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More maximal arcs in Desarguesian projective planes and their geometric structure

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Abstract. In a previous paper R. Mathon gave a new construction method for maximal arcs in finite Desarguesian projective planes via closed sets of conics, as well as giving many new examples of maximal arcs. In the current paper, new classes of maximal arcs are constructed, and it is shown that every maximal arc so constructed gives rise to an infinite class of maximal arcs. Apart from when they are of Denniston type or dual hyperovals, closed sets of conics are shown to give maximal arcs that are not isomorphic to the known constructions. An easy characterisation of when a closed set of conics is of Denniston type is given. Results on the geometric structure of the maximal arcs and their duals are proved, as well as on elements of their collineation stabilisers.

1 Introduction

A *maximal* $\{q(n-1)+n;n\}$*-arc* in a projective plane of order q is a subset of $q(n-1)+n$ points such that every line meets the set in 0 or n points for some $2 \leqslant n \leqslant q$. For such a maximal arc n is called the *degree*. If $\mathscr{K}$ is a maximal $\{q(n-1)+n;n\}$-arc, the set of lines external to $\mathscr{K}$ is a maximal $\{q(q-n+1)/n; q/n\}$-arc in the dual plane called the *dual* of $\mathscr{K}$. It follows that a necessary condition for the existence of a maximal $\{q(n-1)+n;n\}$-arc in a projective plane of order q is that n divides q.

Recently, Ball and Blokhuis [2] completed the classification of maximal arcs in the Desarguesian projective plane of order 16. In [7], R. Mathon gave a construction method for maximal arcs in Desarguesian projective planes that generalised a previously known construction of R. H. F. Denniston ([4]). Using this method several new classes of maximal arcs were then constructed, and a large number of examples given in small order projective planes including the Desarguesian projective plane of order 32. We begin by describing this construction method.

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In the following the order of the fields will always be even. Let $\mathrm{Tr}_{q^m/q}$ be the usual trace map from the finite field $\mathrm{GF}(q^m)$ onto $\mathrm{GF}(q)$. We represent the points of the Desarguesian projective plane, $\mathrm{PG}(2,q)$, via homogeneous coordinates over $\mathrm{GF}(q)$. For $\alpha, \beta \in \mathrm{GF}(q)$ such that the absolute trace $\mathrm{Tr}_{q/2}(\alpha\beta) = 1$, and $\lambda \in \mathrm{GF}(q)$, define $F_{\alpha,\beta,\lambda}$ to be the conic

$$F_{\alpha,\beta,\lambda} = \{(x, y, z) : \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0\}$$

and let $\mathscr{F}$ be the union of all such conics. Note that all the conics in $\mathscr{F}$ have the point $F_0 = (0, 0, 1)$ as their nucleus.

For given $\lambda \neq \lambda'$, define a composition

$$F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'} = F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda+\lambda'}$$

where the operator $\oplus$ is defined by

$$a \oplus b = \frac{\lambda a + \lambda' b}{\lambda + \lambda'}.$$

Given some subset $\mathscr{G}$ of $\mathscr{F}$, we say $\mathscr{G}$ is closed if for every $F_{\alpha,\beta,\lambda} \neq F_{\alpha',\beta',\lambda'} \in \mathscr{G}$, we have that $F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda+\lambda'} \in \mathscr{G}$. In [7], the following theorems are proved.

Theorem 1 ([7, Theorem 2.4]). *Let $\mathscr{G}$ be a closed set of conics with nucleus F_0 in* $\mathrm{PG}(2,q)$, *q even. Then the union of the points of the conics of G together with F_0 form the points of a degree $|\mathscr{G}| + 1$ maximal arc in* $\mathrm{PG}(2,q)$.

Theorem 2 ([7, Theorem 2.5]). *Let A be an additive subgroup of* $\mathrm{GF}(q)$, *q even, with $|A| = d$. Let $p(\lambda) = \sum_{i=0}^{d-1} a_i \lambda^{2^i - 1}$ and $r(\lambda) = \sum_{i=0}^{d-1} b_i \lambda^{2^i - 1}$ be polynomials with coefficients in* $\mathrm{GF}(q)$, *q even. If* $\mathrm{Tr}_{q/2}(p(\lambda)r(\lambda)) = 1$ *for every $\lambda \in A - \{0\}$, then the union of the points of*

$$\mathscr{G} = \{F_{p(\lambda),r(\lambda),\lambda} : \lambda \in A - \{0\}\} \cup \{F_0\}$$

is a degree d maximal arc in $\mathrm{PG}(2,q)$.

