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Solution of the dual reflection equation for $A_{n-1}^{(1)}$ solid-on-solid model

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We obtain a diagonal solution of the *dual reflection* equation for the elliptic $A_{n-1}^{(1)}$ solid-on-solid model. The isomorphism between the solutions of the reflection equation and its dual is studied. © *2004 American Institute of Physics.*
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I. INTRODUCTION

Two-dimensional lattice spin models in statistical mechanics have traditionally been solved by imposing periodic boundary condition. The Yang–Baxter equation[1,2]

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3)=R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2), \tag{1.1}$$

together with such boundary condition then leads to families of commuting row *transfer matrices* and hence solvability.[2] The work of Sklyanin[3] shows that, by using the reflection equation (RE) introduced by Cherednik[4]

$$R_{12}(u_1-u_2)K_1(u_1)R_{21}(u_1+u_2)K_2(u_2)=K_2(u_2)R_{12}(u_1+u_2)K_1(u_1)R_{21}(u_1-u_2), \tag{1.2}$$

it is also possible to construct families of commuting *double-row transfer matrices* for vertex models with open boundary conditions. Then such a scheme has been generalized to *face-type* solid-on-solid (SOS) models.[5,6]

In order to construct the *double-row transfer matrices*, besides the RE, one needs the dual reflection equation whose explicit form is related with the crossing-unitarity relation of the R-matrix.[3,7,5,6] For the $\mathbb{Z}_n$ Belavin model,[8] the dual RE reads[6]

$$R_{12}(u_2-u_1)\tilde{K}_1(u_1)R_{21}(-u_1-u_2-nw)\tilde{K}_2(u_2)=\tilde{K}_2(u_2)R_{12}(-u_1-u_2-nw)\tilde{K}_1(u_1)R_{21}(u_2-u_1), \tag{1.3}$$

where w is the crossing parameter of the R-matrix. Moreover, there exists a *simple-form* isomorphism between the solution of the RE (1.2) and that of its dual (1.3)

$$\tilde{K}(u)=K\left(-u-\frac{nw}{2}\right). \tag{1.4}$$

However, for integrable SOS models, due to the complicated crossing-unitarity relation of R-matrix (Boltzmann weight) (2.18),[9,10] the dual RE (3.2) contains the face-type parameters $\{\lambda_j\}$ in addition to the spectral parameter. A generalized isomorphism between the solutions to the RE and its dual for SOS models, if exists, is yet to be found. In this sense, the dual RE for the

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face-type models has got its own *independent* role in contrast with the vertex model.

The RE of SOS models has been solved to give the diagonal K-matrices for the $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, and $A_{2n+1}^{(2)}$ SOS models.[11] But the generic (nondiagonal) K-matrix is known only for the $A_1^{(1)}$ SOS model.[12,5] However, the dual RE of the face type was solved only for the $A_1^{(1)}$ SOS model.[5] In this article, we consider the dual RE for the $A_{n-1}^{(1)}$ SOS model. After briefly reviewing the face-vertex correspondence between the $\mathbb{Z}_n$ Belavin model and the $A_{n-1}^{(1)}$ SOS model,[14] we construct the isomorphism between the solution of the RE and its dual for the $A_{n-1}^{(1)}$ SOS model in Sec. III. In Sec. IV, we derive a diagonal solution to the dual RE by solving directly. Then we prove that our diagonal solution to the dual RE can be obtained through the isomorphism transformation (3.16) from the diagonal solution[11] of RE by a special choice of the free parameter λ'. The final section is for conclusions.

II. REFLECTION EQUATION AND ITS DUAL FOR $A_{n-1}^{(1)}$ SOS MODEL

A. $\mathbb{Z}_n$ Belavin R-matrix

Let us fix τ such that $\mathrm{Im}(\tau)>0$ and a generic complex number w. Introduce the following elliptic functions:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{m=-\infty}^{\infty} \exp\{\sqrt{-1}\pi[(m+a)^2\tau + 2(m+a)(u+b)]\}, \tag{2.1}$$

$$\theta^{(j)}(u) = \theta\begin{bmatrix} \frac{1}{2}-\frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(u,n\tau), \quad \sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau). \tag{2.2}$$

Among them the σ-function satisfies the following identity:

$$\begin{aligned} &\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) \\ &\quad = \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y), \end{aligned} \tag{2.3}$$

which will be useful in the following. [Our σ-function is the ϑ-function $\vartheta_1(u)$.[13] It has the following relation with the *Weierstrassian* σ-function if denoted by $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2}\sigma(u)$, $\eta_1 = \pi^2\left(\frac{1}{6} - 4\sum_{n=1}^{\infty} nq^{2n}/(1-q^{2n})\right)$ and $q=e^{\sqrt{-1}\tau}$.]

