

Multipartite entanglement and quantum state exchange

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We investigate multipartite entanglement in relation to the process of quantum state exchange. In particular, we consider such entanglement for a certain pure state involving two groups of N trapped atoms. The state, which can be produced via quantum state exchange, is analogous to the steady-state intracavity state of the subthreshold optical nondegenerate parametric amplifier. We show that, first, it possesses *some* $2N$ -way entanglement. Second, we place a lower bound on the amount of such entanglement in the state using a measure called the *entanglement of minimum bipartite entropy*.

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I. INTRODUCTION

Multipartite entanglement is entanglement that crucially involves three or more particles. A well-known example of it occurs in the generalized Greenberger-Horne-Zeilinger (GHZ) state $|\psi\rangle = |0\rangle^{\otimes M} + |1\rangle^{\otimes M}$, where M is an integer greater than two. Multipartite entanglement is an interesting quantum resource for a number of reasons. First, it is a key resource in quantum computation as (i) it has been proved that it is a necessary ingredient in order for a quantum computation to obtain an exponential speedup over classical computation [1] and (ii) it is central to quantum error correction that uses it to encode states, to detect errors, and, ultimately, to help implement fault-tolerant quantum computation (see, for example, Chap. 10 of Ref. [2]). The second reason why multipartite entanglement is interesting is that it can manifest nonclassical correlations such as GHZ-type correlations [4]. Finally, it has been conjectured that multipartite entangled states contain a wealth of interesting and unexplored physics [5].

In order to quantify the amount of multipartite entanglement present in a state, a number of measures have been proposed. First, Vedral *et al.* [6] have suggested a measure of multipartite entanglement for a state ρ that is the minimum relative entropy between ρ and any separable state. For systems with more than two subsystems, they defined a separable state as one in which the state of at least one subsystem can be factored out from that of the others. In addition, Coffman, Kundu, and Wootters [7] have extended the bipartite entanglement measure called the tangle [8] to the *three-tangle* that measures three-way GHZ-type entanglement. Furthermore, Vidal [9] has studied *entanglement monotonies*—quantities whose magnitudes do not increase, on average, under local transformations—and has proposed that *all* of these can be regarded as entanglement measures. Meyer and Wallach [10] have proposed a measure of “global entanglement” for n -qubit pure states which is the sum of a number of terms involving wedge products. Each wedge

product involves computational-basis expansion coefficients for various $(n-1)$ -qubit states obtained by deleting a qubit from the state of interest. Similarly, Wong and Anderson [11] have extended the tangle to an arbitrary even number of qubits for pure states. Finally, Biham, Nielsen, and Osborne [12] have proposed the *Groverian entanglement* for a pure state $|\psi\rangle$ based on how successful Grover’s algorithm [13] performs, given the input $|\psi\rangle$. The Groverian entanglement is equivalent to a measure in Ref. [6]; however, it shows an interesting link between an entanglement measure and the quantum information processing capability of states. In addition to the measures listed above, a number of others have also been proposed. *Quantum state exchange* [14–18] is a newly formulated process by which information is transferred from an electromagnetic field to the vibrational state of one or more trapped atom(s). It is implemented using a stimulated Raman process, which couples the electromagnetic field to the vibrational state and thus transfers information from the former to the latter. We explain it in more detail in Sec. II.

In this paper, we show that quantum state exchange can be used to create an entangled state for $2N$ trapped atoms that is a useful quantum resource. We begin in, Sec. II A, by explaining the process of quantum state exchange via presenting a detailed example of it within a simple system consisting of a harmonically trapped atom interacting with a cavity mode. Next, Sec. II B shows that quantum state exchange can be used to generate an entangled pure state for two groups of N trapped atoms located in two spatially separated far-off-resonance dipole-force traps (FORTs). In Sec. II C, we present a detailed summary of the remaining paper. Secs. III and IV then investigate the nature of the state’s $2N$ -way entanglement; that is, the nature of its entanglement that spans across all $2N$ atoms. In particular, in Secs. III A and IV, we *qualitatively* explore this entanglement by presenting a necessary and sufficient condition for the presence of M -way entanglement for M -partite pure states, and then showing that the state satisfies it. In Sec. IV, we *quantitatively* explore the state’s $2N$ -way entanglement by first presenting a different multipartite entanglement measure for pure states in Sec. IV A. This measure is based on the von Neumann entropies [2,3] for all the reduced density opera-

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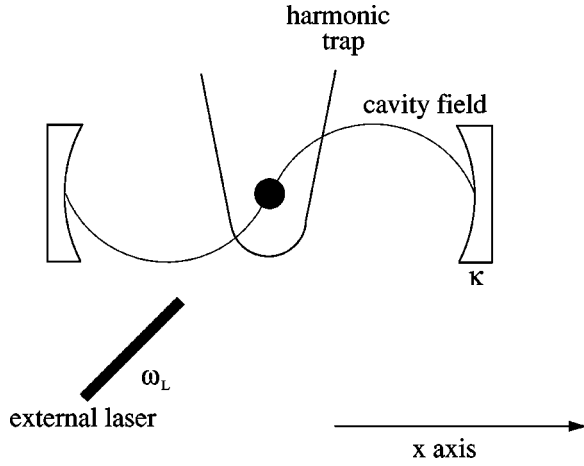


FIG. 1. A schematic diagram of a simple system in which quantum state exchange can occur. A two-level atom (represented by a black circle) lies within a harmonic trap that is itself inside an optical cavity. The cavity, which is aligned along the x axis, supports a mode of frequency ω_c and has one lossy mirror (with damping constant κ) and one ideal one. An external laser beam of frequency ω_L is incident from a direction perpendicular to the x axis.

tors obtainable from some pure state of interest by tracing over some of the subsystems for the state. After defining the measure, we then use it to calculate lower bounds on the amount of $2N$ -way entanglement in the state in Sec. IV B. Finally, we discuss our results in Sec. V.

II. BACKGROUND THEORY

A. Quantum state exchange

Perhaps, the simplest system in which quantum state exchange can occur [14] involves a two-level atom confined within a harmonic trap which, in turn, lies inside a linearly damped optical cavity with one lossy mirror and one ideal one. The atom's vibrational motion is described by the annihilation operators b_x, b_y , and b_z . The two-level atom, which has a transition frequency of ω_a , couples to both, an intracavity electromagnetic field mode of frequency ω_c described by the annihilation operator a and an external laser of frequency ω_L . The cavity and external laser frequencies are chosen so as to drive the Raman transitions that couple adjacent atomic vibrational levels. Furthermore, the cavity's axis coincides with the x axis, while the external laser beam is perpendicular to this axis. Finally, we assume that the harmonic trap is centered on a cavity-field node and thus a schematic diagram of this system is as in Fig. 1.

The system's Hamiltonian is, in a reference frame rotating at frequency ω_L ,

$$H_{\text{total}}^{\text{single}} = H_{\text{sys}} + \kappa(aR^\dagger + a^\dagger R) + H_{\text{res}}, \quad (1)$$

where H_{res} is the free Hamiltonian of the reservoir coupled to the cavity mode, κ is a damping rate, R is a reservoir operator, and H_{sys} is the Hamiltonian for the cavity-atom system which is

$$H_{\text{sys}} = \sum_{j=x,y,z} \hbar \nu_j (b_j^\dagger b_j + 1/2) + \hbar \delta a^\dagger a + \hbar \Delta \sigma_+ \sigma_- + \hbar [\mathcal{E}_L(y,z,t) \sigma_+ + \mathcal{E}_L^*(y,z,t) \sigma_-] + \hbar g_0 \sin(kx) \times (a^\dagger \sigma_- + a \sigma_+), \quad (2)$$

where ν_x, ν_y , and ν_z are the harmonic-oscillator frequencies along the trap's x, y , and z axes, σ_+ and σ_- are atomic raising and lowering operators for the two-level atom, $\delta = \omega_c - \omega_L$, $\Delta = \omega_a - \omega_L$, \mathcal{E}_L is the complex amplitude of the external laser field, $k = 2\pi/\lambda$, where $\lambda = 2\pi c/\omega_c$, $x = \sqrt{\hbar/2m\nu_x}(b_x + b_x^\dagger)$, where m is the mass of the two-level atom, and g_0 ($g_0 \in \mathbb{R}$) is the coupling constant for the atom-field interaction. Observe that $\omega_c - \omega_L = \nu_x$.

The following reasonable assumptions are made about the system, so as to make calculations involving it more tractable

(1) The cavity field and external laser frequencies are appreciably detuned from ω_a , and the two-level atom is initially in the ground state. Thus, the excited internal state is sparsely populated and spontaneous emission effects are negligible and can be ignored.

(2) Vibrational decoherence occurs over a time scale much longer than that of the interaction producing quantum state exchange, as is the case in an ion-trap realization of the system [14]. Consequently, it has a minimal effect over our time scale of interest and is ignored.