Suppose we choose $\alpha \in \mathrm{GF}(q)$ such that $\mathrm{Tr}_{q/2}(\alpha) = 1$, and let A be some additive subgroup of $\mathrm{GF}(q)$. Then the set of conics

$$\{F_{\alpha,1,\lambda} : \lambda \in A - \{0\}\} \cup \{F_0\}$$

is the set of points of a degree $|A|$ maximal arc in $\mathrm{PG}(2,q)$. These maximal arcs were constructed by R. H. F. Denniston in [4]. They are a subset of the *pencil of conics* given by

$$\{F_{\alpha,1,\lambda} : \lambda \in \mathrm{GF}(q) \cup \{\infty\}\}.$$

This pencil partitions the points of the plane into the line $z = 0$, $q - 1$ non-degenerate conics, and the common nucleus $(0, 0, 1)$ of these conics. The line $z = 0$ is often called the *line at infinity* of the pencil and is denoted F_∞.

The Denniston maximal arcs were characterised by Abatangelo and Larato in [1] as exactly those maximal arcs whose point set is the union of elements of such a pencil of conics. Alternatively they were characterised as exactly those maximal arcs whose homography stabiliser admits a cyclic group of order $q + 1$ (the orbits of such a group are exactly the elements of the pencil).

More generally a pencil of conics may be obtained as follows. Suppose F_1 and F_2 are non-degenerate quadratic forms over $\mathrm{GF}(q)$ that have no common zeros, i.e. the conics that they define have no common points. Then the set of polynomials

$$\{\mu F_1 + \gamma F_2 : \mu, \gamma \in \mathrm{GF}(q),\ \mu \text{ and } \lambda \text{ not both zero}\}$$

determine $(q + 1)$ quadratic forms: $q - 1$ pairwise disjoint non-degenerate conics; an exterior line to those conics; and a point that is the nucleus of all the conics. Together these conics partition the points of the plane. Up to isomorphism in $\mathrm{P\Gamma L}(3, q)$ there is a unique such pencil, i.e. up to isomorphism the pencil is independent of the choice of F_1 and F_2. We will call a closed set of conics that is a subset of such a pencil *linear*. It follows immediately from Abatangelo and Larato's result that a linear closed set of conics corresponds to a maximal arc that is isomorphic to one of those of Denniston.

In the next section a new construction of closed sets of conics is given, and it is shown that any closed set of conics is still a closed set of conics in an odd order extension of the underlying field. It is also shown that a closed set of conics of non-Denniston type gives rise to sub-maximal arcs that are not of Denniston type. In Section 6 the geometric structure of the closed sets of conics will be considered. In particular it is shown that the only non-degenerate conics contained in the associated maximal arcs are exactly those of the closed sets of conics, and the Denniston maximal arcs are characterised as exactly those closed sets of conics whose dual maximal arcs contains a regular hyperoval. It is then shown that the maximal arcs of degree $2 < n < q/2$ arising from closed sets of conics are not isomorphic to any of the known classes of maximal arcs, except when they happen to be Denniston. In Section 3.2 the types of collineations that may stabilise a closed set of conics are considered.

2 New maximal arcs in Desarguesian projective planes

The following theorem shows that in some sense no maximal arc arising from closed sets of conics is sporadic.

Theorem 3. *Let $\mathcal{G}$ be a closed set of conics in* $\mathrm{PG}(2, q)$*. Then the equations of the conics of $\mathcal{G}$ give a closed set of conics in* $\mathrm{PG}(2, q^m)$*, for any $m \geqslant 1$, m odd.*

Proof. We first show that the equations of the conics of $\mathcal{G}$ give non-degenerate conics over $\mathrm{GF}(q^m)$. For a conic in $F_{\alpha,\beta,\lambda} \in \mathcal{G}$, the trace $\mathrm{Tr}_{q/2}(\alpha\beta)$ from $\mathrm{GF}(q)$ to $\mathrm{GF}(2)$ is the identity. Now $\mathrm{Tr}_{q^m/2}(\alpha\beta) = \mathrm{Tr}_{q/2} \circ \mathrm{Tr}_{q^m/q}(\alpha\beta)$, but since $\alpha\beta \in \mathrm{GF}(q)$, $\mathrm{Tr}_{q^m/q}(\alpha\beta) = m\alpha\beta$. Hence $\mathrm{Tr}_{q^m/2}(\alpha\beta) = \mathrm{Tr}_{q/2}(m\alpha\beta) = m\,\mathrm{Tr}_{q/2}(\alpha\beta) = 1$, since m is odd and q is even.

