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**Multipartite GRAPH DECOMPOSITION:  
CYCLES AND TRAILS CLOSED**

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This paper surveys results on cycle decompositions of complete multipartite graphs (where the parts are not all of one size, so the graph is not  $K_n$ ). That the houses in the cycle lengths are "small". Cycles up to length  $n$  are considered. When the complete multipartite graph has  $n$  parts, but not hamilton cycles. Properties Which the decompositions may have, such as being gregarious, are anche Mentioned.

**1. Introduction and definitions.**

A great deal of work done on Has Been edge-disjoint decompositions of full of graphs and complete multipartite graphs, where the decomposition is into isomorphic copies of some "small" graph  $G$ . This graph  $G$  may be itself a complete graph  $K_k$  - In which case the complete graphs or decompositionsof of complete multipartite graphs are, rispettivamente, a balanced incomplete block design (of index 1), or a group divisible design with block size  $k$ . A vast amount of work has anche Been Carried Out When the "small" graph  $G$  is a cycle - again Both in cases, When the graph being decomposed is either complete, or complete multipartite. Furthermore, some results anche includes decompositions into closed trails rather than cycles.

In this paper I shall concentrate on reviewing the case of a complete multipartite graph When the decompositionis into copies of a fixed length cycle;

anche certain decompositions into closed trails are Considered here. Particular Which properties to such decomposition may have anche will be considered.

Let us begin with some basic definitions, Which knowledgeable reader may skip over!

A complete multipartite graph  $G = K(a_1, a_2, \dots, a_n)$  Has its vertices No matches grouped into sets, of sizes  $a_1, \dots, a_n$ . There is an edge between any two vertices in different games sets, but no edge between any two vertices in the same games in September. For each  $i$ ,  $1 \leq i \leq n$ , we write  $K_n(m)$  and refer to this graph as the complete *equipartite* graph having  $n$  parts of size  $m$ . Of course if  $m=1$  for all  $i$ , then  $G$  is the complete graph  $K_n$ .

A  $k$ -cycle, written  $(x_1, x_2, \dots, x_k)$  Consists of  $k$  distinct vertices  $x_1, \dots, x_k$ . And  $k$  edges  $\{x_i, x_{i+1}\}$ ,  $1 \leq i \leq k-1$ , and  $\{x_k, x_1\}$ . A  $k$ -cycle system of a simple graph  $G$  is an edge-disjoint decomposition of  $G$  into copies of  $k$ -cycles; equivalently, it can be regarded as a partition of the edge September  $E(G)$  into  $k$ -cycles.

A  $k$ -trail is a closed path of length  $k$ , where the vertices of the trail are not Necessarily distinct. Of course for  $k = 3, 4, 5$ , a  $k$  to  $k$ -trail is anche -cycle, but When  $k \geq 6$  This Is not Necessarily so; for instance 6-trail could be a *bowtie* (That is, two triangles with a common vertex).

From a design-theoretic perspective, an edge-disjoint the decomposition of complete multipartite graph  $K(a_1, a_2, \dots, a_n)$  Into  $k$ -cycles can be regarded as a group-divisible  $k$ -cycle system or design  $(C_k\text{-GDD})$ , With  $n$  groups, of sizes  $a_i$ ,  $1 \leq i \leq n$ . And in the houses  $k = 3$ , When a 3-cycle is anche to block or to triple, we can talk about at 3-GDD of type  $a_1 a_2 \dots a_n$ . Necessary and sufficient conditions for existence of a 3-GDD of arbitrary type  $a_1 a_2 \dots a_n$  are not known, Although in the uniform houses (of type  $n^n$ ) They are, and anche in a very few 'Almost' uniform cases; see Section 2.1.

In the following, Section 2 surveys existence results on comprehensive multi-matches graph decompositions into cycles (including 3-cycles of Which are anche Complete course  $K_3$  blocks), and into closed trails. In Section 3, three extra properties are Considered: *resolvable* cycle decompositions, *colored* cycle described compositions, and so-called *gregarious* cycle decompositions of complete multipartite graphs. Section 4 mentions some work Which Has Been done on *packing* complete multipartite graphs with cycles, When to complete edge-disjoint decomposition is not possible, and the final section includes some open problems. To limit the content, the part sizes in the graph  $K(a_1, \dots, a_n)$  Will not all be one, and so we do not deal here with decompositions of  $K_n$ . Moreover, for actual proofs, the reader is Referred to the original papers.

## 2. Existence results.

### 2.1. 3-cycles, many parts

Since 3-cycle is bipartite Certainly not, any complete multipartite graph  $K$  having an edge-disjoint decomposition into 3-cycles must have Necessarily at least three parts. It is well-known That if the graph  $K$  has three PRECISELY parts, then to decomposition into 3-cycles exists if and only if These three parts are all of the same size. Indeed, it is well-known That such a decomposition of  $K_{n, n, n}$  into 3-cycles is equivalent to the existence of a Latin square  $L$  of order  $n$ : index the vertices in the three parts by the rows, the columns and the entries of  $L$ ; then each filled cell in  $L$  Corresponds to one triangle (or 3-cycle) in the decomposition of  $K_{n, n, n}$ .

In the case of  $K_{n(m)}$  With parts of size  $n$   $m$ , Hanani [28] gave Necessary and sufficient conditions for existence of a decomposition into 3-cycles; he essentially proved the following.

**Theorem 1.** (Hanani [28]) *There is a 3-cycle decomposition of  $K_{n(m)}$  if and only if  $n \geq 3$ , the degree  $m(n-1)$  of any vertex is even, and the number of edges  $\frac{2}{3}n^2m$  is divisible by 3.*

A blackberries accessible one page proof of this Appears as Theorem 3.4 in Colbourn and Pink's "Triple Systems" [21].

