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# Positive-*P* and Wigner representations for quantum-optical systems with nonorthogonal modes

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We generalize the basic concepts of the positive-*P* and Wigner representations to unstable quantum-optical systems that are based on nonorthogonal quasimodes. This lays the foundation for a quantum description of such systems, such as, for example an unstable cavity laser. We compare both representations by calculating the tunneling times for an unstable resonator optical parametric oscillator.

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## INTRODUCTION

The usual quantization procedure for the electromagnetic field involves an infinite set of quantized harmonic oscillators, each associated with an orthogonal field mode. Since the electromagnetic field Hamiltonian is Hermitian this strategy is always possible. On the other hand, if we restrict the field Hamiltonian to a finite volume with an optically unstable geometry this general feature fails. It turns out to be impossible to isolate the system due to the non-Hermitian boundary conditions. Within a classical optics description this represents no fundamental problem since the coupling to the outside reservoir may simply be described by an introduced damping. Within a quantum description the procedure is more delicate since effects such as decoherence have to be included. Usually this is handled by introducing a master equation for the field modes. On the other hand, unstable resonator lasers (for example) are excluded from this general quantization procedure since the resonator will contain nonorthogonal modes. Of course this can, in principle, be passed over by embedding the unstable system in a larger volume and quantizing in this larger closed volume. This treatment is usually known as the "modes of the universe" description [1] but an attempt to use it computationally rapidly exceeds the numerical capacity of any computer. In this paper we present an alternative quantum description based on a set of nonorthogonal quasimodes. Starting from the classical method we generalize the usual quantization procedure and end up with a phenomenological master equation describing the time evolution of the unstable system separately. It is a well-known fact that the nonorthogonality of the cavity modes gives rise to enhanced quantum fluctuations, called excess noise [2,3]. This noise amplification becomes very clear within our description.

This approach has already been used to calculate the spontaneous emission rate of a single atom in an unstable cavity [4,5], finding that the excess-noise factor due to the nonorthogonality of the cavity modes can drastically enhance the spontaneous emission rate into the cavity modes. Siegman's law for the enhanced linewidth of an unstable cavity laser was also recovered.

As a natural next step we extended the theory to the process of parametric down conversion [6,7]. Again we found that the excess-noise factor plays an important role for this genuine quantum-noise-driven process. The intensity of signal photons may be strongly enhanced and the threshold of oscillation is noticeably lowered. On the other hand, the gain in intensity is accompanied by a decrease in field correlations. An enhanced twin-photon-generation rate in a stable resonator has also recently been experimentally demonstrated [8] at the expense of a prolonged photon-coincidence interval (narrower bandwidth of the emitted photons).

In this work we generalize the basic concept of the positive-*P* and Wigner representations to the case of unstable quantum-optical systems that are based on nonorthogonal quasimodes. The generalization to other representations is then straightforward. As an application of the presented phase-space methods we calculate the tunneling times between the two possible steady states of the unstable OPO. It has been shown for stable cavities that the tunneling time predictions can be quite different [9] depending on whether they are calculated using the positive-*P* or truncated Wigner representations. It is, therefore, of interest to calculate how the introduced excess noise in unstable resonators affects the predictions.

## I. CAVITY QED IN TERMS OF NONORTHOGONAL QUASIMODES

For the free electromagnetic field confined to a volume with partially absorbing boundaries it is usually possible to find a complete set of quasimodes $\{u_{\mathbf{n}}(\mathbf{x})\}$, also known as matched modes. The multiple index $\mathbf{n}$ includes all longitudinal, transverse, and polarizational degrees of freedom. The quasimodes are defined as self-reproducing field configurations after one full round trip. Within the paraxial approximation this corresponds to eigenfunctions of Huygens' integral operator, i.e., $L(u_{\mathbf{n}})=\gamma_{\mathbf{n}}u_{\mathbf{n}}$ (see, e.g., Ref. [10]). To require the eigenvalue $\gamma_{\mathbf{n}}$ to be real and positive yields the allowed wave vectors $k_{\mathbf{n}}$. An analytically soluble example is a one-dimensional (1D) symmetric unstable resonator with a Gaussian reflectivity profile [5]. In general, these modes are not necessarily orthogonal, but are biorthogonal to a second set of adjoint modes $\{v_{\mathbf{n}}(\mathbf{x})\}$, such that

 

$$\int_V d\mathbf{x}\, v_{\mathbf{n}}^*(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}) = \delta_{\mathbf{nm}}. \tag{1.1}$$

In fact, the adjoint modes correspond to quasimodes traveling in the opposite direction. Whereas the matched modes can be normalized to unity, Eq. (1.1) gives rise to a normalization constant $K_{\mathbf{n}}$ for the adjoint modes, called the Petermann excess-noise factor [2]. The reason for this name will become clear below. In fact the connection between the excess-noise factor and the adjoint modes was found 10 yr after the prediction of $K$-enhanced laser noise by Siegman [3].

These normalization properties can be concisely defined by

$$\int_V d\mathbf{x}\, u_{\mathbf{n}}^*(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}) = A_{\mathbf{nm}} \quad \text{with} \quad A_{\mathbf{nn}} = 1, \tag{1.2}$$

$$\int_V d\mathbf{x}\, v_{\mathbf{n}}^*(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}) = B_{\mathbf{nm}} \quad \text{with} \quad B_{\mathbf{nn}} = K_{\mathbf{n}}, \tag{1.3}$$

where the integral extends over the resonator volume. Obviously $A$ and $B$ are just inverses. For the usual case of stable geometry the adjoint modes are identical to the matched modes and the overlap matrices $A$ and $B$ become simply identity matrices. A further general property of these quasimodes is that they are complete. Hence every field distribution may be written as

$$A(\mathbf{x},t) = \sum_{\mathbf{n}} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_{\mathbf{n}}}} [a_{\mathbf{n}}(t) u_{\mathbf{n}}(\mathbf{x}) + a_{\mathbf{n}}^\dagger(t) u_{\mathbf{n}}^*(\mathbf{x})], \tag{1.4}$$

$$E(\mathbf{x},t) = i \sum_{\mathbf{n}} \sqrt{\frac{\hbar \omega_{\mathbf{n}}}{2\epsilon_0}} [a_{\mathbf{n}}(t) u_{\mathbf{n}}(\mathbf{x}) - a_{\mathbf{n}}^\dagger(t) u_{\mathbf{n}}^*(\mathbf{x})], \tag{1.5}$$

