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A Framework of Fuzzy Diagnosis

Huaiqing Wang, Mingyi Zhang, Dongming Xu, and Dan Zhang

Abstract—Fault diagnosis has become an important component in intelligent systems, such as intelligent control systems and intelligent eLearning systems. Reiter's diagnosis theory, described by first-order sentences, has been attracting much attention in this field. However, descriptions and observations of most real-world situations are related to fuzziness because of the incompleteness and the uncertainty of knowledge, e.g., the fault diagnosis of student behaviors in the eLearning processes. In this paper, an extension of Reiter's consistency-based diagnosis methodology, Fuzzy Diagnosis, has been proposed, which is able to deal with incomplete or fuzzy knowledge. A number of important properties of the Fuzzy diagnoses schemes have also been established. The computing of fuzzy diagnoses is mapped to solving a system of inequalities. Some special cases, abstracted from real-world situations, have been discussed. In particular, the fuzzy diagnosis problem, in which fuzzy observations are represented by clause-style fuzzy theories, has been presented and its solving method has also been given. A student fault diagnostic problem abstracted from a simplified real-world eLearning case is described to demonstrate the application of our diagnostic framework.

Index Terms—Knowledge representation, fuzzy diagnosis, fault diagnosis, uncertainty reasoning, fuzzy truth function logic, clause-style fuzzy theories.

1 Introduction

The diagnostic tasks deal with the problems of why a correctly designed system is not functioning as it should be, by finding explanations for the faulty behavior. These explanations state how the system is at variance in some ways with its original design. The main diagnosis tasks are to discover the malfunctions in a system, based on the design and the structure of the system and the observations (symptoms, evidence), as well as the root causes of such malfunctions [1], [2], [3].

Fault diagnosis has become an important tool in modern automatic control theory. During the last three decades, an immense amount of research has been done in this field, resulting in a great variety of methods, many of which have been applied on real-world applications [4], [5]. There has also been a rapid movement from traditional methods of signal-based fault diagnosis toward the model-based approach. The core of the so-called model-based approach to fault-diagnosis uses analytical and/or knowledge-based models for residual generation and decision-making methods from artificial intelligence for residual evaluation. The analytical approach to fault diagnosis differs from impractical because it is very difficult to build accurate mathematical models of the target systems. The knowledge about the target system is often incomplete or uncertain, and residual evaluation is a complex logical process that requires the use of intelligent decision-making techniques. A more suitable solution is to use knowledge-based techniques. Knowledge-based techniques for fault diagnosis require a suitable knowledge representation scheme and reasoning facilities. In the area of artificial intelligence, many researchers have argued that using logic as knowledge representation is appropriate for model-based diagnosis. There are two different points of views about logic-based diagnosis in the literature. One is the consistency-based approach to diagnosis, often referred to in literature as diagnosis from first principles [1], [2], [3]. The other approach is based on abductive methods [6], [7], [8], [9], [10], [11], [12], [13], [14], and it often uses heuristic and diagnostic associations derived from experience. In abduction, the diagnostic hypotheses entail observations and are computed by backward chaining from the observations, whereas in the consistency-based approaches, the observations constitute the disjunction of the diagnoses and are computed by forward chaining from the observations. A notable example of the abducitve approach to diagnosis is the MYCIN system [6], while Reiter presents a precise theoretical foundation for consistency-based diagnosis [1].

Knowledge-based diagnosis techniques could be symptom-based and qualitative model-based. If the symptoms are considered in connection with the inputs to the system, the symptom-based approach is being used where knowledge is derived from facts and rules of the system's structure and behavior (the first principle). However, information is incomplete or uncertain in many real-world applications. It is becoming essential to deal with the incomplete knowledge models [15], [16], [17], [18]. Furthermore, the definition of diagnosis as a set of faulty components could be too restrictive since users may want to identify different levels of faults. Reiter's diagnosis theory is not based on uncertain knowledge but is on incomplete knowledge since it is based on first-order sentences. Usually, normality and faultiness of components, obtained from instrument measurements, expert experience, or analysis using probabilistic schemes, cannot be determined accurately. The research on the diagnostic problem for such systems with fuzziness is interesting and important. Zadeh's fuzzy-set theory [19] is a solution for the ideas and approaches for handling

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nonstatistical uncertainty. In the early stage, the fuzzy diagnosis was related to the application of fuzzy set theory in medical diagnosis, and the classifier used fuzzy set principles for solving a medical diagnostic problem [20]. However, such an approach was too simple and lacked theoretical background. Other efforts, such as classical abductive approaches to diagnosis, often focused on a fuzzifying modus, based on various nonclassical logics (fuzzy logic, multiple-valued logic, Lukasiewicz logic, and those of Godel, Goguen, Rescher, etc.) [21], [22], [23]. Recently, Isermann [24] developed a fuzzy diagnostic model as a fuzzy symptom faults map, implemented by using a heuristic knowledge base. The diagnostic reasoning schemes match the proposed fuzzy diagnostic model with the current values of symptoms. In this paper, a fuzzy version of Reiter's consistency-based diagnosis theory has been proposed. Our framework uses Reiter's consistency-based diagnosis methodology and is able to capture the diagnosis problem from systems with fuzziness.

Taking inspiration from Vojitas' work on a formal model of fuzzy logic programming, where notions of a fuzzy theory and its model were introduced [25], we can similarly work on a truth-functional logic, in a narrow sense, based on Hajec's work [28], for building a framework of fuzzy diagnosis. It is worth noting that Vojitas focused on building a procedural and declarative semantics for fuzzy logic programming without negation, proving their soundness and completeness by defining the truth functions of many valued connections and the soundness of many valued modus ponens. In particular, negation cannot occur in any formula. Therefore, this model cannot be applied to fault diagnosis based on consistency, but to threshold computation, abduction, and fuzzy unification based on similarity.

In this paper, the authors present a formal model for fuzzy diagnosis by extending Vojitas' model (allowing occurrence of negation) and defining the notions of consistency and entailability. By comparing with the classical system description and the observation of a system in Reiter's sense, the authors represent their fuzzified extensions by applying fuzzy theories. A framework of fuzzy diagnosis and its properties, similar to that in Reiter's framework, have also been established in such an extended way. Under this framework, computing fuzzy diagnoses is mapped to solving a system of inequalities. Generally, to solve such a system of inequalities is very complex or even impossible. Thus, the authors focus on a set of special cases (e.g., when fuzzy truth values are taken from a finite chain and some classes of fuzzy theories with special forms). In particular, a method for computing fuzzy diagnoses is presented in which the fuzzy system description and the fuzzy observations are clause-style fuzzy theories. A student-fault diagnostic example is given to demonstrate the usefulness of our framework. It is clear that our framework can support the fault diagnosis, either based on Reiter's diagnosis theory with precise knowledge or based on knowledge with fuzziness.

This paper is organized as follows: Reiter's diagnosis theory is introduced and a new characterization of a diagnosis is proposed in Section 2. The framework of fuzzy diagnosis and important properties of fuzzy diagnoses are presented in Section 3. A procedure for finding all diagnoses for any diagnosis problem and a general method for solving a clause-style diagnosis problem are given in Section 4. A student-fault diagnosis problem and its simplified example are described in Section 5. The last section is devoted to the summary and conclusions.

2 Reiter's Theory of Diagnosis

In order to formalize model-based diagnosis, Reiter established a precise theoretical foundation for diagnosis from first principle, using first-order sentences [1]. In this section, we briefly recall the basic notions and results of Reiter's theory of diagnosis, and then present a new characterization of diagnoses.

2.1 Notions and Results of Reiter's Diagnosis Theory

As in [1], a pair (SD, COMPS) of a system has been defined, where SD, the system description, is a set of first-order sentences and COMPS, the system components, is a finite set of constants. In all intended applications, the system description will mention a specific predicate AB(.), interpreted to mean "abnormal." An observation of a system is a finite set of first-order sentences. (SD, COMPS, OBS) for a system (SD, COMPS) with observation, OBS, can be written. A diagnosis for (SD, COMPS, OBS) is a minimal set $\Delta \subseteq \mathrm{COMPS}$ such that

$$\mathrm{SD} \cup \mathrm{OBS} \cup \{\mathrm{AB(c)}|\ \mathrm{c} \in \Delta\} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS\text{-}}\Delta\}$$

is consistent. Reiter gave important properties of a diagnosis for (SD, COMPS, OBS), which are useful for determining the existence of a diagnosis and computing this diagnosis if it exists.

Proposition 2.1 [1]. *A diagnosis exists for (SD, COMPS, OBS) iff (if and only if)* $\mathrm{SD} \cup \mathrm{OBS}$ *is consistent.*

Proposition 2.2 [1]. $\oslash$ *(the empty set) is a diagnosis (and the only diagnosis) for (SD, COMPS, OBS) iff* $\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS}\}$ *is consistent, i.e., if the observation does not conflict with the system, which should be the case if all its components behave correctly.*

Proposition 2.3 [1]. *If* Δ *is a diagnosis for (SD, COMPS, OBS), then for each* $\mathrm{c_i} \in \Delta$,

$$\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS\text{-}}\Delta\} \models \mathrm{AB(c_i)}.$$

Here, the notation "$\models$*" is the classical entailability relation.*

Proposition 2.4 [1]. $\Delta \subseteq \mathrm{COMPS}$ *is a diagnosis for (SD, COMPS, OBS) iff* Δ *is a minimal set such that* $\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS\text{-}}\Delta\}$ *is consistent.*

Reiter pointed out that Proposition 2.3 is rather interesting since it says that the faulty components Δ are logically determined by the normal components $\mathrm{COMPS\text{-}}\Delta$.

2.2 A New Characterization of Diagnoses

Note that the properties above are not enough for designing an algorithm to solve a diagnosis problem since they only characterize necessary or sufficient conditions of the existence of a diagnosis. It can be claimed that the converse of Proposition 2.3 also holds, i.e., it is not only necessary but also sufficient for a set of components to be a diagnosis.