When the size of parts in the complete multipartite graph are allowed to differ, there are a very few papers in the case of three block size; the general situation of a 3-cycle decomposition of  $K_{(n_1, n_2, \dots, n_r)}$  Remains open. In 1992 Colbourn, Hoffman and Rees [19] Showed That the "obvious" Necessary conditions for existence of a 3-cycle decomposition of  $K_{(a, b)}$  The complete multipartite graph with  $n$  groups of size  $a$  and  $b$  one of size, are always sufficient:

**Theorem 2.** (Colbourn, Hoffman and Rees [19]) *Let the edge-set of  $K_{(a, b)}$  not-empty. Then there is a 3-cycle decomposition of  $K_{(a, b)}$  if and only if*

- (I) *When  $n = 2$  there are only three parts, and so  $a = b$ ;*
- (Ii) *in  $b \leq (n - 1)$ ;*
- (Iii) *any vertex in a part of size  $a$  has even degree, ie  $(n - 1) + b$  is even;*
- (Iv) *any vertex in the part of size  $b$  has even degree, ie  $n$  is even;*
- (V) *the number of edges,  $\frac{2}{3}n^2 + Anb$ , is divisible by 3.*

Here condition (ii) is Necessary Because a vertex  $x$  in a part of size  $a$  must be in a 3-cycle with each of the vertices in the part of size  $b$ , and there are at most  $(n - 1)$  other vertices to complete this 3-cycle containing vertex  $x$ . Of course the difficult part is showing sufficiency of These conditions!

The second result on 3-cycle decompositions of complete multipartite graphs having different sized parts Appeared in 1995 (Colbourn, Cusack and Kreher [18]). This paper [18] deals with two different part sizes, and  $a, t \geq 1$ , so with the graph  $K_n(a, t)$  having  $n$  parts of size  $a$  and  $t$  parts of size 1. (From a design theoretic perspective, this can be regarded as a complete graph on  $na + t$  points, having  $n$  holes of size  $a$ .) Again, the "obvious" Necessary conditions are shown to be sufficient.

**Theorem 3.** (Colbourn, Cusack and Kreher [18]) *There is a 3-cycle decomposition of  $K_n(a, t)$  where  $a, n, t$  are positive integers, if and only if*

- (I)  $n$  is odd, and  $n + t$  is odd;
- (ii) if  $n = 1$  then  $t - 1 \geq a$ ;
- (iii) if  $n = 2$  then  $t \geq a$ ;
- (iv) the total number of edges,  $\binom{n}{2}at + Nat + \binom{t}{2}$ , Is divisible by 3.

The necessity here is straightforward: (i) follows Because every vertex must have even degree; (ii) follows Because each vertex in a part of one size must be in a 3-cycle with a vertex in the part of size  $a$ ; for (iii) Consider a vertex  $x$  in one part of  $a$  size - this must Appear in 3-cycles with  $a$  vertices in the Second part of  $a$  size, and to complete These 3-cycles we require  $t \geq a$ . Again, sufficiency Which Proves it is difficult!

Colbourn [17] gives six Necessary conditions for a 3-cycle decomposition of  $K(n_1, n_2, \dots, n_r)$ , And shows sufficiency for order at most 60. The fourth condition listed in [17] has the Following Consequence for the homes of four groups:

**Lemma 1.** *3-cycle decomposition of  $K(n_1, n_2, n_3, n_4)$ , With  $a_1 \geq a_2 \geq a_3 \geq a_4$ , Can not exist Unless at least three of the parts have the same size.*

A simple direct proof follows from considering the four types of triples (Three-cycles). Let  $\alpha$  denote the vertices in size  $n_1$  to the part of size  $n_2$ . If there are  $\alpha$  triples Which Miss Part  $A_4$ ,  $\beta$  triples Which Miss Part  $A_3$ ,  $\gamma$  triples Which Miss Part  $A_2$  and  $\delta$  triples Which Miss Part  $A_1$ , Then

$$\begin{array}{ll} \alpha + \beta = a_1 n_2 & \gamma + \delta = a_1 n_4 \\ \alpha + \delta = a_1 n_3 & \beta + \gamma = a_1 n_4 \\ \alpha + \gamma = a_2 n_3 & \beta + \delta = a_1 n_4 \end{array}$$

These imply That  $\alpha n_2 + \beta n_4 = \alpha n_3 + \beta n_4 = \gamma n_3 + \delta n_4 = \gamma n_4 + \delta n_4$ , And so

$$(n_1 - n_4)(n_2 - n_3) = 0 \text{ and } (a_1 - n_3)(n_2 - n_4) = 0.$$

I know  $t_1 = a_4$  or  $t_2 = a_3$ , And in another  $t_1 = a_3$  or  $t_2 = a_4$ . Hence at least three of the parts have the same size.

Work in the paper [20] by Colbourn et al. includes results (couched in the language of triple systems with holes) on  $K(a_1, t_2, 1, 1, \dots, 1)$  in certain cases, and a recent paper [11] by Bryant and Horsley nicely completes the determination of sufficiency of the conditions for existence of a three-cycle system of  $K(a_1, t_2, 1, 1, \dots, 1)$ . These are That the degrees are all even, the number of edges is a multiple of 3, and there are parts of sufficient size to enable 1 completion of  $a_1$  triangles from a point in the part of size  $t_2$  joining each of the points in the part of size  $t_1$ , Where  $t_1 \geq a_2$  (That is, there are at least  $a_1$  parts of size 1).

Other general results for 3-cycles (or  $K_3$  decompositions) for arbitrary complete multipartite graphs remain open.