where the $\omega_{\mathbf{n}}$ denote the resonance frequencies of the quasimodes. Alternatively one may express the mode operators in terms of field operators

$$a_{\mathbf{n}}(t) = -i \sqrt{\frac{\epsilon_0}{2\hbar \omega_{\mathbf{n}}}} \int dx\, v_{\mathbf{n}}^*(\mathbf{x}) [E(\mathbf{x},t) + i\omega_{\mathbf{n}} A(\mathbf{x},t)], \tag{1.6}$$

$$a_{\mathbf{n}}^\dagger(t) = i \sqrt{\frac{\epsilon_0}{2\hbar \omega_{\mathbf{n}}}} \int d\mathbf{x}\, v_{\mathbf{n}}(\mathbf{x}) [E(\mathbf{x},t) - i\omega_{\mathbf{n}} A(\mathbf{x},t)], \tag{1.7}$$

and consequently find the commutation relations for these operators due to the canonical equal time commutation relations [11],

$$[a_{\mathbf{n}}, a_{\mathbf{m}}^\dagger] = \frac{\omega_{\mathbf{n}} + \omega_{\mathbf{n}}}{2\sqrt{\omega_{\mathbf{n}}\omega_{\mathbf{m}}}} \int d\mathbf{x}\, v_{\mathbf{n}}^*(\mathbf{x}) v_{\mathbf{m}}(\mathbf{x}) \approx B_{\mathbf{nm}}. \tag{1.8}$$

The frequency-dependent prefactor can be neglected for a large range of physically interesting cases as in, for example, the optical or infrared regime. For some purposes it is useful to define a second set of operators corresponding to an expansion in the adjoint modes $\{b_{\mathbf{n}} = A_{\mathbf{nm}} a_{\mathbf{m}}, b_{\mathbf{n}}^\dagger = A_{\mathbf{mn}} a_{\mathbf{m}}^\dagger\}$ for which one finds

$$[b_{\mathbf{n}}, b_{\mathbf{m}}^\dagger] = \frac{\omega_{\mathbf{n}} + \omega_{\mathbf{m}}}{2\sqrt{\omega_{\mathbf{n}}\omega_{\mathbf{m}}}} \int d\mathbf{x}\, u_{\mathbf{n}}^*(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}) \approx A_{\mathbf{nm}}, \tag{1.9}$$

$$[a_{\mathbf{n}}, b_{\mathbf{m}}^\dagger] = \frac{\omega_{\mathbf{n}} + \omega_{\mathbf{m}}}{2\sqrt{\omega_{\mathbf{n}}\omega_{\mathbf{m}}}} \int d\mathbf{x}\, v_{\mathbf{n}}^*(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}) = \delta_{\mathbf{nm}}. \tag{1.10}$$

Using the field expansion of Eqs. (1.4), (1.5) and taking into account the fact that these quasimodes obey the Helmholtz equation, one finds for the free-field Hamiltonian

$$H_F = \frac{1}{2} \int d\mathbf{x} \colon \left( \epsilon_0 E^2(\mathbf{x},t) + \frac{1}{\mu_0} B^2(\mathbf{x},t) \right) \colon \tag{1.11}$$

$$= \sum_{\mathbf{nm}} \hbar \frac{(\omega_{\mathbf{n}} + \omega_{\mathbf{n}})}{2} A_{\mathbf{nm}} a_{\mathbf{n}}^\dagger a_{\mathbf{m}} \tag{1.12}$$

within the same approximation as in Eq. (1.8). In general, there may occur rapidly oscillating terms such as $a_{\mathbf{n}} a_{\mathbf{m}}$, $a_{\mathbf{n}}^\dagger a_{\mathbf{m}}^\dagger$ whose effects vanish in the mean. For systems with backward and forward propagation symmetry they cancel exactly.

So far we have not considered the effect of any losses, but for optically unstable systems this is an unavoidable feature. Since there exists no closed optical path we have a continuous flux of energy towards infinity even for perfectly reflecting mirrors. Physically the energy is scattered into the nonparaxial field and the mode amplitudes decay exponentially with a mean rate $\kappa_{\mathbf{n}}$ (determined by the quasimode eigenvalue $\gamma_{\mathbf{n}}$). In a proper quantum treatment this has to be included by an input-output coupling [12,13]. It is a well-known fact that an open system cannot be described by a single Hamiltonian. Usually it is possible to find an effective, but non-Hermitian, Hamiltonian $H_{\text{eff}}$, which includes the classical mode damping [14]. In addition, a recycling term has to be introduced to preserve the commutation relations. This procedure is known as a master-equation treatment for the density operator $\rho$ that describes the field state. Unfortunately for the system we consider here, i.e., an unstable optical resonator, this procedure is rather involved, since the diffraction losses are indistinguishable from the losses due to mirror transmission in this picture. (Even for perfectly reflecting mirrors the loss rate is still finite). Although a satisfactory derivation of a master equation is to our knowledge, not known or might even be impossible [15], we may give a mathematically and physically clear and consistent procedure describing the system together with any losses. The given form is inspired by the usual master-equation approach and the calculated field Hamiltonian [Eq. (1.12)]. Including some basic requirements we find for a vacuum input

$$\dot{\rho} = -\frac{i}{\hbar}(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + i \sum_{\mathbf{nm}} A_{\mathbf{nm}} (\tilde{\omega}_{\mathbf{m}} - \tilde{\omega}_{\mathbf{n}}^*) a_{\mathbf{m}} \rho a_{\mathbf{n}}^\dagger, \tag{1.13}$$

with

$$H_{\rm eff}=\hbar\sum_{\mathbf{n}}\ \tilde{\omega}_{\mathbf{n}}b_{\mathbf{n}}^{\dagger}a_{\mathbf{n}}, \tag{1.14}$$

and

$$\tilde{\omega}_{\mathbf{n}}=\omega_{\mathbf{n}}-i\kappa_{\mathbf{n}}. \tag{1.15}$$