Proposition 2.5. *Let (SD, COMPS) be a system and OBS be an observation. Given* $\Delta \subseteq \mathrm{COMPS}$, *if* $\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS\text{-}}\Delta\}$ *is consistent, and for any* $\mathrm{c_i} \in \Delta$,

$$\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)}|\ \mathrm{c} \in \mathrm{COMPS\text{-}}\Delta\} \models \mathrm{AB(c_i)},$$

then Δ *is a diagnosis for (SD, COMPS, OBS).*

Combining Propositions 2.3 and 2.5 provides a new characterization of diagnoses, by which we can determine whether any set $\Delta \subseteq \mathrm{COMPS}$ is a diagnosis or not.

Proposition 2.6. $\Delta \subseteq \mathrm{COMPS}$ *is a diagnosis for (SD, COMPS, OBS) iff* $\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)} | \ \mathrm{c} \in \mathrm{COMPS}\text{-}\Delta\}$ *is consistent and for any* $\mathrm{c_i} \in \Delta$,

$$\mathrm{SD} \cup \mathrm{OBS} \cup \{\leftarrow \mathrm{AB(c)} | \ \mathrm{c} \in \mathrm{COMPS}\text{-}\Delta\} \models \mathrm{AB(c_i)}.$$

Remark 2.1. As is well-known, the main idea of a diagnosis by abduction is that users sometimes want the diagnosis not only to be consistent with the observation, but to also predict the outputs given the inputs. Proposition 2.6 shows the observations not only to be consistent with the normal components, but to also explain the faulty assumptions. Conversely, only the subsets of COMPS with this property are diagnoses. So, Proposition 2.6 makes it more understandable that the main difference between the two models of diagnosis is that, in abduction, the diagnoses entail the observations, whereas in the consistency-based model, the observations entail the diagnoses.

Remark 2.2. Proposition 2.6 underlies a new system to compute all diagnoses and to establish complexity. The first step checks consistency of $\mathrm{SD} \cup \mathrm{OBS}$. If it is inconsistent, then there is no diagnosis. If $\mathrm{SD} \cup \mathrm{OBS}$ is consistent, then users may guess a subset of COMPS and test whether it satisfies the conditions in Proposition 2.6. Clearly, checking the consistency of each subset of COMPS takes exponential time. Using a canonical method, as in [14] and [26], it can be established that the problem of existence of diagnoses is Σ_2^{P}-complete.

3 A Framework of Fuzzy Diagnosis

Reiter's diagnosis theory is not based on uncertain knowledge since it is for systems described by first-order sentences. Usually, normality and faultiness of components of a system, which are gotten by instrument measuring, expert experience, or analyses of probability, are not precise but with fuzziness. So, an extension of Reiter's diagnosis theory in a fuzzy sense is necessary and interesting. In this section, a framework of fuzzy diagnosis and some properties of fuzzy diagnoses will be presented, which are extensions of the corresponding results in [1]. First, some basic concepts relative to a theory of fuzzy diagnosis will be formalized. The notions of a fuzzy system and fuzzy observations are introduced, based on the extended notion of a fuzzy theory, by allowing the occurrence of negation; a fuzzy diagnosis is defined by introducing notions of consistency and entailability for fuzzy systems. Second, some important consequences of a fuzzy diagnosis, which are generalizations of Reiter's corresponding results in Reiter's framework [1], are established (detailed derivations are found in the Appendix). Finally, two special subclasses of fuzzy diagnosis will be discussed that are interesting and applicable in many real-life situations.

3.1 Truth-Function Fuzzy Logic in a Narrow Sense

To propose a formal model of fuzzy diagnoses, which is an extension of Reiter's diagnosis theory, truth-function logic in a narrow sense will be outlined in this section.

A multisorted predicate language, with or without function symbols, is used. Recall that a set S of sentences is consistent iff S has a (two-valued) model. S entails a sentence ϕ iff each model of S is also a model of ϕ. If S and ϕ are expressed in a form of first-order clauses, then we can restrict our declarative semantics only on Herbrand models. Since we are interested only in practical diagnoses, we disregard arbitrary interpretations here and base our semantics only on Herbrand interpretations. Following closely Lloyd's presentation and even notation [27], a Herbrand universe of sort A, denoted by $\mathrm{U_L^A}$, consists of all ground terms as crisp. As in [25], let $\mathrm{B_L}$ be the Herbrand base of the language L. All fuzzy predicates will be interpreted by a mapping from $\mathrm{B_L}$ to the unit interval $[0,1]$. We call f: $\mathrm{B_L} \rightarrow [0,1]$ a fuzzy interpretation of our language. For ground atoms $\mathrm{p} \in \mathrm{B_L}$, $\mathrm{f(p)}$ is its truth value. For arbitrary formula ϕ and an evaluation of all sorts of variables $\mathrm{e^A}$: $\mathrm{Va^A} \rightarrow \mathrm{U_L^A}$, the truth value $\mathrm{f}(\phi)[\mathrm{e}]$ is calculated along the complexity of formulas using truth functions of connectives and quantifiers:

$$\begin{aligned}
&\mathrm{f}(\leftarrow \phi) = 1 - \mathrm{f}(\phi),\\
&\mathrm{f}(\phi \vee \psi) = \max(\mathrm{f}(\phi), \mathrm{f}(\psi)),\\
&\mathrm{f}(\phi \vee \psi) = \min(\mathrm{f}(\phi), \mathrm{f}(\psi)),\\
&\mathrm{f}(\phi \rightarrow \psi) = \max(1 - \mathrm{f}(\phi), \mathrm{f}(\psi)),
\end{aligned}$$

$\mathrm{f}(\forall \mathrm{x} \mathrm{f}(\phi)) = \inf\{\mathrm{f}(\phi)[\mathrm{e}'] \ \mathrm{e}' =_{\mathrm{x}} \mathrm{e}\}$, where $\mathrm{e}' =_x \mathrm{e}$ means that e' can differ from e only at x. Finally, let the truth value of a formula ϕ under an interpretation f be the same as that of its generalization and not depend on evolution:

$$\mathrm{f}(\phi) = \inf\{\mathrm{f}(\phi)[\mathrm{e}] | \ \mathrm{e} \text{ arbitrary}\}.$$

Remark 3.1. In Vojitas' approach, negation $\leftarrow$ does not occur in any formula and a many-value modus ponens is defined (it is needed for application to abduction). So, this model is not suitable for capturing a fuzzy fault diagnosis based on consistency. In our framework, occurrence of a negation $\leftarrow$ in a formula is allowed and fuzzified modus ponens is not needed to compute a fuzzy diagnosis based on consistency. Notions of a negative AB-literal $\leftarrow \mathrm{AB(c)}$ and its fuzzy truth-value will be defined in Section 3.2.

Remark 3.2. Proofs of all results in Sections 3.2 and 3.3 are only relevant to the truth of the negation $\leftarrow$. To capture different extensions of classical connectives, as in [25], we can replace the above truth functions of connectives $\vee$, $\wedge$, and $\rightarrow$ with the following connectives in this remark, respectively. In general, the above truth-value function of connectives is often used since it makes computation of diagnoses effective. The clause-style diagnostic problem that we will discuss in Section 4 is just such an example. Of course, it could sometimes lose useful information. A difference between computation and the real-world situation may not just be attributed to the system itself, but in the description of the real situation. In this case, either connectives, or fuzzy predicates, or the rule base can be tuned. So, some of the following types of connectives based on practice or experience can be chosen.

The Lukasiewicz connectives:

$$\begin{aligned}
\vee_{\mathrm{L}} (\mathrm{x}, \mathrm{y}) &= \min(1, \mathrm{x} + \mathrm{y}),\\
\wedge_{\mathrm{L}} (\mathrm{x}, \mathrm{y}) &= \max(0, \mathrm{x} + \mathrm{y} - 1),\\
\leftarrow_{\mathrm{L}} (\mathrm{x}) &= 1 - \mathrm{x},\\
\rightarrow_{\mathrm{L}} (\mathrm{x}, \mathrm{y}) &= \min(1, 1 - \mathrm{x} + \mathrm{y}).
\end{aligned}$$

The Godel intuitionistic connectives:

$$\begin{aligned}\vee_G(x,y) &= \max(x,y),\\ \wedge_G(x,y) &= \min(x,y),\\ \leftarrow_G(0) &= 1,\\ \leftarrow_G(x) &= 0 \text{ for } x>0,\\ \rightarrow_G(x,y) &= y \text{ if } x>y \text{ else } 1.\end{aligned}$$

The product logic:

$$\begin{aligned}\vee_P(x,y) &= x+y-x.y,\\ \wedge_P(x,y) &= x.y,\\ \leftarrow_P(0) &= 1,\\ \leftarrow_P(x) &= 0 \text{ for } x>0,\\ \rightarrow_P(x,y) &= \min(1,y/x) \text{ if } x>y \text{ else } 1.\end{aligned}$$

3.2 Fuzzy Diagnosis

In this section, a framework of fuzzy diagnosis is proposed by developing a fuzzy logic system similar to that in [25]. More specially, notions of a fuzzy theory, consistency, entailability, and an α-level diagnosis are introduced.

A fuzzy theory is a partial mapping T assigned to formula numbers from (0, 1]. Here, partiality of the mapping T is understood as being defined as constantly zero outside of the domain $\mathrm{dom}(T)$. A fuzzy interpretation f is a model of a fuzzy theory T if $f(\phi) \geq T(\phi)$ for all formulas $\phi \in \mathrm{dom}(T)$. This means that the truth value assigned to the axiom is understood as a lower bound of truth values in structures which are models.

Definition 3.1. *A fuzzy system is a pair* (T_{SD}, COMPS), *where* SD *is a set of first-order sentences,* T_{SD} *is a partial mapping from* SD *to (0, 1] (a fuzzy theory), and* COMPS *is a set of constants. A fuzzy observation* T_{OBS} *is a partial mapping from* OBS *to (0, 1] (a fuzzy theory), where* OBS *is a finite set of first-order sentences.*

Definition 3.2. *A collection of fuzzy theories* $\{T_i | i \geq 0\}$ *is consistent if there is an interpretation* f *such that for each* $i \geq 0$ *and for any* $\phi \in \mathrm{dom}(T_i)$, $f(\phi) \geq T_i(\phi)$, *that is,* f *is a common model of* $\{T_i | i \geq 0\}$.

