# Spin squeezing as a measure of entanglement in a two-qubit system

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(Received 26 May 2003; published 5 December 2003)

We show that the two definitions of spin squeezing extensively used in the literature [M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993) and D.J. Wineland *et al.*, Phys. Rev. A **50**, 67 (1994)] give different predictions of entanglement in the two-atom Dicke system. We analyze differences between the definitions and show that the spin squeezing parameter of Kitagawa and Ueda is a better measure of entanglement than the commonly used spectroscopic spin squeezing parameter. We illustrate this relation by examining different examples of a driven two-atom Dicke system in which spin squeezing and entanglement arise dynamically. We give an explanation of the source of the difference using the negativity criterion for entanglement.

DOI: 10.1103/PhysRevA.68.064301 PACS number(s): 03.67.Mn, 42.50.Dv, 42.50.Fx

Spin squeezing results from quantum correlations between atomic spins have received a great deal of attention in recent years [1–9]. The interest in spin squeezing arises not only from the fact that it exhibits reduced fluctuations of the collection of atomic spins below the fundamental spin noise limit, but also from the possibility of interesting novel applications in interferometry and high-precision spectroscopy. Recently, Sørensen *et al.* [10] have proposed spin squeezing as a measure of entanglement in multiatom systems, which opens further applications in quantum information and quantum computation [11]. The advantage of spin squeezing over the well-known entanglement measures, such as concurrence [12] and negativity [13,14] is that spin squeezing can be used as a measure of entanglement in multiatom systems, whereas the former measures can be applied only to two particle (two qubit) systems. Hald *et al.* [15] recently reported preparation of an entangled multiatom state via quantum state transfer from squeezed light to a collection of atomic spins. Kuzmich *et al.* [16] have proposed a scheme to produce spin squeezed states via a quantum nondemolution measurement technique and spin noise reduction using this method has been experimentally observed [17].

There are, however, two different definitions of the spin squeezing parameter frequently used in the literature; the spin squeezing parameter of Kitagawa and Ueda defined as [1]

$$\xi^{S}_{\vec{n}_i} = \frac{2}{S}\langle(\Delta S_{\vec{n}_i})^2\rangle_{\perp}, \quad i=1,2, \tag{1}$$

and the spectroscopic spin squeezing parameter introduced in the context of Ramsey spectroscopy as [2]

$$\xi^{R}_{\vec{n}_i} = \frac{2S\langle(\Delta S_{\vec{n}_i})^2\rangle_{\perp}}{\langle S_{\vec{n}_3}\rangle^2}, \tag{2}$$

where $S$ is the total spin of the system, $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are the three mutually orthogonal unit vectors oriented such that the mean value of one of the spin components, assumed here $\langle S_{\vec{n}_3}\rangle$, is different from zero, while the components $S_{\vec{n}_1}$ and $S_{\vec{n}_2}$ have zero mean values. The variance $\langle(\Delta S_{\vec{n}_i})^2\rangle_{\perp}$ is calculated in the plane orthogonal to the mean spin direction. A multiatom system in a coherent state has variances normal to the mean spin direction, equal to the standard quantum limit of $S/2$. In this case, $\xi^{S}_{\vec{n}_i}=1$. A system with the variance reduced below $S/2$ is characterized by $\xi^{S}_{\vec{n}_i}<1$, that is spin squeezed in a direction normal to the mean spin direction. With the parameter (2), spin squeezing is manifested by $\xi^{R}_{\vec{n}_i}<1$, which indicates a reduction in the frequency noise. Since the mean value $|\langle S_{\vec{n}_3}\rangle| \leq S$, it follows that the parameters (1) and (2) do not describe the same spin squeezing, and that $\xi^{R}_{\vec{n}_i}<1$ implies $\xi^{S}_{\vec{n}_i}<1$, but not vice versa. We note that the spin squeezing parameter proposed by Sørensen *et al.* [10] as a measure of entanglement coincides with the parameter (2). It should also be noted here that in general spin squeezing is sufficient but not necessary conditions for entanglement [18–20].

In studying the relation between entanglement and spin squeezing, we discovered that the two definitions of spin squeezing give somewhat different predictions of entanglement in the two-atom Dicke system. It is the purpose of this Brief Report to point out that the spin squeezing parameter (1) is a better measure of entanglement than the parameter (2). Specifically, we will show that there is a large class of processes for which the parameter (1) is the sufficient and necessary condition for entanglement. It was quite surprising to find this connection, since the parameter (2) is commonly used in the literature to compute spin squeezing and entanglement in multiatom systems. The spin squeezing is currently the widely accepted measure of multiatom entanglement, so we believe that a detailed analysis of the relation between entanglement and these two definitions of spin squeezing is of general interest.