(3) The trap dimensions are small compared to the cavity mode wavelength and thus $\sin(kx) \ll 1$. It follows from this that $\sin kx \approx \eta_x(b_x + b_x^\dagger)$, where $\eta_x = k\sqrt{\hbar/2m\nu_x}$. It also follows that we can arrange things so that the y and z dependences of the external laser field are negligible and thus, assuming \mathcal{E}_L is time independent, that $\mathcal{E}_L(y,z,t) \approx \mathcal{E}e^{-i\phi_L}$, where \mathcal{E} is a real time-independent amplitude.

(4) The damping parameter κ is such that $\nu_x \gg \kappa \gg g_0 \eta_x \mathcal{E}/\Delta$.

Given the assumptions above, and also assuming that $g_0^2/\Delta \ll \delta$ and $g_0 \langle a^\dagger a \rangle \ll \mathcal{E}_L$, $H_{\text{total}}^{\text{single}}$ can be rewritten as [18]

$$H_{\text{total}}^{\text{single}} = \sum_{j=x,y,z} \hbar \nu_j (b_j^\dagger b_j + 1/2) + \hbar \delta a^\dagger a - \frac{\hbar \mathcal{E}^2}{\Delta} - \frac{\hbar g_0 \mathcal{E}}{\Delta} \sin(kx) (a^\dagger e^{-i\phi_L} + a e^{i\phi_L}) + \kappa (aR^\dagger + a^\dagger R) + H_{\text{res}} \quad (3)$$

by adiabatically eliminating the evolution of the internal states. Furthermore, for the system under consideration, it has been shown [14] that in the steady-state regime, the vibrational state of the atom in the x direction is solely determined by the input field (i.e., the light field entering the cavity), such that

$$\tilde{b}_x(\omega) = \frac{\sqrt{2\Gamma}}{i\omega - \Gamma} \tilde{a}_{\text{in}}(\omega), \quad (4)$$

where $\tilde{b}(\omega) = (1/\sqrt{2\pi}) \int_{t=-\infty}^{\infty} dt b_x(t) e^{i(\nu_x - \omega)t}$, $\Gamma = g_0^2 \eta_x^2 \mathcal{E}^2 / (\Delta^2 \kappa)$, and $\tilde{a}_{\text{in}}(\omega) = (1/\sqrt{2\pi}) \int_{t=-\infty}^{\infty} dt e^{-i\omega t} \tilde{a}_{\text{in}}(t)$, where $\tilde{a}_{\text{in}}(t) = (1/\sqrt{2\pi}) \int_{\omega'=-\infty}^{\infty} d\omega' e^{i(\nu_x - \omega')t} c_0(\omega')$, where $c_0(\omega')$ is the value of the reservoir annihilation operator for the frequency ω' at time $t=0$. The proportionality between $\tilde{b}_x(\omega)$ and $\tilde{a}_{\text{in}}(\omega)$ present in Eq. (4) denotes that the “statistics of the input field [have been] . . . “written onto” the state of the oscillator” [14] (i.e., onto the atom’s vibrational state in the x direction). We thus say that quantum state exchange has taken place when this equation holds.

B. System of interest

In this paper, we use quantum state exchange to generate a particular state involving two groups of N trapped atoms which, we later show, contains multipartite entanglement that is a useful quantum resource. Our work follows on from Ref. [15] in which it was *asserted* that for a certain system consisting of two groups of N trapped atoms, quantum state exchange could generate “a highly entangled state of all $2[2N]$ atoms.” The system we consider can be seen as a concrete example of that described in the last paragraph of Ref. [15]—our main original contributions are, first, *demonstrating* that the state within our system is a multipartite entangled one and, second, quantitatively analyzing the multipartite entanglement within it.

The system we consider comprises, first, a subthreshold nondegenerate optical parametric amplifier (NOPA) [19–21] for which the two external output fields first pass through Faraday isolators and then each feed into a different linearly damped optical cavity for which one mirror is perfect and the other one is lossy. The axes of both cavities coincide with the x axis. Each cavity supports an electromagnetic field mode of frequency ω_{jc} that is described by the annihilation operator a_j , where j enumerates the cavities. Within the j th cavity, lie N identical two-level atoms, each possessing an internal transition frequency of ω_a . These are trapped in a linear configuration parallel to the x axis by, first, a one-dimensional FORT [22,23]. This consists of cavity mode of frequency ω_T which exhibits a standing-wave pattern along the x axis. The frequency ω_T is strongly detuned from all atomic resonant frequencies and thus the cavity mode’s field exerts dipole forces on the atoms, trapping each of them near a separate node of the field. In addition, the FORT’s axis is parallel to the x axis and it thus traps the atoms in the x direction. The atoms are also tightly confined in the y and z directions by a two-dimensional far-off resonance optical lattice [24–26] and thus move negligibly in these directions. The combined effect of all the trapping fields is to confine each atom in its own one-dimensional trap parallel to the x axis. Furthermore, the atoms are located such that the mean position of each atom coincides with a node of the cavity field described by a_j . The annihilation operator $b_{jx}^{(m)}$ describes the vibrational motion of the m^{th} atom in the j th trap in the x direction. Finally, external lasers of frequency ω_L whose beams are perpendicular to the x axis are incident on all atoms; Fig. 2 illustrates the system under consideration.

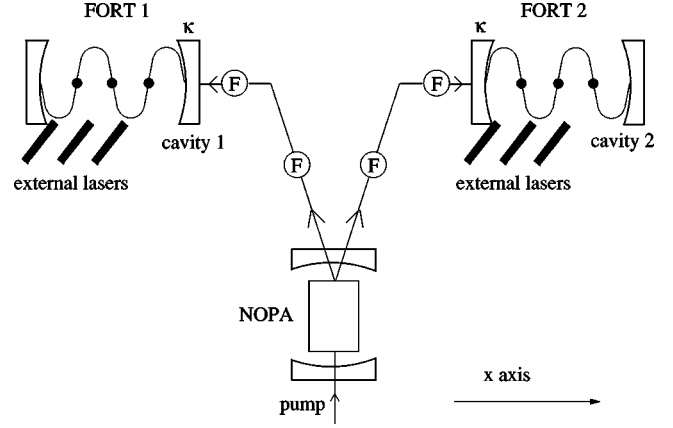


FIG. 2. A schematic diagram for a system involving, first, a subthreshold optical nondegenerate parametric amplifier (NOPA) whose output modes pass through Faraday isolators (represented by F enclosed in a circle) and then feed into linearly damped optical cavities (as indicated by the sinusoidal curves inside both cavities which represent the cavity modes a_1 and a_2). These cavities are aligned along the x axis and both have one ideal mirror and one lossy one (with damping constant κ). Inside each of them is a far-off-resonance dipole-force trap (FORT) that, along with a two-dimensional far-off resonance optical lattice, confines N identical two-level atoms in a linear chain parallel to the x axis. Observe that the FORTs’ trapping modes are not shown in the diagram. External lasers of frequency ω_L are incident on each atom in both traps from a direction perpendicular to the x axis.

The Hamiltonian for the j th

$$H_{j \text{ total}} = H_{j0}^{\text{atom}} + H_{j0} + H_{jI} + \kappa(a_j R_j^\dagger + a_j^\dagger R_j) + H_{j \text{ res}}, \quad (5)$$

where H_{j0}^{atom} is the free Hamiltonian for the vibrational states of the atoms, and H_{j0} is the free Hamiltonian for the cavity field and the atoms’ internal states. The term H_{jI} is the interaction Hamiltonian describing the Raman processes involving the cavity field, the external lasers, and the atoms. To be more specific, $H_{j0}^{\text{atom}} = \hbar \nu_{jx} \sum_{m=1}^N (b_{jx}^{(m)\dagger} b_{jx}^{(m)} + \frac{1}{2})$, where ν_{jx} is the vibrational frequency of the FORT in the j th cavity (which we call the j th FORT) in the x direction. This frequency is equal to $\omega_{jc} - \omega_L$. The Hamiltonian H_{j0} is, in a frame rotating at frequency ω_L ,

$$H_{j0} = \hbar \delta a_j^\dagger a_j + \hbar \Delta \sum_{m=1}^N \sigma_{j+}^{(m)} \sigma_{j-}^{(m)}, \quad (6)$$

where $\delta_j = \omega_{jc} - \omega_L$, $\Delta = \omega_a - \omega_L$, and $\sigma_{j+}^{(m)}$ and $\sigma_{j-}^{(m)}$ are raising and lowering operators for the internal states of the m th atom in the j th trap. The term H_{jI} is

$$H_{jI} = \hbar \sum_{m=1}^N \mathcal{E}_L(y, z, t) \sigma_{j+}^{(m)} + \mathcal{E}_L^*(y, z, t) \sigma_{j-}^{(m)} + \hbar g_0 \sum_{m=1}^N \sin(k_j x_{jm}) (a_j^\dagger \sigma_{j-}^{(m)} + a_j \sigma_{j+}^{(m)}), \quad (7)$$

where \mathcal{E}_L is the complex amplitude for all external lasers, $k_j = \omega_{jc}/c$, and g_0 ($g_0 \in \mathbb{R}$) is the coupling constant for the atom-field interaction. Finally, $H_{j\text{res}}$ is the Hamiltonian for the external reservoir that couples to the j th cavity for which R_j is a reservoir annihilation operator and κ is a damping constant.