To guarantee self-consistency the given master equation is of Lindblad form, it preserves the trace of any operator, such as the density operator, it preserves the commutation relations for all mode operators, such as $a_{\mathbf{n}}$, $a_{\mathbf{n}}^{\dagger}$ and it guarantees the damped oscillation of $a_{\mathbf{n}}$ and $a_{\mathbf{n}}^{\dagger}$ known from the classical model, i.e.,

$$\langle\dot{a}_{\mathbf{n}}\rangle=-(\kappa_{\mathbf{n}}+i\omega_{\mathbf{n}})\langle a_{\mathbf{n}}\rangle, \tag{1.16}$$

$$\langle\dot{a}_{\mathbf{n}}^{\dagger}\rangle=-(\kappa_{\mathbf{n}}-i\omega_{\mathbf{n}})\langle a_{\mathbf{n}}^{\dagger}\rangle. \tag{1.17}$$

In principle, this treatment is very similar to the quantization procedure proposed by Dutra and Nienhuis [15]. They quantize the electromagnetic field for a longitudinal Fabry-Perot resonator by expanding the field in self-reproducing Fox-Li modes. For this specific example one may clearly distinguish between the fields inside and outside the cavity. A similar procedure has to be performed to include the transverse dynamics of unstable optical systems. The full wave equation, without the paraxial approximation, has to be solved, yielding a ''modes of the universe description''. Tracing over the ''nonparaxial'' degrees of freedom then leads to a master equation for the system operators. Although the procedure is very clear in principle, it turns out to be almost impossible to solve in practice.

Let us now explore the consequences of the changed dynamics. Since the free Hamiltonian $H_F$ is Hermitian, it must yield an orthogonal basis of eigenstates. Unfortunately these are not stabilized by the full dynamics that includes the damping. It turns out that the eigenstates of the effective Hamiltonian $H_{\rm eff}$ represent a more adequate basis, a so-called ''damping basis'' [14]. Interestingly the free field Hamiltonian describes the ''real part'' of the non-Hermitian operator $H_{\rm eff}$, i.e.,

$$H_F=\mathrm{Re}\{H_{\rm eff}\}=\frac{1}{2}(H_{\rm eff}+H_{\rm eff}^{\dagger}). \tag{1.18}$$

In general, the eigenstates of the two operators do not coincide, but, again neglecting the variation of $\omega_{\mathbf{n}}$ within one longitudinal set of quasimodes, we find that they do. We note here that the eigenstates of the effective Hamiltonian are very similar to the usual Fock-states, but created using the adjoint mode operators. For a distinct mode $n$ they take the form

$$|N_n\rangle=\frac{b_n^{\dagger N}}{\sqrt{N!}}|0\rangle. \tag{1.19}$$

These states describe field eigenstates containing $N$ quanta of ''energy'' $E_N=\hbar(\omega_n-i\kappa_n)N$ in the quasimode $n$. Hence applying the operators $b_n^{\dagger}$ or $a_n$ from the left corresponds to ''photon'' creation and annihilation, respectively. As an illustrative example of how these eigenstates evolve under the given master equation, we consider an initially $N$-photon state $\rho(0)=|N_k\rangle\langle N_k|$. We find a familiar time evolution since $\rho$ simply decays as

$$\dot{\rho}(0)=-2N\kappa_k\rho(0)+2N\kappa_k|N-1_k\rangle\langle N-1_k|. \tag{1.20}$$

In a similar way to that used to find the new set of eigenstates, we can define generalized Bargmann states in terms of the creation operators. Using a vectorial notation $\vec{b}^{\dagger}\cdot\vec{\alpha}\equiv\Sigma_{\mathbf{n}}b_{\mathbf{n}}^{\dagger}\alpha_{\mathbf{n}}$, we find

$$\|\vec{\alpha}\rangle=\exp\{\vec{b}^{\dagger}\cdot\vec{\alpha}\}\|0\rangle. \tag{1.21}$$

These states can immediately be normalized by including the factor $\exp\{-\vec{\alpha}^{\dagger}\cdot A\cdot\vec{\alpha}/2\}$, thus giving a generalized form of the coherent states, but for convenience we will continue to use the nonnormalized form. As usual these states are eigenstates of the annihilation operators $a_{\mathbf{n}}$ and hence fulfill the relations

$$a_{\mathbf{n}}\|\vec{\alpha}\rangle=\alpha_{\mathbf{n}}\|\vec{\alpha}\rangle, \tag{1.22}$$

$$\frac{\partial}{\partial\alpha_{\mathbf{n}}}\|\alpha_{\mathbf{n}}\rangle=b_{\mathbf{n}}^{\dagger}\|\vec{\alpha}\rangle=\sum_{\mathbf{m}}\ A_{\mathbf{mn}}a_{\mathbf{m}}^{\dagger}\|\vec{\alpha}. \tag{1.23}$$

The goal of the present paper is to generalize existing phase-space techniques involving orthogonal modes to the case of unstable optical systems. We show that this master equation can similarly be transformed into stochastic differential equations suitable for numerical computation. In particular, we consider the positive-*P* and Wigner representations. Other representations may easily be generalized in a similar way. As we shall see the generalized definitions are slightly different from the usual orthogonal-mode analysis (see, e.g., Ref. [16]).

## II. POSITIVE-*P* REPRESENTATION

For a given field-density operator $\rho$, various types of characteristic functions can be defined, from which suitable operator expectation values may be inferred. For the positive-*P* [17] (as well as for the Glauber *P*) distribution the normally ordered characteristic function is used, which gives rise to normally ordered operator expectation values, i.e.,

$$\chi_N(\vec{\eta})=\mathrm{Tr}\{\rho e^{\vec{a}^{\dagger}\cdot\vec{\eta}}e^{-\vec{\eta}^{+}\cdot\vec{a}}\}. \tag{2.1}$$

As usual operator moments correspond to derivatives of the characteristic function, we have, for example,

$$\langle a_{\mathbf{n}}\rangle=-\frac{\partial}{\partial\eta_{\mathbf{n}}^{*}}\chi_N(\vec{\eta}\to 0). \tag{2.2}$$

