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A Poincaré–Birkhoff–Witt commutator lemma for $U_q[\mathrm{gl}(m|n)]$

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We present and prove in detail a Poincaré–Birkhoff–Witt commutator lemma for the quantum superalgebra $U_q[\mathrm{gl}(m|n)]$. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1511828]

I. INTRODUCTION

This paper presents and proves in detail a Poincaré–Birkhoff–Witt (PBW) commutator lemma for the quantum superalgebra $U_q[\mathrm{gl}(m|n)]$. The lemma itself is not new; it dates from a 1993 paper of Rui Bin Zhang[3] on the representation theory of $U_q[\mathrm{gl}(m|n)]$. However, its previous incarnation contained several typographical and other minor errors in its details; and in any case an explicit proof was not supplied. Here, we correct those errors, and supply detailed proofs for our claims.

We mention that we use the phrase ''PBW commutator lemma'' to indicate a result showing commutations sufficient to render any expression within an algebra into a normal form in a PBW basis; for more details for our specific case $U_q[\mathrm{gl}(m|n)]$, we again refer the reader to the original work by Zhang.

II. THE STRUCTURE OF $U_q[\mathrm{gl}(m|n)]$

Following Zhang (Ref. 3, pp. 1237–1238), we provide a full description of $U_q[\mathrm{gl}(m|n)]$ in terms of simple generators and relations. We do so after first introducing the generators and various divers notations.

First, we define a $\mathbb{Z}_2$ grading $[\cdot]$ on the set of $\mathrm{gl}(m|n)$ indices $\{1,\ldots,m+n\}$:

$$[a] \triangleq \begin{cases} 0, & a \leqslant m, \quad \text{even indices}, \\ 1, & a > m, \quad \text{odd indices}, \end{cases}$$

where we use the symbol ''$\triangleq$'' to mean ''is defined as being.'' Throughout, we shall use dummy indices a,b, etc., where meaningful.

A set of generators for the associative superalgebra $U_q[\mathrm{gl}(m|n)]$ is then

$$\{K_a^{\pm}; E_b^a | 1 \leqslant a,b \leqslant m+n, a \neq b\},$$

where the $K_a^{\pm}$ are called ''Cartan generators'' (and of course we intend ''± 1'' where we write ''$\pm$''), and E_b^a is called a ''raising generator'' if $a<b$ and a ''lowering generator'' if $a>b$. We indeed intend that K_a and K_a^{-1} are inverses, that is, that we have relations $K_a K_a^{-1} = K_a^{-1} K_a = \mathrm{Id}$, where Id is the $U_q[\mathrm{gl}(m|n)]$ identity element.

Elements of $U_q[\mathrm{gl}(m|n)]$ are then in general weighted sums of noncommuting products of these generators, where each weight is in general a rational expression of integer-coefficient Laurent polynomials in the polynomial variable q. Under the phrase ''products of generators,'' we include powers of the K_a (see below).

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For various invertible X, we will repeatedly use the notation $\overline{X} \triangleq X^{-1}$; in particular, we set $\overline{q} \triangleq q^{-1}$. Next, for any index a we shall write

$$q_a \triangleq q^{(-)^{[a]}},$$

where we have invoked the shorthand "$(-)$" for "(-1)." For any power N, replacing q with q^N immediately shows that $(q_a)^N = (q^N)_a$, so we may write q_a^N with impunity; in particular, we will write $\overline{q_a} \equiv q_a^{-1}$. Further, we will use the following notation:

$$\Delta \triangleq q - \overline{q}, \quad \Delta_a \triangleq q_a - \overline{q_a} = (-)^{[a]}(q - \overline{q}) = (-)^{[a]}\Delta,$$

$$\overline{\Delta} \triangleq (\Delta)^{-1}, \quad \overline{\Delta}_a \triangleq (\Delta_a)^{-1} = (-)^{[a]}\overline{\Delta}.$$

Now, in terms of q, an equivalent notation for K_a is $q_a^{E_a^a}$. (Here, the exponentiation may be understood in terms of a power series expansion of the $U[\mathrm{gl}(m|n)]$ Cartan generators E_a^a. Strictly speaking, we *could* define these E_a^a as the $U_q[\mathrm{gl}(m|n)]$ Cartan generators, allowing them to appear in infinite sums as exponents of q, but the K_a notation is more convenient.) Thus, powers K_a^N are meaningful, although we will only deal with $N \in \frac{1}{2}\mathbb{Z}$ (that is, integer and half-integer powers). So, we may write $\overline{K_a} \triangleq K_a^{-1}$; indeed the mapping $q \mapsto \overline{q}$ sends K_a^N to $\overline{K_a^N}$, and as expected, for arbitrary powers M,N:

$$K_a^M K_a^N = K_a^{M+N}, \quad \text{where } K_a^0 \equiv \mathrm{Id}.$$

Apart from $N \in \mathbb{N}$, powers (i.e., products) of the non-Cartan generators $(E_b^a)^N$ for $a \neq b$, are not meaningful.

The generators inherit a $\mathbb{Z}_2$ grading from the indices

$$[K_a] \triangleq 0 \quad \text{and } [E_b^a] \triangleq [a] + [b] \pmod 2,$$

so we may also use the terms "even" and "odd" for generators. Elements of $U_q[\mathrm{gl}(m|n)]$ are said to be *homogeneous* if they are linear combinations of generators of the same grading or products of other homogeneous elements; the product XY of homogeneous X,Y has grading $[XY] \triangleq [X] + [Y] (\mathrm{mod}\, 2)$.