Definition 3.3. *A collection* $\{T_i | i \geq 0\}$ *of fuzzy theories entails a fuzzy theory T′, denoted as* $\{T_i | i \geq 0\} \models_F T'$, *if each model of* $\{T_i | i \geq 0\}$ *is also a model of* T'.

Notations. Given any $\Delta \subseteq \mathrm{COMPS}$, let $T_N(\Delta)$ and $T_P(\Delta)$ be two partial mappings (from negative AB-literals and positive AB-literals to (0, 1], respectively) such that

$$\mathrm{dom}(T_N(\Delta)) = \{\leftarrow AB(c) | c \in \text{COMPS-}\Delta\},$$

$$\mathrm{dom}(T_P(\Delta)) = \{AB(c) | c\ \Delta\}.$$

For any $\alpha \in (0,1]$, let $T_N(\Delta,\alpha)$ and $T_P(\Delta,\alpha)$ be two partial mappings such that

$$\mathrm{dom}(T_N(\Delta,\alpha)) = \{\leftarrow AB(c) | c \in \text{COMPS-}\Delta\}$$

and $T_N(\Delta,\alpha)(\leftarrow AB(c)) > 1-\alpha$ for each $c \in \text{COMPS-}\Delta$; $\mathrm{dom}(T_P(\Delta,\alpha)) = \{AB(c)\ c \in \Delta\}$ and $T_P(\Delta)(AB(c)) \geq \alpha$ for each $c \in \Delta$.

Clearly, $T_N(\Delta,\alpha)$ and $T_P(\Delta,\alpha)$ are fuzzy theories.

Definition 3.4. *Let* (T_{SD}, COMPS) *be a fuzzy system and* T_{OBS} *an observation. For any given* $\alpha \in (0,1]$, $\Delta \subseteq \mathrm{COMPS}$ *is defined as an* α-level *diagnosis for* $(T_{SD}, \mathrm{COMPS}, T_{OBS})$ *if* Δ *satisfies the following conditions:*

1. *There are some* $T_N(\Delta,\alpha)$ *and* $T_P(\Delta,\alpha)$ *such that* $\{T_{SD}, T_{OBS}, T_N(\Delta,\alpha), T_P(\Delta,\alpha)\}$ *is consistent.*
2. *For any* $\Delta' \subset \Delta$, *there is no* $T_N(\Delta',\alpha)$ *and* $T_P(\Delta',\alpha)$ *such that* $\{T_{SD}, T_{OBS}, T_N(\Delta',\alpha), T_P(\Delta',\alpha)\}$ *is consistent.*

That is, an α*-level diagnosis for* $(T_{SD}, \mathrm{COMPS}, T_{OBS})$ *is a minimal subset* Δ *of* COMPS *such that*

$$\{T_{SD}, T_{OBS}, T_N(\Delta,\alpha), T_P(\Delta,\alpha)\}$$

is consistent for some $T_N(\Delta,\alpha)$ *and* $T_P(\Delta,\alpha)$.

Here, a diagnosis of a dynamic system is characterized by a common model for $\{T_{SD}, T_{OBS}, T_N(\Delta,\alpha), T_P(\Delta,\alpha)\}$ (if it exists).

To determine the existence of a fuzzy diagnosis and design a method for computing diagnoses, some important properties of a fuzzy diagnosis will be discussed in the next section.

3.2.1 Some Consequences of the Definition

The following important properties corresponding to results in Section 2 can be derived from the previous definitions. They characterize the concept of a fuzzy diagnosis from different points of view.

Theorem 3.1. *An* α*-level diagnosis exists for*

$$(T_{SD}, \mathrm{COMPS}, T_{OBS})$$

iff $\{T_{SD}, T_{OBS}\}$ *is consistent.*

This theorem shows that one can determine if

$$(T_{SD}, \mathrm{COMPS}, T_{OBS})$$

has a diagnosis only by fuzzy system description and observation(s).

Theorem 3.2. *A subset of* COMPS Δ *is an* α*-level diagnosis for* $(T_{SD}, \mathrm{COMPS}, T_{OBS})$ *iff* Δ *is a minimal subset of* COMPS *such that* $\{T_{SD}, T_{OBS}, T_N(\Delta,\alpha)\}$ *is consistent for some* $T_N(\Delta,\alpha)$.

Comparing with Theorem 3.4 in [1], this result shows that the concept of a fuzzy diagnosis, as a fuzzy version of Reiter's concept of a diagnosis, is reasonable. It provides a simpler characterization of a fuzzy diagnosis than does the original Definition 3.4.

Corollary 3.3. $\oslash$ *is an* α*-level diagnosis (and the only diagnosis) iff* $\{T_{SD}, T_{OBS}, T_N(\oslash,\alpha)\}$ *is consistent for some* $T_N(\oslash,\alpha)$.

Theorem 3.4. *Let* $\Delta \subseteq \mathrm{COMPS}$. Δ *is a unique* α*-level diagnosis for* $(T_{SD}, \mathrm{COMPS}, T_{OBS})$ *iff* $\{T_{SD}, T_{OBS}\}$ *is consistent and for any model* f *of* $\{T_{SD}, T_{OBS}\}$, $c \in \Delta \Rightarrow f(AB(c)) > \alpha$.

This theorem gives a simple decision method for a unique α-level diagnosis.

Theorem 3.5. $\Delta \subseteq \mathrm{COMPS}$ *is an* α*-level diagnosis for* $(T_{SD}, \mathrm{COMPS}, T_{OBS})$ *iff for some* $T'_N(\Delta,\alpha)$,

$$\{T_{SD}, T_{OBS}, T'_N(\Delta, \alpha)\}$$

is consistent and for any $T_N(\Delta, \alpha)$,

$$\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\} \models_F T_P(\Delta, =, \alpha),$$

where $T_P(\Delta, =, \alpha)$ *is a partial mapping from* $\{AB(c) | c \in \Delta\}$ *to (0,1] such that* $T_P(\Delta, =, \alpha)(AB(c)) = \alpha$ *for each* $c \in \Delta$.

The above result says that the faulty components are logically determined by the normal components. Based on it, the methods for solving some special problems of fuzzy diagnosis can be designed.

3.2.2 Special Cases

In this section, some special cases of fuzzy diagnoses, abstracted from many real-world situations, will be discussed. There are two types mainly: One is for precisely determined fault components, even if the system description and the observation(s) are fuzzy. In particular, Reiter's diagnosis theory is a special case where a description and observation(s) of a system are precise. Another special case involves computability since infinite truth-values, which leads to checking infinite number of fuzzy interpretations of a fuzzy theory, is not considered from the point of view of computation. Of course, there are real-life situations in which these would not be appropriate, for example, fuzzy diagnosis of clause-style fuzzy systems and observations, where infinite fuzzy truth-value taken in the real unit interval and diagnoses with arbitrary levels are considered (see Section 4).

1. 1-level diagnosis—Δ is a minimal subset of COMPS such that $\{T_{SD}, T_{OBS}, T^*_N(\Delta), T^*_P(\Delta)\}$ is consistent, where

   $$T^*_N(\Delta)(\leftarrow AB(c)) > 0 \text{ for each } c \in \text{COMPS-}\Delta,$$
   $$T^*_P(\Delta)(AB(c)) = 1, \text{ for each } c \in \Delta.$$

   In particular, if we restrict ourselves to considering only the case where fuzzy truth values assigned to formulas are from $\{0, 1\}$, then 1-level diagnoses produce the same results as Reiter's.
2. Let L be a finite subset of (0,1] such that $u \in (0, 1)$ iff $1 - u \in L$. Assume that fuzzy-truth values assigned to formulas are from L.

   Clearly, the above formulas hold for any finite chain L with the complementary operation c such that

   a. $(u^c)^c = u$ for any $u \in L$,
   b. $u \le v \Rightarrow u^c \ge v^c$, and
   c. $u \in L$ iff $u^c \in L$.

4 Computing Diagnosis

In this section, the computability, computational complexity of the fuzzy diagnosis problem and a method for solving a special type of fuzzy diagnosis problem will be discussed.

To begin with, a procedure for solving a fuzzy diagnosis problem will be described.

4.1 Procedure

Given a fuzzy system (T_{SD}, COMPS) and a fuzzy observation set T_{OBS}, it is interesting to note that $\{T_{SD}, T_{OBS}\}$ is consistent and that SD and OBS are finite sets. Let $SD = \{\phi_i | i \le m\}$, $OBS = \{\psi_i | i \le n\}$, $Ran(T_{SD}) = \{_i | r_i = T_{SD}(\phi_i), i \le m)$, and $Ran(T_{OBS}) = \{t_i | t_i = T_{OBS}(\psi_i), i \le n\}$. Using Theorems 3.2 and 3.5, a procedure for finding an α-level diagnosis is given in this section. Briefly speaking, starting from the empty set $\oslash$, check every subset Δ of COMPS according to the order $\le$ on cardinals of subsets. Solving the following system of inequalities on an unknown partial mapping f completes the checking:

$$f(\phi_i) \ge r_i, \text{for each } i \le m, \tag{4.1}$$

$$f(\psi_i) \ge t_i, \text{ for each } i \le n, \tag{4.2}$$

$$f(\leftarrow AB(c)) \le 1 - \alpha \text{ for each } c \in \Delta, \quad \text{or equivalently, } f(AB(c)) \ge \alpha. \tag{4.3}$$

If the system has a solution, then Δ is a diagnosis and its supersets are not checked yet.

Procedure: All Diagnoses

```
begin Diagnoses← ⊘, Candidate← ⊘
    for n = 0 to |COMPS| (the cardinal of COMPS) do
        for each Δ ⊆ COMPS, |Δ| = n and there is no
        Δ′ ∈ Diagnosis such that Δ′ ⊂ Δ do
            begin
                    for each partial mapping f do
                        if f satisfies (4.1)-(4.3) then add Δ to
                        Diagnosis and go to loop*
                    loop**
            end
        loop*
    loop
end
```

4.2 Computability (Decidability) and Computational Complexity

The procedure in the previous section can never end, which implies that the fuzzy diagnosis problem is semidecidable since the number of partial mappings taking values from (0,1] is infinite, and there is no decision algorithm for determining the consistency of first-order formulae. Hence, it is hard to compute diagnoses in most general cases. Nevertheless, there are many practical settings where consistency is decidable, e.g., fuzzy-truth values being taken from a finite chain as in Section 3.2, some class of fuzzy theories with special forms, etc. Even so, combinatorial explosion could be encountered since testing each subset of COMPS with large numbers of components is NP-hard.