Hence the equation of an $F_{\alpha,\beta,\lambda} \in \mathscr{G}$ also gives a non-degenerate conic in $\mathrm{PG}(2, q^m)$. The set $\mathscr{G}$ then clearly gives a closed set of conics in $\mathrm{PG}(2, q^m)$. □

It immediately follows from the theorem that given a degree n maximal arc $\mathscr{K}$ in $\mathrm{PG}(2, q)$ arising from a closed set of conics, there exist degree n maximal arcs $\mathscr{K}_m$ in $\mathrm{PG}(2, q^m)$ for all odd positive integers m. Note that $\mathscr{K}_m$ contains $\mathscr{K}$ in the real sub-plane $\mathrm{PG}(2, q)$ of $\mathrm{PG}(2, q^m)$.

Theorem 4. *Let* $r(\lambda) = \sum_{i=0}^{m-1} b_i \lambda^{2^i - 1}$ *be any polynomial with coefficients* $b_i \in \mathrm{GF}(q^m)$ *such that* $\mathrm{Tr}_{q^m/2}(b_0) = 1$ *and for* $i > 0$, $\mathrm{Tr}_{q^m/q}(b_i) = 0$. *Then the points of*

$$\mathscr{G} = \{F_{1,r(\lambda),\lambda} : \lambda \in \mathrm{GF}(q)^*\} \cup \{F_0\}$$

form a degree q maximal arc in $\mathrm{PG}(2, q^m)$.

Proof. We show that $\mathrm{Tr}_{q^m/2}(r(\lambda)) = 1$ for every $\lambda \in \mathrm{GF}(q)$ and apply Theorem 2.

First note that the trace function is additive, hence

$$\mathrm{Tr}_{q^m/2}(r(\lambda)) = \sum_{i=0}^{m-1} \mathrm{Tr}_{q^m/2}(b_i \lambda^{2^i - 1}).$$

Now for $i > 1$, since $\lambda \in \mathrm{GF}(q)$ we have

$$\begin{aligned} \mathrm{Tr}_{q^m/2}(b_i \lambda^{2^i - 1}) &= \mathrm{Tr}_{q/2} \circ \mathrm{Tr}_{q^m/q}(b_i \lambda^{2^i - 1}) \\ &= \mathrm{Tr}_{q/2}(\lambda^{2^i - 1}\, \mathrm{Tr}_{q^m/q}(b_i)) \\ &= \mathrm{Tr}_{q/2}(\lambda^{2^i - 1}.0) = 0. \end{aligned}$$

Hence $\mathrm{Tr}_{q^m/2}(r(\lambda)) = \mathrm{Tr}_{q^m/2}(b_0) = 1$ for every $\lambda \in \mathrm{GF}(q)$. □

Note that Theorem 3 can also be applied to the maximal arcs of the theorem to get maximal arcs in odd order extensions of the plane.

In Theorem 4 there are q^{m-1} choices for each of the b_i's, $i > 0$, and $q^m - q^{m-1}$ choices for b_0. But distinct polynomials may give isomorphic maximal arcs.

In the theorem the conics defining such a maximal arc were parameterised by elements of $\mathrm{GF}(q)^*$. By taking a subset A of $\mathrm{GF}(q)^*$ such that $A \cup \{0\}$ is closed under addition we may construct a maximal arc whose conics correspond to elements of A. Hence the theorem also implies the existence of degree r maximal arcs in these planes for all r dividing n, though it may be that some of these sub-arcs are of Denniston type. But the following lemma shows that non-Denniston sub-arcs may be obtained from non-Denniston maximal arcs for $r \geqslant 8$.

Lemma 5. *Let $\mathcal{G}$ be a closed set of conics giving rise to a degree $8 \leqslant n < q/2$ maximal arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ that is not of Denniston type. Then there exist maximal arcs of degree r that are not of type Denniston in $\mathrm{PG}(2,q)$ for all $r \geqslant 8$, r dividing n.*

Proof. If we choose $F_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'} \in \mathcal{G}$, then the conics $F_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'}$ and $\mathcal{F}_{\alpha,\beta,\lambda} \oplus \mathcal{F}_{\alpha',\beta',\lambda'}$ are contained in a (unique) pencil $\mathcal{F}$ (see [7, Lemma 2.1]). Since $\mathcal{K}$ is not of Denniston type, the conics of $\mathcal{G}$ are not all contained in a single pencil of conics, so we may choose some $\mathcal{F}_{\alpha'',\beta'',\lambda''}$ not contained in $\mathcal{F}$. Taking the closure under $\oplus$ of $F_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'}$ and $\mathcal{F}_{\alpha'',\beta'',\lambda''}$ then gives rise to a maximal arc $\mathcal{K}_8$ of degree 8 whose conics are not all contained in a single pencil and so is not of Denniston type (this uses the fact proved in the next section that the partition of $\mathcal{K}$ into disjoint conics on a common nucleus is unique). Non-Denniston maximal arcs of degree r, for all $r \geqslant 8$, r dividing n, may then be obtained by extending the set of conics of $\mathcal{K}_8$ by other elements of $\mathcal{G}$. □