We consider the two-atom (two qubit) Dicke system which consists of two identical atoms confined to a volume with dimensions much smaller than the wavelength of the atomic transitions [21,22]. Each atom is assumed to have only two energy levels, ground level $|g_i\rangle$ and excited level

 

$|e_i\rangle$ $(i=1,2)$, which are eigenstates of the energy operator $S_i^z$ with eigenvalues $-1/2$ and $1/2$, respectively.

In the absence of external driving fields, the two-atom Dicke system [21,22] is equivalent to a cascade multilevel system composed of three energy levels $|g\rangle=|g_1\rangle|g_2\rangle, |s\rangle=(|e_1\rangle|g_2\rangle+|g_1\rangle|e_2\rangle)/\sqrt{2}$, and $|e\rangle=|e_1\rangle|e_2\rangle$. The states $|g\rangle$ and $|e\rangle$ are product states of the individual atoms, whereas the state $|s\rangle$ is a maximally entangled state of the system.

In our analysis, we assume that the atoms are driven by two resonant fields: A coherent laser field of the (real) Rabi frequency $\Omega$, and a broadband squeezed vacuum field. We will examine the relation between entanglement and the spin squeezing parameters in three different models of the interaction in which entanglement and spin squeezing arise dynamically.

To calculate the variances and the mean values of the spin components appearing in Eqs. (1) and (2), we apply the master equation of the driven two-atom Dicke system, which in the interaction picture is given by [22]

$$\frac{\partial\hat{\rho}}{\partial t}=\frac{i}{\hbar}[H_s,\hat{\rho}]-\frac{\Gamma}{2}N(S^-S^+\hat{\rho}+\hat{\rho}S^-S^+-2S^+\hat{\rho}S)$$
$$-\frac{\Gamma}{2}(N+1)(S^+S^-\hat{\rho}+\hat{\rho}S^+S^--2S^-\hat{\rho}S^+)$$
$$+\frac{\Gamma}{2}\{M[S^+,[S^+,\hat{\rho}]]+M^*[S^-,[S^-,\hat{\rho}]]\}, \qquad (3)$$

where $\Gamma$ is the spontaneous emission rate of the atoms, $S^\pm=S_1^\pm+S_2^\pm$ are the collective atomic spin operators, and $H_s=-i\hbar(\Omega/2)(S^+-S^-)$ is the interaction Hamiltonian between the atoms and the laser field. The parameters $N$ and $M$ characterize the squeezed field, such that $N$ is the number of photons in the squeezed modes, $M=|M|\exp(i\phi)$ is the magnitude of two-photon correlations between the modes, and $\phi$ is the relative phase between the squeezed and coherent fields. For simplicity, we set the phase $\phi=0$ (or $\pi$) so that $M$ is real.

In order to analyze the relation between entanglement and spin squeezing, we express the parameters (1) and (2) in terms of the density matrix elements of the system. Since the driving fields are on resonance with the atomic transition and $M^*=M$, the stationary off-diagonal density matrix elements (coherences) are real, or equivalently, the Bloch vector has the components $\vec{B}=(\langle S_x\rangle,0,\langle S_z\rangle)$, where $S_x=(S^++S^-)/2$ and $S_z=S_1^z+S_2^z$. Thus, we can study spin squeezing by a single rotation of the nonzero spin components around the $y$ axis. Let $\vec{n}_3$ be the direction of the total spin in the new (rotated) reference frame. Then the variances calculated in the directions $\vec{n}_1$ and $\vec{n}_2$ can be written as

$$\langle(\Delta S_{\vec{n}_1})^2\rangle_\perp=\langle S_z^2\rangle\sin^2\alpha+\langle S_x^2\rangle\cos^2\alpha-\langle S_xS_z\rangle\sin 2\alpha,$$
$$\langle(\Delta S_{\vec{n}_2})^2\rangle_\perp=\langle S_y^2\rangle, \qquad (4)$$

where $\tan\alpha=\langle S_x\rangle/\langle S_z\rangle$.