4.3 Diagnosis for Systems and Observations with Clause-Style Fuzzy Theories

In classical logic, a sentence can be transformed into one with a conjunctive normal form such that determining for the consistency of first-order sentences becomes simpler (of course, this conjunctive transformation problem is still NP-complete). Similarly, in this section, we restrict ourselves to a fuzzy diagnosis for systems with clause-style fuzzy theories, that is, for $(T_{SD}, \text{COMPS } T_{OBS})$, where T_{SD} and T_{OBS} are clause-style fuzzy theories (a clause-style fuzzy theory is a partial mapping T assigned to clause-style formulas; here, each sentence in $\mathrm{SD} \cup \mathrm{OBS}$ is a conjunction of clauses). As in [3], an AB-literal is $\mathrm{AB(c)}$ or $\leftarrow \mathrm{AB(c)}$ for some $\mathrm{c} \in \mathrm{COMPS}$. A literal L is a non-AB-literal if L is not an AB-literal. Suppose that $\{\mathrm{L_1}, \dots, \mathrm{L_k}\}$ is the set of all non-AB-literals occurring in $\mathrm{SD} \cup \mathrm{OBS}$, and that f is an undetermined partial mapping such that $\mathrm{f(L_i)} = 1 - \mathrm{f(L_j)}$ for each pair of complementary literals $\mathrm{L_i}$ and $\mathrm{L_j}$ in $\{\mathrm{L_1}, \dots, \mathrm{L_k}\}$ and such that $\mathrm{f}(\leftarrow \mathrm{AB(c)}) = 1 - \mathrm{f(AB(c))}$ for each $\mathrm{c} \in \mathrm{COMPS}$. Note that the fuzzy-truth value of a conjunction of clauses is not less than some number s from (0,1) iff the fuzzy-truth value of each clause in the conjunction is not less than s. With no loss of generality, it can be assumed that no AB-literals occur in OBS, which conforms to real applications. In fact, (ab)normality of a component cannot be observed. Then, computing diagnoses for systems with clause-style fuzzy theories is translated into solving the following system of inequalities:

$$\begin{aligned}&\mathrm{f(L_{j,1})} \vee \dots \vee \mathrm{f(L_{j,p})} \vee \mathrm{f(AB(c_{j,1}))} \vee \dots \vee \mathrm{f(AB(c_{j,,l}))}\\ &\quad \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,l+1})}) \vee \dots \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,q})}) \geq \mathrm{r_j} \\ &\quad \text{for each } \mathrm{j} : 1 \leq \mathrm{j} \leq \Sigma_{1 \leq \mathrm{i} \leq_{\mathrm{m}} \mathrm{u_i}};\end{aligned} \tag{4.4}$$

$$\mathrm{f(L^*_{j,1})} \vee \dots \vee \mathrm{f(L^*_{j,p})} \geq \mathrm{t_j}, \text{for each } \mathrm{j} : 1 \leq \mathrm{j} \leq \Sigma_{1 \leq \mathrm{i} \leq_{\mathrm{n}} \mathrm{v_i}}; \tag{4.5}$$

$$\begin{aligned}&\mathrm{f}(\leftarrow \mathrm{AB(c)}) \leq 1 - \alpha \text{ for each } \mathrm{c} \in \Delta,\\ &\text{or equivalently, } \mathrm{f(AB(c))} \geq \alpha,\end{aligned} \tag{4.6}$$

where $\mathrm{L_{j,1}}, \dots, \mathrm{L_{j,p}}$,

$$\mathrm{L^*_{j,1}}, \dots, \mathrm{L^*_{j,p}} \in \{\mathrm{L_1}, \dots, \mathrm{L_k}\},$$

$\mathrm{c_{j,1}}, \dots, \mathrm{c_{j,,q}} \in \mathrm{COMPS}$, $\mathrm{r_j} \in \{\mathrm{r_1}, \dots, \mathrm{r_m}\}$, $\mathrm{t_j} \in \{\mathrm{t_1}, \dots, \mathrm{t_n}\}$, $\mathrm{u_i}$ is the number of all clauses (conjunction terms) of ϕ_i for each $\phi_\mathrm{i} \in \mathrm{SD}$ and $\mathrm{v_i}$ is the number of all clauses of ψ_i for each $\psi_\mathrm{i} \in \mathrm{OBS}$, ϕ_i, ψ_i have the forms

$$\begin{aligned}&\mathrm{L_{j,1}} \vee \dots \vee \mathrm{L_{j,p}} \vee \mathrm{AB(c_{j,1})} \vee \dots \vee \mathrm{AB(c_{j,,l})}\\ &\quad \vee \leftarrow \mathrm{AB(c_{j,l+1})} \vee \dots \vee \leftarrow \mathrm{AB(c_{j,q})} \text{ and } \mathrm{L^*_{j,1}} \vee \dots \vee \mathrm{L^*_{j,p}}.\end{aligned}$$

To compute all diagnoses, clearly, it is sufficient to solve the above system of inequalities and then to select all minimal elements from $\{\Delta | \Delta = \{\mathrm{c} | \mathrm{f(AB(c))} \geq \alpha$ for some f satisfying (4.4) and (4.5)$\}$. For solving the system of inequalities (4.4)-(4.6), an enumeration of all candidates for α-level diagnoses is used. That is, for each subset Δ of COMPS, the system of inequalities can be solved. If there is a partial interpretation f such that the system has a solution, then Δ is a candidate for a diagnosis.

Based on properties of operators $\vee$, $\leftarrow$, and relation $\leq$, a general method for solving the system of inequalities constituted by (4.4)-(4.6) is given in this section.

Let Γ be the set of systems of inequalities, which contains all inequalities in (4.4) and all equalities with the form $\mathrm{Ran(f(L))} = (0.1]$ for any literal L occurring in (4.4)-(4.6). For any (AB or non-AB-) literal L, $\overline{Ran(f(L))}$ stands for the complement of $\mathrm{Ran(f(L))}$, i.e., $\overline{[a,b]} = [1-\mathrm{b}, 1-\mathrm{a}]$, $\overline{[a,b)} = (1-\mathrm{b}, 1-\mathrm{a})$, $\overline{(a,b]} = [1-\mathrm{b}, \mathrm{a}]$, $\overline{(a,b)} = (1-\mathrm{b}, 1-\mathrm{a})$ for any $\mathrm{a}, \mathrm{b} \in [0,1]$ and $\mathrm{a} \leq \mathrm{b}$. The following method for solving inequalities is similar to successive elimination by substitution for solving system of linear equalities. Such successive elimination by substitution is based on the following feature: an inequality

$$\begin{aligned}&\mathrm{f(L_1)} \vee \dots \vee \mathrm{f(L)} \vee \mathrm{f}(\leftarrow \mathrm{L}) \vee \dots \vee \mathrm{f(L_m)}) \geq \mathrm{r}\\ &\text{(where L is a (AB or non-AB) literal)}\end{aligned}$$

is equivalent to the inequality

$$\mathrm{f(L_1)} \vee \dots \vee \mathrm{f(L_m)}) \geq \text{ r when } \mathrm{f(L)} \vee \mathrm{f}(\leftarrow \mathrm{L}) < \mathrm{r}$$

or equivalent to the absolute inequality when $\mathrm{f(L)} \vee \mathrm{f}(\vee \mathrm{L}) \geq \mathrm{r}$ (hence, it can be deleted).

Step 1. Delete the disjunction with form $\mathrm{f(L)} \vee \mathrm{f}(\leftarrow \mathrm{L})$ occurring in any inequality of (4.4)-(4.5), where L is a (AB or non-AB) literal. Further, if $\mathrm{f(L)} \vee \mathrm{f}(\leftarrow \mathrm{L}) \geq$ the right part of inequality (in which $\mathrm{f(L)} \vee \mathrm{f}(\leftarrow \mathrm{L})$ occurs), then delete the inequality.

Step 2. Using (4.6), reduce all occurrence of $\mathrm{f(AB(c))}$ and $\mathrm{f}(\leftarrow \mathrm{AB(c)})$.

Substep 1. Reducing $\mathrm{AB(c)}(\mathrm{c} \in \Delta)$.

For each $\mathrm{c} \in \Delta$ and each inequality in Γ with form

$$\begin{aligned}&\mathrm{f(L_{j,1})} \vee \dots \vee \mathrm{f(L_{j,p})} \vee \mathrm{f(AB(c_{j,1}))} \vee \dots \vee \mathrm{f(AB(c))} \vee \dots \vee\\ &\mathrm{f}(\leftarrow \mathrm{AB(c_{j,q})}) \geq \mathrm{r_j},\end{aligned} \tag{$*$}$$

set $\mathrm{Ran(f(AB(c)))} = \mathrm{Ran(f(AB(c)))} \cap [\alpha, 1]$ since c is in the assumed α-level diagnosis Δ.

Case 1. If $\alpha \geq \mathrm{r_j}$, then delete the inequality (*) from Γ.

Case 2. If $\alpha < \mathrm{r_j}$, then replace (*) in Γ with the inequality (* 1), which is obtained by deleting the disjunction $\mathrm{f(AB(c))}$ from (*),

$$\begin{aligned}&\mathrm{f(L_{j,1})} \vee \dots \vee \mathrm{f(L_{j,p})} \vee \mathrm{f(AB(c_{j,1}))} \vee \dots \vee\\ &\mathrm{f}(\leftarrow \mathrm{AB(c_{j,q})}) \geq \mathrm{r_j},\end{aligned} \tag{$*1$}$$

and set $\mathrm{Ran(f(AB(c)))} = \mathrm{Ran(f(AB(c)))} \cap [\alpha, \mathrm{r_j}]$.