The following feasible assumptions are made about the system in order to simplify calculations for it and to focus on its most important aspects

(1) The cavity field and external laser frequencies are appreciably detuned from ω_a , and all two-level ions are initially in the ground state. Thus, the excited internal states are sparsely populated, and spontaneous emission effects are negligible and can be ignored.

(2) Vibrational decoherence occurs over a time scale much longer than that of the interactions producing quantum state exchange and consequently can be ignored.

(3) The wavelength of the cavity mode described by a_j is much greater than the distance that any atom in the j th trap strays from the cavity-field node about which it is trapped. Thus, $\sin(kx_{jm}) \approx kx_{jm} \ll 1$ and hence all atoms experience a potential that is, to a good approximation, harmonic. This justifies the form of H_{j0}^{atom} .

(4) All atoms are tightly confined in the y and z directions. This allows us to ignore the y and z dependences of the external laser fields and thus, assuming \mathcal{E}_L is time independent, it follows that $\mathcal{E}_L(y, z, t) \approx \mathcal{E}e^{-i\phi_L}$, where \mathcal{E} is a real time-independent amplitude.

(5) The damping parameter κ is such that $\nu_{jx} \gg \kappa \gg g_0 \eta_{jx} \sqrt{N} \mathcal{E} / \Delta$, where $\eta_{jx} = k_j \sqrt{\hbar / 2m} \nu_{jx}$, where m is the mass of each atom.

Given the assumptions above, we can write $H_{j\text{total}}$ in terms of normal-mode creation and annihilation operators (by adiabatically eliminating the evolution of the internal states) as

$$H_{j\text{total}} = \hbar \sum_{m=1}^N \nu_{jx} \left(B_{jx}^{\dagger(m)} B_{jx}^{(m)} + \frac{1}{2} \right) + \hbar \delta a^\dagger a - \frac{\hbar N \mathcal{E}^2}{\Delta} - \frac{\hbar g_0 \eta_{jx} \sqrt{N} \mathcal{E}}{\Delta} \left(\sum_{m=1}^N (B_{jx}^{(m)} + B_{jx}^{(m)\dagger}) (a_j^\dagger e^{-i\phi_L} + a_j e^{i\phi_L}) \right) + \kappa (a_j R_j^\dagger + a_j^\dagger R_j) + H_{j\text{res}}, \quad (8)$$

where $B_{jx}^{(m)}$ is the annihilation operator for the m th normal mode for the j th trap in the x direction. For example, $B_{jx}^{(1)}$ is a center-of-mass mode annihilation operator, which is $B_{jx}^{(1)} = 1/\sqrt{N}(b_{jx}^{(1)} + b_{jx}^{(2)} + \dots + b_{jx}^{(N)})$ while $B_{jx}^{(2)}$ is the annihilation operator for the *breathing mode*, which is $B_{jx}^{(2)} = 1/\sqrt{2}(-b_{jx}^{(1)} + b_{jx}^{(2)})$ when $N=2$.

Comparing Eq. (8) to Eq. (3), we see that $B_{jx}^{(1)}$ in Eq. (8) plays an almost identical role to that of b_x in Eq. (3). Given that $\sin(kx) \approx \eta_x(b_x + b_x^\dagger)$ in Eq. (3), the only difference between the forms in which the two operators appear results from a factor of \sqrt{N} appearing in front of \mathcal{E} in Eq. (8). As a consequence, $B_{jx}^{(1)}$ in Eqn (8) couples to the cavity mode a_j

identically—aside from the factor of \sqrt{N} —to the manner in which b_x couples to a . It follows that as quantum state exchange takes place in the system described by $H_{\text{total}}^{\text{single}}$ with information about an input electromagnetic field being transferred to b_x , it also occurs in the system described by $H_{j\text{total}}$ due to the correspondence between the two system's Hamiltonians. Thus, in the latter system, information about the input field is transferred to the center-of-mass mode for the trapped atoms in the x direction *just as if this mode was a vibrational mode for a single harmonically trapped atom*. The only difference between the N -atom case and one described by $H_{\text{total}}^{\text{single}}$ is that the effective coupling in the former case is increased by a factor of \sqrt{N} . This conclusion can also be verified via comparing the Langevin equations for $B_{jx}^{(m)}$ and b_x . Due to symmetry considerations, collective modes other than the center-of-mass modes do not absorb any photons in modes a_1 and a_2 . Thus, assuming that these other modes are initially in vacuum states, they remain so during quantum state exchange.

In Ref. [15], it was shown that we can transfer the intracavity steady state for the subthreshold nondegenerate parametric amplifier which is

$$|\psi\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^n r |n\rangle_1 |n\rangle_2, \quad (9)$$

where the subscripts 1 and 2 denote the two output modes and r is a real squeezing parameter, into the vibrational states in the x direction for two single trapped atoms in different harmonic traps. Using the correspondence between the quantum state exchange processes involving a single harmonically trapped atom and N harmonically trapped atoms demonstrated above, it follows that in the system illustrated in Fig. 2 we can transfer $|\psi\rangle$ into the center-of-mass modes in the x direction for the two sets of N trapped atoms, thus producing in the steady state,

$$|\psi_{\text{C.M.}}\rangle = \frac{1}{\cosh r} \sum_{\mathcal{N}=0}^{\infty} \tanh^{\mathcal{N}} r |\mathcal{N}\rangle_1 |\mathcal{N}\rangle_2, \quad (10)$$

where $|\mathcal{N}\rangle_j$ denotes the *center-of-mass* vibrational number state for the x direction with eigenvalue \mathcal{N} for the atoms in the j th FORT. In writing this state, we have omitted the states of collective modes other than the center-of-mass modes as we have assumed these other modes are in vacuum states throughout the quantum state exchange process.

Importantly, the process of creating $|\psi_{\text{C.M.}}\rangle$ just outlined does not seem to be overly experimentally infeasible. This is so as optical cavities and nondegenerate, optical parametric amplifiers have been widely realized quantum optical laboratories for some time. In addition, neutral atoms have been confined within standing-wave dipole-force traps that, in turn, lie within optical cavities [27]. Relatedly, experiments in which a single harmonically trapped ion has been placed within an optical cavity have been conducted [28].

C. Summary

In this paper, we explore multipartite entanglement in relation to quantum state exchange and in Sec. III, follow on from a multipartite entanglement condition implicit in work by Dür and Cirac [29] by presenting a different condition. The satisfaction of this different condition implies that any pure state comprising of M subsystems is M -way entangled. Here, an M -way entangled state is one possessing entanglement that spans across M subsystems as does the generalized GHZ state $|\psi\rangle = |0\rangle^{\otimes M} + |1\rangle^{\otimes M}$. After presenting this condition, we then use it to show *qualitatively* that $|\psi_{C.M.}\rangle$ is $2N$ -way entangled. In Sec. IV, we *quantitatively* consider the entanglement in $|\psi_{C.M.}\rangle$. We introduce a multipartite entanglement measure for pure states we call the *entanglement of minimum bipartite entropy* or E_{MBE} which is the minimum of the von Neumann entropies of all the reduced density operators obtainable from some pure states of interest by tracing over some of the subsystems for the state. After this, we use E_{MBE} to calculate a *lower bound* for the amount of four-way, six-way, and eight-way entanglements in $|\psi_{C.M.}\rangle$ for $N=2,3,4$, respectively, for a range of r values. Finally, we discuss the nature of our results.

It is interesting to investigate the nature of $|\psi_{C.M.}\rangle$'s $2N$ -way entanglement for a number of reasons. First, it has been claimed—but not demonstrated—that $|\psi_{C.M.}\rangle$ is “entangled state of all $2[N] \cdots$ atoms” [14]. It is thus interesting to investigate $|\psi_{C.M.}\rangle$'s $2N$ -way entanglement in order to see if this implied claim is true. Second, it is interesting to investigate $|\psi_{C.M.}\rangle$'s $2N$ -way entanglement as it is a *massive-particle* state that is important as, to date, mostly *massless* photons have been used to experimentally investigate entanglement. Third, if the claim is true, then it means that $|\psi_{C.M.}\rangle$ is a state consisting of $2N$ entangled harmonic oscillators, each possessing an *infinite-dimensional* Hilbert space as opposed to the two-dimensional Hilbert space of a qubit that is $2N$ -way entangled.