Let $R^B(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the $\mathbb{Z}_n$ Belavin R-matrix[8] given by

$$R^B(u) = \sum_{i,j,k,l} R_{ij}^{kl}(u) E_{ik} \otimes E_{lj}, \tag{2.4}$$

in which E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk}\delta_{il}$. The coefficient functions are[9]

$$R_{ij}^{kl}(u) = \begin{cases} \dfrac{h(u)\sigma(w)\theta^{(i-j)}(u+w)}{\sigma(u+w)\theta^{(i-k)}(w)\theta^{(k-j)}(u)} & \text{if } i+j=k+l \ mod \ n, \\ 0 & \text{otherwise}. \end{cases} \tag{2.5}$$

Here we have set

$$h(u)=\frac{\prod_{j=0}^{n-1}\theta^{(j)}(u)}{\prod_{j=1}^{n-1}\theta^{(j)}(0)}. \tag{2.6}$$

The R-matrix satisfies the quantum Yang–Baxter Eq. (1.1) and the following unitarity and crossing-unitarity relations:[15]

$$\text{Unitarity: } R^{B}{}_{12}(u)R^{B}_{21}(-u)=id, \tag{2.7}$$

$$\text{Crossing-unitarity: } (R^B)^{t_2}_{21}(-u-nw)(R^B)^{t_2}_{12}(u)=\frac{e^{\sqrt{-1}nw}\sigma(u)\sigma(u+nw)}{\sigma(u+w)\sigma(u+nw-w)}id, \tag{2.8}$$

where t_i denotes the transposition in the ith space.

B. $A^{(1)}_{n-1}$ SOS *R*-matrix and face-vertex correspondence

Let $\{\epsilon_i|i=1,2,\ldots,n\}$ be the orthonormal basis of the vector space $\mathbb{C}^n$ such that $\langle\epsilon_i,\epsilon_j\rangle=\delta_{ij}$. The A_{n-1} simple roots are $\{\alpha_i=\epsilon_i-\epsilon_{i+1}|i=1,\ldots,n-1\}$ and the fundamental weights $\{\Lambda_i|i=1,\ldots,n-1\}$ satisfying $\langle\Lambda_i,\alpha_j\rangle=\delta_{ij}$ are given by

$$\Lambda_i=\sum_{k=1}^{i}\epsilon_k-\frac{i}{n}\sum_{k=1}^{n}\epsilon_k.$$

Set

$$\hat{i}=\epsilon_i-\overline{\epsilon},\ \overline{\epsilon}=\frac{1}{n}\sum_{k=1}^{n}\epsilon_k,\ i=1,\ldots,n,\ \text{then}\ \sum_{i=1}^{n}\hat{i}=0. \tag{2.9}$$

For each dominant weight $\Lambda=\sum_{i=1}^{n-1}a_i\Lambda_i$, $a_i\in\mathbb{Z}^+$, there exists an irreducible highest weight finite-dimensional representation V_Λ of A_{n-1} with the highest vector $|\Lambda\rangle$. For example the fundamental vector representation is V_{Λ_1}.

Let $\mathfrak{h}$ be the Cartan subalgebra of A_{n-1} and $\mathfrak{h}^*$ be its dual. A finite-dimensional diagonalizable $\mathfrak{h}$-module is a complex finite-dimensional vector space W with a weight decomposition $W=\oplus_{\mu\in\mathfrak{h}^*}W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $xv=\mu(x)v$, $(x\in\mathfrak{h},v\in W[\mu])$. For example, the fundamental vector representation $V_{\Lambda_1}=\mathbb{C}^n$, the nonzero weight spaces $W[\hat{i}]=\mathbb{C}\epsilon_i$, $i=1,\ldots,n$.