3 The structure of the maximal arcs

3.1 Conics and substructures within the maximal arcs.

Theorem 6. *Let $\mathcal{K}$ be a degree $n < q/2$ maximal arc in $\mathrm{PG}(2,q)$ constructed from a closed set of conics $\mathcal{G}$ with nucleus F_0. Then the point set of $\mathcal{K}$ contains no non-degenerate conics apart from those of $\mathcal{G}$.*

Proof. Suppose $\mathcal{K}$ does contain some conic C other than those of $\mathcal{G}$. Since five points of a non-degenerate conic determine a unique non-degenerate conic, C meets each of $n-1$ conics of $\mathcal{G}$ in at most four points. One of the points of C may be the nucleus F_0 of the other conics. Hence we require that $4(n-1) \geqslant q$, and so $n \geqslant q/4 + 1$. Since n divides q, this implies $n \geqslant q/2$, and the result is proved. □

The case $n = q/2$ really is an exception to the theorem. For instance, the dual of a regular hyperoval may be thought of as a degree $q/2$ maximal arc of Denniston type. But the fact that the collineation stabiliser of the regular hyperoval is transitive on exterior lines shows that the partition of the degree $q/2$ maximal arc into conics on a common nucleus is not unique.

Corollary 1. *Let $\mathcal{K}$ be a degree $n < q/2$ maximal arc in $\mathrm{PG}(2,q)$ constructed from a closed set of conics $\mathcal{G}$ with nucleus F_0. Then any element of the collineation stabiliser of $\mathcal{K}$ permutes the conics of $\mathcal{G}$.*

Proof. A conic in $\mathcal{K}$ must be mapped to a conic contained in $\mathcal{K}$. □

The corollary will be useful in the next subsection when the collineation stabilisers of the maximal arcs are examined.

The following theorem gives an easy geometric characterisation of the Denniston maximal arcs.

Theorem 7. *Let $\mathscr{K}$ be a degree n maximal arc in* $\mathrm{PG}(2,q)$ *constructed from a closed set of conics $\mathscr{G}$ with nucleus F_0. Then $\mathscr{K}$ is of Denniston type if and only if its dual contains a regular hyperoval.*

Proof. Let $\mathscr{K}$ be as in the statement of the theorem. Suppose $\mathscr{K}$ is of type Denniston. Then it is stabilised by a cyclic group of order $q+1$ whose orbits on points are a union of disjoint conics with common nucleus and exterior line. The action on the lines of the plane is the same in that orbits on the lines form a pencil of the same form in the dual plane (see for instance [6]). Hence the orbit of any non-fixed line that is exterior to $\mathscr{K}$ together with the fixed line forms a regular hyperoval in the dual plane. By definition this hyperoval is contained in the dual of $\mathscr{K}$.

Conversely, suppose that the dual maximal arc $\mathscr{K}^D$ of $\mathscr{K}$ does contain a regular hyperoval $\mathscr{H}$ with nucleus N. We show that the conics of $\mathscr{K}$ are contained in a pencil of the required form, and hence $\mathscr{K}$ is of Denniston type.

First notice that since $\mathscr{H} \subset \mathscr{K}^D$ the dual maximal arc $\mathscr{H}^D$ of $\mathscr{H}$ contains $\mathscr{K}$. We show that $\mathscr{H}^D$ can be partitioned into a set of $q/2-1$ conics in a linear pencil with common nucleus F_0, i.e. is of Denniston type.

As noted above a Denniston maximal arc admits a cyclic group of order $q+1$ in its collineation stabiliser, and admitting such a cyclic group characterises the Denniston maximal arcs. The orbits of this group are the points of $q-1$ conics, a single fixed point which is the nucleus of the conics, and the points on a single line. This group has an identical orbit structure in the dual plane, and so the dual of a Denniston maximal arc is also of Denniston type. Now a regular hyperoval admits a collineation stabiliser that is transitive on lines exterior to the hyperoval, and contains cyclic subgroups of order $q+1$. Hence for any exterior line l to a regular hyperoval that we might choose, there exists a cyclic subgroup of order $q+1$ stabilising the hyperoval, and fixing l. It then follows that the dual of a regular hyperoval may be thought of as a degree $q/2$ Denniston maximal arc, all of whose conics have common nucleus, namely the point of the dual plane corresponding to the line l.