**2.2. Even length cycles**

An oft-quoted paper of Dominique Sotteau's [41] deals with complete bipartite graph decompositions, into cycles of some (even Necessarily) fixed length. She Showed:

**Theorem 4.** (Sotteau [41]) *The complete bipartite graph  $K_{a,b}$  can be decomposed into cycles of length  $2k$  if and only if  $a$  and  $b$  are even,  $a \geq k, b \geq k$ , and  $2k$  divides  $ab$ .*

No such general result is known When the complete multipartite graph has blackberries than two parts, even When the restriction is to cycles of even length. In [14], and Cavenagh Billington list certain Necessary conditions for a  $2k$ -cycle decomposition of the complete multipartite graph  $K(t_1, t_2, \dots, t_n)$  To exist, and show sufficiency for 4-, 6- and 8-cycles. If we let  $K(t_1, t_2, \dots, t_n)$  Satisfy  $t_1 \geq a_2 \geq \dots \geq t_n$ , Then These conditions are (see [14], p. 50):

- (I)  $2k$  must divide the total number of edges;
- (ii) the total number of vertices is at least  $2k$ ;
- (Iii) half the degree of a vertex in the smallest sized part,  $\frac{1}{2} \sum_{i=1}^{n-1} t_i$  Must be less than or equal to the total number of cycles (Which of course is the total Number of edges divided by  $2k$ );  $i k \sum_{i=1}^{n-1} t_i \leq \frac{t_1 t_2 \dots t_n}{2k}$   $1 \leq i < j \leq n$ ;
- (Iv) the number of vertices *not* in the largest part must be at least  $k$ ,  $i \sum_{i=1}^n t_i \geq k$ ;
- (V) the degree of each vertex is even;
- (Vi) *in the*  $t_i$  are all of the same parity; if this parity is odd, then  $n$  is odd.

Condition (iv) above is not independent, but is implied by other conditions, and (v) and (vi) are in fact equivalent. However (iii) above is independent of the

other conditions listed. This is illustrated by the complete multipartite graph  $K(6, 6, 2)$  Which fails only condition (iii) in the case  $2k = 12$ . (There is no 12-cycle decomposition of  $K(6, 6, 2)$ .)

Any graph  $K(n_1, n_2, \dots, n_r)$  Satisfying conditions (i) - (vi) above is called  $2k$ -sufficient. It is easy to see That if each  $n_i$  is odd, then  $n \equiv 1 \pmod{4}$  if  $k$  is odd and  $n \equiv 1 \pmod{8}$  if  $k$  is even.

A  $2k$ -small graph  $K(n_1, \dots, n_r)$  Is a  $2k$ -sufficient graph such That for all  $i$ ,  $1 \leq i \leq r$ , there is no positive integer  $t_i$  with

$$t_i \begin{cases} \equiv t_i \pmod{2k} & \text{if } k \text{ is odd} \\ \equiv t_i \pmod{k} & \text{if } k \text{ is even} \end{cases}$$

That such  $K(n_1, \dots, n_{i-1}, t_i, n_{i+1}, \dots, n_r)$  Is anche  $2k$ -sufficient. Existence of a decomposition of any such -small graph into  $2k$ -cycles would suffice, since if  $K(n_1, \dots, n_r)$  Has an edge-disjoint decomposition into  $2k$ -cycles, then so does  $K(n_1, \dots, n_{i-1}, t_i + k, n_{i+1}, \dots, n_r)$ , And if  $k$  is even, so I do  $K(n_1, \dots, n_{i-1}, t_i + 2, n_{i+1}, \dots, n_r)$ , For any  $i$ ,  $1 \leq i \leq r$ .

However,  $2k$ -small graphs can be quite large! For instance,  $K(180, 20, 2)$  is 40-small (the graph  $K(160, 20, 2)$  fails (iii) above).

If one could find  $2k$ -cycle decompositions of all -small graphs  $2k, k \geq 2$  10, this would show sufficiency of conditions (i) - (vi) above for a  $2k$ -cycle decomposition of  $K(n_1, \dots, n_r)$  To exist.

**2.3. Odd length cycles**

Very little is known about decompositions of to complete multipartite graph  $K(n_1, \dots, n_r)$  Into -cycles  $k$  where  $k$  is odd. For 3-cycles, see Subsection 2.1 above. For 5-cycles, even the tripartite graph homes remains open (see the next section), although the equipartite case was solved in [8], along with the  $\lambda$ -fold equipartite graph houses. In two recent preprints [37], [38], decompositions in the case equipartite  $K_n(M)$  for 7-cycles and anche for -cycles  $p$ , where  $p \geq 11$  is a first, are given, When The obvious Necessary conditions hold.

Returning to the non-equipartite homes, there are some partial results for 5-cycle decompositions of complete tripartite graphs, Which we Consider in the next section.

**2.4. Tripartite graph decompositions: cycles**

Suppose the three parts of the tripartite graph under consideration have the same size. Consider the graph  $K_{n, n, n}$ . Cavenagh [12] gave Necessary sufficient conditions for  $K_{n, n, n}$  to have an edge-disjoint decomposition into  $k$ -cycles. These conditions are the obvious necessary ones: the number of vertices,  $3n$ , must be at least  $k$ , and the number of edges,  $3n^2$ , must be a multiple of  $k$ .

The proof of sufficiency splits into the cycle length  $k$  being a multiple of 3, or not. The latter case uses a result Which Enables a closed  $k$ -trail decomposition of  $K_{m, m, m}$  in cui any vertex OCCURS at most  $\frac{m}{k}$  times, to yield a  $k$ -cycle decomposition of  $K_{lm, lm, lm}$ .