For a generic $\lambda\in\mathbb{C}^n$, define

$$\lambda_i=\langle\lambda,\epsilon_i\rangle,\ \lambda_{ij}=\lambda_i-\lambda_j,\ |\lambda|=\sum_{l=1}^{n}\lambda_l,\ i,j=1,\ldots,n. \tag{2.10}$$

Let $R(z,\lambda)\in\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)$ be the R-matrix of the $A^{(1)}_{n-1}$ SOS model given by

$$R(z,\lambda)=\sum_{i=1}^{n}R^{ii}_{ii}(z,\lambda)E_{ii}\otimes E_{ii}+\sum_{i\neq j}\{R^{ij}_{ij}(z,\lambda)E_{ii}\otimes E_{jj}+R^{ji}_{ij}(z,\lambda)E_{ji}\otimes E_{ij}\}. \tag{2.11}$$

The coefficient functions are

$$R^{ii}_{ii}(z,\lambda)=1,\ R^{ij}_{ij}(z,\lambda)=\frac{\sigma(z)\sigma(\lambda_{ij}w-w)}{\sigma(z+w)\sigma(\lambda_{ij}w)}, \tag{2.12}$$

$$R_{ij}^{ji}(z,\lambda)=\frac{\sigma(w)\sigma(z+\lambda_{ij}w)}{\sigma(z+w)\sigma(\lambda_{ij}w)}, \tag{2.13}$$

and λ_{ij} is defined in (2.10). The R-matrix satisfies the dynamical (modified) quantum Yang–Baxter equation

$$\begin{aligned}R_{12}(z_1-z_2,\lambda-h^{(3)})R_{13}(z_1-z_3,\lambda)R_{23}(z_2-z_3,\lambda-h^{(1)})\\ =R_{23}(z_2-z_3,\lambda)R_{13}(z_1-z_3,\lambda-h^{(2)})R_{12}(z_1-z_2,\lambda),\end{aligned} \tag{2.14}$$

with unitarity relation

$$R_{12}(u,\lambda)R_{21}(-u,\lambda)=id. \tag{2.15}$$

We adopt the notation: $R_{12}(z,\lambda-h^{(3)})$ acts on a tensor $v_1\otimes v_2\otimes v_3$ as $R(z,\lambda-\mu)\otimes id$ if $v_3\in W[\mu]$. Let us introduce

$$\tilde{R}(u,\lambda)_{ij}^{kl}=R(u,\lambda)_{ij}^{kl}\left\{\frac{f_2(\lambda;k)}{f_2(\lambda+\hat{k};k)}\frac{f_2(\lambda+\hat{i}+\hat{j};i)}{f_2(\lambda+\hat{j};i)}\right\}, \tag{2.16}$$

$$f_2(\lambda;j)=\prod_{k\neq j}\frac{\sigma(\lambda_{jk}w)}{\sigma(w)}. \tag{2.17}$$

The R-matrix satisfies the following crossing-unitarity relation[6]

$$\sum_{i_2,j_2=1}^{n}\tilde{R}(-u-nw,\lambda-\hat{j}_2)_{j_1\ i_2}^{j_2\ i_1}R(u,\lambda-\hat{j}_2)_{i_3\ j_2}^{i_2\ j_3}=\frac{e^{\sqrt{-1}nw}\sigma(u)\sigma(u+nw)}{\sigma(u+w)\sigma(u+nw-w)}\delta_{i_3}^{i_1}\delta_{j_3}^{j_1}. \tag{2.18}$$

Let us introduce an intertwiner—a n-component column vector $\phi_{\lambda,\lambda-\hat{j}}(u)$ whose kth element is

$$\phi_{\lambda,\lambda-\hat{j}}^{(k)}(u)=\theta^{(k)}(u+nw\lambda_j). \tag{2.19}$$

Using the intertwiner, the face-vertex correspondence can be written as[14]

$$R_{12}^{B}(u_1-u_2)\phi_{\lambda,\lambda-\hat{i}}(u_1)\otimes\phi_{\lambda-\hat{i},\lambda-\hat{i}-\hat{j}}(u_2)=\sum_{kl}R(u_1-u_2,\lambda)_{ij}^{kl}\phi_{\lambda-\hat{l},\lambda-\hat{l}-\hat{k}}(u_1)\otimes\phi_{\lambda,\lambda-\hat{l}}(u_2). \tag{2.20}$$

Then the Yang–Baxter equation of the $\mathbb{Z}_n$ Belavin R-matrix $R^B(u)$ (1.1) is equivalent to the dynamical Yang–Baxter equation of the $A_{n-1}^{(1)}$ SOS R-matrix $R(u,\lambda)$ (2.14).

III. RE AND DUAL RE FOR $A_{n-1}^{(1)}$ SOS MODEL

In this section, using the intertwiner between the $\mathbb{Z}_n$ Belavin R-matrix and that of the $A_{n-1}^{(1)}$ SOS model, we construct the isomorphism between the solution of the RE for the $A_{n-1}^{(1)}$ SOS model and that of its dual from the isomorphism (1.4).