Now the point F_0 corresponds to an external line to $\mathscr{H}$ in the dual plane. So choose the cyclic group of order $q+1$ fixing $\mathscr{H}$ and the line F_0 in the dual plane. The dual of $\mathscr{H}$ is then partitioned into $q/2-1$ conics on common nucleus F_0 and contains $\mathscr{K}$; let $\mathscr{C}_{q/2-1}$ denote the set of these $q/2-1$ conics.

If all of the conics of $\mathscr{K}$ are in $\mathscr{C}_{q/2-1}$, then $\mathscr{K}$ is Denniston and we are done. So suppose some conic $\mathscr{C}$ of $\mathscr{K}$ is not in $\mathscr{C}_{q/2-1}$. Then a nucleus and three points (forming a quadrangle) determine a unique conic on that nucleus. The conics of $\mathscr{C}_{q/2-1}$ and $\mathscr{C}$ all share the nucleus F_0 and so $\mathscr{C}$ meets each conic of $\mathscr{C}_{q/2-1}$ in at most two points. But $\mathscr{C}$ has $q+1 > 2(q/2-1)$ points, giving a contradiction. Hence the only conics contained in $\mathscr{K}$ are some subset of $\mathscr{C}_{q/2-1}$, and $\mathscr{K}$ is Denniston. □

The following corollary answers a question posed in [7].

Corollary 2. *The dual of a non-Denniston maximal arc constructed from a closed set of conics cannot be constructed from a closed set of conics.*

Hence a non-Denniston closed set of conics always gives two non-isomorphic maximal arcs. The case for Dennistons is quite different; the dual of a Denniston maximal arc is always a Denniston maximal arc since both the maximal arc and its dual admit a cyclic group of order $q+1$. In the case that q is a square and the Denniston maximal arc has degree $\sqrt{q}$, the dual is often isomorphic to the original maximal arc. In fact computer calculations based on the results contained in [6] show that the smallest order plane in which a degree $\sqrt{q}$ Denniston maximal arc is not isomorphic to its dual is $\mathrm{PG}(2,256)$. There take the pencil $F_\lambda : x^2 + \alpha xy + y^2 + \lambda z^2 = 0$ where α is a fixed element of $\mathrm{GF}(256)$ such that the quadratic polynomial $\xi^2 + \alpha\xi + 1$ is irreducible, and let

$$\lambda \in \{0, 1, \omega, \omega^2, \omega^{33}, \omega^5, \omega^{36}, \omega^{101}, \omega^{15}, \omega^{47}, \omega^{50}, \omega^{225}, \omega^{138}, \omega^{198}, \omega^{25}, \omega^{26}\}$$

where ω is a generator of $\mathrm{GF}(256)^*$ and satisfies $\omega^{25} + \omega = 1$. Then this describes a degree 16 maximal arcs in $\mathrm{PG}(2,256)$ that is not isomorphic to its dual.

Apart from maximal arcs arising from closed sets of conics the known constructions of degree n maximal arcs in $\mathrm{PG}(2,q)$ are hyperovals ($n=2$) and their duals ($n=q/2$), the construction of Denniston [4], and two constructions of J. A. Thas [8], [9]. The maximal arcs constructed in [9] were shown in [6] to be of Denniston type in $\mathrm{PG}(2,q)$. Those in [8] are sometimes known as Thas '74 maximal arcs and are of two types. The first uses a spread of tangent lines to an elliptic quadric in $\mathrm{PG}(3,q)$ and gives rise to degree q maximal arcs in $\mathrm{PG}(2,q^2)$ for all even q. These were also shown in [6] to be of Denniston type. The second uses a spread of tangent lines to a Tits' ovoid in $\mathrm{PG}(3,q)$ and gives rise to degree q maximal arcs in $\mathrm{PG}(2,q^2)$ for all $q = 2^{2h+1}$, $h \geqslant 1$. It was shown in [6] that up to isomorphism there are two such maximal arcs in $\mathrm{PG}(2,q^2)$ arising using the Tits' ovoid and that these maximal arcs were not of Denniston type. The collineation stabilisers of the maximal arcs were also calculated and shown to have distinct orders. The Thas '74 maximal arcs were characterised in [5] as exactly those degree q maximal arcs in $\mathrm{PG}(2,q^2)$ stabilised by an homology of order $q-1$. It follows that the duals of the Thas '74 maximal arcs are also of Thas '74 type, and since the order of the collineation stabiliser of the Thas '74 maximal arc uniquely identifies that maximal arc it follows that the Thas '74 maximal arcs are isomorphic to their dual maximal arcs. The following corollary is then an immediate consequence of this and the previous corollary.