III. QUALITATIVE RESULTS

A. Negative partial transpose sufficient condition

Assume that for a certain state ρ , we wish to know the answer to the question “Does ρ contain at least some M -way entanglement?” While answering this question does not tell us everything about the nature of ρ 's M -way entanglement, it nevertheless tells us something of interest. One way to answer it, provided that ρ consists of qubits, is to use a condition that can be readily derived from work of Dür and Cirac [29]. This condition involves negative partial transposes (NPTs) [30–32] and thus we name it the NPT *sufficient condition*. It is sufficient for the presence of M -way entanglement for all ρ 's consisting of P qubits, where $P \geq M$, and is based on generalizing the notions of separability and inseparability to many-qubit systems. Before stating the condition, it is useful to mention two things. First, we define an *M-partite split* of ρ [29], to be a division or split of ρ into M parts, where each consists of one or more subsystems. Second, we observe that ρ can always be converted to a state that is diagonal in a certain basis by a “depolarization” pro-

cess consisting of particular local operations [29]. This basis consists of M -qubit generalized GHZ states of the form $|\psi\rangle = 1/\sqrt{2}(|j\rangle|0\rangle \pm |2^{N-1}-j-1\rangle|1\rangle)$, where j is a natural number that we write in binary as $M-1$ bits, i.e. $j \equiv j_1j_2, \dots, j_{M-1}$, where j_x is the x th bit in j 's binary representation. Given these two things, the NPT sufficient condition states that ρ is M -way entangled for a given M -partite split if the diagonal state that it depolarizes to is such that all *bipartite splits* that *contain* the M -partite split have negative partial transposes. A *bipartite split* is one that divides a system into two parts, i.e., a two-partite split. Furthermore, a bipartite split that *contains* an M -partite split is the one that does not separate members of any of the M subsystems onto two different sides of the bipartite split; that is, one that does not cross any of the divisions created by the M -partite split.

B. Result

Following on from the NPT sufficient condition, we propose a necessary and sufficient condition for the existence of M -way entanglement for M -partite pure states. Our condition is based on the traces of the squares of reduced density matrices obtained by tracing over some of the subsystems constituting our system of interest. After formulating it, we use it to demonstrate that $|\psi_{C.M.}\rangle$ contains *some* $2N$ -way entanglements. Our motivations for employing our condition, instead of the NPT sufficient condition, are that (i) it seems to be mathematically simpler to calculate whether or not our condition is satisfied and (ii) as we are concerned with a pure state, our condition is stronger than the NPT sufficient condition in the sense that it is both necessary and sufficient as opposed to just being sufficient.

Our M -way entanglement condition utilizes the fact that when a *pure* state $|\psi\rangle$ for M subsystems is M -way entangled, we *cannot* write it as $|\psi\rangle = |\phi_1\rangle_{Q_j} \otimes |\phi_2\rangle_{\bar{Q}_j}$, where $|\phi_1\rangle_{Q_j}$ and $|\phi_2\rangle_{\bar{Q}_j}$ are the states for the subsystems denoted by Q_j and \bar{Q}_j , respectively, and both Q_j and \bar{Q}_j denote at least one subsystem. To put this in another way, when $|\psi\rangle$ is M -way entangled, there is *no* way to represent it as the tensor product of two pure states. Consequently, excluding all such possibilities suffices to show, and is also, in general, necessary to show, that $|\psi\rangle$ is M -way entangled. This can be done by first checking that no single-subsystem state can be factored out from the state of the remaining $M-1$ subsystems. We do this by checking that the traces of the squares of all the reduced density operators obtainable from $|\psi\rangle$ by tracing over one subsystem are less than one; that is, $\text{Tr}([\rho_{Q_j}]^2) < 1$, where ρ_{Q_j} is the reduced density operator obtained from $|\psi\rangle\langle\psi|$ by tracing over the subsystem denoted by Q_j , for all Q_j denoting just one subsystem. We can then repeat this procedure, considering all Q_j 's corresponding to all pairs of subsystems, then all triples, and so forth until we have considered all Q_j 's corresponding to all sets of R subsystems, where $R = \lfloor M/2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x . It is sufficient to only consider sets of up to those corresponding to $\lfloor M/2 \rfloor$ subsystems as a necessary condition for being able to factor out any larger number of subsystems from $|\psi\rangle$ is the ability to also factor out $\lfloor M/2 \rfloor$ or fewer

subsystems. Underlying the process just described is that of seeing whether or not we can exclude all the ways that $|\psi\rangle$ could fail to be M -way entangled.

Our condition can be formalized as Definition 1 that is as follows.

Definition 1. For a pure state $|\psi\rangle$ for M subsystems, consider the set Q whose members Q_j are themselves sets of subsystems for the system corresponding to $|\psi\rangle$. This set Q contains all sets of P subsystems for this system, where $1 \leq P \leq \lfloor M/2 \rfloor$. Given this, $|\psi\rangle$ is M -way entangled if and only if, for all Q_j , $\text{Tr}([\rho_{Q_j}]^2) < 1$, where ρ_{Q_j} is the reduced density operator obtained by beginning with $|\psi\rangle\langle\psi|$ and tracing over the subsystems Q_j .

To illustrate Definition 1, consider, for example, the GHZ state $|\psi\rangle_{\text{GHZ}} = 1/\sqrt{2}(|000\rangle_{123} + |111\rangle_{123})$, where the subscripts 1, 2, and 3 denote subsystems of $|\psi\rangle_{\text{GHZ}}$. The parameter $P = \lfloor 3/2 \rfloor = 1$ and consequently the set Q comprises all sets of one subsystem and thus $Q = \{\{1\}, \{2\}, \{3\}\}$, where the numbers again denote subsystems for $|\psi\rangle_{\text{GHZ}}$. For the element $\{1\}$, for example, $\text{Tr}([\rho_{\{1\}}]^2) = 1/2$. Calculating $\text{Tr}([\rho_{Q_j}]^2)$ for all of Q 's other elements, we find that it is $1/2$ in all the three cases. Thus, $|\psi\rangle_{\text{GHZ}}$ satisfies Definition 1 and hence is said to be 3-way entangled, as is the case.

To further explain Definition 1, we now apply it to determining whether the following four-party states are four-way entangled: (1) $|\psi_4^{(1)}\rangle = 1/\sqrt{2}(|0000\rangle_{1234} + |1111\rangle_{1234})$, (2) $|\psi_4^{(2)}\rangle = 1/\sqrt{2}|0\rangle_1 \otimes (|000\rangle_{234} + |111\rangle_{234})$, (3) $|\psi_4^{(3)}\rangle = |\phi^+\rangle_{12} \otimes |\phi^+\rangle_{34}$.

Turning to (1), we see that upon tracing over any single subsystem, we produce a reduced density operator of the form $\rho_{Q_j} = 1/2(|000\rangle\langle 000| + |111\rangle\langle 111|)$ for which $\text{Tr}([\rho_{Q_j}]^2) = 1/2$. Similarly, tracing over any two subsystems produces a density operator of the form $\rho_{Q_j} = 1/2(|00\rangle\langle 00| + |11\rangle\langle 11|)$ for which, again, $\text{Tr}([\rho_{Q_j}]^2) = 1/2$. Thus, Definition 1 gives the correct result that $|\psi_4^{(1)}\rangle$ is four-way entangled. For (2), tracing over the first subsystem produces $|\psi_4^{(3)}\rangle = 1/\sqrt{2}(|000\rangle_{234} + |111\rangle_{234})$, which is a pure state and hence $\text{Tr}([\rho_{Q_j}]^2) = 1$ for the corresponding j . Consequently, Definition 1 tells us that $|\psi_4^{(2)}\rangle$ is *not* four-way entangled, as is the case. For (3), tracing over any one subsystem produces the mixed state $\rho = 1/2 \otimes |\phi^+\rangle_{34}\langle\phi^+|$ and so we might be tempted to infer that $|\psi_4^{(3)}\rangle$ is four-way entangled. However, when we trace over subsystems 1 and 2 or subsystems 3 and 4, we produce the pure state $|\phi^+\rangle$ for which $\text{Tr}([\rho_{Q_j}]^2) = 1$. Hence, Definition 1 correctly tells us that $|\psi_4^{(3)}\rangle$ is not four-way entangled.

In applying Definition 1 to $|\psi_{\text{C.M.}}\rangle$, we first write $|\psi_{\text{C.M.}}\rangle$ in terms of vibrational number states for the $2N$ atoms involved as we wish to see if they are $2N$ -way entangled. As a step towards doing so, upon observing that $|\mathcal{N}\rangle_j = ((B_{jx}^{(1)})^{\mathcal{N}_j}/\sqrt{\mathcal{N}!})|0\rangle_j$, we express $|\mathcal{N}\rangle_j$ in terms of vibrational number states in the x direction for *individual* atoms as

$$|\mathcal{N}\rangle_j = \sum_{\text{ac}(\vec{n}, \mathcal{N})} c(\vec{n}, \mathcal{N}) |\vec{n}\rangle_j, \quad (11)$$

where \vec{n} is the N -component vector (n_1, n_2, \dots, n_N) , the state $|\vec{n}\rangle_j = |n_1\rangle_j \otimes |n_2\rangle_j \cdots |n_N\rangle_j$, where $|n_k\rangle_j$ denotes a number state for the k th atom in the j th FORT, and

$$c(\vec{n}, \mathcal{N}) = {}_j\langle \vec{n} | \mathcal{N} \rangle_j = \frac{\binom{\mathcal{N}}{n_1} \binom{\mathcal{N}-n_1}{n_2} \cdots \binom{\mathcal{N}-n_1 \cdots - n_{N-2}}{n_{N-1}}}{\sqrt{\mathcal{N}! \times \mathcal{N}^{\mathcal{N}}}} \times \sqrt{n_1! n_2! \cdots n_N!}. \quad (12)$$