This can be transformed into a quasi-probability distribution for the independent variables $\alpha_{\mathbf{n}}$, $\alpha_{\mathbf{n}}^{+}$ corresponding to the operators $a_{\mathbf{n}}$, $a_{\mathbf{n}}^{\dagger}$,

$$P(\vec{\alpha},\vec{\alpha}^{+})=\int \frac{d^{2}\vec{\eta}}{\pi^{2N}}e^{\vec{\eta}^{+}\cdot\vec{\alpha}-\vec{\alpha}^{\dagger}\cdot\vec{\eta}}\chi_{N}(\vec{\eta}), \tag{2.3}$$

with $N$ denoting the total number of considered modes. We should point out here that, as with the usual definition of the positive-$P$ distribution this integral does not necessarily converge. The positive-$P$ distribution is defined by the expansion of a given density operator in nondiagonal coherent-state projection operators, i.e.,

$$\rho=\int d^{2}\vec{\alpha}d^{2}\vec{\alpha}^{+}\frac{\|\vec{\alpha}\rangle\langle\vec{\alpha}^{+}\|}{\langle\vec{\alpha}^{\dagger}\|\vec{\alpha}\rangle}P(\vec{\alpha},\vec{\alpha}^{\dagger}), \tag{2.4}$$

with

$$\langle\vec{\alpha}^{+}\|\vec{\alpha}\rangle=e^{\vec{\alpha}^{+}\cdot A\cdot\vec{\alpha}}. \tag{2.5}$$

This nondiagonal representation is chosen because, for many interesting nonlinear optical processes, the Glauber $P$ leads to Fokker-Planck equations with nonpositive-definite diffusion matrices. At the expense of the variables $\alpha_{\mathbf{n}}$ and $\alpha_{\mathbf{n}}^{\dagger}$ no longer remaining complex conjugates except in the mean of a large number of integrations of a stochastic differential equation, the phase-space doubling used in the positive-$P$ always allows for a positive-definite diffusion matrix [17]. We should note here that Eq. (2.3) is only one possible choice for the given distribution. Although Eq. (2.3) is not a general expression and does not exist in some cases we will use it to make clear the analogy to other representations such as the Glauber $P$ or Wigner. However, for this definition one may easily show that similar transformation rules are valid, as in the usual orthogonal mode case. Taking into account the coherent state properties Eqs. (1.22), (1.23) we find

$$a_{\mathbf{n}}\rho\rightarrow\alpha_{\mathbf{n}}P(\vec{\alpha},\vec{\alpha}^{+}), \tag{2.6}$$

$$\rho a_{\mathbf{n}}^{\dagger}\rightarrow\alpha_{\mathbf{n}}^{+}P(\vec{\alpha},\vec{\alpha}^{+}), \tag{2.7}$$

$$\rho a_{\mathbf{n}}\rightarrow\left(\alpha_{\mathbf{n}}-\sum_{\mathbf{m}}B_{\mathbf{nm}}\frac{\partial}{\partial\alpha_{\mathbf{m}}^{+}}\right)P(\vec{\alpha},\vec{\alpha}^{+}), \tag{2.8}$$

$$a_{\mathbf{n}}^{\dagger}\rho\rightarrow\left(\alpha_{n}^{+}-\sum_{\mathbf{m}}B_{\mathbf{mn}}\frac{\partial}{\partial\alpha_{\mathbf{m}}}\right)P(\vec{\alpha},\vec{\alpha}^{+}). \tag{2.9}$$

The only limitation is that $P(\vec{\alpha},\vec{\alpha}^{+})$ must fall off sufficiently fast for large $\alpha_{\mathbf{n}}$, $\alpha_{\mathbf{n}}^{+}$, which is also a limitation for the normal positive-$P$. For a given master equation such as Eq. (1.13) one may now immediately deduce the corresponding Fokker-Planck-equation via these operator correspondences. Using standard techniques this may further be mapped onto stochastic differential equations. For example, the master equation Eq. (1.13) yields a zero-diffusion matrix. This gives rise to differential equations without noise terms

$$\dot{\alpha}_{\mathbf{n}}=-(\kappa_{\mathbf{n}}+i\omega_{\mathbf{n}})\alpha_{\mathbf{n}}, \tag{2.10}$$

$$\dot{\alpha}_{\mathbf{n}}^{+}=-(\kappa_{\mathbf{n}}-i\omega_{\mathbf{n}})\alpha_{\mathbf{n}}^{+}. \tag{2.11}$$

The whole procedure is very similar to the stable-resonator case. We can even give an illustrative example where the positive-$P$ distribution is exactly the same. From the expansion Eq. (2.4) for a coherent state $|\vec{\beta}\rangle$, the $P$ distribution can be chosen as

$$P(\vec{\alpha},\vec{\alpha}^{+})=\delta^{2N}(\vec{\alpha}-\vec{\beta})\delta^{2N}(\vec{\alpha}^{+}-\vec{\beta}^{+}), \tag{2.12}$$

identical to that in the stable case.

### Existence proof

To show that a positive-$P$ representation exists for any arbitrary state we construct an orthogonal or "canonical" set of operators corresponding to a "modes of the universe" description. Since the overlap matrices are Hermitian, positive and inverse, they may always be written as $A=C^{\dagger}\cdot C$, $B=C^{-1}\cdot(C^{\dagger})^{-1}$. The annihilation operators in this new basis simply take the form

$$\vec{c}=C\cdot\vec{a}=C^{\dagger-1}\cdot\vec{b}. \tag{2.13}$$

It may easily be shown that those operators fulfill the canonical commutation relations. Hence the operators $\vec{c}^{\dagger}$ correspond to photon creation and a canonical Bargmann state may be written as

$$\|\vec{\alpha}_{c}\rangle_{c}=e^{\vec{c}^{\dagger}\cdot\vec{\alpha}_{c}}|0\rangle=e^{\vec{b}^{\dagger}\cdot C^{-1}\cdot\vec{\alpha}_{c}}|0\rangle=\|C^{-1}\cdot\vec{\alpha}_{c}\rangle. \tag{2.14}$$

Also a coherent state transforms in the same way

$$|\vec{\alpha}_{c}\rangle_{c}=|C^{-1}\cdot\vec{\alpha}_{c}\rangle. \tag{2.15}$$

In this basis a positive-$P$ function ($P_c$) always exists, i.e.,

$$\rho=\int d^{2}\vec{\alpha}_{c}d^{2}\vec{\alpha}_{c}^{+}P_{c}(\vec{\alpha}_{c},\vec{\alpha}_{c}^{+})\|\vec{\alpha}_{c}\rangle_{cc}\langle\vec{\alpha}_{c}^{+}\|e^{-\vec{\alpha}_{c}^{+}\cdot\vec{\alpha}_{c}}. \tag{2.16}$$