Now, the full set of generators includes some redundancy; in that its elements may be expressed in terms of a subset of them, that is the following $U_q[\mathrm{gl}(m|n)]$ *simple generators*:

$$\{K_a^{\pm}; E_a^{a+1}, E_{a+1}^a | 1 \leq a, a+1 \leq m+n\};$$

note that there are only two *odd* simple generators: E_m^{m+1} (lowering) and E_{m+1}^m (raising). In the $\mathrm{gl}(m|n)$ case, the remaining *nonsimple* (non-Cartan) generators satisfy the same commutation relations as the simple generators. However, for $U_q[\mathrm{gl}(m|n)]$, the nonsimple generators are instead recursively defined in terms of weighted sums of products of simple generators [Ref. 2, p. 1971, (3)] and [Ref. 3, p. 1238, (2)]. Writing $S_b^a \triangleq \mathrm{sign}(a-b)$, the elements of the set of nonsimple generators $\{E_b^a | \, |a-b| > 1\}$ may be defined by

$$E_b^a \triangleq E_c^a E_b^c - q_c^{S_b^a} E_b^c E_c^a, \tag{1}$$

where we intend c to be an *arbitrary* index strictly between a and b; we do *not* intend a sum here.

Last, the *graded commutator* $[\cdot,\cdot]$ is defined for homogeneous X,Y by

$$[X,Y] \triangleq XY - (-)^{[X][Y]} YX, \tag{2}$$

and extended by linearity. As $U_q[\mathrm{gl}(m|n)]$ is an *associative* superalgebra, we have the following useful identities involving homogeneous elements:

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$$\begin{aligned} &(\mathrm{a}) \quad [XY,Z]=X[Y,Z]+(-)^{[Y][Z]}[X,Z]Y, \\ &(\mathrm{b}) \quad [X,YZ]=[X,Y]Z+(-)^{[X][Y]}Y[X,Z]. \end{aligned} \tag{3}$$

A. $U_q[\mathrm{gl}(m|n)]$ relations

In terms of the set of simple generators, that is

$$\{K_a^{\pm};E_a^{a+1},E_{a+1}^{a}|1\leqslant a,a+1\leqslant m+n\},$$

our algebra $U_q[\mathrm{gl}(m|n)]$ satisfies the following relations.

(1) The Cartan generators commute, that is for $M,N\in\{\pm 1\}$,

$$K_a^M K_b^N = K_b^N K_a^M. \tag{4}$$

(2) The Cartan generators commute with the simple raising and lowering generators in the following manner:

$$K_a E_{b\pm 1}^b = q_a^{(\delta_b^a - \delta_{b\pm 1}^a)} E_{b\pm 1}^b K_a. \tag{5}$$

(3) The non-Cartan simple generators satisfy

$$[E_{a+1}^a,E_b^{b+1}]=\delta_b^a\overline{\Delta}_a(K_a\overline{K}_{a+1}-\overline{K}_aK_{a+1}). \tag{6}$$

and, for $|a-b|>1$, we have the commutations

$$E_a^{a+1}E_b^{b+1}=E_b^{b+1}E_a^{a+1} \quad \text{and} \quad E_{a+1}^aE_{b+1}^b=E_{b+1}^bE_{a+1}^a. \tag{7}$$

(4) The squares of the odd simple generators are zero

$$(E_{m+1}^m)^2=(E_m^{m+1})^2=0. \tag{8}$$

(5) If neither m nor n is 1, we have the $U_q[\mathrm{gl}(m|n)]$ *Serre relations* (else if either m or n is 1, omit them). Most succinctly expressed in terms of the nonsimple generators, for $a\neq m$, we have

$$\begin{aligned} &(\mathrm{a}) \quad E_a^{a+1}E_a^{a+2}=q_aE_a^{a+2}E_a^{a+1}, \\ &(\mathrm{b}) \quad E_{a+1}^aE_{a+2}^a=q_aE_{a+2}^aE_{a+1}^a, \\ &(\mathrm{c}) \quad E_{a-1}^{a+1}E_a^{a+1}=q_aE_a^{a+1}E_{a-1}^{a+1}, \\ &(\mathrm{d}) \quad E_{a+1}^{a-1}E_{a+1}^a=q_aE_{a+1}^aE_{a+1}^{a-1}, \end{aligned} \tag{9}$$

and also

$$[E_m^{m+1},E_{m-1}^{m+2}]=[E_{m+1}^m,E_{m+2}^{m-1}]=0.$$

The interested reader may use (1) to expand these into expressions involving only the simple generators; however the results are cumbersome and unedifying.

B. Useful results from the $U_q[\mathrm{gl}(m|n)]$ relations

(1) From (4), it immediately follows that all powers of the Cartan generators commute; that is, for any powers $M,N\in\frac{1}{2}\mathbb{Z}$:

$$K_a^M K_b^N = K_b^N K_a^M. \tag{10}$$

(2) Lemma 2 of Ref. 1 shows that (5) may be much strengthened to cover all non-Cartan generators and all powers of Cartan generators:

$$K_a^N E_c^b = q_a^{N(\delta_b^a-\delta_c^a)} E_c^b K_a^N, \tag{11}$$

that is, where b,c are *any* meaningful indices (i.e., even including the case $b=c$), and $N\in\frac{1}{2}\mathbb{Z}$ is any power.

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The proof of our PBW commutator lemma uses these results, and also calls on Lemma 1 of Ref. 3, which we now cite, with some slight notational changes and simplifications:

Lemma 1: Where $a<b$, we have the following two results.