Substep 2. Reducing $\leftarrow \mathrm{AB(c)}(\mathrm{c} \in \Delta)$.

For each inequality in Γ with the form

$$\begin{aligned}&\mathrm{f(L_{j,1})} \vee \dots \vee \mathrm{f(L_{j,p})} \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,1})}) \vee \dots \vee\\ &\mathrm{f}(\leftarrow \mathrm{AB(c)}) \vee \dots \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,q})}) \geq \mathrm{r_j},\end{aligned} \tag{$**$}$$

set $\mathrm{Ran(f(\leftarrow AB(c)))} = \mathrm{Ran(f(\leftarrow AB(c)))} \cap) 0, 1-\alpha]$ since c is in the assumed α-level diagnosis Δ.

Case 1. If $1 - \alpha < \mathrm{r_j}$, then replace (**) in Γ with the following inequality obtained by deleting the disjunction $\mathrm{f}(\leftarrow \mathrm{AB(c)})$ from (**),

$$\begin{aligned}&\mathrm{f(L_{j,1})} \vee \dots \vee \mathrm{f(L_{j,p})} \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,1})})\\ &\quad \vee \dots \vee \mathrm{f}(\leftarrow \mathrm{AB(c_{j,q})}) \geq \mathrm{r_j}.\end{aligned}$$

Case 2. If $1 - \alpha \geq \mathrm{r_j}$, then Γ is replaced its variants Γ_1 and Γ_2. Γ_1 is as in Case 1 except setting

$$\mathrm{Ran(f(\leftarrow AB(c)))} = \mathrm{Ran(f(\leftarrow AB(c)))} \cap (0, r_j).$$

Γ_2 is obtained by deleting (**) from Γ and setting $\mathrm{Ran(f(\leftarrow AB(c)))} = \mathrm{Ran(f(\leftarrow AB(c)))} \cap [\mathrm{r_j}, 1-\alpha]$.

Substep 3. Reducing $\mathrm{AB(c)}(\mathrm{c} \in \text{COMPS-}\Delta)$.

For each $\mathrm{c} \in \text{COMPS-}\Delta$ and each inequality in Γ (its variants) with the form

$$f(L_{j,1}) \vee \ldots \vee f(L_{j,p}) \vee f(AB(c_{j,1})) \vee \ldots \vee f(AB(c)) \vee \ldots \vee f(\leftarrow AB(c_{j,q})) \geq r_j, \quad (***)$$

set $Ran(f(AB(c))) = Ran(f(AB(c))) \cap (0, \alpha]$ since c in not in the assumed α-level diagnosis Δ.

Case 1. If $\alpha \geq r_j$, then Γ replaces its variants Γ_1 and Γ_2. Γ_1 is obtained by deleting the inequality (***) from Γ and setting $Ran(f(AB(c))) = Ran(f(AB(c))) \cap [r_j, \alpha]$. Γ_2 is obtained by replacing (***) in Γ with the following inequality:

$$f(L_{j,1}) \vee \ldots \vee f(L_{j,p}) \vee f(AB(c_{j,1})) \vee \ldots \vee f(\leftarrow AB(c_{j,q})) \geq r_j \quad (***1)$$

and setting $Ran(f(AB(c))) = Ran(f(AB(c))) \cap (0, r_j)$, where (*** 1) is obtained by deleting the disjunction $f(AB(c))$ from (***).

Case 2. If $\alpha < r_j$, then replace (***) in Γ with the following inequality obtained by deleting the disjunction $f(AB(c))$ from (***),

$$f(L_{j,1}) \vee \ldots \vee f(L_{j,p}) \vee f(AB(c_{j,1})) \vee \ldots \vee f(\leftarrow AB(c_{j,q})) \geq r_j.$$

Substep 4. Reducing $\leftarrow AB(c)(c \in COMPS\text{-}\Delta)$.

For each inequality in Γ (or its variants) with form

$$f(L_{j,1}) \vee \ldots \vee f(L_{j,p}) \vee f(\leftarrow AB(c_{j,1})) \vee \ldots \vee f(\leftarrow AB(c)) \vee \ldots \vee f(\leftarrow AB(c_{j,q})) \geq r_j, (****)$$

set $Ran(f(\leftarrow AB(c))) = (1 - \alpha, 1)$ since c is not in the assumed α-level diagnosis Δ.

Case 1. If $1 - \alpha < r_j$, then Γ replaces its variants Γ_1 and Γ_2. Γ_1 is obtained by deleting (****) from Γ and setting $Ran(f(\leftarrow AB(c))) = Ran(f(\leftarrow AB(c))) \cap [r_j, 1]$. Γ_2 is obtained by replacing (****) in Γ with the following inequalities:

$$f(L_{j,1}) \vee \ldots \vee f(L_{j,p}) \vee f(\leftarrow AB(c_{j,1})) \vee \ldots \vee f(\leftarrow AB(c_{j,q})) \geq r_j \quad (****1)$$

and setting

$$Ran(f(\leftarrow AB(c))) = Ran(f(\leftarrow AB(c))) \cap [1 - \alpha, r_j).$$

Case 2. If $1 - \alpha \geq r_j$, then delete (****) from Γ.

By the above reduction, no AB-literal occurs in each system of inequalities in Γ (its variants). Note that in the reduction, for Γ (or its variant), if there is some $c \in COMPS$ such that $\overline{Ran(f(AB(c)))} \cap Ran(f(\leftarrow AB(c))) = \oslash$, then delete Γ (or its variant).

By adding (4.5) to Γ and its each variant, Γ^* and corresponding variants can be obtained.

Step 3. For each inequality, e.g.,

$$f(L_1) \vee \ldots \vee f(L_n) \vee f(L) \geq r(n \geq 1),$$

in Γ^* (corresponding variants), if $f(L) \geq s$ $(r, s \in [0, 1])$ is not a member of Γ^*, then replace Γ^* with Γ_1^* and Γ_2^*: Γ_1^* is obtained by deleting the inequalit

$$f(L_1) \vee \ldots \vee f(L_n) \vee f(L) \geq r$$

from Γ^* and setting $Ran(L) = Ran(L) \cap [r, 1]$; and Γ_2^* is obtained by replacing $f(L_1) \vee \ldots \vee f(L_n) \vee f(L) \geq s$ with $f(L_1) \vee \ldots \vee f(L_n) \geq s$ and setting $Ran(L) = Ran(L) \cap (0, r]$.

Repeat the above procedure until no inequality with form $f(L_1) \vee \ldots \vee f(L_n) \geq r(n \geq 2)$ occurs in Γ^* and its each variant. Then, for each inequality with form $f(L) \geq p$, set $Ran(f(L)) = Ran(f(L)) \cap [p, 1]$. As in Step 2, Γ^*, its variant, is deleted, in which there is a pair of complementary literals L and $\leftarrow$ L such that $\overline{Ran(f(L))} \cup Ran(f(\leftarrow L)) = \oslash$.

Step 4. If there is some $\Gamma^* \neq \oslash$ (or its variant is not empty), then the system (4.4)-(4.6) has a solution; otherwise, it is unsolvable.

Example A. Solve the system of inequalities

$$\begin{aligned} &f(AB(c_1)) \vee f(P) \vee f(Q) \geq 0.6, \\ &f(AB(c_2)) \vee f(\leftarrow P) \geq 0.8, \end{aligned} \quad (4.4*)$$

$$f(P) \vee f(\leftarrow Q) \geq 0.55, \quad (4.5*)$$

$$f(AB(c_1)) < 0.7,\ f(AB(c_2)) < 0.7. \quad (4.6*)$$

First, set

$$\Gamma = \{f(AB(c_1)) \vee f(P) \vee f(Q) \geq 0.6, f(AB(c_2)) \vee f(\leftarrow P) \geq 0.8\} \text{ and}$$

$$Ran(f(AB(c_1)) = Ran(f(AB(c_2))) = (0, 0.7).$$

Reducing occurrences of $f(AB(c_1)$ and $f(AB(c_2))$ in Γ, the following is obtained:

$$\begin{aligned} \Gamma_1 = \{&f(\leftarrow P) \geq 0.8, Ran(f(AB(c_1))) = [0.6, 0.7), \\ &Ran(f(AB(c_2))) = (0, 0.7)\} \end{aligned}$$

$$\begin{aligned} \Gamma_2 = \{&f(P) \vee f(Q) \geq 0.6, f(\leftarrow P) \geq 0.8, \\ &Ran(f(AB(c_1))) = (0, 0.6), Ran(f(AB(c_2))) = (0, 0.7)\}. \end{aligned}$$

By adding (4.5*) to Γ, it can be obtained:

$$\begin{aligned} \Gamma_1^* = \{&f(\leftarrow P) \geq 0.8, Ran(f(AB(c_1))) = [0.6, 0.7), \\ &Ran(f(AB(c_2))) = [0, 0.7), f(P) \vee f(\leftarrow Q) \geq 0.55\}, \end{aligned}$$

$$\begin{aligned} \Gamma_2^* = \{&f(P) \vee f(Q) \geq 0.6, f(\leftarrow P) \geq 0.8, \\ &Ran(f(AB(c_1))) = [0.6, 0.7), Ran(f(AB(c_2))) = (0, 0.7), \\ &f(P) \vee f(\leftarrow Q) \geq 0.55\}. \end{aligned}$$

Considering the inequality $f(P) \vee f(\leftarrow Q) \geq 0.55$, since $f(\leftarrow Q) \geq s$ is not in Γ_1^* for any $s \in (0, 1]$, we have

$$\begin{aligned} \Gamma_1^{**} = \{&f(\leftarrow P) \geq 0.8, Ran(f(AB(c_1))) = [0.6, 0.7), \\ &Ran(f(AB(c_2))) = (0, 0.7), Ran(f(\leftarrow Q)) = (0.55, 1]\} \\ \Gamma_2^{**} = \{&f(\leftarrow P) \geq 0.8, f(P) \geq 0.55, \\ &Ran(f(AB(c_1))) = [0.6, 0.7), Ran(f(AB(c_2))) = (0, 0.7), \\ &Ran(f(\leftarrow Q)) = (0, 0.55)\}. \end{aligned}$$

Note that a contradiction $(\overline{Ran(f(P))} \cap Ran(f(\leftarrow P)) = \oslash)$ appears in Γ_2^{**}, so $\Gamma_2^{**} = \oslash$.