Transforming back to the original biorthogonal basis, i.e., $\vec{\alpha}=C^{-1}\cdot\vec{\alpha}_{c}$, we immediately find

$$\begin{aligned}\rho=&\int d^{2}\vec{\alpha}d^{2}\vec{\alpha}^{+}[\det A]^{2}P_{c}(C\cdot\vec{\alpha},\vec{\alpha}^{+}\cdot C^{\dagger})\|C\cdot\vec{\alpha}\rangle_{c}\\&\times{}_{c}\langle\vec{\alpha}^{+}\cdot C^{\dagger}\|e^{-\vec{\alpha}^{+}\cdot C^{\dagger}\cdot C\cdot\vec{\alpha}}\\=&\int d^{2}\vec{\alpha}d^{2}\vec{\alpha}^{+}[\det A]^{2}P_{c}(C\cdot\vec{\alpha},\vec{\alpha}^{+}\cdot C^{\dagger})\|\vec{\alpha}\rangle\langle\vec{\alpha}^{+}\|e^{-\vec{\alpha}^{+}\cdot A\cdot\vec{\alpha}}.\end{aligned} \tag{2.17}$$

Thus a generalized $P$ function is given in terms of a positive-$P$ representation in a "mode of the universe" basis, i.e.,

$$P(\vec{\alpha},\vec{\alpha}^{+})=[\det A]^{2}P_{c}(C\cdot\vec{\alpha},\vec{\alpha}^{+}\cdot C^{\dagger}). \tag{2.18}$$

After we have shown that a positive-$P$ representation always exists we give a constructive example of one possible representation. Of course this form is, as in the canonical case, not unique. Here we have [17]

$$P_c(\vec{\alpha},\vec{\alpha}^+)=\frac{1}{(2\pi)^{2N}}e^{-|\vec{\alpha}_-|^2}\,{}_c\langle\vec{\alpha}_+|\rho|\vec{\alpha}_+\rangle_c, \tag{2.19}$$

with $\vec{\alpha}_+=(\vec{\alpha}+\vec{\alpha}^+)/2$, $\vec{\alpha}_-=(\vec{\alpha}-\vec{\alpha}^+)/2$. Applying the transformation rule we find

$$P(\vec{\alpha},\vec{\alpha}^\dagger)=\frac{[\det A]^2}{(2\pi)^{2N}}e^{-\vec{\alpha}_-^\dagger\cdot C^\dagger\cdot C\cdot\vec{\alpha}_-}\,{}_c\langle C\cdot\vec{\alpha}_+|\rho|C\cdot\vec{\alpha}_+\rangle_c$$

$$=\frac{[\det A]^2}{(2\pi)^{2N}}e^{-\vec{\alpha}_-^\dagger\cdot A\cdot\vec{\alpha}_-}\langle\vec{\alpha}_+|\rho|\vec{\alpha}_+\rangle. \tag{2.20}$$

## III. WIGNER REPRESENTATION

For the Wigner distribution [18] we use the symmetrically ordered characteristic function, i.e.,

$$\chi(\vec{\eta})=\mathrm{Tr}\{\rho e^{\vec{a}^\dagger\cdot\vec{\eta}-\vec{\eta}^\dagger a}\} \tag{3.1}$$

$$=\chi_N(\vec{\eta})e^{-\vec{\eta}^\dagger\cdot B\cdot\vec{\eta}/2}. \tag{3.2}$$

Again this can be transformed into a quasi-probability distribution for the variables $\alpha_{\mathbf{n}}$, $\alpha_{\mathbf{n}}^*$ corresponding to the operators $a_{\mathbf{n}}$, $a_{\mathbf{n}}^\dagger$, i.e.,

$$W(\vec{\alpha},\vec{\alpha}^\dagger)=\int\frac{d^2\vec{\eta}}{\pi^{2N}}e^{\vec{\eta}^\dagger\cdot\vec{\alpha}-\vec{\eta}\cdot\vec{\alpha}^\dagger}\chi(\vec{\eta}). \tag{3.3}$$

But, unlike the positive-*P* variables, $\alpha_{\mathbf{n}}$, $\alpha_{\mathbf{n}}^*$ are now complex conjugate to each other. As in the stable case the Wigner distribution is not necessarily positive. The appropriate operator correspondences are

$$a_{\mathbf{n}}\rho\rightarrow\left(\alpha_{\mathbf{n}}+\frac{1}{2}\sum_{\mathbf{m}}B_{\mathbf{nm}}\frac{\partial}{\partial\alpha_{\mathbf{m}}^*}\right)W(\vec{\alpha},\vec{\alpha}^\dagger), \tag{3.4}$$

$$\rho a_{\mathbf{n}}\rightarrow\left(\alpha_{\mathbf{n}}-\frac{1}{2}\sum_{\mathbf{m}}B_{\mathbf{nm}}\frac{\partial}{\partial\alpha_{\mathbf{m}}^*}\right)W(\vec{\alpha},\vec{\alpha}^\dagger), \tag{3.5}$$

$$\rho a_{\mathbf{n}}^\dagger\rightarrow\left(\alpha_{\mathbf{n}}^*+\frac{1}{2}\sum_{\mathbf{m}}B_{\mathbf{mn}}\frac{\partial}{\partial\alpha_{\mathbf{m}}}\right)W(\vec{\alpha},\vec{\alpha}^\dagger), \tag{3.6}$$

$$a_{\mathbf{n}}^\dagger\rho\rightarrow\left(\alpha_{\mathbf{n}}^*-\frac{1}{2}\sum_{\mathbf{m}}B_{\mathbf{mn}}\frac{\partial}{\partial\alpha_{\mathbf{m}}}\right)W(\vec{\alpha},\vec{\alpha}^\dagger). \tag{3.7}$$