First, if $a,b\neq c,c+1$, then

$$\begin{aligned} &(\mathrm{a}) \quad [E^a_b, E^c_{c+1}] = 0, \\ &(\mathrm{b}) \quad [E^b_a, E^{c+1}_c] = 0. \end{aligned} \tag{12}$$

Second, if $a\neq c$ or $b\neq c+1$, then

$$\begin{aligned} &(\mathrm{a}) \quad [E^a_b, E^{c+1}_c] = \delta^{c+1}_b K_c \overline{K}_{c+1} E^a_c - \delta^a_c (-)^{[E^{c+1}_c]} E^{c+1}_b \overline{K}_c K_{c+1}, \\ &(\mathrm{b}) \quad [E^b_a, E^c_{c+1}] = \delta^c_a K_c \overline{K}_{c+1} E^b_{c+1} - \delta^b_{c+1} (-)^{[E^c_{c+1}]} E^c_a \overline{K}_c K_{c+1}. \end{aligned} \tag{13}$$

C. The algebra antiautomorphism ω

Again following Zhang,[3] we introduce an *ungraded* $U_q[\mathrm{gl}(m|n)]$ algebra antiautomorphism ω, defined for *simple* generators E^a_b by

$$\omega(E^a_b) \triangleq E^b_a, \quad \omega(K_a) \triangleq \overline{K}_a, \quad \omega(q) \triangleq \overline{q}, \tag{14}$$

where by $\omega(q)=\overline{q}$, we intend the more intelligible $\omega(q\,\mathrm{Id})=\overline{q}\,\mathrm{Id}$. Declaring ω to be an ungraded antiautomorphism means that we intend

$$\omega(XY)=\omega(Y)\omega(X) \quad \text{and} \quad \omega(X+Y)=\omega(X)+\omega(Y); \tag{15}$$

observe that ω does indeed preserve grading, that is for homogeneous X, we have $[\omega(X)] = [X]$. Then, for homogeneous X,Y, we have, using (2),

$$\omega([X,Y]) = [\omega(Y),\omega(X)]. \tag{16}$$

The expression $\omega(E^a_b)=E^b_a$ in fact holds for *all* E^a_b; the generalization to nonsimple generators follows from the application of ω to their definition in (1). Moreover, we have immediately from (14) the following useful results:

$$\omega(K^N_a) = \overline{K}^N_a, \quad \omega(q^N) = \overline{q}^N, \quad \omega(q^N_a) = \overline{q}^N_a, \quad \omega(\Delta_a) = -\Delta_a.$$

Zhang goes on to define a set of "generalized Lusztig automorphisms," but we do not require these. In fact, it appears to be impossible to define them consistently for superalgebras (as claimed in Ref. 3), hence invalidating their use in the proof of the PBW commutator lemma.

III. THE PBW COMMUTATOR LEMMA

Using the above machinery, we are now ready to state and prove the $U_q[\mathrm{gl}(m|n)]$ PBW commutator lemma. To whit, we will prove the following, which is slightly different from the original (Lemma 2 of Ref. 3).

Lemma 2: We have the following commutations.

First, (6) *generalizes to the case of nonsimple generators, that is*

$$[E^a_b, E^b_a] = \overline{\Delta}_a (K_a \overline{K}_b - \overline{K}_a K_b) \quad \text{all} \quad a,b. \tag{17}$$

Second, where there are three distinct indices, we have

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$$[E^a_c, E^c_b] = \begin{cases} \text{(a)} & \overline{K}_b K_c E^a_b, \quad c<b<a, \\ \text{(b)} & E^a_b K_a \overline{K}_c, \quad c<a<b, \\ \text{(c)} & E^a_b \overline{K}_a K_c, \quad b<a<c, \\ \text{(d)} & K_b \overline{K}_c E^a_b, \quad a<b<c, \end{cases} \tag{18}$$

$$[E^c_a, E^c_b] = [E^a_c, E^b_c] = 0, \quad a<c<b \text{ or } b<c<a, \tag{19}$$

$$[E^c_a, E^c_b] = \begin{cases} \text{(a)} & (-)^{[E^c_b]} q_c E^c_b E^c_a, \quad a<b<c, \\ \text{(b)} & (-)^{[E^c_a]} q_c E^c_b E^c_a, \quad c<a<b, \end{cases}$$
$$E^a_c E^b_c = \begin{cases} \text{(c)} & (-)^{[E^b_c]} q_c E^b_c E^a_c, \quad a<b<c, \\ \text{(d)} & (-)^{[E^a_c]} q_c E^b_c E^a_c, \quad c<a<b. \end{cases} \tag{20}$$

Third, we describe the situation where there are no common indices, where we have $a<b$ and $c<d$. For $i,j \in \mathbb{N}$, let $S(i,j)$ denote the set $\{i, i+1, \ldots, j\}$. Then, if $S(a,b)$ and $S(c,d)$ are either disjoint or one is totally contained within the other, that is if $a<c<d<b$. $a<b<c<d$. $c<a<b<d$ or $c<d<a<b$, we have a total of 16 cases:

$$[E^a_b, E^c_d] = [E^a_b, E^d_c] = [E^b_a, E^c_d] = [E^b_a, E^d_c] = 0. \tag{21}$$

More interestingly, if there is some other overlap between the sets $S(a,b)$ and $S(c,d)$, that is if $a<c<b<d$ or $c<a<d<b$, then we have the eight cases

$$[E^a_b, E^c_d] = \begin{cases} \text{(a)} & +\Delta_b E^a_d E^c_b, \quad a<c<b<d, \\ \text{(b)} & -\Delta_d E^a_d E^c_b, \quad c<a<d<b, \end{cases}$$
$$[E^b_a, E^d_c] = \begin{cases} \text{(c)} & +\Delta_b E^d_a E^b_c, \quad a<c<b<d, \\ \text{(d)} & -\Delta_d E^d_a E^b_c, \quad c<a<d<b, \end{cases} \tag{22}$$

$$[E^a_b, E^d_c] = \begin{cases} \text{(a)} & -\Delta_b \overline{K}_b K_c E^a_c E^d_b, \quad a<c<b<d, \\ \text{(b)} & +\Delta_d E^d_b E^a_c \overline{K}_a K_d, \quad c<a<d<b, \end{cases}$$
$$[E^b_a, E^c_d] = \begin{cases} \text{(c)} & -\Delta_c E^b_d E^c_a \overline{K}_c K_b, \quad a<c<b<d, \\ \text{(d)} & +\Delta_a \overline{K}_d K_a E^c_a E^b_d, \quad c<a<d<b. \end{cases} \tag{23}$$

