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Simple operational interpretation of the fidelity of mixed states

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This Brief Report presents a corollary to Uhlmann's theorem which provides a simple operational interpretation of the fidelity of mixed states.

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For any states ρ and σ, the *fidelity* is defined by $F(\rho,\sigma)\equiv \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$. (See [1] for other expressions for the fidelity and extensive references.) When applied to pure states $|\psi\rangle$ and $|\phi\rangle$, this expression reduces to $F(|\psi\rangle,|\phi\rangle) = |\langle\psi|\phi\rangle|$ which has an operational interpretation in terms of the distinguishability of the two states: when testing whether or not $|\phi\rangle$ is the same as $|\psi\rangle$, the probability that $|\phi\rangle$ passes the test is $F(|\psi\rangle,|\phi\rangle)^2$. Unfortunately, such an interpretation is lacking when the input states are mixed. In this Brief Report we present a result which gives the mixed-state fidelity a simple operational interpretation. It is a corollary to Uhlmann's theorem [2,3], which provides a formula for the fidelity in terms of purifications.

Uhlmann's theorem. Let ρ and σ be two states of a quantum system Q, and let E be a second system with dimension greater than or equal to the dimension of Q. Then

$$F(\rho,\sigma)=\max|\langle\psi_0|\phi_0\rangle|, \tag{1}$$

where the maximization runs over all $|\psi_0\rangle$ and $|\phi_0\rangle$ which are *purifications of* ρ *and* σ in EQ.

The corollary gives an alternative formula for the fidelity in which the purifying systems of Uhlmann's theorem have been replaced by quantum operations.

Corollary. Let ρ and σ be two states of a quantum system Q. Then

$$F(\rho,\sigma)=\max|\langle\psi|\phi\rangle|, \tag{2}$$

where the maximization runs over all pure states $|\psi\rangle$ and $|\phi\rangle$ which are *taken to* ρ and σ *by some quantum operation* $\mathcal{E}$, that is, $\mathcal{E}(\psi)=\rho$, and $\mathcal{E}(\phi)=\sigma$. [Note: a quantum operation is a completely positive trace-preserving map [4], and we use the shorthand $\mathcal{E}(\alpha)\equiv\mathcal{E}(|\alpha\rangle\langle\alpha|)$.]

An interpretation suggested by this characterization is the following: If ρ and σ are the (potentially mixed) output states of a noisy channel, then the fidelity $F(\rho,\sigma)$ is an upper bound on the overlap of the input states, assuming they were pure.

Proof of the corollary. Denote the quantity on the right-hand side of Eq. (2) by $F'(\rho,\sigma)$. We aim to show $F'(\rho,\sigma)=F(\rho,\sigma)$ using Uhlmann's theorem. The crucial step is a relationship between quantum operations and purifications. Suppose Q is initially in the state $|\beta\rangle$, and $\mathcal{E}$ takes $|\beta\rangle$ to τ. We can always introduce a second system E (assumed to be in the initial state $|0\rangle$, and with dimension equal to the dimension of Q squared) so that

$$\tau=\mathcal{E}(\beta)=\mathrm{tr}_E[U|0\rangle|\beta\rangle\langle 0|\langle\beta|U^\dagger] \tag{3}$$

for some unitary U which acts on EQ. Comparing the two sides of this equation, we see that $U|0\rangle|\beta\rangle$ purifies τ in the joint system EQ.

So, for every pair $|\psi\rangle$, $|\phi\rangle$, the condition "$\exists\ \mathcal{E}:\mathcal{E}(\psi) = \rho$ and $\mathcal{E}(\phi)=\sigma$" can be rewritten as "$\exists U: U|0\rangle|\psi\rangle$ and $U|0\rangle|\phi\rangle$ purify ρ and σ, respectively." Therefore, from Uhlmann's theorem, we have

$$F(\rho,\sigma)\geqslant\max|\langle 0|\langle\psi|U^\dagger U|0\rangle|\phi\rangle|=\max|\langle\psi|\phi\rangle|, \tag{4}$$

where the maximization runs over states $|\psi\rangle$, $|\phi\rangle$ satisfying the condition above. But the quantity on the right-hand side is just $F'(\rho,\sigma)$, so we have $F(\rho,\sigma)\geqslant F'(\rho,\sigma)$.

Furthermore, every pair $|\psi_0\rangle$, $|\phi_0\rangle$ in the composite space EQ can be obtained by a unitary transformation on a pair $|0\rangle_E|\psi\rangle_Q$, $|0\rangle_E|\phi\rangle_Q$, provided $\langle\psi|\phi\rangle=\langle\psi_0|\phi_0\rangle$. Therefore the maximization in Eq. (4) runs over the same values as the maximization in Eq. (1), so we have $F(\rho,\sigma)=F'(\rho,\sigma)$, which completes the proof.

The value of this corollary lies in providing an operational interpretation for the fidelity. Furthermore, this approach may be valuable for suggesting new approaches to problems. As an example, this characterization suggests a simple, intuitive proof of the well-known fact that no quantum operation can increase the distinguishability of two quantum states [5]. In terms of the fidelity, this means that for any quantum operation $\mathcal{G}$ which acts on Q, $F[\mathcal{G}(\rho),\mathcal{G}(\sigma)]\geqslant F(\rho,\sigma)$ for all states ρ and σ of Q. The proof is immediate from the corollary: Choose $|\psi\rangle$, $|\phi\rangle$ and $\mathcal{E}$ such that $\mathcal{E}(\psi)=\rho$, $\mathcal{E}(\phi)=\sigma$, and $F(\rho,\sigma)=|\langle\psi|\phi\rangle|$. Then $\mathcal{G}\circ\mathcal{E}$ takes $|\psi\rangle$ and $|\phi\rangle$ to $\mathcal{G}(\rho)$ and $\mathcal{G}(\sigma)$, respectively. Therefore, by the corollary, $F[\mathcal{G}(\rho),\mathcal{G}(\sigma)]\geqslant|\langle\psi|\phi\rangle|=F(\rho,\sigma)$.

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