In the case of the tripartite graph  $K_{r, s, t}$ , When the three parts have possibly arbitrary different sizes, Necessary and sufficient conditions for to decomposition  $k$  into  $k$ -cycles is not known in general. As remarked above, for 3-cycles, the three parts must have the same size. For  $2k$ -cycles,  $k \geq 2$ , an exact decomposition will only be possible if all three parts have even size (and Satisfy (i) - (vi) Section 2.2 above). It is still an open problem to verify That the Necessary conditions in Section 2.2 are sufficient for existence of a decomposition into  $k$ -cycles  $2k, 2k \geq 10$ , even When there are only three parts.

When the cycle length  $2k+1$  is odd and greater than 3, the problem of determining Necessary and sufficient conditions for a decomposition of  $K_{r, s, t}$  into  $2k+1$ -cycles remains open. Indeed, even in the case of 5-cycles, determination of a 5-cycle decomposition of  $K_{r, s, t}$  Whenever the "obvious" Necessary conditions hold, is incomplete. This problem was Considered by Mahmoodian and Mirzakhani [36], where the Necessary conditions for a decomposition of  $K_{r, s, t}$  ( $R \leq s \leq t$ ) into 5-cycles were listed:

- (I)  $r, s$  and  $t$  are all even or all odd;
- (ii)  $rs + rt + st$  is divisible by 5;
- (iii)  $t \leq 4rs / (r + s)$ .

It is easy to See That These conditions are Necessary: condition (i) Ensures That every vertex has even degree; condition (ii) Ensures That the total number of edges is a multiple of 5; and condition (iii) follows from the fact That the number of edges between the two smallest parts (of sizes  $r$  and  $s$ ) must be greater than or equal to the total number of 5-cycles (Which is one fifth of the total number of edges).

Mahmoodian Mirzakhani and [36] deal with the case when  $r, s, t$  are all 0 (Mod 5) and and they offer a prize of 100,000 Iranian Rials for the proving of the sufficiency of These three conditions. Cavenagh and Billington [15], [13] deal with further Top cases, know the current state of play Is that the Necessary conditions

(i), (ii), (iii) above are sufficient When Two (or more) of the matches have sets the same size, or When all parts have even size. So the remaining open houses is When All games sets are odd, and of three different sizes. It is Likely That the method used in [15] this will work for open houses, but it will be long and tedious! This method basically exploits the connection between a graph and a tripartite kind of latin rectangle. The edges in the graph  $K_{r,s,t}$  Where  $r \leq s \leq t$ , can be Represented by the entries in the so-called "Latin rectangle" shown in Figure

2.1. The entries in  $[AB]$  are latin row in the  $t$  symbols; the entries in  $\begin{bmatrix} TO \\ C \end{bmatrix}$  are latin column. Each filled cell in  $A$  Corresponds to a triangle in  $K_{r,s,t}$ ; each entry  $B$  (and  $B$  notes That need not be latin column) Corresponds to an edge between parts of sizes  $r$  and  $t$ , while each entry in  $C$  Corresponds to an edge between parts of size  $r$  and  $s$ .

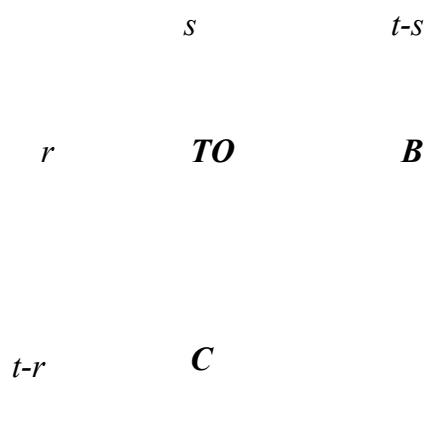


Figure 2.1

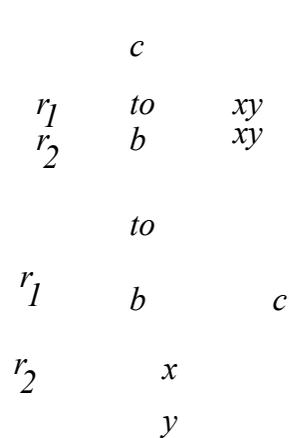


Figure 2.2

Figure 2.2 gives an example of how some of the entries in a latin representation, consisting of two triangles and four further Top edges, can be "traded" to give two 5-cycles. The edges are Explicitly listed in the table below.

Edges from entries in latin representation Edges reconfigured as 5-cycles  
 $(r_1, A, c), (r_2, B, c), r_1c, r_1d, r_2c, r_2d$   $(r_1, C, b, r_2, C), (r_1, A, c, r_2, D)$

By judicious partitioning of a suitable latin representation into various trades like the one illustrated in Figure 2.2 (but Often Considerably blackberries complications

cated!), results on decomposition of tripartite graphs  $K_{r, s, t}$  into various cycles It is achieved.

A precursor of this method was used in [2], where  $K_{r, s, t}$  is decomposed into specified numbers of 3- and 4-cycles; it was fully exploited in the 5-cycle papers [15], [13].

### 2.5. Tripartite graph decompositions: closed trails

Some recent work by Billington and Cavenagh [3] has Dealt with the decomposition of a complete tripartite graph with equal sized parts into any

number of any length closed trails. Balister [1] That Showed a complete graph  $K_n$  (When  $n$  is odd) or  $K_n - F$  (where  $F$  is a 1-factor in the case  $n$  is even) It can be decomposed into circuits of lengths  $m_1, m_2, \dots, m_t$  Whenever  $m_i \geq 3$  ( $1 \leq i \leq t$ ) and  $\sum_{i=1}^t m_i$  equals the number of edges in  $K_n$  ( $N$  odd) or  $K_n - F$  ( $N$  even). The paper [3] does likewise for the graph  $K_{n, n, n}$ . In Particular, the Following is proved.