A. RE and its dual for SOS model

The RE of the K-matrix $\mathcal{K}(\lambda|u)$ for the face-type SOS model was given as follows:[5,12,16,17]

$$\sum_{i_1,i_2}\sum_{j_1,j_2} R(u_1-u_2,\lambda)^{i_0\ j_0}_{i_1\ j_1}\mathcal{K}(\lambda+\hat{j}_1+\hat{i}_2|u_1)^{i_1}_{i_2}R(u_1+u_2,\lambda)^{j_1\ i_2}_{j_2\ i_3}\mathcal{K}(\lambda+\hat{j}_3+\hat{i}_3|u_2)^{j_2}_{j_3}$$
$$=\sum_{i_1,i_2}\sum_{j_1,j_2}\mathcal{K}(\lambda+\hat{j}_1+\hat{i}_0|u_2)^{j_0}_{j_1}R(u_1+u_2,\lambda)^{i_0\ j_1}_{i_1\ j_2}\mathcal{K}(\lambda+\hat{j}_2+\hat{i}_2|u_1)^{i_1}_{i_2}R(u_1-u_2,\lambda)^{j_2\ i_2}_{j_3\ i_3}. \qquad (3.1)$$

The dual RE of the K-matrix $\tilde{\mathcal{K}}(\lambda|u)$ was written down by[5,6]

$$\sum_{i_1,i_2}\sum_{j_1,j_2} R(u_2-u_1,\lambda)^{i_0\ j_0}_{i_1\ j_1}\tilde{\mathcal{K}}(\lambda+\hat{j}_1+\hat{i}_1|u_1)^{i_1}_{i_2}\tilde{R}(-u_1-u_2-nw,\lambda)^{j_1\ i_2}_{j_2\ i_3}\tilde{\mathcal{K}}(\lambda+\hat{j}_2+\hat{i}_3|u_2)^{j_2}_{j_3}$$
$$=\sum_{i_1,i_2}\sum_{j_1,j_2}\tilde{\mathcal{K}}(\lambda+\hat{j}_0+\hat{i}_0|u_2)^{j_0}_{j_1}\tilde{R}(-u_1-u_2-nw,\lambda)^{i_0\ j_1}_{i_1\ j_2}\tilde{\mathcal{K}}(\lambda+\hat{j}_2+\hat{i}_1|u_1)^{i_1}_{i_2}R(u_2-u_1,\lambda)^{j_2\ i_2}_{j_3\ i_3}, \qquad (3.2)$$

where $\tilde{R}(u,\lambda)$ is defined in (2.16) for the $A^{(1)}_{n-1}$ SOS model. The explicit expressions of $\tilde{R}(u,\lambda)$ for other types of SOS models were given in Ref. 6. Because of the *nontrivial* dependence on the face-type parameters $\{\lambda_j\}$, the dual RE of SOS models should be treated separately in contrast with those of the vertex models.

As in the Sklyanin scheme for the vertex models, one can construct families of commuting *double-row transfer matrices* for the SOS model with open boundary condition in terms of the K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$.[5,6]

B. Isomorphism between the solutions of the RE and its dual for $A^{(1)}_{n-1}$ SOS model

Thanks to the face-vertex correspondence between the $\mathbb{Z}_n$ Belavin vertex model and the $A^{(1)}_{n-1}$ SOS model (2.20), we can construct the isomorphism between the solutions of the RE and its dual for the $A^{(1)}_{n-1}$ SOS model from the isomorphism (1.4) of the $\mathbb{Z}_n$ Belavin vertex model.

Let us introduce other types of intertwiners $\bar{\phi}$ and $\tilde{\phi}$ satisfying the following orthogonality conditions:

$$\sum_k \bar{\phi}^{(k)}_{\lambda,\lambda-\hat{i}}(u)\phi^{(k)}_{\lambda,\lambda-\hat{j}}(u)=\delta_{ij}, \qquad (3.3)$$

$$\sum_k \tilde{\phi}^{(k)}_{\lambda+\hat{i},\lambda}(u)\phi^{(k)}_{\lambda+\hat{j},\lambda}(u)=\delta_{ij}. \qquad (3.4)$$