Corollary 3. *The Thas* '74 *maximal arcs arising from a spread of tangent lines to a Tits' ovoid cannot be constructed from a closed set of conics.*

We conclude this section by showing that a non-linear closed set of conics cannot contain pairs of "large" linear closed sets of conics.

Theorem 8. *Let $\mathscr{K}$ be a degree n maximal arc in* $\mathrm{PG}(2,q)$ *arising from a closed set of conics. Suppose $\mathscr{K}$ contains two maximal arcs $\mathscr{K}_1$ and $\mathscr{K}_2$ of degrees $n_1 \geqslant 4$ and $n_2 \geqslant 4$, that are of Denniston type and are contained in distinct pencils, then $n_1 n_2 \leqslant 2n$.*

Proof. Define an incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ as follows. The points $\mathscr{P}$ are the $n-1$ conics of the closed set of conics of $\mathscr{K}$. The blocks are the sets of the form $\{C_1, C_2, C_1 \oplus C_2\}$ for $C_1, C_2 \in \mathscr{P}$, $C_1 \neq C_2$, and incidence is containment. Then it is easily shown that the incidence structure is a $2-(n-1,3,1)$ design. We show that it is a projective space of dimension $(\log_2 n) - 1$ over GF(2) by showing that it satisfies the axiom of Pasch.

For any distinct conics C_1, C_2 and C_3 in a closed set, it is readily verified that $(C_1 \oplus C_2) \oplus (C_3 \oplus C_2) = C_1 \oplus C_3$. Also note that $\oplus$ is commutative. Let $\{C_1, C_2, C_3\}$ and $\{C_1, C_2', C_3'\}$ be distinct blocks on C_1. Then $C_2 \oplus C_3' = (C_3 \oplus C_1) \oplus (C_2' \oplus C_1) = C_3 \oplus C_2'$. Hence the block joining C_2 to C_3' meets the block joining C_3 to C_2' in a point of the set $\mathscr{P}$, and so the design is a projective space $\mathrm{PG}((\log_2 n) - 1, 2)$ [3, Section 1.4].

Notice that any closed subset of conics of $\mathscr{K}$ corresponds to a subspace of $\mathrm{PG}((\log_2 n) - 1, 2)$. Hence $\mathscr{K}_1$ and $\mathscr{K}_2$ correspond to subspaces of dimension $(\log_2 n_1) - 1$ and $(\log_2 n_2) - 1$ respectively. Now a closed triple of conics determines a unique pencil of conics, and so for $\mathscr{K}_1$ and $\mathscr{K}_2$ to be contained in distinct pencils, their corresponding subspaces must meet in at most a point in $\mathrm{PG}((\log_2 n) - 1, 2)$. Hence if m is the dimension of the span of the subspaces corresponding to $\mathscr{K}_1$ and $\mathscr{K}_2$, we require that the dimension of intersection $(\log_2 n_1) - 1 + (\log_2 n_2) - 1 - m$ is at most 0. But m is at most $(\log_2 n) - 1$. Hence we get the condition that $(\log_2 n_1) - 1 + (\log_2 n_2) - 1 - (\log_2 n) + 1 \leqslant 0$, i.e. $n_1 n_2 \leqslant 2n$. □

Corollary 4. *A degree $n \geqslant 16$ maximal arc arising from a non-Denniston closed set of conics contains at most one degree $n/2$ Denniston maximal arc.*

Proof. Suppose a degree n maximal arc arising from a non-Denniston closed set of conics contains two degree $n/2$ Denniston maximal arcs. Then these maximal arcs must be in distinct pencils. Applying the theorem then gives $\frac{n}{2}\frac{n}{2} \leqslant 2n$ and so $n \leqslant 8$. □

The bound given in the corollary can be tight. Seven of the degree 16 maximal arcs in PG(2, 64) constructed in [7] are shown there to contain unique degree 8 Denniston maximal arcs. Taking the odd order extensions of these maximal arcs via Theorem 3 will also give further examples.

3.2 Collineations of the maximal arcs. In [6], the collineation stabilisers of the Denniston maximal arcs were calculated in terms of the additive subgroups of $\mathrm{GF}(q)$ that defined them. In this section we examine the collineation stabilisers of closed sets of conics that are not of Denniston type. The general problem of calculating the collineation stabilisers seems at present to be intractable, but results may be obtained for particular cases.

In the previous subsection we saw that for $n < q/2$, the only conics contained within a maximal arc constructed from a closed set of conics were those of the closed set of conics. We use this fact to prove results about the collineation stabilisers.