A simple calculation using Eq. (4) shows that the spin squeezing parameter (1) becomes

$$\xi_{\vec{n}_1}^S=2(1-\rho_{ss})\sin^2\alpha+(1+\rho_{ss}+2\rho_{eg})\cos^2\alpha,$$
$$\xi_{\vec{n}_2}^S=1+\rho_{ss}-2\rho_{eg}, \qquad (5)$$

whereas the parameter (2) takes the form

$$\xi_{\vec{n}_1}^R=[2(1-\rho_{ss})\sin^2\alpha+(1+\rho_{ss}+2\rho_{eg})\cos^2\alpha]/U^2,$$
$$\xi_{\vec{n}_2}^R=(1+\rho_{ss}-2\rho_{eg})/U^2, \qquad (6)$$

where

$$U=(\rho_{ee}-\rho_{gg})\cos\alpha+2^{-1/2}(\rho_{es}+\rho_{sg}+\rho_{se}+\rho_{gs})\sin\alpha.$$

From the structure of Eqs. (5) and (6), it is clear that the necessary condition to obtain spin squeezing is to create two-photon coherences $\rho_{eg}$. For $\rho_{eg}<0$, the right-hand sides of $\xi_{\vec{n}_1}^S$ and $\xi_{\vec{n}_1}^R$ can be less than 1, whereas the right-hand sides of $\xi_{\vec{n}_2}^S$ and $\xi_{\vec{n}_2}^R$ are always greater than 1. Thus, spin squeezing can be observed only in $\xi_{\vec{n}_1}^S$ and $\xi_{\vec{n}_1}^R$ components. On the other hand, for $\rho_{eg}>0$, the right-hand sides of only $\xi_{\vec{n}_2}^S$ and $\xi_{\vec{n}_2}^R$ can be less than 1.

Having introduced the spin squeezing parameters in terms of the density matrix elements, we now turn to our central problem to determine which of the spin squeezing parameters is a better measure of entanglement. Consider first the two-atom Dicke system driven by the squeezed field alone $(\Omega=0)$. In this case, the master equation (3) leads to the following nonzero steady-state solutions for the density matrix elements [22]

$$\rho_{ee}=[N^2(2N+1)-(2N-1)|M|^2]/W,$$
$$\rho_{ss}=(2N+1)[N(N+1)-|M|^2]/W,$$
$$\rho_{eg}=\rho_{ge}=|M|/W, \qquad (7)$$

where

$$W=(2N+1)(3N^2+3N+1-3|M|^2).$$

Since the one-photon coherences are zero, we can easily verify that $\langle S_z\rangle\neq 0$ and $\langle S_x\rangle=\langle S_y\rangle=0$. This implies that we can determine spin squeezing in the $xy$ plane without any rotation $(\alpha=0)$. In this case $(n_1,n_2,n_3)=(x,y,z)$.

Given the steady-state density matrix of the system, it is possible to calculate the stationary entanglement between the atoms. To quantify the degree of entanglement, we use the negativity criterion for entanglement [13,14] and find that the eigenvalues of the partial transposition of the density matrix with the nonzero matrix elements (7) are

$$\lambda_{1\pm}=\tfrac{1}{2}\rho_{ss}\pm|\rho_{eg}|,$$

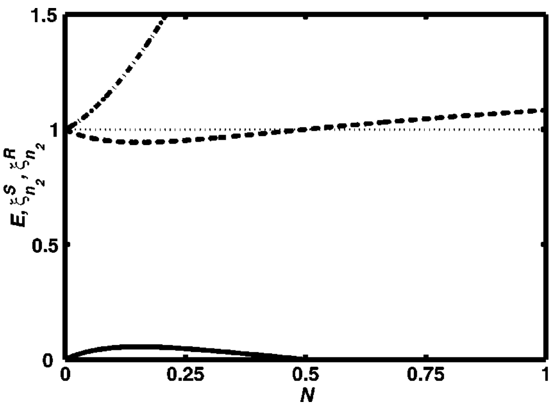


FIG. 1. Entanglement measure $E$ (solid line) and the spin squeezing parameters $\xi^S_{\vec{n}_2}$ (dashed line) and $\xi^R_{\vec{n}_2}$ (dashed-dotted line) as a function of $N$ for the classical squeezed field with $M=N$.

$$\lambda_{2\pm}=\tfrac{1}{2}\{(\rho_{ee}+\rho_{gg})\pm[(\rho_{ee}-\rho_{gg})^2+\rho_{ss}^2]^{1/2}\}. \qquad (8)$$

From this it readily follows that $\lambda_{1+}$ and $\lambda_{2+}$ are always positive. Moreover, it is easily verified that with the solution (7), the eigenvalue $\lambda_{2-}$ is positive for all values of the parameters involved. Thus, the system exhibits entanglement when $|\rho_{eg}|>\rho_{ss}/2$, and then the degree of entanglement is

$$E=\max(0,-2\lambda_{1-})=2|\rho_{eg}|-\rho_{ss}. \qquad (9)$$

It is evident by comparison of Eq. (9) with Eqs. (5) and (6) that the condition for entanglement $(E>0)$ is completely equivalent to the condition for spin squeezing predicted by $\xi^S_{\vec{n}_2}$, and there is a simple relationship

$$E=1-\xi^S_{\vec{n}_2}. \qquad (10)$$

A value of $\xi^S_{\vec{n}_2}<1$ indicates spin squeezing and at the same moment there is entanglement $(E>0)$ between the atoms. In addition, the amount of entanglement which can be obtained is equal to the degree of spin squeezing. Thus, we conclude that the parameter (1) is the sufficient and necessary condition for entanglement induced by a squeezed vacuum field.