The sum $\sum_{\text{ac}(\vec{n}, \mathcal{N})}$ denotes the sum over *all combinations* of n_1, n_2, \dots, n_N such that $\sum_{j=1}^N n_j = \mathcal{N}$ [33]. Using Eq. (11) to represent $|\psi_{\text{C.M.}}\rangle$ in terms of vibrational number states for individual atoms, we obtain

$$|\psi_{\text{C.M.}}\rangle = \frac{1}{\cosh r} \sum_{\mathcal{N}=0}^{\infty} \tanh^{\mathcal{N}} r \left(\sum_{\text{ac}(\vec{n}, \mathcal{N})} c(\vec{n}, \mathcal{N}) |\vec{n}\rangle_1 \right) \otimes \left(\sum_{\text{ac}(\vec{m}, \mathcal{N})} c(\vec{m}, \mathcal{N}) |\vec{m}\rangle_2 \right). \quad (13)$$

We now show that the right-hand side of Eq. (13) satisfies Definition 1 and thus $|\psi_{\text{C.M.}}\rangle$ is $2N$ -way entangled. We do this by first writing $|\psi_{\text{C.M.}}\rangle$ as the most general bipartite state possible involving vibrational number states for individual atoms. Next, we show that, upon tracing over the atoms in the half of the bipartite split containing the lesser number of atoms and then finding the trace of the square of the resulting reduced density operator, this is less than one. It follows that, for all j , $\text{Tr}([\rho_{Q_j}]^2) < 1$. Hence, we satisfy Definition 1 and so $|\psi_{\text{C.M.}}\rangle$ is $2N$ -way entangled.

Dividing the atoms in $|\psi_{\text{C.M.}}\rangle$ into two subsystems A and B containing, respectively, R and $2N - R$ atoms ($R \neq 0$), we can write $|\psi_{\text{C.M.}}\rangle$ as

$$|\psi_{\text{C.M.}}\rangle = \sum_{i=0}^{\infty} c_i |f_i\rangle_A \otimes |g_i\rangle_B, \quad (14)$$

where $\| |f_i\rangle_A \| = \| |g_i\rangle_B \| = 1$ and the $|f_i\rangle_A$, but not necessarily the $|g_i\rangle_B$, are mutually orthogonal. (As we can always write $|\psi_{\text{C.M.}}\rangle$ in biorthogonal form [34], there exist $|g_i\rangle_B$ that are mutually orthogonal. However, we are not concerned with this form in the current calculation and so do not consider such a decomposition of $|\psi_{\text{C.M.}}\rangle$.) To give an example, when $N=2$ and A contains the first atom in the first trap

$$|\psi_{\text{C.M.}}\rangle = \frac{1}{\cosh r} |0\rangle_A \otimes \left(|000\rangle_B + \frac{\tanh r}{2} |101\rangle_B + \frac{\tanh r}{2} |110\rangle_B + \frac{\tanh^2 r}{4} |202\rangle_B + \frac{\sqrt{2} \tanh^2 r}{4} |211\rangle_B + \frac{\tanh^2 r}{4} |220\rangle_B + \cdots \right) + \frac{1}{\cosh r} |1\rangle_A \otimes \left(\frac{\tanh r}{2} |001\rangle_B + \frac{\tanh r}{2} |010\rangle_B + \frac{\tanh^2 r}{\sqrt{8}} |102\rangle_B \right)$$

$$\begin{aligned}
 & + \frac{\sqrt{2}\tanh^2 r}{2} |111\rangle_B + \frac{\tanh^2 r}{2} |120\rangle_B + \dots \Big) \\
 & + \frac{1}{\cosh r} |2\rangle_A \otimes \left(\frac{\tanh^2 r}{4} |002\rangle_B + \frac{\sqrt{2}\tanh^2 r}{8} |011\rangle_B \right. \\
 & \left. + \frac{\tanh^2 r}{4} |020\rangle_B + \dots \right) + \dots, \quad (15)
 \end{aligned}$$

where $|x\rangle_A = |n_1=x\rangle_A$ and $|x_1 x_2 x_3\rangle_B = |n_2=x_1, m_1=x_2, m_2=x_3\rangle_B$. Here, for example, $c_0 = 1/\cosh r$, $|f_0\rangle_A = |0\rangle_A$, $c_1 = 1/\cosh r$, $|f_1\rangle_A = |1\rangle_A$,

$$\begin{aligned}
 |g_0\rangle_B = \frac{1}{\sqrt{\mathcal{M}_0}} & \left(|000\rangle_B + \frac{\tanh r}{2} |101\rangle_B + \frac{\tanh r}{2} |110\rangle_B \right) \\
 & + \frac{\tanh^2 r}{4} |202\rangle_B + \frac{\sqrt{2}\tanh^2 r}{4} |211\rangle_B + \frac{\tanh^2 r}{4} |220\rangle_B \\
 & + \dots \Big)
 \end{aligned}$$

and

$$\begin{aligned}
 |g_1\rangle_B = \frac{1}{\sqrt{\mathcal{M}_1}} & \left(\frac{\tanh r}{2} |001\rangle_B + \frac{\tanh r}{2} |010\rangle_B + \frac{\tanh^2 r}{\sqrt{8}} |102\rangle_B \right. \\
 & \left. + \frac{\sqrt{2}\tanh^2 r}{2} |111\rangle_B + \frac{\tanh^2 r}{2} |120\rangle_B + \dots \right),
 \end{aligned}$$

where \mathcal{M}_0 and \mathcal{M}_1 normalize $|g_0\rangle_B$ and $|g_1\rangle_B$. Upon tracing over A in Eq. (14) and squaring the resulting reduced density operator ρ_{Q_A} , we obtain

$$[\rho_{Q_A}]^2 = \sum_{i,j=0}^{\infty} c_i^2 c_j^2 |g_i\rangle\langle g_i| |g_j\rangle\langle g_j|. \quad (16)$$

Calculating the trace of $[\rho_{Q_A}]^2$ yields

$$\text{Tr}([\rho_{Q_A}]^2) = \sum_{i,j=0}^{\infty} c_i^2 c_j^2 |d_{ij}|^2, \quad (17)$$

where $d_{ij} = \langle g_i | g_j \rangle$. As the trace of a density operator is always one, we know that

$$\sum_{i,j=0}^{\infty} c_i^2 c_j^2 = \left(\sum_{i=0}^{\infty} c_i^2 \right) \left(\sum_{j=0}^{\infty} c_j^2 \right) = 1. \quad (18)$$

It thus follows from Eq. (17) that, as $c_i \neq 0$ for all i , if $|d_{ij}|^2 < 1$ for at least one d_{ij} then $\text{Tr}([\rho_{Q_A}]^2) < 1$.

As the center-of-mass state $|\mathcal{N}\rangle_1 |\mathcal{N}\rangle_2$ has an even number of center-of-mass phonons in total ($2\mathcal{N}$), when we express it as a sum of vibrational number states for individual atoms, these states all contain an even number of individual phonons in total. Furthermore, because $|\psi_{\text{C.M.}}\rangle$ contains the state $|\mathcal{N}=0\rangle_1 |\mathcal{N}=0\rangle_2$, $|f_i\rangle_A$ in Eq. (14) for one particular value of i , which we denote by $|f_i^{\text{zero}}\rangle_A$, is a tensor product of

ground states for some of the $2N$ atoms in $|\psi_{\text{C.M.}}\rangle$. For example, in Eq. (15), $|f_i^{\text{zero}}\rangle_A = |0\rangle_A$. Given that, in general, $|f_i^{\text{zero}}\rangle_A$ contains zero individual phonons, only states with an *even* number of individual phonons in total are present in the $|g_i\rangle_B$ with the same index i , which we denote by $|g_i^{\text{zero}}\rangle_B$. This is so as we require the total number of individual phonons in $|f_i^{\text{zero}}\rangle_A \otimes |g_i^{\text{zero}}\rangle_B$ to be even.

In addition to $|f_i^{\text{zero}}\rangle_A$, because $|\psi_{\text{C.M.}}\rangle$ includes the term $|\mathcal{N}=1\rangle_1 |\mathcal{N}=1\rangle_2$, there also exists an $|f_i\rangle_A$ in Eq. (13) containing just one individual phonon, which we denote as $|f_i^{\text{one}}\rangle_A$. For example, in Eq. (15) $|f_1^{\text{one}}\rangle_A = |1\rangle_A$. In general, the $|g_i\rangle_B$ with the same index i as $|f_i^{\text{one}}\rangle_A$, which we denote by $|g_i^{\text{one}}\rangle_B$, comprises states with an *odd* number of individual phonons in total as dictated by the requirement that the total number of individual phonons for $|f_i^{\text{one}}\rangle_A \otimes |g_i^{\text{one}}\rangle_B$ is even. Thus, $|g_i^{\text{one}}\rangle_B$ is orthogonal to $|g_i^{\text{zero}}\rangle_B$ and the corresponding $|d_{ij}|^2 = |\langle g_i^{\text{zero}} | g_i^{\text{one}} \rangle|^2 = 0$. Returning to the right-hand side of Eq. (17), this means that $\text{Tr}([\rho_{Q_A}]^2) < 1$ for Q_A and thus Definition 1 is satisfied. This allows us to infer that $|\psi_{\text{C.M.}}\rangle$ is $2N$ -way entangled and consequently we have verified the assertion that $|\psi_{\text{C.M.}}\rangle$ is an “entangled state of all $2[N] \cdots$ atoms”—except, of course, when $r=0$.