In the above, we disagree with the results published in Ref. 3 in several places. First (11) shows that (18a) and (18d) are actually equivalent to the published results

$$[E^a_c, E^c_b] = \begin{cases} \text{(a)} & q_b E^a_b K_c \overline{K}_b, \quad c<b<a, \\ \text{(d)} & \overline{q}_b E^a_b K_b \overline{K}_c, \quad a<b<c. \end{cases}$$

However, for all the commutators involving no common indices, we differ in substance. The published results for (22) are

$$[E^a_b, E^c_d] = +\Delta_b E^a_d E^c_b, \quad a<c<b<d, \quad c<a<d<b,$$

$$[E^b_a, E^d_c] = -\Delta_b E^b_c E^d_a, \quad a<c<b<d, \quad c<a<d<b,$$

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and for (23) are

$$[E_b^a,E_c^d]=\begin{cases}\text{(a)} & +\Delta_b E_b^d E_c^a \overline{K}_b K_a, \quad a<c<b<d,\\ \text{(b)} & +\Delta_a E_c^a E_b^d \overline{K}_a K_d, \quad c<a<d<b,\end{cases}$$

$$[E_a^b,E_d^c]=\begin{cases}\text{(c)} & -\Delta_b \overline{K}_a K_b E_a^c E_d^b, \quad a<c<b<d,\\ \text{(d)} & -\Delta_a K_a \overline{K}_d E_d^b E_a^c, \quad c<a<d<b.\end{cases}$$

We mention that it was the discovery of errors in *computations* while working on material described in Ref. 1 that led us to check and correct these PBW results, and consequently rediscover and debug the proof.

Proof of Lemma 2: We prove the components of the lemma in a different order to that in which we state them. This is to ensure consistency as later parts of the proof recycle results previously shown.

(21) These are the 16 commutators involving $a<b$ and $c<d$, with no overlap betwen $S(a,b)$ and $S(c,d)$.

First, in the cases $a<b<c<d$ and $a<c<d<b$, in evaluating $[E_b^a,E_d^c]$, we may use (1) to recursively expand the raising generator E_d^c into a sum of products of simple raising generators, and then apply (3b) until we have a weighted sum of terms all involving commutators of the form $[E_b^a,E_{e+1}^e]$, where $a,b\neq e,e+1$, all of which are necessarily 0 by (12a), thus $[E_b^a,E_d^c]=0$ for these two cases.

Second, swapping $a\leftrightarrow c$ and $b\leftrightarrow d$ in these two cases, and rearranging then yields $[E_b^a,E_d^c]=0$ for the cases $c<d<a<b$ and $c<a<b<d$.

Third, the four cases $[E_b^a,E_c^d]=0$ follow by a similar argument, calling on (13a) rather than (12a).

Last, the remaining eight cases $[E_a^b,E_d^c]=0$ and $[E_a^b,E_c^d]=0$ follow by the application of ω to the first eight cases, and reversing the commutators.

(19) Initially, we show (19a), that is for the case $a<c<b$ we show $[E_a^c,E_b^c]=0$. If in fact $a=c-1$, then the result is already known from (13a), so we assume otherwise, that is we consider the case $a<c-1<c<b$,

$$\begin{aligned}[E_a^c,E_b^c] &\overset{(1)}{=}[E_{c-1}^c E_a^{c-1},E_b^c]-q_{c-1}[E_a^{c-1}E_{c-1}^c,E_b^c]\\ &\overset{(3a)}{=} E_{c-1}^c[E_a^{c-1},E_b^c]+(-)^{[E_a^{c-1}][E_b^c]}[E_{c-1}^c,E_b^c]E_a^{c-1}\\ &\quad -q_{c-1}(E_a^{c-1}[E_{c-1}^c,E_b^c]+(-)^{[E_{c-1}^c][E_b^c]}[E_a^{c-1},E_b^c]E_{c-1}^c)\\ &\overset{(21)}{=}(-)^{[E_a^{c-1}][E_b^c]}[E_{c-1}^c,E_b^c]E_a^{c-1}-q_{c-1}E_a^{c-1}[E_{c-1}^c,E_b^c]\overset{(13a)}{=}0.\end{aligned}$$

Swapping $a\leftrightarrow b$ and reversing the commutator then yields $[E_a^c,E_b^c]=0$ for the case $b<c<a$. Taking ω of these two cases yields $[E_c^a,E_c^b]=0$ for the cases $a<c<b$ and $b<c<a$.

(17) We show the result for $a<b$ using strong mathematical induction, that is, we assume it true for all a',b' such that $|a'-b'|<|a-b|$, and use this to show that it is then necessarily true for our a,b. To this end, we already know from (6) that it is true for $|a-b|=1$. (If $|a-b|\leqslant 1$, the result is already true, indeed trivially so if $a=b$.) To whit, where $a<b$, and $b-a>1$, that is $a<b-1<b$, we have

