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## Integrable variant of the one-dimensional Hubbard model

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A new integrable model which is a variant of the one-dimensional Hubbard model is proposed. The integrability of the model is verified by presenting the associated quantum  $R$ -matrix which satisfies the Yang–Baxter equation. We argue that the new model possesses the  $SO(4)$  algebra symmetry, which contains a representation of the  $\eta$ -pairing  $SU(2)$  algebra and a spin  $SU(2)$  algebra. Additionally, the algebraic Bethe ansatz is studied by means of the quantum inverse scattering method. The spectrum of the Hamiltonian, eigenvectors, as well as the Bethe ansatz equations, are discussed. © 2002 American Institute of Physics. [DOI: 10.1063/1.1481956]

### I. INTRODUCTION

Since the discovery of high temperature superconductivity in cuprates,<sup>1</sup> a tremendous effort has been made to uncover the mystery of this phenomenon. It is generally believed that the strongly correlated electron systems behaving as non-Fermi liquids are closely related to superconducting materials. This has caused an intense study in strongly correlated electron systems.<sup>2–7</sup> These systems possess various physical characteristics which are decisively dominated by the competing interactions; e.g., the Coulomb interaction in the Hubbard model, spin fluctuations through the antiferromagnetic coupling for the super-symmetric t-J model and current-density correlated interaction inducing hole pairs of Cooper type superconductors in the one-dimensional (1D) Bariev model. The 1D Hubbard model as a prototype among the strongly correlated electron systems has attracted a substantial deal of interest in the study of integrable quantum field theory, mathematical physics and condensed matter physics since its exact solution was achieved by Lieb and Wu<sup>8</sup> in 1968. Towards a complete understanding of the mathematical structure of the 1D Hubbard model in the framework of the quantum inverse scattering method (QISM), a fundamental advance was achieved by Shastry<sup>9</sup> in demonstrating the integrability of the model. Specifically, it was shown that a two-dimensional statistical covering model of two coupled symmetric six vertex models provides a one parameter family of transfer matrices commuting with the Hamiltonian of the 1D Hubbard model. The algebraic formulation with respect to the integrability leads to the quantum  $R$ -matrix<sup>9–12</sup> which facilitates not only the algebraic Bethe ansatz solution,<sup>13</sup> but also the construction of the boost operator<sup>14</sup> for the model. Remarkably, the Hamiltonian of the Hubbard model was proved to exhibit the  $SO(4)$  symmetry by Yang and Zhang<sup>15</sup> (see also Ref. 16). Besides the spin  $SU(2)$  algebra, the  $SO(4)$  algebra contains the  $\eta$ -pair  $SU(2)$  algebra with the

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raising operator creating an on-site pair of electrons with opposite spins. This can be interpreted as a localized Cooper pair. A complete set of eigenstates of the Hamiltonian can be obtained by exploiting the  $SO(4)$  symmetry.<sup>17</sup>

The 1D Hubbard Hamiltonian with more competing interactions may also be considered. Along this line, many extended Hubbard models have been constructed in the literature, such as a  $u(2|2)$  extended Hubbard model,<sup>5</sup> supersymmetric  $U_q(\mathfrak{osp}(2|2))$  electronic systems<sup>18</sup> and  $SU(N)$  Hubbard models.<sup>19</sup> In this article, we present an alternative 1D Hubbard model such that the Hamiltonian has off-site Coulomb interaction instead of the on-site one of the standard Hubbard model. The integrability of this model is verified by presenting the associated quantum  $R$ -matrix which fulfills the Yang–Baxter equation (YBE). We show that the model exhibits the  $SO(4)$  symmetry with new representations of the  $\eta$ -pairing  $SU(2)$  algebra and the  $\zeta$ -pairing spin  $SU(2)$  algebra. Moreover, the algebraic Bethe ansatz is formulated by means of the QISM. Though the model exhibits the same spectrum as the standard Hubbard model on a periodic lattice, the new quantum  $R$ -matrix, the hidden nesting structure associated with an asymmetric isotropic six-vertex model and the Bethe eigenvectors do distinguish this model from the standard one.<sup>9,10</sup> The essential differences between the two models manifest in the open lattice versions, which we will discuss in more depth in the conclusion.

The article is organized as follows. In Sec. II, we introduce a Lax operator associated with the new Hubbard model and construct a nontrivial higher conserved quantity commuting with the Hamiltonian. In Sec. III, we present the  $R$ -matrix associated with the model by solving the Yang–Baxter relation. The  $SO(4)$  symmetry is verified too. In Sec. IV, we formulate the algebraic Bethe ansatz solutions for the model with periodic boundary conditions. The eigenvectors and eigenvalues of the Hamiltonian are presented explicitly. Section V is devoted to a discussion and conclusion.