Theorem 9. *Let $\mathscr{K}$ be a non-Denniston maximal arc of degree $n \neq q/2$ in* $\mathrm{PG}(2, q)$

arising from a closed set of conics. Then there is at most one non-identity element of $\mathrm{PGL}(3,q)_{\mathscr{K}}$ *that fixes each of the conics of the closed set. Hence the subgroup of* $\mathrm{P\Gamma L}(3,q)$ *fixing each of the conics of the closed set has order dividing* $2h$, *where* $q = 2^h$.

Proof. Suppose that $\mathscr{K}$ contains the three conics $C_1, C_2, C_1 \oplus C_2$ in some linear pencil $\mathscr{F}$, and where $C_1 \oplus C_2$ is the composition of C_1 and C_2 as defined in the introduction. Then it follows that the union of the conics of $\mathscr{K}$ in $\mathscr{F}$ with the nucleus is the set of points of a degree at least 4 maximal arc $\mathscr{K}_{\mathrm{sub}}$ that is of type Denniston. Now since $\mathscr{K}$ is not of Denniston type there exists some conic C' of $\mathscr{K}$ that is not contained in $\mathscr{F}$. The closed set of conics $C_1, C', C_1 \oplus C'$ is in some linear pencil $\mathscr{F}'$, and the conics of $\mathscr{F}'$ that are contained in $\mathscr{K}$ together with the nucleus again determine a maximal arc $\mathscr{K}'_{\mathrm{sub}}$ of Denniston type.

We first show that the line at infinity of the pencil $\mathscr{F}$ is distinct from the line at infinity of $\mathscr{F}'$. Suppose we have some non-degenerate conic with quadratic form F_C, and suppose there is a line l exterior to the conic and whose points are the zeros of some (degenerate) quadric F_l. Then, just as two non-intersecting conics, taking $\mathrm{GF}(q)$ linear combinations of F_C and F_l determines a unique pencil of conics. Since $\mathscr{F} \neq \mathscr{F}'$ share the conic C_1, it follows that the line at infinity of $\mathscr{F}$ must be distinct from that of $\mathscr{F}'$.

The full collineation stabiliser of a pencil of conics of the form we are interested in is isomorphic to $\mathrm{GF}(q^2)^* \rtimes \mathrm{Aut}(\mathrm{GF}(q^2))$, and the homography stabiliser is isomorphic to $\mathrm{GF}(q^2)^* \rtimes C_2$, where C_2 is an elation group of order 2 (see [6]). In [6] it is shown that the homography/collineation stabiliser of any Denniston maximal arc of degree n, $2 < n < q/2$, is a subgroup of such a group. It is also shown that if you fix each of the conics in the Denniston maximal arc then this homography stabiliser is isomorphic to the semidirect product $C_{q+1} \rtimes C_2$ of a cyclic group, C_{q+1} of order $q+1$, whose orbits are the elements of the pencil with an elation of order 2. Now every non-trivial element of C_{q+1} fixes a unique line, that line being the line at infinity of the pencil. So such an element cannot fix both the line at infinity of $\mathscr{F}$ and $\mathscr{F}'$. It follows that no non-trivial element of the homography stabiliser of $\mathscr{F}$ can stabilise each of the elements of $\mathscr{F}'$, apart from possibly an element of order 2. □

Note that in the proof the elation group C_2 stabilising a (sub) Denniston maximal arc has centre on the line at infinity of that maximal arc, and axis the line joining the nucleus to the centre. Hence for such an element to stabilise the non-Denniston maximal arc, the lines at infinity of all the sub maximal arcs must be concurrent.

We now examine the case where the collineation stabiliser of a closed set of conics is known to be a subgroup of some Denniston maximal arc that it contains.

Theorem 10. *Let* $\mathscr{F}$ *be a non-linear closed set of conics in* $\mathrm{PG}(2,2^h)$ *and* $\mathscr{K}$ *the associated maximal arc. Let* $\mathscr{F}_{\mathrm{sub}}$ *be a linear closed subset of* $\mathscr{F}$ *and let* $\mathscr{K}_{\mathrm{sub}}$ *be the associated maximal arc, where* $|\mathscr{F}_{\mathrm{sub}}|$ *and* $|\mathscr{F}|$ *are coprime,* $|\mathscr{F}_{\mathrm{sub}}| \geqslant 4$. *Suppose that the collineation stabiliser* $\mathrm{P\Gamma L}(3,q)_{\mathscr{K}}$ *of* $\mathscr{K}$ *fixes* $\mathscr{K}_{\mathrm{sub}}$, *then* $|\mathrm{P\Gamma L}(3,q)_{\mathscr{K}}|$ *divides* $2h$.