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$$
\begin{aligned}
[E^a_b,E^b_a] &\overset{(1)}{=} [E^a_b,E^b_{b-1}E^{b-1}_a - q_{b-1}E^{b-1}_aE^b_{b-1}] \\
&\overset{(3b)}{=} [E^a_b,E^b_{b-1}]E^{b-1}_a + (-)^{[E^a_b][E^b_{b-1}]}E^b_{b-1}[E^a_b,E^{b-1}_a] \\
&\quad - q_{b-1}[E^a_b,E^{b-1}_a]E^b_{b-1} - (-)^{[E^a_b][E^a_{b-1}]}q_{b-1}E^{b-1}_a[E^a_b,E^b_{b-1}],
\end{aligned} \tag{24}
$$

where the factors $[E^a_b]\equiv[a]+[b]$ within the parity factors are redundant. In (24), we thus require the evaluation of the commutators $[E^a_b,E^b_{b-1}]$ and $[E^a_b,E^{b-1}_a]$. To this end, we have first

$$
[E^a_b,E^b_{b-1}] \overset{(13a)}{=} K_{b-1}\overline{K}_bE^a_{b-1}, \tag{25}
$$

and second

$$
\begin{aligned}
[E^a_b,E^{b-1}_a] &\overset{(1)}{=} [E^a_{b-1}E^{b-1}_b - \overline{q}_{b-1}E^{b-1}_bE^a_{b-1},E^{b-1}_a] \\
&\overset{(3a)}{=} E^a_{b-1}[E^{b-1}_b,E^{b-1}_a] + (-)^{[E^{b-1}_b][E^{b-1}_a]}[E^a_{b-1},E^{b-1}_a]E^{b-1}_b \\
&\quad - \overline{q}_{b-1}E^{b-1}_b[E^a_{b-1},E^{b-1}_a] - \overline{q}_{b-1}(-)^{[E^{b-1}_a]}[E^{b-1}_b,E^{b-1}_a]E^a_{b-1} \\
&\overset{(19)}{=} [E^a_{b-1},E^{b-1}_a]E^{b-1}_b - \overline{q}_{b-1}E^{b-1}_b[E^a_{b-1},E^{b-1}_a].
\end{aligned}
$$

Using the strong inductive assumption, we then have

$$
\begin{aligned}
[E^a_b,E^{b-1}_a] &= \overline{\Delta}_a\begin{pmatrix} (K_a\overline{K}_{b-1} - \overline{K}_aK_{b-1})E^{b-1}_b \\ -\overline{q}_{b-1}E^{b-1}_b(K_a\overline{K}_{b-1} - \overline{K}_aK_{b-1}) \end{pmatrix} \\
&\overset{(11)}{=} \overline{\Delta}_aE^{b-1}_b\begin{pmatrix} \overline{q}_{b-1}K_a\overline{K}_{b-1} - q_{b-1}\overline{K}_aK_{b-1} \\ -\overline{q}_{b-1}K_a\overline{K}_{b-1} + \overline{q}_{b-1}\overline{K}_aK_{b-1} \end{pmatrix} \\
&= -\overline{\Delta}_aE^{b-1}_b\overline{K}_aK_{b-1}(q_{b-1} - \overline{q}_{b-1}) \\
&= -\overline{\Delta}(-)^{[a]}\Delta(-)^{[b-1]}E^{b-1}_b\overline{K}_aK_{b-1} \\
&= -(-)^{[E^{b-1}_a]}E^{b-1}_b\overline{K}_aK_{b-1}.
\end{aligned} \tag{26}
$$

Now substitute (25) and (26) into (24),

$$
\begin{aligned}
[E^a_b,E^b_a] &= K_{b-1}\overline{K}_bE^a_{b-1}E^{b-1}_a - (-)^{[E^b_{b-1}]}(-)^{[E^{b-1}_a]}E^b_{b-1}E^{b-1}_bK_{b-1}\overline{K}_a \\
&\quad + (-)^{[E^{b-1}_a]}q_{b-1}E^{b-1}_bK_{b-1}\overline{K}_aE^b_{b-1} - (-)^{[E^{b-1}_a]}q_{b-1}E^{b-1}_aK_{b-1}\overline{K}_bE^a_{b-1} \\
&= (E^a_{b-1}E^{b-1}_a - (-)^{[E^{b-1}_a]}E^{b-1}_aE^a_{b-1})K_{b-1}\overline{K}_b - (-)^{[E^a_b]}(E^b_{b-1}E^{b-1}_b \\
&\quad - (-)^{[E^{b-1}_b]}E^{b-1}_bE^b_{b-1})K_{b-1}\overline{K}_a \\
&\overset{(2)}{=} [E^a_{b-1},E^{b-1}_a]K_{b-1}\overline{K}_b - (-)^{[E^a_b]}[E^b_{b-1},E^{b-1}_b]K_{b-1}\overline{K}_a \\
&= \overline{\Delta}_a(K_a\overline{K}_{b-1} - \overline{K}_aK_{b-1})K_{b-1}\overline{K}_b - (-)^{[E^a_b]}\overline{\Delta}_b(K_b\overline{K}_{b-1} - \overline{K}_bK_{b-1})K_{b-1}\overline{K}_a
\end{aligned}
$$

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$$=\overline{\Delta}_a(K_a\overline{K}_b-\overline{K}_aK_{b-1}^2\overline{K}_b-K_b\overline{K}_a+\overline{K}_bK_{b-1}^2\overline{K}_a)$$

$$=\overline{\Delta}_a(K_a\overline{K}_b-\overline{K}_aK_b).$$

Thus, we have shown (17) for general $a<b$. The case $a>b$ then follows by swapping $a\leftrightarrow b$ in the above, and rearranging.