One can derive the "completeness" relations from the above conditions

$$\sum_k \bar{\phi}^{(i)}_{\lambda,\lambda-\hat{k}}(u)\phi^{(j)}_{\lambda,\lambda-\hat{k}}(u)=\delta_{ij}, \qquad (3.5)$$

$$\sum_k \tilde{\phi}^{(i)}_{\lambda+\hat{k},\lambda}(u)\phi^{(j)}_{\lambda+\hat{k},\lambda}(u)=\delta_{ij}, \qquad (3.6)$$

and the following relation between the intertwiners $\bar{\phi}$ and $\tilde{\phi}$ from their definitions (3.3) and (3.4):[6]

$$\bar{\phi}_{\lambda+\hat{j},\lambda}(u)=\frac{\sigma\left(u+w|\lambda|-\frac{n-1}{2}-w\right)}{\sigma\left(u+w|\lambda|-\frac{n-1}{2}\right)}\left\{\prod_{k\neq j}\frac{\sigma(\lambda_{jk}w)}{\sigma(\lambda_{jk}w+w)}\right\}\tilde{\phi}_{\lambda+\hat{j},\lambda}(u-nw). \qquad (3.7)$$

Noting the fact $\langle\bar{\epsilon},\epsilon_j\rangle=1/n$ and the definition of the intertwiner (2.19), one can derive the following relations: for $\forall\alpha\in\mathbb{C}$

$$\phi_{\lambda+\alpha\bar{\epsilon},\lambda+\alpha\bar{\epsilon}-\hat{j}}(u) = \phi_{\lambda,\lambda-\hat{j}}(u+\alpha w), \tag{3.8}$$

$$\bar{\phi}_{\lambda+\alpha\bar{\epsilon},\lambda+\alpha\bar{\epsilon}-\hat{j}}(u) = \bar{\phi}_{\lambda,\lambda-\hat{j}}(u+\alpha w), \tag{3.9}$$

$$\tilde{\phi}_{\lambda+\alpha\bar{\epsilon},\lambda+\alpha\bar{\epsilon}-\hat{j}}(u) = \tilde{\phi}_{\lambda,\lambda-\hat{j}}(u+\alpha w). \tag{3.10}$$

Define

$$\mathcal{K}(\lambda|u)_i^j = \sum_{s,t} \tilde{\phi}^{(s)}_{\lambda-\hat{i}+\hat{j},\lambda-\hat{i}}(u) K(u)_t^s \phi^{(t)}_{\lambda,\lambda-\hat{i}}(-u), \tag{3.11}$$

$$\tilde{\mathcal{K}}(\lambda|u)_i^j = \sum_{s,t} \bar{\phi}^{(s)}_{\lambda,\lambda-\hat{j}}(-u) \tilde{K}(u)_t^s \phi^{(t)}_{\lambda-\hat{j}+\hat{i},\lambda-\hat{j}}(u). \tag{3.12}$$

Then we have

Theorem 1 (Ref. 6): *The above relations (3.11) and (3.12) map the solutions $K(u)$ and $\tilde{K}(u)$ to the RE (1.2) and the dual (1.3) for the $\mathbb{Z}_n$ Belavin R-matrix to the solutions $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ to the RE (3.1) and the dual (3.2) for the $A^{(1)}_{n-1}$ SOS R-matrix, and vice versa.*

Using the relations (3.5) and (3.6), one can invert (3.11)

$$K(u)_t^s = \sum_{i,j} \phi^{(s)}_{\lambda-\hat{i}+\hat{j},\ \lambda-\hat{i}}(u) \mathcal{K}(\lambda|u)_i^j \bar{\phi}^{(t)}_{\lambda,\lambda-\hat{i}}(-u). \tag{3.13}$$

Using the isomorphism (1.4) between the solutions of the RE and the dual RE for the $\mathbb{Z}_n$ Belavin R-matrix, the relations (3.5), (3.6), and (3.12), we have

$$\begin{aligned}
\tilde{\mathcal{K}}(\lambda|u)_\mu^\nu &= \sum_{s,t} \bar{\phi}^{(s)}_{\lambda,\lambda-\hat{\nu}}(-u) \tilde{K}(u)_t^s \phi^{(t)}_{\lambda-\hat{\nu}+\hat{\mu},\lambda-\hat{\nu}}(u) \\
&= \sum_{s,t} \bar{\phi}^{(s)}_{\lambda,\lambda-\hat{\nu}}(-u) K\left(-u-\frac{nw}{2}\right)_t^s \phi^{(t)}_{\lambda-\hat{\nu}+\hat{\mu},\ \lambda-\hat{\nu}}(u) \\
&= \sum_{i,j}\sum_{s,t} \bar{\phi}^{(s)}_{\lambda,\ \lambda-\hat{\nu}}(-u) \phi^{(s)}_{\lambda'-\hat{i}+\hat{j},\ \lambda'-\hat{i}}\left(-u-\frac{nw}{2}\right) \mathcal{K}\left(\lambda'|-u-\frac{nw}{2}\right)_i^j \\
&\quad \times \bar{\phi}^{(t)}_{\lambda',\ \lambda'-\hat{i}}\left(u+\frac{nw}{2}\right) \phi^{(t)}_{\lambda-\hat{\nu}+\hat{\mu},\ \lambda-\hat{\nu}}(u) \\
&= \sum_{i,j} M(\lambda,\lambda'-\hat{i}|-u)_j^\nu \mathcal{K}\left(\lambda'|-u-\frac{nw}{2}\right)_i^j M\left(\lambda',\lambda-\hat{\nu}|u+\frac{nw}{2}\right)_\mu^i,
\end{aligned} \tag{3.14}$$