**Theorem 5.** ([3]) *The complete tripartite graph  $K_{n, n, n}$  has an edge-disjoint decomposition into closed trails of (not Necessarily distinct) lengths  $m_1, m_2, \dots, m_t$  if and only if  $m_i \geq 3$  for  $1 \leq i \leq t$  and  $\sum_{i=1}^t m_i \equiv 2 \pmod{3}$ .*

The method used to show sufficiency of the obvious Necessary conditions Involves a back-circulant latin square of order  $n$ , Which itself Represents the edges of  $K_{n, n, n}$  as a set of  $n^2$  triangles (with row, column, being the entry index sets of the three parts in  $K_{n, n, n}$ ). By judicious use of "trades", working, generally speaking, down pairs of columns (or three columns in one instance When  $n$  is odd), the Latin square is partitioned up into closed trails of the required lengths. On the whole, These trails consist of linked cycles of lengths 3, 4 and 5 (Depending upon Their length form 3) Although some cases require further Top refinement (see [3] for details).

Two straightforward observations help with the proof. One Is that a collection of 3-cycles in  $K_{n, n, n}$  Arising from entries in a latin square of order  $n$  will form a connected circuit provided That the entries can be ordered with adjacent ones being (i) in the same row, or (ii) in the same column, or (iii) the same symbol. That Another is any set of integers (being potential circuit lengths)  $P = \{x_1, x_2, \dots, x_m\}$  Satisfying  $\sum_{i=1}^m x_i \equiv 2 \pmod{3}$ , Can be partitioned into subsets  $P_j$ , Each containing at most three of the  $x_i$  So THAT the numbers in each  $P_j$  Sum to a multiple of 3, and no subset of  $P_j$  sums to a multiple of 3.

For example, if  $P = \{4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 7, 8, 9, 10, 10, 17\}$  (These numbers sum to  $3 \times 6 = 18$ ), In the case of the graph  $K_{n, n, n}$ , A possible partition of

$P$  is given by

6, 6, 6

$\{4, 4, 4\}, \{4, 4, 4\}, \{9\}, \{4, 4, 10\}, \{5, 5, 5\}, \{7, 8\}, \{10, 17\}.$

Then in backcirculant latin square of order 6 is appropriately partitioned up know  
 That connected entries give rise to trails of lengths in  $P_j$ , For each  $P$  in September  
 partition of  $P$ . For instance, Consider  $\{7, 8\}$  above; Consisting of trails  $C_3 \cup C_4$   
 and  $C_3 \cup C_5$  have lengths 7 and 8, and arise from cells (the five 3-cycles) such  
 Those as in Figure 2.3 below.

There is an obvious corollary to Theorem 5, since a closed trail of length  
 less than 6 is a cycle:

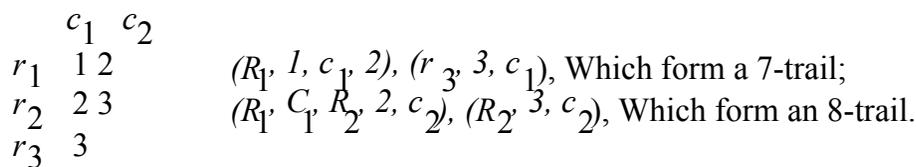


Figure 2.3

**Corollary 5.1.** *Let  $\alpha, \beta, \gamma$  be non-negative integers such That  $3 + \alpha 4 \beta + \gamma = 5$   
 $3 n^2$ . Then there exists an edge-disjoint decomposition of  $K_{n, n, n}$  cycles into  $\alpha 3,$   
 $\beta$  and  $\gamma 4$ -cycles 5-cycles.*

**ADCT graphs**

Even if a connected graph  $G$  contains a closed trail of length  $m$  For  $1 \leq$   
 $i \leq t$ , and  $\sum_{i=1}^t m_i = |E(G)|$ , then  $G$  is said to be *arbitrarily decomposable*  
*into closed trails* (ADCT) if  $G$  has an edge-disjoint decomposition into closed  
 trails of lengths  $m_i, 1 \leq i \leq t$ .

Using this terminology, Balister [1] Showed That  $K_n$  ( $N$  odd) and  $K_n - F$   
 ( $N$  even,  $F-1$  factor) Both are ADCT, and Billington and Cavenagh [3] Showed  
 That  $K_{n, n, n}$  ADCT is. In the bipartite case, the graph  $K_{r, s}$  Both with  $r$  and  $s$   
 even was shown by Hornak and Wozniak [32] to be ADCT (here, Necessarily,  
 each trail length  $m_i$  must be even). It can be easily verified That if the general  
 tripartite graph  $K_{r, s, t}$  ADCT is, then you match the sizes are  $1, 1, 3$  or  $1, 1, 5$ , or else  
 $r = s = t$ . Hence [3] completes the determination of ADCT tripartite graphs.

### 3. Extra properties on the decomposition.

#### 3.1. Resolvable cycle decompositions of complete multipartite graphs

The requirement of *resolvability* imposed on a cycle decomposition means that the cycles in the decomposition are able to be partitioned into *resolution classes*, where each class contains resolution, ounces each, all the vertices in the whole graph. In graph theoretic terms, this means that the cycle decomposition forms a 2-factorization of the complete multipartite graph, where the 2-factors consist of a number of  $k$ -cycles.

Following the example illustrates an extra which property to decomposition may have, which will be reviewed in Sections 3.3 and 3.4.

**Example 3.1.** Two resolvable 4-cycle decompositions of  $K(2, 2, 2, 2)$ .

Figure 3.1 (a) shows a resolvable 4-cycle decomposition of  $K(2, 2, 2, 2)$ , with the three parallel classes (or 2-factors), while Figure 3.1 (b) shows one which is not only resolvable but anche *gregarious*, fatto che each cycle has its vertices in different parts (see Section 3.3).