(**18**) We first show (18a), that is for the case $c<b<a$,

$$[E_c^a,E_b^c]\overset{(1)}{=}[E_b^aE_c^b,E_b^c]-q_b[E_c^bE_b^a,E_b^c]$$

$$\overset{(3a)}{=}E_b^a[E_c^b,E_b^c]+(-)^{[E_c^b]}[E_b^a,E_b^c]E_c^b-q_bE_c^b[E_b^a,E_b^c]-(-)^{[E_b^a][E_b^c]}q_b[E_c^b,E_b^c]E_b^a$$

$$\overset{(19)}{=}E_b^a[E_c^b,E_b^c]-q_b[E_c^b,E_b^c]E_b^a$$

$$\overset{(17)}{=}\overline{\Delta}_b(E_b^a(K_b\overline{K}_c-\overline{K}_bK_c)-q_b(K_b\overline{K}_c-\overline{K}_bK_c)E_b^a)\overset{(11)}{=}\overline{\Delta}_b(q_bK_b\overline{K}_c-\overline{q}_b\overline{K}_bK_c-q_bK_b\overline{K}_c$$
$$+q_b\overline{K}_bK_c)E_b^a$$
$$=\overline{K}_bK_cE_b^a.$$

A parallel proof yields (18c) for the case $b<a<c$,

$$[E_c^a,E_b^c]\overset{(1)}{=}[E_c^a,E_a^cE_b^a]-q_a[E_c^a,E_b^aE_a^c]$$

$$\overset{(3b)}{=}[E_c^a,E_a^c]E_b^a+(-)^{[E_c^a]}E_a^c[E_c^a,E_b^a]-q_a[E_c^a,E_b^a]E_a^c-(-)^{[E_c^a][E_b^a]}q_aE_b^a[E_c^a,E_a^c]$$

$$\overset{(19)}{=}[E_c^a,E_a^c]E_b^a-q_aE_b^a[E_c^a,E_a^c]$$

$$\overset{(17)}{=}\overline{\Delta}_a((K_a\overline{K}_c-\overline{K}_aK_c)E_b^a-q_aE_b^a(K_a\overline{K}_c-\overline{K}_aK_c))$$

$$\overset{(11)}{=}\overline{\Delta}_aE_b^a(q_aK_a\overline{K}_c-\overline{q}_a\overline{K}_aK_c-q_aK_a\overline{K}_c+q_a\overline{K}_aK_c)=E_b^a\overline{K}_aK_c.$$

Taking ω of (18a) yields

$$[E_c^b,E_a^c]\overset{(15,16)}{=}E_a^bK_b\overline{K}_c,\quad c<b<a,$$

and swapping $a\leftrightarrow b$ then yields (18b),

$$[E_c^a,E_b^c]=E_b^aK_a\overline{K}_c,\quad c<a<b.$$

Similarly, taking ω of (18c) yields

$$[E_c^b,E_a^c]\overset{(15,16)}{=}K_a\overline{K}_cE_a^b,\quad b<a<c,$$

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and swapping $a \leftrightarrow b$ then yields (18d),

$$[E_c^a, E_b^c] = K_b \overline{K}_c E_b^a \quad a<b<c.$$

(**20**) In a sense, these results are really glorified Serre relations. We first prove (20a), that is for the case $a<b<c$. Initially assume that $b \neq c-1$ that is $a<b<c-1<c$. Then we have

$$E_a^c E_b^c \overset{(1)}{=} E_a^c (E_{c-1}^c E_b^{c-1} - q_{c-1} E_b^{c-1} E_{c-1}^c) \overset{(21)}{=} E_a^c E_{c-1}^c E_b^{c-1} - (-)^{[E_b^{c-1}]} q_{c-1} E_b^{c-1} E_a^c E_{c-1}^c. \tag{27}$$

Thus, we must investigate $E_a^c E_{c-1}^c$. To this end, observe that our assumption that $b \neq c-1$ means that we have already assumed that $a \neq c-2$, that is, that we safely have $a<c-2<c-1<c$, hence

$$E_a^c E_{c-1}^c \overset{(1)}{=} (E_{c-2}^c E_a^{c-2} - q_{c-2} E_a^{c-2} E_{c-2}^c) E_{c-1}^c \overset{(21)}{=} E_{c-2}^c E_{c-1}^c E_a^{c-2} - q_{c-2} E_a^{c-2} E_{c-2}^c E_{c-1}^c. \tag{28}$$

So now, we must investigate $E_{c-2}^c E_{c-1}^c$, and this falls into two cases. In the general case, if $c \neq m+1$, the Serre relation of (9c) gives us $E_{c-2}^c E_{c-1}^c = q_{c-1} E_{c-1}^c E_{c-2}^c$. On the other hand, if $c = m+1$, then we have

$$E_{m-1}^{m+1} E_m^{m+1} \overset{(1)}{=} (E_m^{m+1} E_{m-1}^m - q_m E_{m-1}^m E_m^{m+1}) E_m^{m+1} \overset{(8)}{=} E_m^{m+1} E_{m-1}^m E_m^{m+1},$$

$$E_m^{m+1} E_{m-1}^{m+1} \overset{(1)}{=} E_m^{m+1} (E_m^{m+1} E_{m-1}^m - q_m E_{m-1}^m E_m^{m+1}) \overset{(8)}{=} -q_m E_m^{m+1} E_{m-1}^m E_m^{m+1},$$

hence $E_{m-1}^{m+1} E_m^{m+1} = -\overline{q}_m E_m^{m+1} E_{m-1}^{m+1}$. Taken together, we have for *any* c,

$$E_{c-2}^c E_{c-1}^c = (-)^{[E_{c-1}^c]} q_c E_{c-1}^c E_{c-2}^c. \tag{29}$$

Installing (29) into (28), we have

$$\begin{aligned} E_a^c E_{c-1}^c &= (-)^{[E_{c-1}^c]} q_c (E_{c-1}^c E_{c-2}^c E_a^{c-2} - q_{c-2} E_a^{c-2} E_{c-1}^c E_{c-2}^c) \\ &\overset{(21)}{=} (-)^{[E_{c-1}^c]} q_c E_{c-1}^c (E_{c-2}^c E_a^{c-2} - q_{c-2} E_a^{c-2} E_{c-2}^c) \\ &\overset{(1)}{=} (-)^{[E_{c-1}^c]} q_c E_{c-1}^c E_a^c. \end{aligned} \tag{30}$$