## II. THE MODEL

Let us begin by introducing a variant of the 1D Hubbard model with the Hamiltonian

$$H = \sum_{j=1}^L \{(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + (\sigma \rightarrow \tau)\} + \frac{U}{4} \sum_{j=1}^L \sigma_j^z \tau_{j+1}^z. \quad (1)$$

Above  $\sigma_j$  and  $\tau_j$  are the two commuting species of Pauli matrices acting on site  $j$ , and  $U$  is a Coulomb coupling constant. Above and throughout, periodic boundary conditions are imposed on all summations evaluated over the lattice length  $L$ . The difference from the standard Hubbard model is that the model (1) exhibits the off-site Coulomb interaction instead of the on-site one. We shall see that it not only breaks the spin reflection symmetry but also specifies a new representation of  $\eta$ -pairing  $SU(2)$  algebra and spin  $SU(2)$  algebra in order to maintain the  $SO(4)$  symmetry. To verify the integrability of the model, we, at first, identify a relation between the Hamiltonian (1) and the transfer matrix which is defined by

$$\tau(u) = \text{Tr}_0 T(u) \quad (2)$$

with

$$T(u) = L_{0L}(u) \cdots L_{01}(u). \quad (3)$$

The local Lax operators associated with model (1) have to be alternatively chosen as

$$L_{0j}(u) = L_{0j}^\sigma(u) I_0^2 L_{0j}^\tau(u) \tag{4}$$

$$= \begin{pmatrix} e^{h(u)} P_j^+ Q_j^+ & e^{h(u)} P_j^+ \tau_j^- & e^{-h(u)} \sigma_j^- Q_j^+ & e^{-h(u)} \sigma_j^- \tau_j^- \\ e^{-h(u)} P_j^+ \tau_j^+ & e^{-h(u)} P_j^+ Q_j^- & e^{h(u)} \sigma_j^- \tau_j^+ & e^{h(u)} \sigma_j^- Q_j^- \\ e^{h(u)} \sigma_j^+ Q_j^+ & e^{h(u)} \sigma_j^+ \tau_j^- & e^{-h(u)} P_j^- Q_j^+ & e^{-h(u)} P_j^- \tau_j^- \\ e^{-h(u)} \sigma_j^+ \tau_j^+ & e^{-h(u)} \sigma_j^+ Q_j^- & e^{h(u)} P_j^- \tau_j^+ & e^{h(u)} P_j^- Q_j^- \end{pmatrix}, \tag{5}$$

where

$$P_j^\pm = w_4(u) \pm w_3(u) \sigma^\mp;$$

$$Q_j^\pm = w_4(u) \pm w_3(u) \tau^\mp$$

with a parametrization  $\gamma(u) = w_4(u) - w_3(u) = \sin(u)$ ;  $\alpha(u) = w_4(u) + w_3(u) = \cos(u)$ . We would like to mention that the Lax operators

$$L_{0j}^\sigma(u) = w_4(u) + w_3(u) \sigma_j^\mp \sigma_0^\mp + \sigma_j^+ \sigma_0^- + \sigma_0^+ \sigma_j^-, \tag{6}$$

$$L_{0j}^\tau(u) = w_4(u) + w_3(u) \tau_j^\mp \tau_0^\mp + \tau_j^+ \tau_0^- + \tau_0^+ \tau_j^-, \tag{7}$$

$$I_0 = \cosh \frac{h(u)}{2} + \sigma_0^\mp \tau_0^\mp \sinh \frac{h(u)}{2}, \tag{8}$$

have been chosen the same as that for the Hubbard model.<sup>9-11</sup> It follows that the Hamiltonian (1) is related to the transfer matrix (2) in the following way:

$$\ln \tau(u) = \ln \tau(0) + Hu + \frac{1}{2!} Ju^2 + \dots, \tag{9}$$

above the Hamiltonian  $H = \sum_{j=1}^L H_{j(j+1)}$  with the Hamiltonian density

$$H_{j(j+1)} = L_{0(j+1)}(0) L'_{0j}(0) L_{0j}^{-1}(0) L_{0(j+1)}^{-1}(0), \tag{10}$$

and the second higher conserved current can be given as

$$J = \sum_{j=1}^L J_{j(j+1)(j+2)} \tag{11}$$

with

$$J_{j(j+1)(j+2)} = B_{j(j+1)} - H_{j(j+1)}^2 - [H_{j(j+1)}, H_{(j+1)(j+2)}], \tag{12}$$

$$B_{j(j+1)} = L_{0(j+1)}(0) L''_{0j}(0) L_{0j}^{-1}(0) L_{0(j+1)}^{-1}(0). \tag{13}$$