Finally, the solution

$$\begin{aligned} \{&0.6 \leq f(AB(c_1)) < 0.7, \\ &f(AB(c_2)) < 0.7, f(P) \leq 0.2, f(Q) \leq 0.45\} \end{aligned}$$

can be obtained.

Example B. Solve the system of inequalities obtained by replacing (4.6*) in the above system with the following:

$$f(AB(c_1)) < 0.6, f(AB(c_2)) < 0.6. \quad (4.6**)$$

Similarly, by using (4.6**) to reduce occurrences of $f(AB(c_1))$ and $f(AB(c_2))$, it can be obtained:

$$\Gamma^* = \{f(P) \vee f(Q) \geq 0.6, f(\leftarrow P) \geq 0.8, f(P) \vee f(\leftarrow Q) \geq 0.55, \\ Ran(f(AB(c_1))) = Ran(f(AB(c_1))) = (0, 0.6)\}.$$

Finally, by using $f(\leftarrow P) \geq 0.8$ to reduce all occurrences of $f(P)$ in Γ^*,

$$\Gamma^* = \{f(\leftarrow P) \geq 0.8, f(Q) \geq 0.6, (\leftarrow Q) \geq 0.55, \\ Ran(f(AB(c_1))) = Ran(f(AB(c_1))) = (0, 0.6)\}$$

can be obtained, which contains the contradiction $\{f(Q) \geq 0.6, (\leftarrow Q) \geq 0.55\}$. So, the system of inequalities has no solution.

Example C. Solve the system of inequalities obtained by replacing (4.6*) in the above system with the following:

$$f(AB(c_1)) \geq 0.6, f(AB(c_2)) < 0.6. \quad (4.6***)$$

Set

$$\Gamma = \{f(AB(c_1)) \vee f(P) \vee f(Q) \geq 0.6, \\ f(AB(c_2)) \vee f(\leftarrow P) \geq 0.8, \\ Ran(f(AB(c_1)) = [0.6, 1]).$$

By using $f(AB(c_1)) \geq 0.6$ to reduce all occurrences of $f(AB(c_1))$ in Γ,

$$\Gamma = \{f(AB(c_2)) \vee f(\leftarrow P) \geq 0.8, Ran(f(AB(c_1)) = [0.6, 1])$$

can be obtained. By using $f(AB(c_2)) < 0.6$ to reduce all occurrences of $f(AB(c_2))$ in Γ,

$$\Gamma = \{Ran(f(AB(c_1))) = [0.6, 1], \\ Ran(f(AB(c_2))) = (0, 0.6), f(\leftarrow P) \geq 0.8\}$$

can be obtained. Adding (4.5*) to Γ and considering the inequality $f(P) \vee f(\leftarrow Q) \geq 0.55$, since $f(P) \geq s$ is in Γ for any $s \in (0, 1]$, the following can be obtained:

$$\Gamma_1^* = \{Ran(f(AB(c_1))) = [0.6, 1], Ran(f(AB(c_2))) = (0, 0.6), \\ Ran(f(P)) = [0, 0.55), f(\leftarrow P) \geq 0.8, f(\leftarrow Q) \geq 0.55\}$$

and

$$\Gamma_1^{**} = \{Ran(f(AB(c_1))) = [0.6, 1], Ran(f(AB(c_2))) = (0, 0.6), \\ Ran(f(P)) = [0.55, 1], f(\leftarrow P) \geq 0.8\}.$$

Note that a contradiction $(\overline{Ran(f(P))} \cap Ran(f(\leftarrow P)) = \oslash)$ appears in Γ_1^{**}, so $\Gamma_1^{**} = \oslash$. Finally, the solution

$$\{0.6 \leq f(AB(c_1)), f(AB(c_2)) \leq 0.6, f(P) \leq 0.2, f(Q) \leq 0.45\}$$

can be obtained.

The above method is NP-hard since it could encounter a composition explosion caused by reduction of non-AB-literals. Some skills for solving a system of inequalities are suggested as follows: An extended version of strategies for searching all subsets of COMPS and for dealing with a composition explosion in solving a system of inequalities will be discussed in a subsequent paper:

1. Reducing with a pair of complementary literals: Users simultaneously proceed to reduce a pair of complementary literals such that an unsolvable system of inequalities could be pruned as early as possible.
2. Depth-first: Users always give priority to a system including an inequality with form $f(AB(c)) < \alpha$ when reducing an AB-literal since we are only interested in a minimal set $\{c | c \in COMPS, f(AB(c)) \geq \alpha$ and f is a partial interpretation such that (4.4) is satisfied$\}$.

5 An Example

In this section, a substantial application of our approach has been illustrated, followed by a simplified example for the computing method.

5.1 Student-Fault Diagnosis Problem

In this section, an eLearning fault diagnosis problem, based on a student online education system, Intelligent eLearning System (IeLS) [29], is explained briefly:

1. Language: A set of constants COMP={Under (short for understanding capability), Creat (creativity), Intel (intelligence), Lmem (long-term memory), Smem (short-term memory), Selfl (self-learning ability), Prereq i (knowledge level of prerequisites i, $1 \leq i \leq n$), Seef $(\cdot)$ (self-efficacy), Caref (carefulness)}, and other sets of constants, e.g., $STUDENT = \{st1, \ldots, st\ m)$, a distinguished unary predicate $AB(\cdot)$ which domain is COMP and other predicates, e.g., $Sex(\cdot)$, $Age(\cdot)$, and $Effort(\cdot)$ which domain is STUDENT and $Per(\cdot)$ (performance), $Satis(\cdot)$ (satisfiability), $Time_j(\cdot)$ (spent time for j: $1 \leq j \leq k$) which domain is COURSE, etc. Clearly, most of the predicates, e.g., $AB(c)(c \in COPMS)$, $Effortt(\cdot)$, $Satis(\cdot)$, and $Seef(\cdot)$, etc., are linguistic terms with fuzziness. Hence, fuzzifying them is a more adequate approach.
2. Fuzzy system description TSD: This describes how the system components normally behave by appearing to the distinguished predicate AB. Each sentence in T_{SD} is a given fact on the structure of the system or an IF-THEN rule based on the pedagogical models, experts' experiences, data analysis (data mining), and axioms for lattice theory over [0,1], etc. In the problem system description, 23 rules are included, which are transformed into 51 clauses. For example, a rule "normally an older student, who has worse long-term memory and better knowledge on prerequisites, or a student who studies with average effort, could slowly or imprecisely take a quiz" can be represented as the following formula:

$$(x)((Older(x) \wedge (AB(Lmem)) \wedge AB(Prereq1) \\ \wedge \ldots \wedge AB(Prereq3)) \vee aEffort(x)) \rightarrow Slow \wedge Perf$$

with a partial mapping T satisfying the conditions:

$$\mathrm{T(Older(x))} \geq 0.6, 0.5 \leq \mathrm{T(AB(Lmem))} < 0.7,$$
$$\mathrm{T(\leftarrow AB(Prereq1))} \geq 0.6, \mathrm{T(\leftarrow AB(Prereq2))} \geq 0.75,$$
$$\mathrm{T(\leftarrow AB(Plereq3))} \geq 0.8, 0.5 \leq \mathrm{T(aEffort(x))} < 0.7,$$
$$\mathrm{T(Slow)} \geq 0.6, 0.3 \leq \mathrm{T(Perf)} \leq 0.6.$$

Here, the predicates Older, Effort, and Slow are determined by Age, Time_1 (spent by a quiz), and Time_2 (spent by learning), respectively.

3. Fuzzy Observation $\mathrm{T_{OBS}}$: This is obtained based on learning profiles of students, performance, spent time, or questionnaire data analysis, etc. For example, given a student and a course, his performance, learning time, spent time, etc., are observable. Intuitively, for a given student and a course, the components must be determined, which, when assumed to be functioning abnormally, will explain the discrepancy between the observed and correct system behavior. In the eLearning student-fault diagnosis problem, there are $8 + n$ (usually, $n \geq 3$) components (depending on the number of prerequisite courses for different real-world situations) and more than seven predicates. So, it is necessary to solve a system of inequalities, constituted from 51 inequalities, for each of 2^{8+n} subsets of COMPS. From the general method in Section 4, it is easy to see that reducing non-AB-literals also can lead to combination explosion. Hence, in general, computing diagnoses could lead to exponential complexity; an extension version of strategies searching through subsets of COMPS and non-AB-literals will be presented in a subsequent paper.

5.2 A Simplified Example

As a simplified example of the student diagnosis problem, assume there is an eLearning system (SD, COMPS), where system description SD contains two rules:

1. "Normally, an intelligent student can precisely or speedily answer questions."
2. "Normally, a negligent student cannot precisely answer questions."

The system components are intelligence and negligence, denoted as c_1 and c_2, respectively. Now, assume it is observed that someone could precisely or slowly answer questions. Preciseness and speediness of answering questions are represented by the propositions P and Q, respectively. In the classical logic, the system (SD, COMPS) and the observation OBS are obtained, where $\mathrm{SD} = \{\leftarrow \mathrm{AB}(c_1) \rightarrow \mathrm{P} \vee \mathrm{Q}, \ \leftarrow \mathrm{AB}(c_2) \rightarrow \leftarrow \mathrm{P}\}$ (its clause-style description is $\mathrm{SD} = \{\mathrm{AB}(c_1) \vee \mathrm{P} \vee \mathrm{Q}, \mathrm{AB}(c_2)) \vee \leftarrow \mathrm{P}\}$), $\mathrm{COMPS} = \{c_1, c_2\}$ and $\mathrm{OBS} = \{\mathrm{P} \vee \leftarrow \mathrm{Q}\}$.