Installing (30) into (27), we obtain the required (20a) for the special case $a<b<c-1<c$,

$$\begin{aligned} E_a^c E_b^c &= (-)^{[E_{c-1}^c]} q_c (E_{c-1}^c E_a^c E_b^{c-1} - (-)^{[E_b^{c-1}]} q_{c-1} E_b^{c-1} E_{c-1}^c E_a^c) \\ &\overset{(21)}{=} (-)^{[E_{c-1}^c]} (-)^{[E_b^{c-1}]} q_c (E_{c-1}^c E_b^{c-1} - q_{c-1} E_b^{c-1} E_{c-1}^c) E_a^c \\ &\overset{(1)}{=} (-)^{[E_b^c]} q_c E_b^c E_a^c. \end{aligned}$$

If in fact $b=c-1$, then if also $a \neq c-2$, then (30) covers our result, and if $a=c-2$, then (29) covers it. Together, we have (20a) for all $a<b<c$. A parallel proof covers (20b), that is, the case $c<a<b$; but we omit this. Before proceeding, we condense our notation. We have

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$$E^c_a E^c_b = \begin{cases} (-)^{[E^c_b]} q_c E^c_b E^c_a, & a<b<c, \\ (-)^{[E^c_a]} q_c E^c_b E^c_a, & c<a<b. \end{cases}$$

Combining these two results, we may write, for $a<b$,

$$E^c_a E^c_b = (-)^{[E^c_{z(a,b,c)}]} q_c E^c_b E^c_a \quad \text{if } z(a,b,c) \neq c, \tag{31}$$

where $z(a,b,c)$ is a little function which picks out the median element of the set of natural numbers $\{a,b,c\}$. Applying ω to (31) and cross multiplying yields

$$E^a_c E^b_c \overset{(15)}{=} (-)^{[E^{z(a,b,c)}_c]} q_c E^b_c E^a_c \quad \text{if } z(a,b,c) \neq c,$$

which is immediately seen to cover (20c) and (20d),

$$E^a_c E^b_c = \begin{cases} (-)^{[E^b_c]} q_c E^b_c E^a_c, & a<b<c, \\ (-)^{[E^a_c]} q_c E^b_c E^a_c, & c<a<b. \end{cases}$$

(**22**) Beginning with the case $a<c<b<d$, we have

$$\begin{aligned} [E^a_b, E^c_d] &\overset{(2)}{=} E^a_b E^c_d - (-)^{[E^a_b][E^c_d]} E^c_d E^a_b \\ &\overset{(1)}{=} E^a_b (E^c_b E^b_d - \overline{q}_b E^b_d E^c_b) - (-)^{[E^c_b]} (E^c_b E^b_d - \overline{q}_b E^b_d E^c_b) E^a_b \\ &= (E^a_b E^c_b E^b_d - (-)^{[E^c_b]} E^c_b E^b_d E^a_b) - \overline{q}_b (E^a_b E^b_d E^c_b - (-)^{[E^c_b]} E^b_d E^c_b E^a_b). \end{aligned}$$

Now, for $a<c<b$, by (20c), we have $E^a_b E^c_b = (-)^{[E^c_b]} q_b E^c_b E^a_b$. Installing this, we quickly obtain (22a),

$$\begin{aligned} [E^a_b, E^c_d] &= (-)^{[E^c_b]} E^c_b (q_b E^a_b E^b_d - E^b_d E^a_b) - \overline{q}_b (E^a_b E^b_d - \overline{q}_b E^b_d E^a_b) E^c_b \\ &\overset{(1)}{=} (-)^{[E^c_b]} q_b E^c_b E^a_d - \overline{q}_b E^a_d E^c_b \overset{(21)}{=} E^a_d E^c_b (q_b - \overline{q}_b) = \Delta_b E^a_d E^c_b. \end{aligned}$$

Swapping $a \leftrightarrow c$ and $b \leftrightarrow d$ in (22a) then yields

$$[E^c_d, E^a_b] = \Delta_d E^c_b E^a_d, \quad c<a<d<b. \tag{32}$$

Reversing both the commutator and the RHS product yields

$$-(-)^{[E^c_d][E^a_b]} [E^a_b, E^c_d] \overset{(21)}{=} (-)^{[E^c_b][E^a_d]} \Delta_d E^a_d E^c_b,$$

but for $c<a<d<b$, in fact $[E^c_d][E^a_b] = [E^c_b][E^a_d] = [E^a_d]$, yielding (22b),

$$[E^a_b, E^c_d] = -\Delta_d E^a_d E^c_b, \quad c<a<d<b.$$

Next, applying ω to (22a) yields

$$[E^d_c, E^b_a] \overset{(15,16)}{=} -\Delta_b E^b_c E^d_a, \quad a<c<b<d.$$

Reversing both the commutator and the RHS product yields (22c),

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$$[E^b_a, E^d_c] \overset{(21)}{=} \Delta_b E^d_a E^b_c, \quad a<c<b<d.$$

Last, applying ω to (32) yields (22d),

$$[E^b_a, E^d_c] \overset{(15,16)}{=} -\Delta_d E^d_a E^b_c, \quad c<a<d<b.$$

(**23**) We first show (23a), that is for the case $a<c<b<d$. We have

$$\begin{aligned}
[E^a_b, E^d_c] &\overset{(1)}{=} [E^a_b, E^d_b E^b_c] - q_b [E^a_b, E^b_c E^d_b] \\
&\overset{(3b)}{=} [E^a_b, E^d_b] E^b_c + (-)^{[E^d_b][E^a_b]} E^d_b [E^a_b, E^b_c] - q_b([E^a_b, E^b_c] E^d_b + (-)^{[E^a_b][E^b_c]} E^b_c [E^a_b, E^d_b]) \\
&\overset{(19)}{=} E^d_b [E^a_b, E^b_c] - q_b [E^a_b, E^b_c] E^d_b \\
&\overset{(18d)}{=} E^d_b K_c \overline{K}_b E^a_c - q_b K_c \overline{K}_b E^a_c E^d_b \\
&\overset{(11,21)}{=} -\Delta_b \overline{K}_b K_c E^a_c E^d_b .
\end{aligned}$$