The above considerations are illustrated in Fig. 1, where we plot the entanglement measure and the spin squeezing parameters for a classical squeezed field with the correlations $M=N$. The figure shows that $\xi^R_{\vec{n}_2}>1$ for all $N$, but $\xi^S_{\vec{n}_2}$ is less than 1 for $N<1/2$, and also an entanglement appears in the same range of $N$. This shows that $\xi^S_{\vec{n}_2}$ correctly predicts entanglement, while with the parameter $\xi^R_{\vec{n}_2}$, one could observe entanglement without spin squeezing.

In Fig. 2, we plot $E$ and the spin squeezing parameters for a quantum squeezed field with perfect correlations $M^2=N(N+1)$. Since in this case $\rho_{ss}=0$ and $\rho_{eg}>0$, both $\xi^S_{\vec{n}_2}$ and $\xi^R_{\vec{n}_2}$ are less than 1 for the entire range of $N$. Thus, both parameters predict entanglement and spin squeezing for all $N$. However, the amount of entanglement is equal to the degree of spin squeezing predicted by $\xi^S_{\vec{n}_2}$.

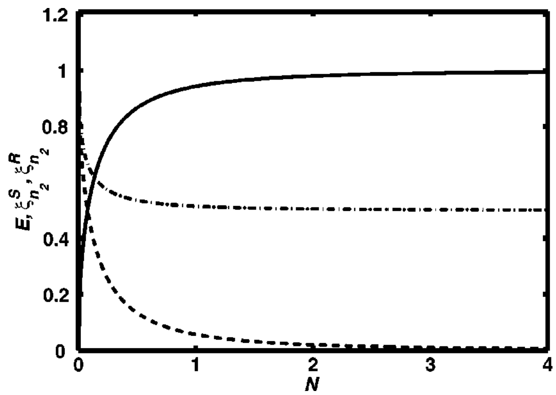


FIG. 2. Negativity $E$ (solid line) and the spin squeezing parameters $\xi^S_{\vec{n}_2}$ (dashed line), $\xi^R_{\vec{n}_2}$ (dashed-dotted line) as a function of $N$ for the quantum squeezed field with $M=\sqrt{N(N+1)}$.

It is easy to show that the entanglement created by the quantum squeezed field is related to the pure two-atom squeezed state [23,24]. Under the squeezed field excitation, there are entangled states generated which can be found by the diagonalization of the density matrix

$$|\Psi_+\rangle=[(\Pi_+-\rho_{ee})|g\rangle+\rho_{eg}|e\rangle]/\mathcal{N}_+,$$

$$|\Psi_-\rangle=[\rho_{ge}|g\rangle+(\Pi_--\rho_{gg})|e\rangle]/\mathcal{N}_-, \qquad (11)$$

where $\mathcal{N}_\pm$ are the normalization constants, and

$$\Pi_\pm=\tfrac{1}{2}(\rho_{gg}+\rho_{ee})\pm\tfrac{1}{2}[(\rho_{gg}-\rho_{ee})^2+4\rho_{eg}^2]^{1/2} \qquad (12)$$

are the populations of the entangled states.

It is evident from Eq. (11) that the two-photon coherences create entangled states which are linear superpositions of the states $|g\rangle$ and $|e\rangle$. Note that the steady state with the classical squeezed field is a mixed state with the populations $\rho_{ss}\neq 0$ and $\Pi_\pm\neq 0$, whereas for the quantum squeezed field $\rho_{ss}=0$, $\Pi_-=0$, and then the stationary state of the system reduces to the pure state $|\Psi_+\rangle$.