By fuzzifying the linguistic terms with fuzziness—intelligence, negligence, preciseness, and speediness, a fuzzy system $(\mathrm{T_{SD}}, \mathrm{COMPS})$ and an observation OBS are obtained, where SD, COMPS, and OBS are as previous, and the mappings $\mathrm{T_{SD}}$ (the fuzzy system description) and $\mathrm{T_{OBS}}$ (the fuzzy observation) are as follows:

$$\mathrm{T_{SD}(P)} = 0.2, \mathrm{T_{SD}(Q)} = 0.6, \mathrm{T_{SD}(AB(c_1))} = 0.3,$$
$$\mathrm{T_{SD}(AB(c_2))} = 0.25; \mathrm{T_{SD}(AB(c_1) \vee P \vee Q)} = 0.6,$$
$$\mathrm{T_{SD}(AB(c_2)) \vee \leftarrow P)} = 0.8; \mathrm{T_{OBS}(P)} = 0.55,$$
$$\mathrm{T_{OBS}(Q)} = 0.8; \mathrm{T_{OBS}(P \vee \leftarrow Q)} = 0.55.$$

In general, a literal is used to characterize a property enjoyed by an object. A partial mapping f evaluates a number f(L) in $(0, 1]$ to a (AB or non-AB) literal L. The value of f(L) represents the truth level of L. For example, the meanings of f(L) are as follows:

- 1—completely to be normal,
- 0.8—very likely to be normal,
- 0.7—likely to be normal,
- 0.6—little bit to be normal,
- 0.5—neutral,
- 0.4—little bit to be abnormal.
- 0.3—likely to be abnormal, and
- 0.2—very likely to be abnormal.

In real world applications, such values can be determined using statistical sampling methods or data mining techniques.

5.2.1 Application of a General Method

Given $\alpha = 0.7$, α-level fuzzy diagnoses is obtained. That is, it is necessary to find a partial interpretation f such that

$$\begin{aligned} &\mathrm{f(AB(c_1)) \vee f(P) \vee f(Q)} \geq 0.6, \\ &\mathrm{f(AB(c_2)) \vee f(\leftarrow P)} \geq 0.8, \mathrm{f(P) \vee f(\leftarrow Q)} \geq 0.55 \end{aligned} \tag{I}$$

and such that the set $\{c | c \in \mathrm{COMPS}, \mathrm{f(AB(c))} \geq 0.7\}$ is minimal.

At first, assume that $\Delta = \oslash$, we get the following system of inequalities:

$$\mathrm{(AB(c_1)) \vee f(P)} \vee f(Q) \geq 0.6, \mathrm{f(AB(c_2)) \vee f(\leftarrow P)} \geq 0.8,$$
$$\mathrm{f(P) \vee f(\leftarrow Q)} \geq 0.55, \mathrm{f(AB(c_1))} < 0.7, \mathrm{f(AB(c_2))} < 0.7.$$

From the Example A, it can be known that $\oslash$ is a unique 0.7-level diagnosis. This result shows that for the observed student, both of his/her foolish level (the negation of intelligent) and his/her careful level (the negation of negligent) are lower than 0.7.

For an α-level diagnosis {c2} and {c1}, the selection of the value α is important. The 0.7-level diagnosis means that users want the diagnosis to be "likely true." If users choose $\alpha < 0.6$, it means that users want the diagnosis to be "a litter bit likely true" or even more uncertainly.

In conclusion, such diagnoses are very useful for improving teaching quality. For instance, based on the example above, teachers can help the student to avoid the negligence, based on the diagnosis $\{c_1\}$. Note that if c_2 is in an α-level diagnosis, it means that the level of carefulness is at least α, i.e., the level of negligence is at most $1 - \alpha$.

5.2.2 Skills for Solving System (I)

As stated in Section 4.3.2, by reducing pairs of complementary literals $\{\mathrm{P}, \leftarrow \mathrm{P}\}$ and $\{\mathrm{Q}, \leftarrow \mathrm{Q}\}$ in (I), two systems of inequalities can be obtained:

$$\mathrm{f(\leftarrow P)} \geq 0.8, \mathrm{f(AB(c_1) \vee Q)} \geq 0.6, \mathrm{f(\leftarrow Q)} \geq 0.55, \tag{I, 1}$$

$$f(\leftarrow P) < 0.8, f(AB(c_1) \vee P \vee Q) \geq 0.6, \\ f(AB(c_2)) \geq 0.8, f(P \vee \leftarrow Q) \geq 0.55. \tag{I, 2}$$

Further, five systems of inequalities will be obtained:

$$f(\leftarrow P) \geq 0.8, f(\leftarrow Q) \geq 0.55, f(AB(c_1)) \geq 0.6; \tag{I, 1, 1}$$

$$f(P) \geq 0.55, f(Q) \geq 0.6, f(AB(c_2)) \geq 0.8; \tag{I, 2, 1}$$

$$f(P) \geq 0.6, f(AB(c_2)) \geq 0.8; \tag{I, 2, 2}$$

$$f(P) \geq 0.55, f(AB(c_1)) \geq 0.6, f(AB(c_2) \geq 0.8; \tag{I, 2, 3}$$

$$0.2 < f(P) < 0.45, f(Q) \leq 0.55, f(AB(c_1)) \geq 0.6, f(AB(c_2) \\ \geq 0.8. \tag{I, 2, 4}$$

So, system $(T_{SD}, COMPS, T_{OBS})$ has α-level diagnoses $\{c_2\}$ and $\{c_1\}$ when $\alpha \leq 0.6$, while it has the α-level diagnosis $\oslash$ when $\alpha > 0.6$. As a faulty component, c_2 (i.e., negligence) is more certain than c_1 (intelligence).

From the above, one can see that computing diagnoses without resorting to solving the inequalities for each subset of COMPS is possible in certain situations, which will be discussed in a subsequent paper. For example, it is suitable when at most four AB-literals with distinct components occur in each clause of T_{SD} since reducing three AB-literals will generate 2^3 systems of inequalities with occurrence of at most one AB-literal. And, solving a system without constraint on α-level can give more information than that in Section 5.2.1, e.g., $(T_{SD}, COMPS, TOBS)$ has α-level$(\alpha \leq 0.6)$ diagnosis $\{c_1\}$ when $T_{OBS}(P) \leq 0.2$ and $T_{OBS}(Q) < 0.6$; it has α-level $(\alpha \leq 0.8)$ diagnosis $\{c_2\}$ for $T_{OBS}(P)$ close to 1; it has α-level $(\alpha \leq 0.6)$ diagnosis $\{c_1\}$ for both $T_{OBS}(P)$ and $T_{OBS}(Q)$ close to 0, and has the diagnosis $\oslash$ for $T_{OBS}(P)$ close to 0 and $T_{OBS}(Q)$ close to 1. Based on such information, one can immediately derive a diagnosis from observation(s) and verify validity of the system description.

6 Conclusions and Further Work

Considering most real-world situations where description and observations are with fuzziness, an extension of Reiter's theory of diagnosis from first principle is important and necessary. In this paper, such a framework has been proposed and some of its important properties have been established. To accomplish this, a truth-functional fuzzy logic (based on [25]) without any logical axiom has been studied, which is able to satisfy very general situations with almost arbitrary connectives [25]. Based on this approach, the notions of consistency and entailability for a fuzzy theory are given. Then, Reiter's framework of diagnosis to fuzzy diagnosis are extended. The notion of α-level fuzzy diagnosis is introduced, and its properties similar to that in [1] are obtained. Further, computing diagnoses is mapped to solving a system of inequalities. In general, to solve such a system of inequalities is very complex or even impossible. Some special important cases are discussed. In particular, we focus on the case where T_{SD} and T_{OBS} are clause-style fuzzy theories, that is, each sentence in $SD \cup OBS$ is a conjunction of clauses. A generic method for computing diagnoses is derived when T_{SD} and T_{OBS} are clause-style fuzzy theories. Some problem-solving strategies are also given. A real-world case about the student fault diagnosis in eLearning processes is described to demonstrate the usefulness of our framework.

Reiter and de Kleer et al. characterized diagnosis by conflict set, implicate, and prime implicate, respectively [1], [3]. A corresponding fuzzy version will be explored in the future. A new special case for effectively computing diagnosis and strategies of searching through subsets of COMPS will also be discussed in a subsequent paper. The authors are also considering applying the framework on real-world applications.

Appendix

Proposition 2.5. *Let* $(SD, COMPS)$ *be a system and* OBS *an observation. Given a* $\Delta \subseteq COMPS$ *if* $SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\}$ *is consistent and for any* $c_i \in \Delta$,

$$SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\} \models AB(c_i),$$

then Δ *is a diagnosis for* $(SD, COMPS, OBS)$. *Here, the notation "*$\models$*" is the classical entailability relation.*

Proof. The case where $\Delta = \oslash$ is obvious using Proposition 2.2. Assume that $\Delta \neq \oslash$. It is clear that

$$SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\} \cup \{AB(c) | c \in \Delta\}$$

is consistent since

$$SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\}$$

is consistent and $SD \cup OBS\{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\} \models AB(c_i)$ for any $c_i \in \Delta$. We show minimality of Δ. If there is a proper subset Δ' of Δ such that Δ' satisfies the conditions of the proposition, then

$$SD\ OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\} \\ \cup \{\leftarrow AB(c) | c \in \Delta - \Delta'\}$$

is consistent. This contradicts the hypothesis that for any $c_i \in \Delta$,

$$SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\} \models AB(c_i).$$

So, Δ is a minimal set of COMPS such that $SD \cup OBS \cup \{\leftarrow AB(c) | c \in COMPS\text{-}\Delta\}$ is consistent. Using Proposition 2.4, Δ is a diagnosis for (SD, COMPS, OBS). □

Theorem 3.1. *An* α-level *diagnosis exists for*

$$(T_{SD}, COMPS, T_{OBS})$$

iff $\{T_{SD}, T_{OBS}\}$ *is consistent.*

Proof. "$\Rightarrow$" The proof is obvious by Definitions 3.2 and 3.4.