For the master equation Eq. (1.13) one finds a nontrivial diffusion matrix that gives rise to the stochastic differential equations

$$\dot{\alpha}_{\mathbf{n}}=-(\kappa_{\mathbf{n}}+i\omega_{\mathbf{n}})\alpha_{\mathbf{n}}+\xi_{\mathbf{n}}(t), \tag{3.8}$$

$$\dot{\alpha}_{\mathbf{n}}^*=-(\kappa_{\mathbf{n}}-i\omega_{\mathbf{n}})\alpha_{\mathbf{n}}^\dagger+\xi_{\mathbf{n}}^*(t), \tag{3.9}$$

where the complex Gaussian noise terms have the correlations

$$\langle\xi_{\mathbf{n}}(t)\xi_{\mathbf{m}}^*(t')\rangle=\frac{1}{2}(\vec{\kappa}_{\mathbf{n}}+\tilde{\kappa}_{\mathbf{m}}^*)B_{\mathbf{nm}}\delta(t-t'), \tag{3.10}$$

$$\langle\xi_{\mathbf{n}}(t)\xi_{\mathbf{m}}(t')\rangle=0, \tag{3.11}$$

$$\langle\xi_{\mathbf{n}}^*(t)\xi_{\mathbf{m}}^*(t')\rangle=0, \tag{3.12}$$

with $\vec{\kappa}_{\mathbf{n}}=\kappa_{\mathbf{n}}+i\omega_{\mathbf{n}}$. This discussion shows that for a biorthogonal system the amount of noise fluctuations can be strongly enhanced by the adjoint overlap matrix *B*. Considering a single mode separately this gives rise to an excess-noise factor of $B_{nn}=K_n$ as predicted by Petermann. We would like to mention at this point that the Wigner representation gives exactly the same predictions as the positive-*P* representation when solving the stochastic differential equations Eqs. (2.10), (2.11) and (3.8), (3.9).

As an analytical example we give the Wigner distribution for a pure coherent state $|\vec{\beta}\rangle$ [cf. Eq. (2.12)]. One has to perform the complex Fourier transform of the exponential factor in Eq. (3.2) to obtain

$$W(\vec{\alpha},\vec{\alpha}^\dagger)=\left(\frac{2}{\pi}\right)^N\det A\,e^{-2(\vec{\alpha}-\vec{\beta})^\dagger\cdot A\cdot(\vec{\alpha}-\vec{\beta})}. \tag{3.13}$$

### Existence proof

Similarly to the positive-*P* representation the Wigner function may be obtained from the always existing Wigner function in a ‘‘modes of the universe’’ basis. In order to see this we first transform the characteristic function from one basis to the other, i.e.,

$$\chi_c(\vec{\eta}_c)=\mathrm{Tr}\{\rho e^{-\vec{\eta}_c^\dagger\cdot\vec{c}+\vec{c}^\dagger\cdot\vec{\eta}_c}\} \tag{3.14}$$

$$=\mathrm{Tr}\{\rho e^{-\vec{\eta}_c^\dagger\cdot C\cdot\vec{a}+\vec{a}^\dagger\cdot C^\dagger\cdot\vec{\eta}_c}\}=\chi(C^\dagger\vec{\eta}_c). \tag{3.15}$$

In the canonical basis the Wigner function is given by

$$W_c(\vec{\alpha}_c,\vec{\alpha}_c^+)=\int\frac{d^2\vec{\eta}_c}{\pi^{2N}}e^{\vec{\eta}_c^\dagger\cdot\vec{\alpha}_c-\vec{\alpha}_c^+\cdot\vec{\eta}_c}\chi_c(\vec{\eta}_c). \tag{3.16}$$

Changing the integration variables $\vec{\eta}=C^\dagger\cdot\vec{\eta}_c$ and using the transformation rule for the characteristic function leads to

$$W_c(\vec{\alpha}_c,\vec{\alpha}_c^+)=\int\frac{d^2\vec{\eta}}{\pi^{2N}}\det(A^{-1})\times e^{\vec{\eta}^\dagger\cdot C^{-1}\cdot\vec{\alpha}_c-\vec{\alpha}_c^+\cdot C^{-1\dagger}\cdot\vec{\eta}}\chi_c(C^{\dagger-1}\cdot\vec{\eta})$$

$$=\int\frac{d^2\vec{\eta}}{\pi^{2N}}\det(A^{-1})e^{\vec{\eta}^\dagger\cdot C^{-1}\cdot\vec{\alpha}_c-\vec{\alpha}_c^+\cdot C^{-1\dagger}\cdot\vec{\eta}}\chi(\vec{\eta}). \tag{3.17}$$

Comparing with the definition of the Wigner function [Eq. (3.3)] and again changing the variables to $\vec{\alpha}=C^{-1}\cdot\vec{\alpha}_c$, we obtain immediately

$$W(\vec{\alpha},\vec{\alpha}^\dagger)=\det A\,W_c(C\cdot\vec{\alpha},\vec{\alpha}^\dagger\cdot C^\dagger). \tag{3.18}$$

We would like to remark here that the Wigner function for a coherent state Eq. (3.13) may be obtained from the well-known Wigner function in an orthogonal basis applying this general transformation rule.

## IV. TUNNELING TIMES FOR AN UNSTABLE OPO

As we have seen the positive-$P$ and the Wigner distributions are equivalent representations of a given density operator. Hence it seems arbitrary which representation is used. This is of course correct. Problems may occur when transforming a given master equation into stochastic differential equations since the normal procedure is to neglect third or higher-order derivatives within the corresponding Fokker-Planck equation for nonlinear processes, although methods have recently been developed to avoid this problem [19]. As a demonstrative example we consider the parametric oscillator in an unstable cavity. Using the positive-$P$ representation only derivatives up to second order occur, so that it may be treated exactly. On the other hand the Fokker-Planck equation using the Wigner representation contains third-order derivatives. Following a widely used procedure, we can truncate the resulting Fokker-Planck equation at second order, a treatment that gives evolution equations equivalent to those of the semiclassical theory of stochastic electrodynamics [20]. Although following this procedure gives exactly the same predictions for the signal and pump intensities as does the positive-$P$ [7], there are differences for the tunneling times. As is well known, this discrepancy also exists for a stable cavity without excess noise [9,21].