IV. QUANTIFYING THE AMOUNT OF $2N$ -WAY ENTANGLEMENT IN $|\psi_{\text{C.M.}}\rangle$

A. Theory

In the preceding section, we presented a qualitative result that showed that $|\psi_{\text{C.M.}}\rangle$ possessed some $2N$ -way entanglement. However, we would also like to know how much $2N$ -way entanglement $|\psi_{\text{C.M.}}\rangle$ contains. For this reason, we present *quantitative* measure of M -way entanglement for M -partite *pure states*, for an arbitrary M . This measure is based on the von Neumann entropies of reduced density operators produced by considering all bipartite splits for some state of interest. We call it the *entanglement of minimum bipartite entropy* or E_{MBE} , which we soon define. After this, we then argue that it is a plausible measure and finally, use it to calculate a *lower bound* on the amount of $2N$ -way entanglement in $|\psi_{\text{C.M.}}\rangle$.

For a pure state $|\psi\rangle$ with M subsystems, E_{MBE} is

$$E_{\text{MBE}}(|\psi\rangle) = \min(\mathbf{S}_{\text{all}}), \quad (19)$$

where \mathbf{S}_{all} is the set containing the von Neumann entropies of *all* the reduced density operators obtained from $|\psi\rangle\langle\psi|$ by tracing over a set of P subsystems in $|\psi\rangle$, where $1 \leq P \leq \lfloor M/2 \rfloor$. The function $\min(X)$ returns the smallest element of the set X . Thus, as the von Neumann entropies of both sides of any bipartite split of $|\psi\rangle$ are equal [2], \mathbf{S}_{all} contains the von Neumann entropies for *all* the reduced states that we can generate from $|\psi\rangle$. For example, when $|\psi\rangle = |\psi\rangle_{\text{GHZ}} = 1/\sqrt{2}(|000\rangle_{123} + |111\rangle_{123})$, the sets of subsystems containing P members that we trace over in obtaining \mathbf{S}_{all} are $\{1\}$, $\{2\}$, and $\{3\}$, where the numbers denote either the “1,” “2,” or “3” subsystems of $|\psi\rangle_{\text{GHZ}}$. As the von Neumann entropy of the state ρ is $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ [2,3], the von Neumann

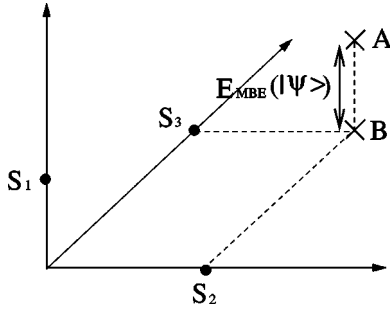


FIG. 3. Coordinate space illustrating that E_{MBE} can be seen as a distance-based entanglement measure. The distance between A [the point representing $\mathbf{S}_{\text{all}}(|\phi\rangle) = \{S_1, S_2, S_3\}$, where $|\phi\rangle$ is three-way entangled and $S_1 < S_2, S_3$] and B [the point representing $\mathbf{S}_{\text{all}}(|\phi_{\text{zero}}\rangle) = \{0, S_2, S_3\}$, where $|\phi_{\text{zero}}\rangle$ appears to be the closest pure state with no three-way entanglement to $|\phi\rangle$] is E_{MBE} . The quantities S_1 , S_2 , and S_3 are dimensionless.

entropy for the reduced density operator generated from $|\psi\rangle_{\text{GHZ}}\langle\psi|$ upon tracing over the subsystem denoted by any one of these sets is 1. Hence $\mathbf{S}_{\text{all}} = \{1, 1, 1\}$ and so $E_{\text{MBE}}(|\psi\rangle_{\text{GHZ}}) = 1$. Thus, we say that $|\psi\rangle_{\text{GHZ}}$ has one unit of three-way entanglement.

To provide some insight into E_{MBE} , it is now shown that it can be thought of as a *distance-based* measure of M -way entanglement; that is, as measuring the distance between $|\psi\rangle$ and the closest pure state with zero M -way entanglement given a certain metric. To understand this, observe that, naively, it seems reasonable to think that there exists a pure state $|\psi_{\text{zero}}\rangle$ with zero M -way entanglement that has an identical \mathbf{S}_{all} to $|\psi\rangle$'s except for one element. This element corresponds to the smallest element of $\mathbf{S}_{\text{all}}(|\psi\rangle)$ and is zero. The next step in comprehending the distance-based nature of E_{MBE} is representing $\mathbf{S}_{\text{all}}(|\psi\rangle)$ and $\mathbf{S}_{\text{all}}(|\psi_{\text{zero}}\rangle)$ by points A and B , respectively, in a coordinate space for which each coordinate denotes the possible values of an element of either $\mathbf{S}_{\text{all}}(|\psi\rangle)$ or $\mathbf{S}_{\text{all}}(|\psi_{\text{zero}}\rangle)$; that is, a space that graphically represents $\mathbf{S}_{\text{all}}(|\psi\rangle)$ and $\mathbf{S}_{\text{all}}(|\psi_{\text{zero}}\rangle)$. For such a space, we observe that no pure state with zero M -way entanglement is represented by a point closer to A than B . It is in this sense that we think of $|\psi_{\text{zero}}\rangle$ as being the closest pure state to $|\psi\rangle$ with zero M -way entanglement. Finally, the distance-based nature of $E_{\text{MBE}}(|\psi\rangle)$ can be seen by observing that the distance between A and B is $E_{\text{MBE}}(|\psi\rangle)$. This point is illustrated in Fig. 3 for the three-way entangled state $|\phi\rangle$ comprising three subsystems for which $\mathbf{S}_{\text{all}}(|\phi\rangle) = \{S_1, S_2, S_3\}$, where $S_1 < S_2, S_3$ and $S_1, S_2, S_3 \neq 0$. Naively, the closest pure state to $|\phi\rangle$ with no three-way entanglement $|\phi_{\text{zero}}\rangle$ seems to be such that $\mathbf{S}_{\text{all}}(|\phi_{\text{zero}}\rangle) = \{0, S_2, S_3\}$. Representing $\mathbf{S}_{\text{all}}(|\phi\rangle)$ and $\mathbf{S}_{\text{all}}(|\phi_{\text{zero}}\rangle)$ graphically in the manner described above by points A and B in Fig. 3, we observe that the distance between these is $S_1 = E_{\text{MBE}}$. Generalizing this notion, we see that E_{MBE} can be viewed as measuring the distance between $|\psi\rangle$ and the nearest pure state with zero M -way entanglement. This distance seems to be a plausible measure of $|\psi\rangle$'s M -way entanglement and thus E_{MBE} appears to have an underlying intuitive motivation.

To further highlight the plausibility of E_{MBE} , consider the following analogy. Imagine an ordinary chain with M links. If $M - 1$ of these are strong and the other one is weak, then the chain is close to breaking and so only has a small amount of “nonbrokenness”—even though all but one of the links are solid. This is so as nonbrokenness is a wholistic property that is a manifestation of the nature of all M links. Relating this to E_{MBE} , just as nonbrokenness is a wholistic property, so E_{MBE} measures a wholistic property, namely, M -way entanglement, that relates to the nature of all M subsystems of M -partite states. In analogy with a chain with just one weak link, an M -partite pure state for which all members of \mathbf{S}_{all} are large, except for one, is very close to possessing no M -way entanglement. In this way, we see that E_{MBE} and, in particular, the presence of the min function in it seems plausible.

Another interesting feature of E_{MBE} is that it satisfies three well-known desiderata for *bipartite* entanglement measures [35], as we now show. (It seems plausible that these should also be desiderata for *multipartite* entanglement measures.) They are the following:

- (1) the proposed entanglement measure is zero for all product states;
- (2) the proposed entanglement measure is invariant under local unitaries;
- (3) the proposed entanglement measure does not increase, on average, under local operations, classical communication (LOCC) and division into subensembles.

Beginning with (1), if the state of interest is a product state, where we define a product state to be one for which we can factor out the state of at least one of the subsystems, then at least one member of \mathbf{S}_{all} is zero and so E_{MBE} is also zero, as we desire. Turning to (2), we note that for a general bipartite split, the von Neumann entropy of the reduced density matrix obtained by tracing over the subsystems on the side of the split with the lesser number of particles is invariant under unitary transformations that act on only one subsystem. Consequently, if we define local unitaries to be those which act just on a single subsystem, then E_{MBE} satisfies (2).