Here the prime denotes the derivative with respect to spectral parameter  $u$ . After a straightforward calculation, the equation (10) does provide us with the expression (1), whereas the second conserved quantity (11) has the form

$$\begin{aligned}
 J_{j(j+1)(j+2)} = & \frac{U}{2} \{ [-\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+] \tau_{j+1}^z + [-\tau_j^+ \tau_{j+1}^- + \tau_{j+1}^+ \tau_j^-] + \sigma_j^z [-\sigma_j^+ \sigma_{j+1}^- \\
 & + \sigma_j^- \sigma_{j+1}^+] \tau_{j+2}^z + [-\tau_{j+1}^+ \tau_{j+2}^- + \tau_{j+2}^+ \tau_{j+1}^-] \sigma_j^z \} + [-\sigma_{j+2}^+ \sigma_j^- + \sigma_j^+ \sigma_{j+2}^-] \sigma_{j+1}^z \\
 & + [-\tau_{j+2}^+ \tau_j^- + \tau_j^+ \tau_{j+2}^-] \tau_{j+1}^z. \tag{14}
 \end{aligned}$$

Here we would like to stress that both the Hamiltonian (1) and the conserved quantity (11) should be understood as global operators. It is meant that  $[H, J] = 0$  rather than  $[H_{j(j+1)}, J_{j(j+1)(j+2)}] = 0$ . The mutual commutativity of  $H$  and  $J$  convinces us of the existence of a quantum  $R$ -matrix associated with the model (1). We shall present a rigorous proof of the integrability of the model in the next section.

### III. INTEGRABILITY OF THE MODEL

It has long been clarified that the existence of the quantum  $R$ -matrix which fulfills the Yang–Baxter relation is desirable for constructing integrable quantum chains. This suggests to us a way to verify the integrability of the model presented above. Indeed, following Ref. 11, we, after a cumbersome algebraic calculation, can find a class of solutions to the Yang–Baxter relation

$$\overset{\vee}{R}(u, v) L_{0j}(u) \otimes L_{0j}(v) = L_{0j}(v) \otimes L_{0j}(u) \overset{\vee}{R}(u, v), \tag{15}$$

which is given as

$$\overset{\vee}{R}(u, v) = \begin{pmatrix} \rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_2^- & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_2^+ & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_5 & 0 & 0 & \rho_6^+ & 0 & 0 & \rho_6^- & 0 & 0 & \rho_8 & 0 & 0 \\ 0 & \rho_{10} & 0 & 0 & \rho_2^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_6^- & 0 & 0 & \rho_3 & 0 & 0 & \rho_7 & 0 & 0 & \rho_6^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2^- & 0 & 0 & 0 & 0 & 0 & \rho_{10} & 0 \\ 0 & 0 & \rho_{10} & 0 & 0 & 0 & 0 & 0 & \rho_2^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_6^+ & 0 & 0 & \rho_7 & 0 & 0 & \rho_3 & 0 & 0 & \rho_6^- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2^+ & 0 & 0 & \rho_{10} \\ 0 & 0 & 0 & \rho_8 & 0 & 0 & \rho_6^- & 0 & 0 & \rho_6^+ & 0 & 0 & \rho_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & \rho_2^- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1 \end{pmatrix}, \tag{16}$$

with the Boltzmann weights

$$\rho_1 = (\cos u \cos v e^l + \sin v \sin u e^{-l}) \rho_2,$$

$$\begin{aligned} \rho_4 &= (\cos u \cos v e^{-l} + \sin v \sin u e^l) \rho_2, \\ \rho_9 &= (\sin u \cos v e^{-l} - \sin v \cos u e^l) \rho_2, \\ \rho_{10} &= (\sin u \cos v e^l - \sin v \cos u e^{-l}) \rho_2, \\ \rho_2^+ &= e^l \rho_2, \quad \rho_2^- = e^{-l} \rho_2, \\ \rho_3 &= \frac{(\cos u \cos v e^l - \sin v \sin u e^{-l})}{\cos^2 u - \sin^2 v} \rho_2, \\ \rho_5 &= \frac{(\cos u \cos v e^{-l} - \sin v \sin u e^l)}{\cos^2 u - \sin^2 v} \rho_2, \\ \rho_6^+ &= \frac{(\cos u \sin u e^{-l} - \sin v \cos v e^l)}{\cos^2 u - \sin^2 v} \rho_2, \\ \rho_6^- &= \frac{(\cos u \sin u e^l - \sin v \cos v e^{-l})}{\cos^2 u - \sin^2 v} \rho_2 \end{aligned}$$

and

$$\begin{aligned} \rho_8 &= \rho_3 - \rho_1, \\ \rho_7 &= \rho_5 - \rho_4, \\ l = h(u) - h(v), \quad i &= \frac{\sinh 2h(u)}{\sin 2u} = \frac{U}{2}, \end{aligned} \tag{17}$$

which enjoy the following identities:

$$\begin{aligned} \rho_4 \rho_1 + \rho_9 \rho_{10} &= 1, \\ \rho_1 \rho_5 + \rho_3 \rho_4 &= 2, \\ \rho_6^+ \rho_6^- &= \rho_3 \rho_5 - 1. \end{aligned}$$