(to)

(B)

Figure 3.1

Given Sotteau's result [41] for bipartite graphs, Example 3.1 (a) perhaps Seems more 'natural', and is Certainly the easier one to find! We shall meet Example 3.1 (b) in Section 3.4; That known each 4-cycle has all its vertices in equipartite different parts of the graph  $K(2, 2, 2, 2)$ .

Thus a cycle decomposition of  $G = K(a_1, \dots, a_n)$  is said to be *resolvable* if it forms a partition of the edge-set of  $G$  into 2-factors, with each 2-factor in union of cycles. If the cycles are all of the same length  $k$ , then this is also called to  $C_k$ -factorization of  $G$ . In Liu [34], [35], the problem of finding such a  $C_k$ -factorization is posed as a generalization of the famous Oberwolfach problem ([27] Guy 1967 - see Liu), and of the spouse-avoiding variant of this ([33] Huang, Kotzig and Rosa). For the  $C_k$ -factorization case, there are *no* Delegations, with  $a_i$  the people in the delegation. These  $\sum_{i=1}^n a_i = n$  people are to be seated at a number of round tables seating  $t_1, \dots, t_s$  people (where each table seats  $k$  people for the case of  $k$ -cycles), and where  $\sum_{i=1}^s t_i = \frac{n}{k}$ . For a number (Which can be Calculated!) Of different meals, I know That every person sits next to *not* every person in his or her delegation, exactly once.

This is the same as finding a 2-factorization of  $K(n_1, \dots, n_m)$  Where each Consists of two-factor  $s$  cycles, lengths of  $t_1, \dots, t_s$ .

In 1991, Hoffman and Schellenberg [30] That Showed  $K_n(2)$  (Having  $n$  parts of size 2) has a  $C_k$ -factorization Whenever the Necessary conditions (even degree, and total number of vertices multiple of  $k$ ) hold, except there is That *no*  $C_3$ -factorization of  $K_3(2)$  or of  $K_6(2)$

Also in 1991, Piotrowski [39] Dealt with the bipartite houses, and Showed That  $K_{m,m}$  has a  $C_k$ -factorization if and only if  $m$  and  $k$  are even and  $k \mid 2m$ , except That  $K_{6,6}$  has no  $C_6$ -factorization.

In 1993 Rees [40] That proved  $K_{n(m)}$  has a  $C_3$ -factorization if and only if the degree,  $m(n-1)$ , is even, and the total number of vertices,  $mn$ , is divisible by 3, That except  $K_{2,2}, K_{6,6}$ , and  $K_{6(2)} = K(2, 2, 2, 2, 2, 2)$  have no  $C_3$ -factorization.

In 2000 and in 2003, Liu [34], [35] That Showed for  $k \geq 3, n \geq 3$ , the equipartite complete graph  $K_{n(m)}$  has a  $C_k$ -factorization if and only if the degree is even and  $k \mid nm$ , except for the four cases above Mentioned:  $K_{6,6}$  has no  $C_6$ -factorization, and  $K_{2,2}, K_{6,6}$ , and  $K_{6(2)}$  have no  $C_3$ -factorization. Thus for equipartite the homes, completed by Liu [34], [35], holds the Following:

**Theorem 6.** *When  $k \geq 3$  and  $n \geq 2$ , there is a resolvable  $k$ -cycle decomposition of  $K_{n(m)}$  (i.e. a  $C_k$ -factorization of the complete graph having equipartite No parts of size  $m$ ) if and only if*

$$k \mid mn, \quad m(n-1) \text{ is even,} \quad k \text{ is even if } n = 2,$$

and there is no resolvable 3-cycle decomposition of  $K_{2,2}$ ,  $K_{6,6}$ , or  $K_{6(2)}$ . Nor any resolvable 6-cycle decomposition of  $K_{6,6}$ .

In the case of different cycle lengths, some (partial) results have also been obtained. Indeed, Liu [35] uses the result that  $K_{n(4)}$  has a  $\{C_3, C_5\}$ -factorization for  $n \geq 3$  and  $n = 7, 10, 11$ , in order to evidence his main result on  $C_t$ -factorizations. However, the Generalised Oberwolfach problem, Delegations of different sizes with  $a_1, \dots, a_n$  and / or with different sized tables  $t_1, \dots, t_s$ , clearly is very difficult, and remains open.

Recent work by Hoffman and SH Holliday (see [31], [29]) looks at equipartite homes minus the 1-factor, and gives a resolution into  $2k$ -cycles. In particular, they test (using a delightfully named "cracked easter egg" approach) the following.

**Theorem 7.** ([31], [29]) *There is a resolvable  $2k$ -cycle decomposition of  $K_{n(m)} - F$ , where  $F$  is a 1-Factor, if and only if  $m$  is odd,  $n$  is even, and  $2k \mid mn$ .*

### 3.2. Colouring cycle decompositions of complete multipartite graphs

An  $m$ -cycle decomposition of a graph  $G$  is said to be *equitably  $k$ -colored* if the vertices of  $G$  are colored with  $k$  colors  $c_1, \dots, c_k$  in such a way for that each cycle  $C$  in the decomposition, the number of vertices colored  $c_i$  differs by at most 1 from the number of vertices colored  $c_j$ , for  $1 \leq i, j \leq k$ . Most work on equitable coloring has been done for decompositions of complete graphs, or of complete graphs minus 1-factor, but Waterhouse [42] deals with equitable 2-colorings of complete multipartite graphs into cycles. She shows that a 3-cycle decomposition of  $K_{n(m)}$  has an equitable two-coloring if and only if (besides the usual requirements on the number of edges, even degree, at least three parts) there are only 3 or 4 parts altogether.