We consider a geometrically unstable 1D cavity with symmetric mirrors with a Gaussian reflectivity profile in order to use analytically given expressions for the matched and adjoint modes [22]. For the sake of simplicity we assume a uniform classical pump field of frequency $\omega_P$ and a longitudinally thin [23] but transversely large crystal. Besides the free cavity dynamics described by the phenomenological master equation [Eq. (1.13)] discussed above, we have to include the interaction with the pump field described by the Hamiltonian [7]

$$H=H_{\text{eff}}+H_P+H_{\text{ext}}+H_{\text{int}} \tag{4.1}$$

with

$$H_{\text{eff}}=\hbar\sum_n \tilde{\omega}_n b_n^\dagger a_n, \tag{4.2}$$

$$H_P=\hbar\omega_P A_P^\dagger A_P, \tag{4.3}$$

$$H_{\text{ext}}=i(A_P\varepsilon_{in}^*-A_P^\dagger\varepsilon_{in}), \tag{4.4}$$

$$H_{\text{int}}=\frac{i\hbar g}{2}\sum_n\left(A_P\sqrt{K_n}b_n^{\dagger 2}-\frac{A_P^\dagger}{\sqrt{K_n}}a_n^2\right), \tag{4.5}$$

where $\varepsilon_{in}$ is the pump strength, $g$ is the coupling constant and the integral extends over the volume of the nonlinear medium, which is assumed to be transversally very large compared with the mode width $w$. We would like to mention here that although each single term of the interaction Hamiltonian shows a clear asymmetry between upconversion and downconversion the asymmetry vanishes exactly during summation over $n$. Alternatively the sum may be written as $\Sigma_n A_P a_n^{\dagger 2}/\sqrt{K_n}$, recovering the obviously Hermitian form. But we still keep the asymmetric form since the operators $b_n^\dagger, a_n$ actually correspond to photon creation and annihilation in the subharmonic field. Finally the pump field losses $\kappa_P$ are treated by a standard reservoir coupling to give

$$\begin{aligned}\dot{\rho}=&-\frac{i}{\hbar}[H\rho-\rho H^\dagger]\\&+\kappa_P(2A_P\rho A_p^\dagger-A_P^\dagger A_P\rho-\rho_P A_P^\dagger A)\\&+\sum_{nm}A_{nm}\{(\tilde{\kappa}_n^*+\tilde{\kappa}_m)a_m\rho a_n^\dagger-\tilde{\kappa}_m a_n^\dagger a_m\rho-\tilde{\kappa}_n^*\rho_n^{a\dagger}a_m\},\end{aligned} \tag{4.6}$$

with $\tilde{\kappa}_n=\kappa_n+i\Delta_n$ and $\Delta_n=\omega_n-\omega_P/2$. For the sake of simplicity we will restrict the subsequent calculations to resonance. Using the positive-$P$ representation this can be turned into a Fokker-Planck equation using the operator correspondences of Eqs. (2.6)–(2.9), giving

$$\begin{aligned}\dot{P}(\alpha_n,\alpha_n^+,a_P,a_P^+)=\Bigg\{&-\sum_n\frac{\partial}{\partial\alpha_n}\left(-\tilde{\kappa}_n\alpha_n+ga_P\sqrt{K_n}\sum_m A_{mn}\alpha_m^+\right)-\sum_n\frac{\partial}{\partial\alpha_n^+}\left(-\tilde{\kappa}_n^*\alpha_n^++ga_P^+\sqrt{K_n}\sum_m A_{nm}\alpha_m\right)\\&-\frac{\partial}{\partial a_p}\left(-\kappa_p a_p-\frac{g}{2}\sum_n\frac{\alpha_n^2}{\sqrt{K_n}}+\varepsilon_m\right)-\frac{\partial}{\partial a_p^\dagger}\left(-\kappa_p a_p^+-\frac{g}{2}\sum_n\frac{\alpha_n^{+2}}{\sqrt{K_n}}+\varepsilon_{in}^*\right)+\frac{1}{2}\sum_n\frac{\partial^2}{\partial\alpha_n^2}ga_P\sqrt{K_n}\\&+\frac{1}{2}\sum_n\frac{\partial^2}{\partial\alpha_n^{+2}}ga_P^+\sqrt{K_n}\Bigg\}P(\alpha_n,\alpha_n^+,a_P,a_P^+).\end{aligned} \tag{4.7}$$

This equation can then be mapped onto a set of stochastic differential equations which includes the real Gaussian noise sources $\eta_n,\eta_n^+$ associated with $\alpha_n$ and $\alpha_n^+$, respectively. The noise correlations

$$\langle\eta_n(t)\eta_m(t')\rangle=ga_P\sqrt{K_n}\delta_{nm}\delta(t-t'), \tag{4.8}$$

$$\langle\eta_n^+(t)\eta_m^+(t')\rangle=ga_P^+\sqrt{K_n}\delta_{nm}\delta(t-t'), \tag{4.9}$$

$$\langle \eta_n(t)\eta_m^+(t')\rangle=0, \tag{4.10}$$

may be derived immediately from the nonzero diffusion terms in Eq. (4.7). Interestingly the dynamics include a cross-mode coupling due to the nonorthogonality of the cavity modes and the noise amplitude is directly enhanced by the excess noise. Again this lends support to the interpretation of excess noise as a local enhancement of the vacuum quantum fluctuations.

Using the Wigner representation and hence the transformation rules [Eqs. (3.4)–(3.7)] we find exactly the same drift terms representing the deterministic part of the stochastic differential equations. Differences occur within the second-order derivatives, giving rise to different noise correlations, and there are also third-order derivatives present,

$$\begin{aligned}\dot{W}(\alpha_n,\alpha_n^*,a_P,a_P^*)=\Bigg\{&-\sum_n \frac{\partial}{\partial\alpha_n}\left(-\tilde{\kappa}_n\alpha_n+ga_P\sqrt{K_n}\sum_m A_{mn}\alpha_m^*\right)-\sum_n\frac{\partial}{\partial\alpha_n^*}\left(-\tilde{\kappa}_n^*\alpha_n^++ga_P^*\sqrt{K_n}\sum_m A_{nm}\alpha_m\right)\\&-\frac{\partial}{\partial a_P}\left(-\kappa_p a_p-\frac{g}{2}\sum_n\frac{\alpha_n^2}{\sqrt{K_n}}+\varepsilon_{in}\right)-\frac{\partial}{\partial a_p^*}\left(-\kappa_p a_p{*}-\frac{g}{2}\sum_n\frac{\alpha_n^{*2}}{\sqrt{K_n}}+\varepsilon_{in}^*\right)\\&+\sum_{nm}\frac{\partial^2}{\partial\alpha_n\partial\alpha_m^*}\frac{1}{2}(\tilde{\kappa}_n+\tilde{\kappa}_m^*)B_{nm}+\frac{\partial^2}{\partial\alpha_P\partial\alpha_P^*}\kappa_P+\frac{1}{8}\sum_n\frac{\partial^3}{\partial\alpha_n^2\partial\alpha_P^*}g\sqrt{K_n}\\&+\frac{1}{8}\sum_n\frac{\partial^3}{\partial\alpha_n^{*2}\partial\alpha_P}g\sqrt{K_n}\Bigg\}W(\alpha_n,\alpha_n^*,a_P,a_P^*).\end{aligned} \tag{4.11}$$