In considering (3), it is important to remember that E_{MBE} is only for pure states and thus we ignore local operations that convert $|\psi\rangle$ to a mixed state. For example, we do not consider local operations that transform $|\psi\rangle$ to a state that is close to a maximally mixed state and thus has large values for the von Neumann entropies of all its reduced states. We choose this example as such local operations increase the value of $\min(\mathbf{S}_{\text{all}})$ for a system of interest. However, they manifestly do not increase its M -way entanglement, but instead transform its state into one for which E_{MBE} is not applicable. With this constraint in mind, we define a local operation to be one that involves just one subsystem such as a projective measurement on a single subsystem. Given this definition, it can be shown that for *bipartite* pure states LOCC and division into subensembles cannot increase the average entanglement of any state as measured by the von Neumann entropy of its reduced states (entropy of entanglement) [25]. It follows that they also cannot increase any member of $\mathbf{S}_{\text{all}}(|\psi\rangle)$, on average, as these faithfully measure the bipartite entanglement in $|\psi\rangle$ given some bipartite split

for it. Thus, E_{MBE} also cannot increase, on average, under LOCC and division into subensembles and so E_{MBE} satisfies (3).

Another well-known desideratum for a bipartite entanglement measure is that it is *additive over tensor products* [35]. However, it can be shown that E_{MBE} is *superadditive*; that is, the M -way entanglement of a combined state generated from two states with a and b units of M -way entanglement can be greater than $a+b$ (but, importantly, not when $M=2$). It is an open question as to whether or not multipartite entanglement is additive and so we do not know if the superadditivity of E_{MBE} represents a flaw.

For E_{MBE} to be a reasonable measure, it ought to reduce to the standard pure state bipartite entanglement measure of the entropy of entanglement. For E_{MBE} , when $N=1$, we have $E_{\text{MBE}} = \min(\mathbf{S}_{\text{all}}) = S_{\{1\}}$, where $S_{\{1\}}$ is the von Neumann entropy for the reduced density operator $\rho_{Q_{\{1\}}}$ $= \text{Tr}_1(|\psi\rangle\langle\psi|)$, and so we recover the desired measure, namely, the entropy of entanglement. Finally, E_{MBE} seems to be plausible as for $|\psi\rangle = \sqrt{c}|0\rangle^{\otimes N} + \sqrt{1-c}|1\rangle^{\otimes N}$, where $c \in [0,1]$ and N is a positive integer, $E_{\text{MBE}} = -c \log_2 c - (1-c) \log_2 (1-c)$. This expression increases monotonically in the interval $c \in [0,1/2]$ and attains its maximum value of one for $c=1/2$. Such behavior seems reasonable.

B. Results

In this section, we use E_{MBE} to calculate *lower bounds* on the amount of $2N$ -way entanglement present in $|\psi_{\text{C.M.}}\rangle$ for $N=2,3,4$, for a range of r values. We obtain these lower bounds by, first, calculating $\text{Tr}([\rho_{Q_j}]^2)$ for a general Q_j . Next, we determine the *linear entropy* $S_L(\rho_{Q_j})$ [36] from the relation $S_L(\rho_{Q_j}) = 1 - \text{Tr}([\rho_{Q_j}]^2)$ and then use the fact that $S_L(\rho)/\log_2 e \leq S(\rho)$ to obtain our lower bounds. We calculate a lower bound rather than E_{MBE} itself as it is computationally infeasible to calculate E_{MBE} due to the fact that it is computationally infeasible to calculate the required von Neumann entropies of reduced density operators given the infinite-dimensional bases of the harmonic oscillators comprising $|\psi_{\text{C.M.}}\rangle$. This is so as these are generally calculated by first diagonalizing ρ , and it is computationally infeasible to do this, in general, when ρ is a square matrix of infinite dimensions.

We begin with the initial density operator $\rho_{\text{C.M.}} = |\psi_{\text{C.M.}}\rangle\langle\psi_{\text{C.M.}}|$ that can be written in the center-of-mass number-state basis as

$$\rho_{\text{C.M.}} = \sum_{\mathcal{N}, \mathcal{N}'}^{\infty, \infty} f(\mathcal{N}, \mathcal{N}') |\mathcal{N}\rangle_1 |\mathcal{N}\rangle_2 \langle \mathcal{N}'|_1 \langle \mathcal{N}'|, \quad (20)$$

where $f(\mathcal{N}, \mathcal{N}') = \tanh^{\mathcal{N}+\mathcal{N}'} r / \cosh^{2r}$. To obtain a general ρ_{Q_j} , we trace over the first T atoms in the first FORT and the first V in the second one, arriving at

$$\rho_{Q_j} = \sum_{\vec{P}=0}^{\infty} \sum_{\mathcal{N}, \mathcal{N}'}^{\infty, \infty} f(\mathcal{N}, \mathcal{N}') \langle \vec{P} | \mathcal{N}\rangle_1 |\mathcal{N}\rangle_2 \langle \mathcal{N}'|_1 \langle \mathcal{N}'| \vec{P}\rangle, \quad (21)$$

where \vec{P} is a dummy variable given by $\vec{P} = (p_1^{(1)}, p_2^{(1)}, \dots, p_T^{(1)}, p_1^{(2)}, \dots, p_V^{(2)})$. Here $p_\alpha^{(j)}$ denotes a vibrational number state for the α th atom in the x direction in the j th FORT, $\vec{0} = (0_{(1)}, 0_{(2)}, 0_{(3)}, \dots, 0_{(T+V)})$, and $\vec{\infty} = (\infty_{(1)}, \infty_{(2)}, \infty_{(3)}, \dots, \infty_{(T+V)})$, where a bracketed subscript enumerates the elements of $\vec{0}$ or $\vec{\infty}$. We adopt a notation such that a sum of the form $\sum_{\vec{X}=\vec{0}}^{\vec{Y}}$, where \vec{X} and \vec{Y} are the F -component vectors (X_1, X_2, \dots, X_F) and (Y_1, Y_2, \dots, Y_F) , respectively, denotes the set of sums $\sum_{X_1=0}^{Y_1} \sum_{X_2=0}^{Y_2} \sum_{X_3=0}^{Y_3} \dots \sum_{X_F=0}^{Y_F}$. Furthermore, we also assume that a state of the form $|\vec{X}\rangle$ denotes the state $|X_1\rangle \otimes |X_2\rangle \dots |X_F\rangle$. Note that due to an exchange symmetry for atoms in the same group of atoms, it is sufficient to just consider the reduced density operators denoted by Eq. (21) to deal with all possible ρ_{Q_j} 's. That is, we do not need to consider, say, tracing over the first and third atoms in the first FORT and the second one in the second FORT. This is so as the ρ_{Q_j} , this yields, is identical to that produced by tracing over the first two atoms in the first FORT and the first one in the second FORT.

We now find $[\rho_{Q_j}]^2$ and then trace over the remaining $2N - (T+V)$ atoms, producing

$$\begin{aligned} \text{Tr}([\rho_{Q_j}]^2) = & \text{Tr} \left(\sum_{\vec{P}=0}^{\infty} \sum_{\vec{P}'=0}^{\infty} \sum_{\mathcal{N}, \mathcal{N}', \mathcal{M}, \mathcal{M}'}^{\infty, \infty, \infty, \infty} f(\mathcal{N}, \mathcal{N}') f(\mathcal{M}, \mathcal{M}') \right. \\ & \times \langle \vec{P} | \mathcal{N}\rangle |\mathcal{N}\rangle \langle \mathcal{N}' | \mathcal{N}'\rangle \langle \vec{P}' | \mathcal{M}\rangle |\mathcal{M}\rangle \\ & \left. \times \langle \mathcal{M}' | \mathcal{M}'\rangle \langle \vec{P} | \mathcal{M}\rangle \right), \quad (22) \end{aligned}$$

where, in analogy with \vec{P} , \vec{P}' is a dummy variable given by $\vec{P}' = (\mathcal{P}_1^{(1)}, \mathcal{P}_2^{(1)}, \dots, \mathcal{P}_T^{(1)}, \mathcal{P}_1^{(2)}, \dots, \mathcal{P}_V^{(2)})$ where $\mathcal{P}_\alpha^{(j)}$ denotes a vibrational number state in the x direction for the α th atom in the j th FORT.

Using Eq. (22), we now numerically determine $S_L(\rho_{Q_j})$ for particular values of N and r for arbitrary T and V values. Our results provide lower bounds for $S(\rho_{Q_j})$ as $S_L(\rho)/\log_2 e \leq S(\rho)$ as can be verified by considering a power-series expansion for $S(\rho)$. Hence, knowing $S_L(\rho_{Q_j})$ for all bipartite splits of $|\psi_{\text{C.M.}}\rangle$ allows us to infer a lower bound for $\min(\mathbf{S}_{\text{all}})$ and hence one for E_{MBE} . We thus calculate all $S_L(\rho_{Q_j})$ for $N=2,3,4$ for a range of r values numerically using straightforward C++ code. These results are then used to place lower bounds on $E_{\text{MBE}}(|\psi_{\text{C.M.}}\rangle)$ for four-way, six-way, and eight-way entanglements which appear in Figs. 4(a) and 4(b).