Proof. Let $\mathscr{F}, \mathscr{K}, \mathscr{F}_{\text{sub}}$ and $\mathscr{K}_{\text{sub}}$ be as in the statement of the theorem. Since $\mathscr{K}_{\text{sub}}$ is fixed by $\mathrm{P\Gamma L}(3,q)_{\mathscr{K}}$ and so by $\mathrm{PGL}(3,q)_{\mathscr{K}}$, it follows as in the previous theorem that $\mathrm{PGL}(3,q)_{\mathscr{K}}$ is a subgroup of the stabiliser of the Denniston maximal arc $\mathscr{K}$, and so is isomorphic to a group of the form $\mathrm{GF}(q^2)^* \rtimes C_2$. Now $\mathrm{GF}(q^2)^* \rtimes C_2$ is isomorphic to $G = (C_{q+1} \times C_{q-1}) \rtimes C_2$ where C_{q+1} is as in the proof of the previous theorem and C_{q-1} is a (cyclic) homology group of order $q-1$. In [6] it is shown that the homology group has centre the nucleus of $\mathscr{K}_{\text{sub}}$ and axis the line at infinity of the pencil associated with $\mathscr{K}_{\text{sub}}$.

Let g be an non-identity element of $\mathrm{PGL}(3,q)_{\mathscr{K}}$. There are two cases to consider.

(i) $g \in C_{q+1} \times C_{q-1}$, $g \neq 1$. Suppose there exists i such that $g^i \in C_{q-1}$, $g^i \neq 1$. Now g^i stabilises $\mathscr{K}$. The orbits of non-fixed points of g^i all have the same length, l say. On a line through the nucleus there are $|\mathscr{F}|$ points of $\mathscr{K}$, hence l divides $|\mathscr{F}|$. But g^i also stabilises $\mathscr{K}_{\text{sub}}$ and so l divides $|\mathscr{F}_{\text{sub}}|$. This contradicts our assumption that $|\mathscr{F}|$ and $|\mathscr{F}_{\text{sub}}|$ are coprime. Hence there does not exist i with $g^i \in C_{q-1}$ and $g^i \neq 1$. Since $q+1$ and $q-1$ are coprime it follows that $g \in C_{q+1}$.

Suppose $g \neq 1$. Now as previously mentioned the orbits of C_{q+1} are the conics of the pencil that make up $\mathscr{K}_{\text{sub}}$, together with the nucleus and the line at infinity. So the orbits of the group generated by g not on the nucleus or the line at infinity are subsets of conics of size at least three. But three such points and the nucleus determine a unique conic, and this conic is an orbit of C_{q+1}. Hence since g stabilises $\mathscr{K}$ it follows that $\mathscr{K}$ is a union of conics of the pencil determined by C_{q+1}. Applying Abatangelo and Larato's characterisation gives that $\mathscr{K}$ is then Denniston, contradicting our hypothesis. Hence $g = 1$.

(ii) $g \notin C_{q+1} \times C_{q-1}$. Then $g^2 \in C_{q+1} \times C_{q-1}$ and arguing as in (i) gives that $g^2 = 1$. So g has order 2 and so is an elation. Suppose two distinct elations in G stabilise both $\mathscr{K}$ and $\mathscr{K}_{\text{sub}}$. Then it is easily shown that their product is in $C_{q+1} \times C_{q-1}$ and is the identity if and only if they are equal. It follows that at most one elation in G fixes $\mathscr{K}$ and $\mathscr{K}_{\text{sub}}$. Hence the homography stabiliser of $\mathscr{K}$ has order at most 2, and the collineation stabiliser has order at most $2h$. □

As mentioned at the end of the previous section, seven of non-Denniston degree 16 maximal arcs in $\mathrm{PG}(2,64)$ given in [7] contain a unique Denniston degree 8 maximal arc. Hence they have collineation stabilisers of order dividing 12. In fact in the paper it is shown that five of them have collineation stabiliser of order 4, and three have collineation stabiliser of order 2. But the above theorem and the details of the proof explain the structure of the groups.

Finally, taking extensions of maximal arcs satisfying this theorem via Theorem 3 will also give maximal arcs that also have small order collineation stabilisers.

4 Conclusion

In this paper and in [7] several constructions of closed sets of conics that are non-linear, i.e. give rise to maximal arcs which are not of Denniston type, have been given. The largest known maximal arcs that are part of a class are of degree 2^{m+1} in $\mathrm{GF}(2^{2m})$ ([7, Theorem 3.3]). It would be interesting to know what the largest n is such that in

PG$(2, q)$ there exists a non-linear closed set of conics giving rise to a maximal arc of degree n.

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