This  $R$ -matrix with more distinct Boltzmann weights is indeed different from the one for the standard Hubbard model<sup>9-11</sup> and a twisted version<sup>20</sup> which is associated with the Hubbard model with chemical potential terms. Running a Maple program we may check that the  $R$ -matrix satisfies the Yang–Baxter equation

$$R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v). \tag{18}$$

So far we have built up the QISM mechanism for the alternative Hubbard model and concluded the integrability of the model as well. On the other hand, a fermionic model is always favorable in the study of the condensed matter physics due to the clear distinction between the fermionic degrees of freedom and bosonic degrees of freedom. By performing the Jordan–Wigner transformations,<sup>11,21</sup> one may obtain the Hamiltonian of a fermionic model which is equivalent to the Hubbard model (1):

$$H = - \sum_{j=1}^{N-1} \sum_s (a_{(j+1)s}^\dagger a_{js} + a_{js}^\dagger a_{(j+1)s}) + U \sum_{j=1}^N \left( n_{j\uparrow} - \frac{1}{2} \right) \left( n_{(j+1)\downarrow} - \frac{1}{2} \right). \tag{19}$$

Above  $a_{js}^\dagger$  and  $a_{js}$  are creation and annihilation operators with spins ( $s = \uparrow$  or  $\downarrow$ ) at site  $j$  satisfying the anti-commutation relations

$$\{a_{js}, a_{j's'}\} = \{a_{js}^\dagger, a_{j's'}^\dagger\} = 0, \tag{20}$$

$$\{a_{js}, a_{j's'}^\dagger\} = \delta_{jj'} \delta_{ss'}, \tag{21}$$

and  $n_{js} = a_{js}^\dagger a_{js}$  is the density operator. The integrability of the fermionic model (19) requires that the graded Lax operator related to the Hamiltonian (19),

$$\mathcal{L}_{0j}(u) = \begin{pmatrix} -e^{h(u)} f_{j\uparrow} f_{j\downarrow} & -e^{h(u)} f_{j\uparrow} a_{j\downarrow} & ie^{-h(u)} a_{j\uparrow} g_{j\downarrow} & ie^{-h(u)} a_{j\uparrow} a_{j\downarrow} \\ -ie^{-h(u)} f_{j\uparrow} a_{j\downarrow}^\dagger & e^{-h(u)} f_{j\uparrow} g_{j\downarrow} & e^{h(u)} a_{j\uparrow} a_{j\downarrow}^\dagger & ie^{h(u)} a_{j\uparrow} g_{j\downarrow} \\ e^{h(u)} a_{j\uparrow}^\dagger f_{j\downarrow} & e^{h(u)} a_{j\uparrow}^\dagger a_{j\downarrow} & e^{-h(u)} g_{j\uparrow} f_{j\downarrow} & e^{-h(u)} g_{j\uparrow} a_{j\downarrow} \\ -ie^{-h(u)} a_{j\uparrow}^\dagger a_{j\downarrow}^\dagger & e^{-h(u)} a_{j\uparrow}^\dagger g_{j\downarrow} & ie^{h(u)} g_{j\uparrow} a_{j\downarrow}^\dagger & -e^{h(u)} g_{j\uparrow} g_{j\downarrow} \end{pmatrix}, \tag{22}$$

must generate the graded Yang–Baxter relation

$$\overset{\vee}{\mathcal{R}}(u, v) \overset{\vee}{\mathcal{L}}_{0j}(u) \otimes \overset{\vee}{\mathcal{L}}_{0j}(v) = \overset{\vee}{\mathcal{L}}_{0j}(v) \otimes \overset{\vee}{\mathcal{L}}_{0j}(u) \overset{\vee}{\mathcal{R}}(u, v), \tag{23}$$

with the graded  $R$ -matrix which is given by

$$\overset{\vee}{\mathcal{R}}(u, v) = \overset{\vee}{W} R(u, v) \overset{\vee}{W}^{-1}, \tag{24}$$

where

$$W = \sigma^z \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes I \tag{25}$$