We now consider the second model in which the system is driven by the coherent field $(\Omega\neq 0)$ in the absence of the squeezed field $(N=M=0)$. This is an interesting example where one can create spin squeezing and entanglement with the linear Hamiltonian $H_s$. Typical schemes considered for the generation of spin squeezing involve quadratic Hamiltonians [1–10]. After straightforward but lengthy calculations, we find the following steady-state solutions for the density matrix elements

$$\rho_{ee}=\Omega^4/D, \quad \rho_{ss}=(\Omega^4+2\Gamma^2\Omega^2)/D,$$

$$\rho_{es}=\rho_{se}=\sqrt{2}\Gamma\Omega^3/D, \quad \rho_{eg}=\rho_{ge}=2\Gamma^2\Omega^2/D,$$

$$\rho_{sg}=\rho_{gs}=\sqrt{2}\Gamma\Omega(\Omega^2+2\Gamma^2)/D, \qquad (13)$$

where $D=3\Omega^4+4\Gamma^2\Omega^2+4\Gamma^4$.

Proceeding as above, we again make use of the negativity criterion for entanglement. There are obviously four eigenvalues of the partial transposition of the density matrix of the

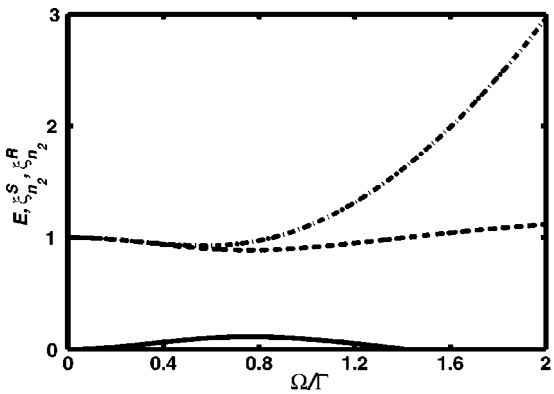


FIG. 3. Negativity $E$ (solid line) and the spin squeezing parameters $\xi^S_{\vec{n}_2}$ (dashed line), $\xi^R_{\vec{n}_2}$ (dashed-dotted line) as a function of $\Omega/\Gamma$.

system. It is straightforward to show that one of the eigenvalues is equal to $\lambda_{1-}$, whereas the remaining eigenvalues are the three roots of the cubic equation

$$p^3-(1-\tfrac{1}{2}\rho_{ss}+\rho_{eg})p^2+[(1-\rho_{ss})(\tfrac{1}{2}\rho_{ss}+\rho_{eg})+\rho_{ee}\rho_{gg} -\tfrac{1}{4}\rho_{ss}^2-\rho_{es}^2-\rho_{sg}^2]p+\rho_{gg}\rho_{es}^2+\rho_{ee}\rho_{sg}^2-(\tfrac{1}{2}\rho_{ss}+\rho_{eg}) \times(\rho_{ee}\rho_{gg}-\tfrac{1}{4}\rho_{ss}^2)-\rho_{ss}\rho_{sg}\rho_{es}=0. \quad (14)$$

It is easily verified that the roots are real and positive for all values of $\Omega$.

Thus, we conclude that the system is entangled when $|\rho_{eg}|>\rho_{ss}/2$, and again the entanglement is related to the spin squeezing parameter (1). Figure 3 shows $E$ and the squeezing parameters as a function of $\Omega$. An entanglement appears for $\Omega<\sqrt{2}\Gamma$ and, as predicted, corresponds to the spin squeezing predicted by $\xi^S_{\vec{n}_2}$.

Finally, we turn to the third model in which the atoms are driven simultaneously by coherent and squeezed vacuum fields. Similar to the second case, all the density matrix elements are real. Hence, the condition for entanglement $|\rho_{eg}|>\rho_{ss}/2$ holds. However, it can be shown that one of the roots of Eq. (14) can be negative indicating that one can observe entanglement without spin squeezing. We have checked numerically that this can happen for $M>0$. For $M<0$ the roots are positive for all values of $\Omega$ and $N$. Thus, the condition for entanglement, $|\rho_{eg}|>\rho_{ss}/2$, also holds in this model and, according to Eq. (5), coincides with the condition for spin squeezing predicted by $\xi^S_{\vec{n}_2}$.

In summary, we have examined the relationship between entanglement and spin squeezing parameters in the two-atom Dicke system. Characterizing the spin squeezing parameters by the density matrix elements, we have examined simple models of driven two-atom Dicke systems in which spin squeezing and entanglement arise dynamically. We have found that the spin squeezing parameter of Kitagawa and Ueda is a better measure of entanglement than the spectroscopic spin squeezing parameter. For the models discussed we have established that the parameter of Kitagawa and Ueda is the sufficient and necessary condition for entanglement. The arguments considered here cannot be extended for systems composed of a large number of atoms as no definite measure of entanglement exists for number of atoms $n$ larger than 2. Nevertheless, it is possible to extend the arguments to two atoms of the $n>2$ atoms [25].

This research was supported in part by Malaysia IRPA research Grant No. 09-02-08-0203-EA002.

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