"$\Leftarrow$" Assume that $\{T_{SD}, T_{OBS}\}$ is consistent. Then, there is an interpretation f such that $f(\phi) \geq T_{SD}(\phi)$ for any $\phi \in SD$, and that $f(\psi) \geq T_{OBS}(\psi)$ for any $\psi \in OBS$. Let

$$\Delta_f = \{c \in COMPS | f(AB(c)) \geq \alpha\}. \tag{3.1}$$

Clearly, f is a model of $\{T_{SD}, T_{OBS}, T_N(\Delta_f, \alpha), T_P(\Delta_f, \alpha)\}$, where

- $T_N(\Delta_f, \alpha)(\leftarrow AB(c)) = f(\leftarrow AB(c)$ for each

$$c \in \text{COMPS-}\Delta_f,$$

and

- $T_P(\Delta_f, \alpha)(AB(c)) = f(AB(c))$ for each $c \in \Delta_f$.

If $\Delta_f = \oslash$ (empty set), then it is clear that $\oslash$ is an α-level diagnosis. Assume that $\Delta_f \neq \oslash$. Set $S(\Delta) = \{\Delta_f | f$ is a model of $\{T_{SD}, T_{OBS}\}$ and Δ_f is defined as in (3.1)$\}$. Using Zorn's lemma, $S(\Delta)$ has a minimal element under inclusion of sets. We claim that each of such minimal elements is just an α-level diagnosis for

$$(T_{SD}, \text{COMPS}, T_{OBS})$$

and vice versa. In fact, if Δ^* is a minimal element, then its corresponding model f^* is also a model of $\{T_{SD}, T_{OBS}, T_N(\Delta_{f^*}, \alpha), T_P(\Delta_{f^*}, \alpha)\}$. If there is $\Delta' \cup \Delta^*$ such that $\{T_{SD}, T_{OBS}, T_N(\Delta', \alpha), T_P(\Delta', \alpha)\}$ is consistent for some $T_N(\Delta', \alpha)$ and $T_P(\Delta', \alpha)$, then any model f' of $\{T_{SD}, T_{OBS}, T_N(\Delta', \alpha), T_P(\Delta', \alpha)\}$ is also a model of $\{T_{SD}, T_{OBS}\}$. It is easy to verify that $\Delta'_{f'} = \Delta'$, where $\Delta'_{f'}$ is defined as in (3.1). In fact, we have that $\Delta' \subseteq \Delta_{f'}$ since $f(AB(c)) > \alpha$ for each $c \in \Delta'$. Conversely, if there is some $c \in \Delta'_{f'} - \Delta'$, then we have that 1) $f'(AB(c)) \geq \alpha$ by the definition of $\Delta'_{f'}$ and 2) $f'(\leftarrow AB(c)) > 1 - \alpha$, i.e., $f'(AB(c)) < \alpha$, since f' is a model of

$$\{T_{SD}, T_{OBS}, T_N(\Delta', \alpha), T_P(\Delta', \alpha)\}.$$

This contradiction shows that $\Delta'_{f'} \subseteq \Delta'$. Hence, $\Delta'_{f'} = \Delta'$. Therefore, $\Delta' \in S(\Delta)$. But, this contradicts the minimality of Δ^*. So, Δ^* is an α-level diagnosis. Similarly, we have that if Δ^* is an α-level diagnosis, then Δ^* is a minimal element of $S(\Delta)$. □

Theorem 3.2. *A subset of* COMPS Δ *is an* α-level *diagnosis for* $(T_{SD}, \text{COMPS}, T_{OBS})$ *iff* Δ *is a minimal subset of* COMPS *such that* $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\}$ *is consistent for some* $T_N(\Delta, \alpha)$.

Proof. "$\Rightarrow$" If Δ is an α-level diagnosis for

$$(T_{SD}, \text{COMPS}, T_{OBS}),$$

then there is some $T_N(\Delta, \alpha)$ and $T_P(\Delta, \alpha)$ such that $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha), T_P(\Delta, \alpha)\}$ is consistent. So is $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\}$. Now, we show that Δ is minimal. If there is $\Delta' \subseteq \Delta$ such that $\{T_{SD}, T_{OBS}, T_N(\Delta', \alpha)\}$ is consistent for some $T_N(\Delta', \alpha)\}$, then

$$\{T_{SD}, T_{OBS}, T_N(\Delta', \alpha)\}$$

has a model f. So, $f(\leftarrow AB(c)) > 1 - \alpha$ for each $c \in \text{COMPS} - \Delta'$. In particular, $f(\leftarrow AB(c)) > 1 - \alpha$ for each $c \in \Delta - \Delta'$. On the other hand, Let

$$\Delta'_f = \{c \in \text{COMPS} | f(AB(c)) \geq \alpha\}.$$

It is easy to check that $\Delta'_f \subseteq \Delta'$ and f is also a model of $\{T_{SD}, T_{OBS}, T_N(\Delta'_f, \alpha), T_P(\Delta'_f, \alpha)\}$. So, $\Delta'_f \subseteq \Delta$ is a subset of COMPS such that $\{T_{SD}, T_{OBS}, T_N(\Delta'_f, \alpha), T_P(\Delta'_f, \alpha)\}$ is consistent—a contradiction. Thus, minimality of Δ is proven.

"$\Leftarrow$" If Δ is a minimal subset of COMPS such that $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\}$ is consistent for some $T_N(\Delta, \alpha)$, then, in a way similar to the above, we have that a model f of $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\}$ is also one of $\{T_{SD}, T_{OBS}, T_N(\Delta'_f, \alpha), T_P(\Delta'_f, \alpha)\}$, i.e.,

$$\{T_{SD}, T_{OBS}, T_N(\Delta'_f, \alpha), T_P(\Delta'_f, \alpha)\}$$

is consistent. We claim that Δ is a minimal set with this property. In fact, if there is a set $\Delta^* \subseteq \Delta$ and some $T_N(\Delta^*_f, \alpha)$ and $T_P(\Delta^*_f, \alpha)$ such that

$$\{T_{SD}, T_{OBS}, T_N(\Delta^*, \alpha), T_P(\Delta^*, \alpha)\}$$

is consistent, then so is $\{T_{SD}, T_{OBS}, T_N(\Delta^*, \alpha)\}$. This contradicts the minimality of Δ. So, Δ is an α-level diagnosis. □

Theorem 3.4. *Let* $\Delta \subseteq$ COMPS. Δ *is a unique* α-level *diagnosis for* $(T_{SD}, \text{COMPS}, T_{OBS})$ *iff* $\{T_{SD}, T_{OBS}\}$ *is consistent and for any model* f *of* $\{T_{SD}, T_{OBS}\}, c \in \Delta \Rightarrow f(AB(c)) > \alpha$.

Proof. "Only If" Consistency of $\{T_{SD}, T_{OBS}\}$ is clear. From the proof of Theorem 3.1, it is immediately apparent that $\Delta \subseteq \Delta_f$ for any model f of $\{T_{SD}, T_{OBS}\}$ since Δ is a unique α-level diagnosis. So, $c \in \Delta \Rightarrow f(AB(c)) \geq \alpha$.

"IF" Assume that Δ_f and $S(\Delta)$ are defined as in the proof of Theorem 3.1.

Since $\{T_{SD}, T_{OBS}\}$ is consistent and with the hypothesis that $c \in \Delta \Rightarrow f(AB(c)) \geq \alpha$, then $\Delta \subseteq \Delta_f$ for any model f of $\{T_{SD}, T_{OBS}\}$. So, Δ is a unique minimal element of $S(\Delta)$ and is also a unique α-level diagnosis for $(T_{SD}, \text{COMPS}, T_{OBS})$. □

Theorem 3.5. $\Delta \subseteq$ COMPS *is an* α-level *diagnosis for* $(T_{SD}, \text{COMPS}, T_{OBS})$ *iff for some*

$$T'_N(\Delta, \alpha), \{T_{SD}, T_{OBS}, T'_N(\Delta, \alpha)\}$$

is consistent and for any

$$T_N(\Delta, \alpha), \{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\} \models_F T_P(\Delta, =, \alpha),$$

where $T_P(\Delta, =, \alpha)$ *is a partial mapping from* $\{AB(c) | c \in \Delta\}$ *to* $(0, 1]$ *such that* $T_P(\Delta, =, \alpha)(AB(c)) = \alpha$ *for each* $c \in \Delta$.

Proof. "$\Rightarrow$" The case where $\Delta = \oslash$ is true vacuously. Suppose that $\Delta \neq \oslash$. There is some $T'_N(\Delta, \alpha)$ such that $\{T_{SD}, T_{OBS}, T'_N(\Delta, \alpha)\}$ is consistent since Δ is an α-level diagnosis. For any partial mapping $T_N(\Delta, \alpha)$, if f is a model of $\{T_{SD}, T_{OBS}, T_N(\Delta, \alpha)\}$, then $f(\leftarrow AB(c)) > 1 - \alpha$ for each $c \in \text{COMPS} - \Delta$. By applying the minimality of Δ, $f(\leftarrow (AB(c)) \leq 1 - \alpha$ for each $c \in \Delta$. So, $f(AB(c)) \geq \alpha$ for each $c \in \Delta$, which implies that f is a model of $T_P(\Delta, =, \alpha)$.

"$\Leftarrow$" It is sufficient to show minimality of Δ by Definition 3.4. If, on the contrary, Δ is not an α-level diagnosis, then there is $\Delta^* \subseteq \Delta$ and Δ^* is an α-level diagnosis. So, there is some $T^*_N(\Delta^*, \alpha)$ such that $\{T_{SD}, T_{OBS}, T^*_N(\Delta^*, \alpha)\}$ is consistent. Assume that f is a model of

$$\{T_{SD}, T_{OBS}, T^*_N(\Delta^*, \alpha)\}.$$

Then, $f(\leftarrow AB(c)) > 1 - \alpha$, i.e., $f(AB(c)) < \alpha$, for each $c \in \Delta - \Delta^*$. On the other hand, f is also a model of $\{T_{SD}, T_{OBS}, T_N^*(\Delta, \alpha)\}$, where $T_N^*(\Delta, \alpha)$ is the restriction to $\{\leftarrow AB(c) | c \in \Delta\}$. Based on hypothesis $f(AB(c)) \geq \alpha$ for each $c \in \Delta - \Delta^*$, a contradiction occurs. □

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