and

$$f_{js} = \sin u - (\sin u - i \cos u) n_{js}, \quad g_{js} = \cos u - (\cos u + i \sin u) n_{js}, \tag{26}$$

with the grading  $P(1) = P(4) = 0, P(2) = P(3) = 1$ . The monodromy matrix is defined by

$$\mathcal{T}(u) = \overset{\vee}{\mathcal{L}}_{0L}(u) \cdots \overset{\vee}{\mathcal{L}}_{01}(u), \tag{27}$$

such that the transfer matrices

$$\tau(u) = \text{str}_0 \mathcal{T}(u) \tag{28}$$

commute each other for different values of the parameter  $u$ . It can be verified that an expansion of the logarithm of the transfer matrix (28) in powers of  $u$  will lead to the Hamiltonian (19) as well as higher conserved quantities.

We would like to remark that the model possesses the  $SO(4)$  symmetry if we consider a new representation of the  $\eta$ -pair  $SU(2)$  algebra,

$$\eta = \sum_{i=1}^L (-1)^i a_{i\uparrow} a_{(i+1)\downarrow}, \quad \eta^\dagger = (\eta)^\dagger, \quad \eta_z = \frac{1}{2} \sum_{i=1}^L (n_{i\uparrow} + n_{i\downarrow}) - \frac{1}{2} L, \tag{29}$$

and the  $\zeta$ -pair spin  $SU(2)$  algebra

$$\zeta = \sum_{i=1}^L a_{i\uparrow}^\dagger a_{(i+1)\downarrow}, \quad \zeta^\dagger = (\zeta)^\dagger, \quad \zeta_z = \frac{1}{2} \sum_{i=1}^L (n_{(i+1)\downarrow} - n_{i\uparrow}), \tag{30}$$

which comprise the SO(4) algebra. Taking into account the globality of these operators, one may show that the Hamiltonian (19) commutes with the generators of the above two SU(2) algebras. This symmetry could be expected to complete all eigenstates of the Hubbard model like the case in the standard Hubbard model. Here the  $\eta$ -pairing raising operator creating a pair of electrons with opposite spin on different sites could be interpreted as a delocalized Cooper pair.

#### IV. ALGEBRAIC BETHE ANSATZ

Towards an exact solution of an integrable model, the algebraic Bethe ansatz seems to have more utility than the coordinate Bethe ansatz because the former not only provides us with the spectrum of all conserved quantities, but makes a close connection to the finite temperature properties of the model. There have been a lot of papers devoted to the study of the nested algebraic Bethe ansatz<sup>22</sup> for the multistate integrable models with Lie algebra (or Lie superalgebra) symmetry. Following the so-called ABCDF approach to solve the Hubbard-like models,<sup>13,23</sup> we shall formulate the algebraic Bethe ansatz for the model in that which follows. To this end, as usual, we have to perform the ansatz step by step. However, it is not necessary to restate all of the calculations used in solving our model because of the similarity to the routine proposed in Ref. 13.

In order to carry out the algebraic Bethe ansatz for this Hubbard model, we first need to find the eigenvalues and eigenvectors of the transfer matrix (28):

$$\tau|\Phi_n\rangle = \lambda|\Phi_n\rangle. \tag{31}$$

Following the prescription in Ref. 13, the eigenvectors of the transfer matrix are given by

$$|\Phi_n\rangle = \Phi_n \cdot \mathcal{F}|0\rangle, \tag{32}$$

where the components of  $\mathcal{F}$  are coefficients of an arbitrary linear combination of vectors  $\Phi_n$  and  $|0\rangle$  is the pseudovacuum state, chosen here as the standard ferromagnetic one

$$|0\rangle = \otimes_{j=1}^N |0\rangle_j, \tag{33}$$

where

$$|0\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \tag{34}$$

which corresponds to the doubly occupied state. We write the monodromy matrix  $\mathcal{T}(u)$  in (27) as

$$\mathcal{T}(u) = \begin{pmatrix} B(u) & B_1(u) & B_2(u) & F(u) \\ C_1(u) & A_{11}(u) & A_{12}(u) & E_1(u) \\ C_2(u) & A_{21}(u) & A_{22}(u) & E_2(u) \\ C_3(u) & C_4(u) & C_5(u) & D(u) \end{pmatrix} \tag{35}$$

such that the necessary commutation relations between the diagonal fields and the creation fields can be derived from the Yang–Baxter algebra

$$\mathcal{R}_{12}(u,v) \overset{1}{\mathcal{T}}(u) \overset{2}{\mathcal{T}}(v) = \overset{2}{\mathcal{T}}(v) \overset{1}{\mathcal{T}}(u) \mathcal{R}_{12}(u,v). \tag{36}$$