where $\lambda' \in \mathbb{C}^n$ is arbitrary and a *crossing matrix* $M(\lambda,\lambda'|u)_j^\nu$ is defined by

$$M(\lambda,\lambda'|u)_j^\nu = \sum_t \bar{\phi}^{(t)}_{\lambda,\ \lambda-\hat{\nu}}(u) \phi^{(t)}_{\lambda'+\hat{j},\lambda'}\left(u-\frac{nw}{2}\right). \tag{3.15}$$

Finally, we obtain

Theorem 2: *The solutions to the RE (3.1) and the dual (3.2) for the $A^{(1)}_{n-1}$ SOS R-matrix have the following isomorphism:*

$$\tilde{\mathcal{K}}(\lambda|u)_\mu^\nu = \sum_{i,j} M(\lambda,\lambda'-\hat{i}|-u)_j^\nu \mathcal{K}\left(\lambda'|-u-\frac{nw}{2}\right)_i^j M\left(\lambda',\lambda-\hat{\nu}|u+\frac{nw}{2}\right)_\mu^i, \tag{3.16}$$

where $\lambda' \in \mathbb{C}^n$ is arbitrary.

We remark that the crossing matrix (3.15) is generally *nondiagonal*. Hence, the corresponding

$\tilde{\mathcal{K}}(\lambda|u)$ of the solution to the dual RE (3.2) obtained by the isomorphism (3.16) from the diagonal solution[11] to RE is generally nondiagonal, too, except for the case that a special choice of "*moduli*" parameter λ' is chosen as (4.5) (this special case will be clarified later in the next section). However, in order to diagonalize the corresponding *double-row transfer matrices* for the $A_{n-1}^{(1)}$ SOS model by the algebraic Bethe ansatz method, one needs $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ both diagonal.[18,19] In the next section, we shall search for a diagonal $\tilde{\mathcal{K}}(\lambda|u)$.

IV. DIAGONAL SOLUTION OF THE DUAL RE FOR $A_{n-1}^{(1)}$ SOS MODEL

In this section we look for the diagonal solution to the dual RE (3.2) for the $A_{n-1}^{(1)}$ SOS model, namely, the K-matrix $\tilde{\mathcal{K}}(\lambda|u)$ of following form:

$$\tilde{\mathcal{K}}(\lambda|u)_i^j = \tilde{k}(\lambda|u)_i \delta_i^j, \tag{4.1}$$

where $\{\tilde{k}(\lambda|u)_i\}$ are the functions of the face parameters $\{\lambda_j\}$ and the spectral parameter u. From directly solving the Eq. (3.2), we have

Theorem 3: For

$$\tilde{k}(\lambda|u)_i = \left\{ \prod_{k\neq i} \frac{\sigma(\lambda_{ik}w - w)}{\sigma(\lambda_{ik}w)} \right\} \frac{\sigma\left(\lambda_i w + \bar{\xi} + u + \frac{nw}{2}\right)}{\sigma\left(\lambda_i w + \bar{\xi} - u - \frac{nw}{2}\right)} f(u,\lambda), \tag{4.2}$$

in which $\bar{\xi}$ is a free parameter and $f(u,\lambda)$ is any nonvanishing function of λ and u, the diagonal K-matrix $\tilde{\mathcal{K}}(\lambda|u)$ with entries (4.1) and (4.2) is a solution to the dual RE (3.2) for the $A_{n-1}^{(1)}$ SOS model.