As $|\psi_{\text{C.M.}}\rangle$ is the sum of an infinite number of state vectors, to calculate S_L , in practice, we truncate the sum over \mathcal{N} in the definition of $|\psi_{\text{C.M.}}\rangle$ at a finite value. This induces errors in our lower bounds for $E_{\text{MBE}}(|\psi_{\text{C.M.}}\rangle)$ for which upper bounds can be derived. For all data points in Figs. 4(a)

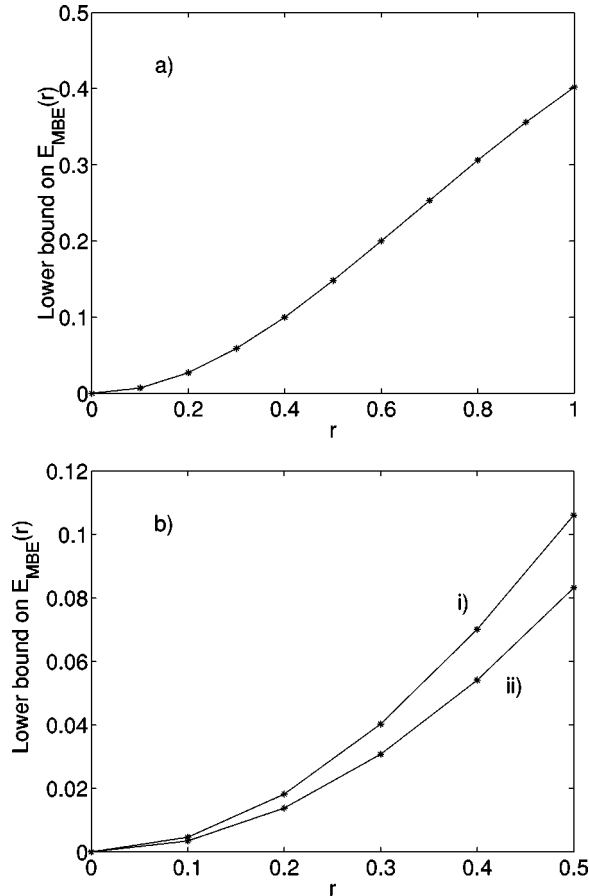


FIG. 4. Plots of lower bounds on E_{MBE} (dimensionless) for $|\psi_{\text{C.M.}}\rangle$ as a function of r (dimensionless) for (a) $N=2$ (four-way entanglement), and (b) (i) $N=3$ (six-way entanglement) and (ii) $N=4$ (eight-way entanglement). Note that in all figures, we have linearly interpolated between points 0.1 units apart on horizontal axes. Numerical errors are less than 10^{-3} for all data points.

and 4(b), the errors on our lower bounds for $E_{\text{MBE}}(|\psi_{\text{C.M.}}\rangle)$ have been calculated to be less than 10^{-3} and hence are negligible.

Two interesting features of Figs. 4(a) and 4(b) are that, first, for a given r value our lower bound on E_{MBE} decreases for increasing N . It is *possible* that we can understand this behavior by observing that for constant r we initially have a fixed entanglement resource, namely, the entangled output of the NOPA. It is conceivable that the decrease under consideration results from this fixed resource being spread amongst a larger number of subsystems as we increase N thus, perhaps, causing it to distribute less bipartite entanglement to any given bipartite split of $|\psi_{\text{C.M.}}\rangle$. In turn, this may decrease the S_L of both halves of an arbitrary split, thus explaining the decrease in our lower bound for E_{MBE} for increasing N . The second interesting feature of Figs 4(a) and 4(b) is that as we increase r E_{MBE} increases as expected, given that an increased r means that we have more center-of-mass entanglement.

V. DISCUSSION

Throughout the paper, we have emphasized that $|\psi_{\text{C.M.}}\rangle$ contains $2N$ -way entanglement. However, for this entangle-

ment to be meaningful, it must have observable effects. One feature of the system under consideration that makes its entanglement conducive to producing such an effect is the fact that the atoms in the system are spatially separated and thus, in principle, are individually accessible. Thus, for example, we could shine a sufficiently narrow laser beam on one of the atoms and, provided it did not propagate perpendicular to the x axis, implement a local displacement on the vibrational state of the atom in the x direction. Furthermore, accessing individual atoms is made easier by the fact that neighboring atoms do not have to be located at successive cavity-field nodes. Instead, they can occupy every second, third etc. node, thus increasing their spatial separation and making it easier to address them one at a time. Another advantageous consequence of the fact that each atom is individually accessible is that it permits us to perform measurements on the vibrational states of single atoms, perhaps by employing a certain quantum-optical technique used to measure the position of individual trapped atoms by having them interact strongly with a low-photon number cavity mode [37].

In light of the considerations of the preceding paragraph, some possible applications of the entanglement in $|\psi_{\text{C.M.}}\rangle$ are as follows.

a. *Violations of inequalities based on local realism.* A number of such inequalities for an arbitrary number of quantum systems have been formulated [38]. Given the close connection between violations of these inequalities and entanglement, $|\psi_{\text{C.M.}}\rangle$ is the sort of state we might expect to violate at least some $2N$ -party inequalities based on local realism. However, as the Hilbert space for the vibrational motion each atom is infinite dimensional and not two dimensional (as is the case for qubits) the violations may require discretizing or “binning” measurement results of a continuous variable such as quadrature phase amplitude.

b. *Solving quantum communication complexity problems (distributed quantum computing).* Quantum communication complexity problems [39] involve a number of parties attempting to evaluate some function f for a particular input string. Each party is given part of the input string and then uses shared prior entanglement, local classical computation, and public communication in attempting to evaluate f . In such a scenario, the prior entanglement can allow the evaluation to be performed in a superior manner to that attainable classically. As the entanglement in $|\psi\rangle_{\text{C.M.}}$ is such that every atom in the corresponding system is with every other one, it is a quantum resource seemingly well suited to being of use to in solving quantum communication complexity problems better than can be done classically.

c. *Continuous-variable quantum computation.* Continuous-variable quantum computation [40] involves quantum computing with infinite-dimensional quantum systems as opposed to the usual two-dimensional qubits. The most obvious way to perform this sort of computation with the system under consideration would be to, first, consider each atom in it as a qudit in the limit of $d \rightarrow \infty$. After this, we would then need to implement two-qudit gates by having different atoms interact with each other in a pairwise manner. One method by which this might be accomplished is by using a scheme [41] employed in optical lattices to get two

spatially separated trapped neutral atoms of different species to interact with one another. This is done by varying the polarizations of the electromagnetic fields trapping the atoms which has the effect of varying the potentials that the atoms see in such a manner that they move towards each other. Once together, the atoms interact via a dipole-dipole coupling. It is conceivable that this method could be applied to implement two-qudit gates in the system of interest. One complication, however, in utilizing this scheme is that it necessitates that we modify our system by having the $2N$ atoms in it comprised two different species, perhaps, with the species of atom alternating as we move along each linear configuration. Nevertheless, while the system under consideration may not be the most natural one in which to do continuous-variable quantum computation, there is some possibility that the entanglement in it could be used to do this.

A. Qualitative results

The thinking underlying Definition 1 is the same as that which underlies the NPT sufficient condition for M -way entanglement. However, there are significant differences between the two. First, Definition 1 involves arbitrary dimensional subsystems, whereas the NPT sufficient condition deals only with qubits. Second, the NPT sufficient condition is a sufficient but not a necessary condition for M -way entanglement, whereas the satisfaction of Definition 1 is both necessary and sufficient for pure states. Third, the NPT sufficient condition uses the partial transpose to determine the presence of M -way entanglement, whereas Definition 1 uses the mathematically simpler entity the trace of the square of a reduced density operator. Observe that Definition 1 is narrower than the NPT sufficient condition in the sense that it only applies to pure states, while the NPT sufficient condition is applicable to both pure and mixed states.

B. Quantitative results

A number of issues surround E_{MBE} , which we now discuss.

What does E_{MBE} tell us about what quantum resource we have? Ideally, we would like to be able to relate E_{MBE} to one or more quantum tasks or protocols such as distributed quantum computation with E_{MBE} telling us something valuable about how well we can perform these tasks. This is so as if we could do this, then it would increase E_{MBE} 's utility. Unfortunately, however, this has not yet been accomplished.

Can we tractably calculate E_{MBE} ? For an entanglement measure to be useful, it must be tractable and able to be calculated in practice. Unfortunately, E_{MBE} seems to be difficult to calculate, at least for the state considered.

Although E_{MBE} has the two above negative features we note that, first, further research may eliminate them and, second, we should consider them alongside the positive features of E_{MBE} which are that it is a reasonable measure and that it helps us to understand the nature of the entanglement in $|\psi_{\text{C.M.}}\rangle$ and also the capabilities of quantum state exchange. Our results contribute to the understanding of multipartite entanglement involving massive particles and infinite-dimensional Hilbert spaces within a context that is not overly experimentally infeasible.

To conclude, we have shown that quantum state exchange can be used to produce the state $|\psi_{\text{C.M.}}\rangle$ for two sets of trapped atoms in spatially separated FORTs. We have also shown that $|\psi_{\text{C.M.}}\rangle$ is a $2N$ -way entangled state and, in addition, have placed a lower bound on the amount of such entanglement that it possesses. Finally, we have discussed quantum information processing tasks that the $2N$ -way entanglement in $|\psi_{\text{C.M.}}\rangle$ could be used to help perform.

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