In the above,



$$\mathcal{R}_{12}(u,v) = \overset{\vee}{\mathcal{P}}R(u,v).$$

Here  $\mathcal{P}$  is the graded permutation operator. Let us first display an important commutation role, which reveals to us a hidden nesting structure and the symmetry of eigenvectors,

$$\begin{aligned} \vec{B}(u)\vec{B}(v) &= \frac{\rho_4(u,v)}{\rho_1(u,v)}\vec{B}(v)\vec{B}(u) \cdot \hat{r}(u,v) + \frac{i}{\rho_8(u,v)\rho_1(u,v)}F(v)B(u)\vec{\xi}_1(u,v) \\ &+ \frac{i}{\rho_8(u,v)}F(u)B(v)\vec{\xi}_2(u,v), \end{aligned} \tag{37}$$

where

$$\begin{aligned} \vec{\xi}_1(u,v) &= (0, f_1(u,v), f_2(u,v), 0); \quad \vec{\xi}_2(u,v) = (0, \rho_6^+(u,v), \rho_6^-(u,v), 0), \\ \hat{r}(u,v) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(u,v) & b(u,v) & 0 \\ 0 & c(u,v) & d(u,v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{38}$$

with

$$\begin{aligned} f_1(u,v) &= \rho_6^-(u,v)\rho_8(u,v) - \rho_6^+(u,v)\rho_5(u,v), \\ f_2(u,v) &= \rho_6^-(u,v)\rho_5(u,v) - \rho_6^+(u,v)\rho_8(u,v), \\ a(u,v) &= \frac{\rho_3(u,v)\rho_8(u,v) - \rho_6^+(u,v)^2}{\rho_4(u,v)\rho_8(u,v)}, \\ d(u,v) &= \frac{\rho_3(u,v)\rho_8(u,v) - \rho_6^-(u,v)^2}{\rho_4(u,v)\rho_8(u,v)}, \\ b(u,v) = c(u,v) &= \frac{\rho_6^+(u,v)\rho_6^-(u,v) - \rho_8(u,v)\rho_7(u,v)}{\rho_4(u,v)\rho_8(u,v)}. \end{aligned}$$

It turns out that the auxiliary matrix  $\hat{r}(u,v)$  is nothing but a gauged rational  $R$ -matrix of an isotropic six-vertex model. If we adopt the parametrization introduced in Ref. 13 or 24, explicitly,

$$\tilde{x} = -\frac{\sin x}{\cos x}e^{-2h(x)} + \frac{\cos x}{\sin x}e^{2h(x)}, \quad x = u, v, \tag{39}$$

one may find that

$$\begin{aligned} a(u,v) &= -\frac{Ue^{-\theta(u,v)}}{\tilde{u} - \tilde{v} - U}, \quad d(u,v) = -\frac{Ue^{\theta(u,v)}}{\tilde{u} - \tilde{v} - U}, \\ b(u,v) = c(u,v) &= \frac{\tilde{u} - \tilde{v}}{\tilde{u} - \tilde{v} - U}, \end{aligned}$$

with

$$e^{-\theta(u,v)} = \frac{\cos v \sin u}{\sin v \cos u}.$$

We shall see that the  $\hat{r}$ -matrix (38) is related to the one of the isotropic six-vertex model via a proper gauge transformation, which does not change the spectrum of the spin sector. This seems to provide a new version of the  $R$ -matrix, which does not have the difference property, for the isotropic six-vertex model. In view of the commutation relation (37), the creation operators  $\vec{B}_a$ ,  $\vec{E}_a$  do not intertwine. So it is reasonable that the eigenvectors of the transfer matrices are generated only by the creation operators  $\vec{B}_a(u)$  and  $F(u)$ . Following the argument in Ref. 13, we may find that the  $n$ -particle vector can be determined recursively by the following relation:

$$\begin{aligned} \Phi_n(v_1, \dots, v_n) = & \vec{B}(v_1) \otimes \Phi_{n-1}(v_2, \dots, v_n) + \sum_{j=2}^n \frac{1}{i\rho_8(v_1, v_j)} \prod_{k \neq j}^n \frac{\rho_1(v_k, v_j)}{i\rho_9(v_k, v_j)} [\vec{\xi}_2(v_1, v_j) \\ & \otimes F(v_1) \Phi_{n-2}(v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_n) B(v_j)] \prod_{k=2}^{j-1} \frac{\rho_4(v_k, v_j)}{\rho_1(v_k, v_j)} \hat{r}_{k, k+1}(v_k, v_j). \end{aligned} \tag{40}$$