She also shows that a 5-cycle decomposition of  $K_{n(m)}$  exists with an equitable 2-coloring if and only if the usual conditions (even degree, and number of edges in multiple of 5) hold.

In the case of a 4-cycle decomposition of  $K_{n(1, \dots, t, \dots, n)}$ , Waterhouse shows that one with an equitable 2-coloring exists if and only if each  $n_i$  is even; similarly, a 6-cycle decomposition with an equitable 2-coloring exists if and only if the necessary conditions for decomposition to hold and all the parts have even size.

### 3.3. Gregarious cycle decompositions

The first mention of a cycle decomposition in a multipartite graph being *gregarious* appeared in [6] in 2003. The word is chosen for its usual meaning of "Outgoing", or "reaching out". Basically, a cycle is said to be *gregarious* if its vertices occur in as many different parts of the multipartite graph as possible; I know provided there are as many parts as there are vertices in the cycle, every vertex will appear in a different part of the graph. Figure 3.1 illustrates the difference between a 4-cycle decomposition of  $K(2, 2, 2, 2)$  which is *gregarious* (in (b)) and which is certainly not *gregarious* (in (a)). Since no "zig-zagging" to and fro between a pair of parts is allowed, even in the case of 4-cycle decomposition is considerably harder, with the condition of being *gregarious* imposed.

The paper [6] deals with a *gregarious* 4-cycle decomposition of  $K_{r, s, t}$  and gives necessary and sufficient conditions for existence. Since there are only three parts to this graph, each 4-cycle is required to meet all three parts, and necessarily will have two of its four vertices lying in the same part.

**Theorem 8.** ([6]) *There exists a gregarious 4-cycle decomposition of  $K_{r, s, t}$  if and only if*

- (I)  $r \equiv s \equiv t \equiv 0 \pmod{2}$ ;
- (ii)  $s(r + t) - rt \geq 8$ ;

(iii)  $r(s + t) - st \geq 8$  or  $r(s + t) - st = 0$ .

The necessity of conditions (ii) and (iii) is not immediately obvious. These follow from counting the three possible types of cycles (see Figure 3.2), which yields

$$t \geq \frac{rs}{r+s}, S \geq \frac{rt}{r+t}, R \geq \frac{st}{s+t}.$$

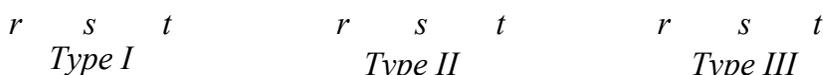


Figure 3.2

Recall That  $r \leq s \leq t$ . Then if  $r = s$ , we have *not* bounded above by  $rs / (r + s)$ , while if  $r < s$  then  $t$  is unbounded. Also the number of 4-cycles of each type is either 0 or at least 2. I know (in order to cover all edges PRECISELY in  $K_{r, s, t}$ ) We have  $2 \leq$  number of type II  $\leq$  number of type III. The necessity follows.

When at least two of  $r, s, t$  are  $0 \pmod{4}$ , existence of a gregarious decomposition follows from existence of a gregarious decomposition of the *path* tripartite graph with vertex sets of half the size. Then by doubling the number of vertices, each gregarious path gives rise to two gregarious 4-cycles in this tripartite houses; see Figure 3.3.

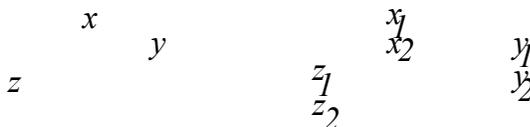


Figure 3.3

However, When Two or more of  $r, s, t$  are  $2 \pmod{4}$ , we still use gregarious paths, but the method requires a "latin representation" approach again, related to That described in Section 2.4 above, but with a  $3 \times 3$  "hole"; see [6] for details!

If we now Consider the case of gregarious 4-cycle decompositions of complete multipartite graphs with *blackberries* than three parts, each vertex of every 4-cycle will lie in a different games in September For four parts, it is straightforward to That verify  $K_{(i_1, i_2, i_3, i_4)}$  Has a gregarious 4-cycle decomposition if and only if  $i_1 = A_2 = A_3 = A_4$  and  $i_1$  are even.

The present state of knowledge on gregarious 4-cycle decompositions includes the Following for equipartite and almost equipartite graphs.

**Theorem 9.** ([7]) *If conditions are right for 4-cycle decomposition of  $K_{n(m)}$  or  $K_{n(m)}$ , Then a gregarious 4-cycle decomposition of These graphs anche exists, provided  $t m \leq (n - 1) / 2$ .*

When the number of parts is  $No. 1 \pmod{8}$ , then a 4-cycle system of  $K_n$  can be taken, and the points "blown up"  $m$ -fold to Obtain a gregarious 4-cycle decomposition of  $K_{n(m)}$  Otherwise, if  $n \equiv 1 \pmod{8}$  Necessarily all parts must have even size. I know the houses  $K$  Dealt with is, from Which a suitable

gregarious decomposition of  $K_n(m)$  (2 for 10) is. However the non-equipartite homes is less straightforward, and incomplete at the moment, Although certain cases (All but one part the same size) sono stati Dealt with in [7].

It is perhaps worth remarking That any group divisible design with block size 5 will give rise to a gregarious 5-decomposition cycle, since of course each block of size 5 will give rise to two 5-cycles with all five vertices in different parts or groups. (The same applies to anche  $p$ -cycles for any odd first  $p$ .) However, there will be gregarious 5-cycle decompositions of complete multipartite graphs in cases When a  $K_5$ -decomposition is not possible.

### 3.4. Resolvable and gregarious cycle decompositions

Since work on cycle decompositions with the extra property of being Gregory garious is Relatively new, very little Has Been done on Requiring the decomposed Both sition to be resolvable *and* gregarious. The paper [9] is a start in this direction.

