\rangle \tag{22}$$

$$= e^{-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}|\gamma'\rangle. \tag{23}$$

The matrix $e^{-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$ represents a rotation of 2θ around the $\hat{\mathbf{n}}$ axis, Eq. (20), on the pseudospin Bloch sphere. Initially there is a rotation by an angle of $\theta_i=2\theta$. Each application of Q' leads to the rotation by a further $\theta_r=4\theta$. Every rotation is applied in the same plane, orthogonal to $\hat{\mathbf{n}}$. This is shown in Fig. 2.

After s applications of the Grover iteration, the state of the system is

$$|\psi_s\rangle = e^{-(2s+1)\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}|\gamma'\rangle \tag{24}$$

$$= \cos[(2s+1)\theta]|\gamma'\rangle + \frac{U_{\tau\gamma}}{|U_{\tau\gamma}|}\sin[(2s+1)\theta]U^\dagger|\tau\rangle. \tag{25}$$

The probability of success P_s is simply given by the absolute value squared of the amplitude of $U^\dagger|\tau\rangle$. We then find

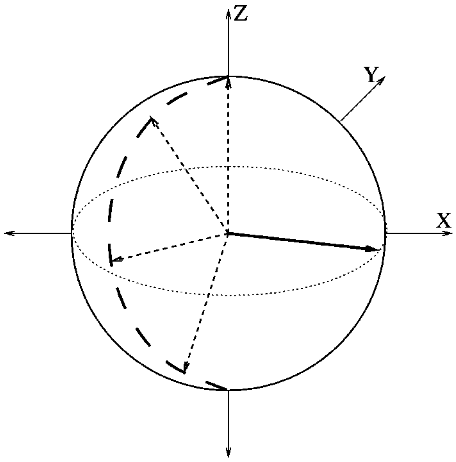

FIG. 2. Bloch sphere showing axis of rotation.

$$P_s = \sin^2[(2s+1)\theta], \tag{26}$$

where s is the number of applications of the Grover iteration and $\sin\theta = |U_{\tau\gamma}|$. Specifically in the case under consideration by Feng using X rotations, we have $U = W_n$, Eq. (3). For an arbitrary prepared state $|\gamma\rangle$ and arbitrary marked state $|\tau\rangle$, we obtain

$$\sin\theta = |\langle\tau|W_n|\gamma\rangle| = \frac{1}{\sqrt{N}}, \tag{27}$$

where $N = 2^n$ and n is the number of qubits. The probability of success given in Eqs. (26) and (27) is the same as may be obtained for Grover's algorithm based on the Hadamard transformation.

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