Proof: Substituting $\tilde{\mathcal{K}}(\lambda|u)$ of form (4.1) into the dual RE (3.2) for the $A_{n-1}^{(1)}$ SOS model, one finds the only nontrivial conditions of $\tilde{k}(\lambda|u)_i$ are

$$\begin{aligned} &R(u_2-u_1,\lambda)_{ji}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_1)_j\tilde{R}(-u_1-u_2-nw,\lambda)_{ji}^{ij}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_2)_j \\ &\quad + R(u_2-u_1,\lambda)_{ij}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_1)_i\tilde{R}(-u_1-u_2-nw,\lambda)_{ji}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_2)_j \\ &\quad = R(u_2-u_1,\lambda)_{ji}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_1)_i\tilde{R}(-u_1-u_2-nw,\lambda)_{ij}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_2)_i \\ &\qquad + R(u_2-u_1,\lambda)_{ji}^{ij}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_1)_j\tilde{R}(-u_1-u_2-nw,\lambda)_{ji}^{ji}\tilde{k}(\lambda+\hat{i}+\hat{j}|u_2)_i, \ i\neq j. \end{aligned}$$

Substituting (2.16) and (4.2) into the above equation, the dual RE (3.2) is equivalent to the following equation:

$$\begin{aligned} &\left\{ \sigma(u_-+\lambda_{ij}w)\sigma(u_+) - \sigma(u_-)\sigma(u_+-\lambda_{ij}w)\frac{\sigma(\lambda_j w+\bar{\xi}'-u_1')\sigma(\lambda_i w+\bar{\xi}'+u_1')}{\sigma(\lambda_j w+\bar{\xi}'+u_1')\sigma(\lambda_i w+\bar{\xi}'-u_1')} \right\} \\ &\quad \times \frac{\sigma(\lambda_j w+\bar{\xi}'-u_2')\sigma(\lambda_i w+\bar{\xi}'+u_2')}{\sigma(\lambda_j w+\bar{\xi}'+u_2')\sigma(\lambda_i w+\bar{\xi}'-u_2')} \\ &\quad = \sigma(u_++\lambda_{ij}w)\sigma(u_-) - \sigma(u_+)\sigma(u_--\lambda_{ij}w)\frac{\sigma(\lambda_j w+\bar{\xi}'-u_1')\sigma(\lambda_i w+\bar{\xi}'+u_1')}{\sigma(\lambda_j w+\bar{\xi}'+u_1')\sigma(\lambda_i w+\bar{\xi}'-u_1')}, \end{aligned} \tag{4.3}$$

where $u_-=u_1'-u_2'$, $u_+=u_1'+u_2'$, $u_i'=-u_i-nw/2$, $\bar{\xi}'=\bar{\xi}+(n-2)/nw$. Equation (4.3) is a consequence of the identity (2.3). Then we complete our proof.

Now we shall study the relation between our solution of the dual RE and the diagonal solution of RE which was given as follows:[11]

$$\mathcal{K}(\lambda|u)_i^j = k(\lambda|u)_i\delta_i^j = g(u,\lambda)\frac{\sigma(\lambda_i w+\xi-u)}{\sigma(\lambda_i w+\xi+u)}\delta_i^j. \tag{4.4}$$

Here, $g(\lambda|u)$ is any nonvanishing function of λ and u, and ξ is a free parameter. Let us choose

$$\lambda' = \lambda + \frac{n}{2}\bar{\epsilon} \Rightarrow \lambda_i' = \lambda_i + \tfrac{1}{2}, \tag{4.5}$$

the vector $\bar{\epsilon}$ is defined in (2.9). Using the relation (3.8), the crossing matrix $M(\lambda,\lambda+(n/2)\bar{\epsilon}-\hat{i}|u)_i^\nu$ defined in (3.15) becomes simple

$$M\left(\lambda,\lambda+\frac{n}{2}\bar{\epsilon}-\hat{i}|u\right)_i^\nu = \sum_t \bar{\phi}^{(t)}_{\lambda,\lambda-\hat{\nu}}(u)\,\phi^{(t)}_{\lambda+(n/2)\bar{\epsilon},\lambda+(n/2)\bar{\epsilon}-\hat{\nu}}\left(u-\frac{nw}{2}\right) = \sum_t \bar{\phi}^{(t)}_{\lambda,\lambda-\hat{i}}(u)\,\phi^{(t)}_{\lambda,\lambda-\hat{i}}(u) = \delta_i^\nu. \tag{4.6}$$

The resulting solution to the dual RE by the isomorphism transformation (3.16) from the diagonal solution to RE is

$$\tilde{\mathcal{K}}(\lambda|u)_\mu^\nu = k\left(\lambda+\frac{n}{2}\bar{\epsilon}\,\middle|\,-u-\frac{nw}{2}\right)_\nu M\left(\lambda+\frac{n}{2}\bar{\epsilon},\lambda-\hat{\nu}\middle|u+\frac{nw}{2}\right)_\mu^\nu. \tag{4.7}$$