Explicitly, the two-particle eigenvector reads

$$\Phi_2(v_1, v_2) = \vec{B}(v_1) \otimes \vec{B}(v_2) + \vec{\xi}_2(v_1, v_2) \otimes F(v_1) B(v_2) \frac{1}{i\rho_8(v_1, v_2)}. \tag{41}$$

From the commutation relation (37), we can conclude that  $\Phi_n(v_1, \dots, v_n)$  satisfies an exchange symmetry relation

$$\Phi_n(v_1, \dots, v_j, v_{j+1}, \dots, v_n) = \frac{\rho_4(v_j, v_{j+1})}{\rho_1(v_j, v_{j+1})} \Phi_n(v_1, \dots, v_{j+1}, v_j, \dots, v_n) \cdot \hat{r}_{j, j+1}(v_j, v_{j+1}) \tag{42}$$

based on the following identity:

$$\begin{aligned} & \frac{\rho_4(v_j, v_{j+1})}{\rho_1(v_{j+1}, v_j) \rho_8(v_{j+1}, v_j) \rho_1(v_j, v_{j+1})} \vec{\xi}_1(v_{j+1}, v_j) \cdot \hat{r}(v_j, v_{j+1}) \\ & = - \frac{1}{\rho_8(v_j, v_{j+1})} \vec{\xi}_2(v_j, v_{j+1}). \end{aligned} \tag{43}$$

In the above expressions,  $\vec{\xi}$  plays the role of forbidding two spin up or two spin down electrons at same site. Also,  $F(u)$  creates a local hole pair with opposite spins. In order to manipulate the eigenvalue of the transfer matrix (28) we need the commutation roles involving the diagonal fields over the creation fields. After some algebra, from the Yang–Baxter relation (36) we have

$$B(u) \vec{B}_a(v) = \frac{\rho_1(v, u)}{i\rho_9(v, u)} \vec{B}_a(v) B(u) - \frac{1}{i\rho_9(v, u)} \vec{B}_a(u) B(v) \cdot \hat{\eta}_1(v, u), \tag{44}$$

$$\begin{aligned} D(u) \vec{B}_a(v) = & \frac{i\rho_{10}(u, v)}{\rho_8(u, v)} \vec{B}_a(v) D(u) - \frac{1}{\rho_8(v, u)} F(v) \vec{C}_{a+3}^*(u) \cdot \hat{\eta}_1(u, v) \\ & + \frac{\rho_5(u, v)}{\rho_8(u, v)} F(u) \vec{C}_{a+3}^*(v) + \frac{i}{\rho_8(u, v)} \vec{\xi}_2(u, v) \cdot (\vec{E}^*(u) \otimes \hat{A}(v)), \end{aligned} \tag{45}$$

$$\begin{aligned} \hat{A}_{ab}(u)\vec{B}_a(v) &= \frac{i\rho_4(u,v)}{\rho_9(u,v)}\vec{B}(v)\otimes\hat{A}(u)\cdot\hat{r}(u,v) - \frac{i}{\rho_9(u,v)}\hat{\eta}_2(u,v)\cdot\vec{B}(u)\otimes\hat{A}(v) \\ &+ \frac{1}{\rho_9(u,v)\rho_8(u,v)}\{F(v)\vec{C}_{3-a}(u)\otimes\vec{\xi}_1(u,v) + \hat{\eta}_2(u,v)\cdot F(u)\vec{C}_{3-a}(v) \\ &\otimes\vec{\xi}_2(u,v)\} + \frac{1}{\rho_8(u,v)}\vec{E}^*(u)B(v)\otimes\vec{\xi}_2(u,v). \end{aligned} \tag{46}$$

Above we introduced the notations

$$\begin{aligned} \hat{\eta}_1(u,v) &= \begin{pmatrix} \rho_2^+(u,v) & 0 \\ 0 & \rho_2^-(u,v) \end{pmatrix}, \\ \hat{\eta}_2(u,v) &= \begin{pmatrix} \rho_2^-(u,v) & 0 \\ 0 & \rho_2^+(u,v) \end{pmatrix}, \\ \hat{A}(u) &= \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix}, \\ \vec{B} &= (B_1, B_2), \quad \vec{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \\ \vec{C}^* &= (C_4, C_5), \quad \vec{E}^* = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \end{aligned} \tag{47}$$