In [9], Billington, Hoffman and Rodger investigated  $n$ -cycle decomposition tions of the complete equipartite graph  $K_n(m)$  Which Both are gregarious *and* resolvable. The main result is:

**Theorem 10.** ([9]) *There exists an edge-disjoint decomposition of  $K_n(m)$  into Both  $n$ -cycles Which is gregarious and resolvable if and only if  $m$  is not Odds When  $n$  is even, and  $(m, n) = (2, 3), (6, 3)$ .*

A gregarious resolvable decomposition of  $K_3(2)$  or  $K_3(6)$  into 3-cycles is not possible (for These two cases would imply existence of a pair of orthogonal latin squares of orders 2 and 6).

Liu's [34], [35] resolvable decompositions (or 2-factorizations!) Of  $K_n(m)$  Generally are not gregarious, know His results did not help in [9].

### 4. Maximum Packings.

When the Necessary conditions for an edge-disjoint decomposition of  $K_n(m)$  into  $k$ -cycles fail, it is natural to ask for a *packing* of this complete multipartitegraph with  $k$ -cycles. A  $k$ -cycle packing of  $G = K_n(m)$  is a set of edge-disjoint  $k$ -cycles in  $G$ . The packing is *maximum* if its number of cycles is not less than the number of cycles in any other packing of  $G$  with  $k$ -cycles. The edges of  $G$  not occurring in any  $k$ -cycle are Referred to as the *leave* of the packing. Obviously a maximum packing will have a minimum leave.

Not much work done on Has Been packing complete multipartite graphs  $K$  with cycles, in the homes That  $K$  is "genuine", That is, When  $K$  is *not* in full graph (all parts of size 1).

For 3-cycles, maximum packings in the equipartite homes,  $K_{n(m)}$  For all  $n$  and  $m$ , are Dealt with in [10].

In [4], the problem of finding a maximum packing of  $K_{(n_1, \dots, n_r)}$  With 4-cycles is completely solved, and the leaves are given minimum. (These minimum leaves can be quite large!) Using this result, a natural generalization was to determinates in maximum packing of the  $\lambda$ -fold graph  $\lambda K_{(n_1, \dots, n_r)}$  With 4-cycles; Appears this in [5].

For 6-cycles, the problem of finding a maximum packing in the equipartite Been homes has recently completed [23]. In this paper was to find and Huang maximum packing of the complete equipartite graph  $K_{m(n)}$  with edge-disjoint 6-cycles, and they give the minimum leaves. (They also find a minimum *covering*, where every edge of  $K_{m(n)}$  Appears in almeno 6-cycle, and where the "Excess" edges used in blackberries than one 6-cycle - sometimes called the padding - Form in September as small as possible.)

As remarked in Section 2.2 above, Necessary and sufficient conditions are known [14] for the existence of a decomposition (with empty leave) of  $K_{(n_1, \dots, n_r)}$  Into 4-cycles, 6 and 8-cycles-cycles. As do as I am aware, no packing results for cycles in multipartite graphs are known other than Those Mentioned above for 3-, 4- and 6-cycles.

## 5. Conclusions and some open problems.

There are several related topics Which are Not mentioned above, such as coverings, and hamilton decompositions. Decompositions into cycles of

different lengths have also ignored Been here Because of space considerations, as have  $\lambda$ -fold decompositions. See [22], [16] for examples in the bipartite houses, and [2] in the tripartite houses, with cycles of differing lengths Within the one decomposition.

Other papers deal with group divisible designs Which allow  $\lambda_1$  edges between pairs of points in the *same* group, and  $\lambda_2$  edges between pairs of points in different groups. Cycles of lengths 3 and 4 sono stati Dealt with in this way; see papers [24] and [26] by Fu, Rodger and Sarvate for the 3-cycle houses, and was and Rodger [25] for the 4-cycle houses.

Appended below are some of the open problems Mentioned here.

**Problem 2.1.**

Find Necessary and sufficient conditions on  $K(i_1, \dots, t_0, n)$  For it to have an edge-disjoint decomposition into 3-cycles. (See Colbourn [17] for six conditions Which are shown sufficient for orders up to 60.)

Find Links: partial results in this direction; see [11], [20], [17], [18], [19].

**Problem 2.2.**

That shows a graph  $K(i_1, \dots, t_0, n)$  Which is  $2k$ -sufficient has to decomposition into  $2k$ -cycles, for  $2k \geq 10$ .

**Problem 2.3.**

Evidence That the Necessary conditions for a 5-cycle decomposition of  $K_{r, s, t}$  are sufficient in the remaining houses, When  $r, s, t$  are all odd and all different.

**Problem 2.4.**

Find Necessary and sufficient conditions for  $K_{2r, 2s, 2t}$  to have a decomposition into  $2k$ -cycles, for any  $k \geq 5$ . notes That this is really a subset of Problem 2.2 above.

**Problem 2.5.**

That shows the graph  $K_{r, s} - F$ , where  $r, s$  are odd and  $F$  is a (smallest possible) spanning subgraph of odd degree, is ADCT. (See Section 2.5.)

**Problem 3.2.**

Investigate equitable  $k$ -colourings of complete multipartite graph decomposition cycle positions, for  $k > 2$ .

**Problem 4.1.**

Investigate maximum packings of complete multipartite graphs with small cycles. In particular:

- (1) Investigate packing  $K_{r, s}$  with 3- and 4-cycles (see [2] for the case of an leave empty, and a specified number of 3- and 4-cycles).
- (2) Consider packing tripartite graphs with 5-cycles. With methods used in [14], [13] this evidence could no harder than 2.3 Problem!

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