As mentioned above, we will neglect the third-order terms in order to perform the numerical integration of the stochastic equations. The nontrivial correlations for the complex Gaussian noise terms $\xi_n,\xi_p$ associated with the signal and pump field take the form

$$\langle \xi_n(t)\xi(t')_m^*\rangle=\frac{1}{2}(\tilde{\kappa}_n+\tilde{\kappa}_m^*)B_{nm}\delta(t-t'), \tag{4.12}$$

$$\langle \xi_p(t)\xi_p^*(t')\rangle=\kappa_p\delta(t-t'). \tag{4.13}$$

Obviously the third-order derivatives also survive without excess noise ($K_n\to 1$). It has been already pointed out in Refs. [9,21] that for stable cavity geometries the predicted tunneling times $t_T$ between the two possible steady state values,

$$\alpha_{ss}=\pm\left[\frac{2\kappa_p\kappa}{g^2}\left(\frac{|\varepsilon_{in}|}{\varepsilon_{th}}-1\right)\right]^{1/2}, \tag{4.14}$$

may differ strongly between the positive-*P* and truncated Wigner representations. Starting initially with one eigenstate the time evolution will be given as

$$\langle a\rangle_t=\langle a\rangle_0 e^{-t/t_T}, \tag{4.15}$$

and one ends up with a statistical mixture of the two possible states. This is a genuine quantum noise-driven effect since without noise $\langle a\rangle_t$ would not decay at all. Taking the effects of excess noise into account one also finds that the threshold of oscillation is shifted downwards by the excess-noise factor so that we have [7]

$$\varepsilon_{th}\approx\frac{\kappa_P\kappa}{\sqrt{K}g}. \tag{4.16}$$

In the following we investigate the influence of excess noise on the tunneling times $t_T$. For this purpose we continuously change the curvature of the mirrors from the stable to the unstable regime. Scaling the horizontal axis in Fig. 1 to the excess-noise factor, we see that $t_T$ increases approxi-

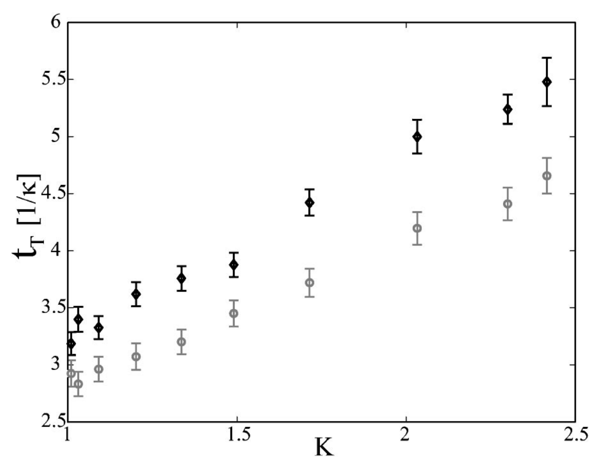


FIG. 1. This picture compares the tunneling times $t_T$ using the positive-*P* representation (diamonds) with the results of a truncated Wigner simulation (circles). We find that for both methods *K* enters approximately linearly, so that the ratio is independent of the excess noise. Here we are well above the stable threshold $\varepsilon_{in}=1.5\kappa\kappa_p/g$. For the other parameters we have chosen $g=\kappa_P=\kappa$. The error bars correspond to the sampling errors due to the finite number of trajectories (10 000).

mately linearly with *K*. Whereas the positive-*P* representation has been proven to give accurate results [21] for the case of a stable cavity, the values corresponding to the Wigner simulations could be quite different. We see in Fig. 1 that this is also the case here, but the ratio between the predicted values does not seem to depend on *K*. Furthermore, the growth of the tunneling times can be directly attributed to a shifted oscillation threshold. Since we have kept the pump strength constant at 50% above the *stable* threshold value ($\varepsilon_{in}=1.5\kappa\kappa_P/g$ the signal field effectively interacts with a stronger pump field for increasing excess noise. For the other parameters we have chosen $\kappa_P=\kappa$ and $g=\kappa$. Since $\kappa$ is strongly increasing when changing from a stable to an unstable cavity configuration one would have to increase the coupling strength to compensate. Of course, in practice it would be very difficult to reach this strong coupling regime in unstable resonators (e.g., $g(L/f=-0.2)\approx 7.8g(L/f=0.2)$). Nevertheless the result still has some physical meaning since the ratio between the methods does not depend on the coupling strength [21]. The big advantage of this rather "unphysical" assumption is that it clearly demonstrates the effect of the excess noise.

To produce these results we considered the analytically soluble model of a 1D unstable resonator with symmetric spherical mirrors of Gaussian reflectivity profile [7]. We changed the ratio between cavity length and focal length such that $0.2\geqslant L/f\geqslant -0.2$, continously switching from the optically stable to the optically unstable regime. The transverse cavity extension was fixed with a Fresnel number of $N=20$. We found that considering 10 transverse modes was sufficient to obtain convergence in the solutions.

## CONCLUSIONS

We have generalized two standard phase-space methods widely used in quantum optics, the positive-*P* and Wigner representations, to the case of unstable resonators featuring nonorthogonal modes. We have developed the operator correspondences connecting the underlying density operator with the corresponding Fokker-Planck equations. These equations may be easily mapped onto stochastic differential equations that may form a basis for more extensive studies of unstable optical systems. The usual case of orthogonal modes is found as a straightforward limit of our equations.

As an illustrative example we have calculated the tunneling times of an unstable OPO, as these are known to be different in the two representations. Well above threshold and within the strong-coupling regime we obtained clear differences. The Fokker-Planck equation for the positive-*P* representation contains derivatives up to the second order and can thus be mapped directly onto stochastic differential equations. On the other hand, when using the Wigner equations one usually neglects the third-order derivatives, which causes easily visible discrepancies for the predicted tunneling times. We have shown that the ratio between the predictions of the two methods is essentially unaffected by the excess noise. The differences are identical to the differences that occur for corresponding stable parameters. However, the predictions for the field intensities are exactly the same for both methods even with included excess noise.

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