Applying ω to (23a) yields

$$[E^c_d, E^b_a] \overset{(15,16)}{=} \Delta_b E^b_d E^c_a \overline{K}_c K_b, \quad a<c<b<d, \tag{33}$$

and swapping $a \leftrightarrow c$ and $b \leftrightarrow d$ then yields (23b),

$$[E^a_b, E^d_c] = \Delta_d E^d_b E^a_c \overline{K}_a K_d, \quad c<a<d<b.$$

Next, reversing the commutator in (33) yields

$$[E^b_a, E^c_d] \overset{(16)}{=} -(-)^{[E^b_a][E^c_d]} \Delta_b E^b_d E^c_a \overline{K}_c K_b .$$

However, for the case $a<c<b<d$, we have $[E^b_a][E^c_d] = [E^b_c]$, thus, $(-)^{[E^b_a][E^c_d]} \Delta_b = (-)^{[E^b_c]} (-)^{[b]} \Delta = (-)^{[c]} \Delta = \Delta_c$, yielding (23c),

$$[E^b_a, E^c_d] = -\Delta_c E^b_d E^c_a \overline{K}_c K_b, \quad a<c<b<d.$$

Last, applying ω to (23c) yields

$$[E^d_c, E^a_b] \overset{(15,16)}{=} \Delta_c \overline{K}_b K_c E^a_c E^d_b, \quad a<c<b<d,$$

and then swapping $a \leftrightarrow c$ and $b \leftrightarrow d$ yields (23d),

$$[E^b_a, E^c_d] = \Delta_a \overline{K}_d K_a E^c_a E^b_d, \quad c<a<d<b.$$

$\square$

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IV. DISCUSSION

Of some interest is that we may use our PBW commutator lemma to show that (8) in fact generalizes to the nonsimple odd generators, that is

$$(E_b^a)^2 = 0,$$

for any indices a, b such that $[a] \neq [b]$. The proof of this statement is left as an (easy) exercise involving (20).

Now that it is established, we may concentrate the notation of our lemma—this is useful for encoding purposes.

The entirety of (19) and (20) may be summarized by

$$E_c^a E_c^b = \kappa E_c^b E_c^a \quad \text{and} \quad E_a^c E_b^c = \kappa E_b^c E_a^c, \quad \text{any} \quad a \neq b \neq c,$$

where

$$\kappa \triangleq \begin{cases} 1 & \text{if} \quad z(a,b,c) = c, \\ (-)^{[E_c^{z(a,b,c)}]} \overline{q}_c^{S_b^a} & \text{otherwise} \end{cases}$$

and where $z(a,b,c)$ is our little function which picks out the median element of the set of three distinct natural numbers $\{a,b,c\}$. (The 1 factor follows as $[E_c^a][E_c^b] = 0$ for c strictly between a and b.)

The entirety of (21) to (23) may be summarized by

$$[E_b^a, E_d^c] = \begin{cases} +\Delta_b E_d^a E_b^c, & a < c < b < d, \\ -\Delta_d E_d^a E_b^c, & c < a < d < b, \\ +\Delta_a E_b^c E_d^a, & b < d < a < c, \\ -\Delta_c E_b^c E_d^a, & d < b < c < a, \\ -\Delta_b \overline{K}_b K_d E_d^a E_b^c, & a < d < b < c, \\ +\Delta_c E_b^c E_d^a \overline{K}_a K_c, & d < a < c < b, \\ -\Delta_c E_d^a E_b^c \overline{K}_c K_a, & b < c < a < d, \\ +\Delta_b \overline{K}_d K_b E_b^c E_d^a, & c < b < d < a, \\ 0, \quad a \neq b \neq c \neq d & \text{otherwise.} \end{cases}$$

Finally, we mention that the consistency (if not the veracity) of our lemma is also supported by extensive computer tests using MATHEMATICA. By this, we mean that we confirm that

$$\texttt{NormalOrder}(XY) = \texttt{NormalOrder}(\texttt{ExpandNS}(XY)), \tag{34}$$

for a range of $U_q[\mathrm{gl}(m|n)]$ nonsimple generators X, Y, where $\texttt{NormalOrder}(X)$ is a function which renders X in a normal form, and $\texttt{ExpandNS}(X)$ is a function which recursively expands all nonsimple generators in X, using (1).

To be more specific, let the *height* of generator $X \equiv E_b^a$ be $|a-b|$; this is a measure of its distance from simplicity. For $U_q[\mathrm{gl}(m|n)]$, it varies from 0 (for Cartan generators), to 1 (for simple non-Cartan generators); and then for the nonsimple generators from a minimum of 2 to a maximum of $m+n-1$ for the maximally nonsimple E_1^{m+n} and E_{m+n}^1.

Then, we confirm that our code satisfies (37), for all $U_q[\mathrm{gl}(m|n)]$ generators X,Y of height at most $m+n-1$ for all m,n such that $m+n \leqslant 5$; at most 3 for $m+n \leqslant 10$; and at most 2 for $m+n \leqslant 18$. The computational expense in performing these checks rises at least exponentially with

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height, so we have to abandon our calculations at this point. However, our results do amount to a complete consistency check of our lemma, for all $U_q[\mathrm{gl}(m|n)]$ such that $m+n \leqslant 5$.

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