The relations (3.7) and (4.5) enable us to further simplify the expression of the crossing matrix $M(\lambda+(n/2)\bar{\epsilon},\lambda-\hat{\nu}|u+nw/2)_\mu^\nu$:

$$\begin{aligned}
M\left(\lambda+\frac{n}{2}\bar{\epsilon},\lambda-\hat{\nu}\middle|u+\frac{nw}{2}\right)_\mu^\nu &= \frac{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n-2}{2}w-\frac{n-1}{2}\right)}{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n}{2}w-\frac{n-1}{2}\right)}\left\{\prod_{k\neq\nu}\frac{\sigma(\lambda_{\nu k}w-w)}{\sigma(\lambda_{\nu k}w)}\right\} \\
&\quad\times\sum_t \tilde{\phi}^{(t)}_{\lambda+(n/2)\bar{\epsilon},\lambda+(n/2)\bar{\epsilon}-\hat{\nu}}\left(u-\frac{nw}{2}\right)\phi^{(t)}_{\lambda-\hat{\nu}+\hat{\mu},\lambda-\hat{\nu}}(u) \\
&= \frac{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n-2}{2}w-\frac{n-1}{2}\right)}{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n}{2}w-\frac{n-1}{2}\right)}\left\{\prod_{k\neq\nu}\frac{\sigma(\lambda_{\nu k}w-w)}{\sigma(\lambda_{\nu k}w)}\right\} \\
&\quad\times\sum_t \tilde{\phi}^{(t)}_{\lambda,\lambda-\hat{\nu}}(u)\,\phi^{(t)}_{\lambda-\hat{\nu}+\hat{\mu},\lambda-\hat{\nu}}(u) \\
&= \frac{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n-2}{2}w-\frac{n-1}{2}\right)}{\sigma\left(u+|\lambda-\hat{\nu}|w+\frac{n}{2}w-\frac{n-1}{2}\right)}\left\{\prod_{k\neq\nu}\frac{\sigma(\lambda_{\nu k}w-w)}{\sigma(\lambda_{\nu k}w)}\right\}\delta_\mu^\nu.
\end{aligned}$$

Finally, the resulting solution to the dual RE by the isomorphism transformation (3.16) from the diagonal solution to RE is given by

$$\tilde{\mathcal{K}}(\lambda|u)_\mu^\nu = \frac{\sigma\left(u + |\lambda - \hat{\nu}|w + \frac{n-2}{2}w - \frac{n-1}{2}\right)}{\sigma\left(u + |\lambda - \hat{\nu}|w + \frac{n}{2}w - \frac{n-1}{2}\right)} \left\{ \prod_{k \neq \nu} \frac{\sigma(\lambda_{\nu k} w - w)}{\sigma(\lambda_{\nu k} w)} \right\} k\left(\lambda + \frac{n}{2}\bar{\varepsilon} - u - \frac{nw}{2}\right)_\nu \delta_\mu^\nu. \tag{4.8}$$

Substituting the diagonal solution of RE (4.4) into the above equation and after redefining the boundary parameter $\bar{\xi}$ and the free nonvanishing function $f(u,\lambda)$, one finds that the resulting diagonal solution (4.8) to the dual RE is exactly the same as (4.2).

V. CONCLUSION AND COMMENTS

By using the face-vertex correspondence (2.20) and the isomorphism (1.4) between the solutions to the RE and its dual for the $\mathbb{Z}_n$ Belavin R-matrix, we construct the isomorphism between the solutions to the RE and its dual for the $A_{n-1}^{(1)}$ SOS R-matrix. By directly solving the equation, we obtain a diagonal solution to the dual RE. Our solution to the dual RE can also be obtained through the isomorphism transformation (3.16) from the diagonal solution to RE obtained in Ref. 11 by a special choice of the free parameter λ' (4.5). Furthermore, the diagonal $\tilde{\mathcal{K}}(\lambda|u)$ obtained in this article enables us to diagonalize the *double-row transfer matrices* of the $\mathbb{Z}_n$ Belavin model with open boundary condition described by the diagonal $\mathcal{K}(\lambda|u)$ and the diagonal $\tilde{\mathcal{K}}(\lambda|u)$.[19]

Alternatively in Ref. 20, the very isomorphism with the special choice of the free parameter λ' (4.5) from the diagonal solution of RE to the diagonal solution of the dual RE was constructed by fusion procedure. However, our *generic* isomorphism transformation (3.16) gives a way to construct a *nondiagonal* solution of the dual RE with additional free parameters $\{\lambda_i'\}$.

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