In order to determine the eigenvalue of the transfer matrix (28) acting on the multi-particle eigenstates we need to consider the commutation relations for the creation field  $F(u)$ :

$$B(u)F(v) = -\frac{\rho_1(v,u)}{\rho_8(v,u)}F(v)B(u) + \frac{\rho_5(v,u)}{\rho_8(v,u)}F(u)B(v) - \frac{i}{\rho_8(v,u)}[\vec{B}(u)\otimes\vec{B}(v)]\cdot\vec{\xi}_2^t(v,u), \tag{48}$$

$$D(u)F(v) = -\frac{\rho_1(v,u)}{\rho_8(v,u)}F(v)D(u) + \frac{\rho_5(v,u)}{\rho_8(v,u)}F(u)D(v) + \frac{i}{\rho_8(v,u)}\vec{\xi}_2(v,u)\cdot[\vec{E}^*(u)\otimes\vec{E}^*(v)], \tag{49}$$

$$\begin{aligned} \hat{A}(u)F(v) &= \left[1 - \frac{\rho_2^+(u,v)\rho_2^-(u,v)}{\rho_9(u,v)\rho_{10}(u,v)}\right]F(v)\hat{A}(u) + \frac{1}{\rho_9(u,v)\rho_{10}(u,v)}\hat{\eta}_2(u,v)\cdot F(u)\hat{A}(v)\cdot\hat{\eta}_2(u,v) \\ &+ \frac{1}{i\rho_9(u,v)}\hat{\eta}_2(u,v)\cdot\vec{B}(u)\otimes\vec{E}^*(v) - \frac{1}{i\rho_{10}(u,v)}\vec{E}^*(u)\otimes\vec{B}(v)\cdot\hat{\eta}_2(u,v), \end{aligned} \tag{50}$$

$$\vec{B}(u)F(v) = \frac{i\rho_9(u,v)}{\rho_1(u,v)}F(v)\vec{B}(u) + \frac{1}{\rho_1(u,v)}\hat{\eta}_2(u,v)\cdot\vec{B}(v)F(u), \tag{51}$$

$$F(u)\vec{B}(v) = -\frac{i\rho_{10}(u,v)}{\rho_1(u,v)}\vec{B}(v)F(u) + \frac{1}{\rho_1(u,v)}\hat{\eta}_1(u,v)\cdot F(v)\vec{B}(u). \tag{52}$$

Finally, if we adopt the variables  $z_{\pm}(v_i)$  used in Ref. 13, i.e.,

$$z_-(v_i) = \frac{\cos v_i}{\sin v_i} e^{2h(v_i)}, \quad z_+(v_i) = \frac{\sin v_i}{\cos v_i} e^{2h(v_i)}, \tag{53}$$

and make a shift on the spin rapidity  $\tilde{\lambda}_j = \tilde{\lambda}_j + U/2$ , the eigenvalue of the transfer matrix (28) is given as (up on a common factor)

$$\begin{aligned} \tau(u) |\Phi_n(v_1, \dots, v_n)\rangle = & \left\{ [z_-(u)]^L \prod_{i=1}^n \frac{\sin u(1+z_-(v_i)/z_+(u))}{\cos u(1-z_-(v_i)/z_-(u))} \right. \\ & + [z_+(u)]^L \prod_{i=1}^n \frac{\sin u(1+z_-(v_i)z_-(u))}{\cos u(1-z_-(v_i)z_+(u))} \\ & - \prod_{i=1}^n \frac{\sin u(1+z_-(v_i)/z_+(u))}{\cos u(1-z_-(v_i)/z_-(u))} \prod_{l=1}^M \frac{(\tilde{u}-\tilde{\lambda}_l+U/2)}{(\tilde{u}-\tilde{\lambda}_l-U/2)} \\ & \left. + \prod_{i=1}^n \frac{\sin u(1+z_-(v_i)z_-(u))}{\cos u(1-z_-(v_i)z_+(u))} \prod_{l=1}^M \frac{(\tilde{u}-\tilde{\lambda}_l-3U/2)}{(\tilde{u}-\tilde{\lambda}_l-U/2)} \right\} |\Phi_n(v_1, \dots, v_n)\rangle, \tag{54} \end{aligned}$$

provided that

$$[z_-(v_i)]^L = \prod_{l=1}^M \frac{(\tilde{v}_i - \tilde{\lambda}_l + U/2)}{(\tilde{v}_i - \tilde{\lambda}_l - U/2)}, \tag{55}$$

$$\prod_{i=1}^n \frac{(\tilde{\lambda}_j - \tilde{v}_i + U/2)}{(\tilde{\lambda}_j - \tilde{v}_i - U/2)} = - \prod_{\substack{l=1, \\ l \neq j}}^M \frac{(\tilde{\lambda}_j - \tilde{\lambda}_l + U)}{(\tilde{\lambda}_j - \tilde{\lambda}_l - U)}, \tag{56}$$

where

$$j = 1, \dots, M, \quad i = 1, \dots, n.$$

If we express the variable  $z_-(u_i)$  in terms of the (hole) momenta  $k_i$  by  $z_-(u_i) = e^{ik_i}$ , from the relation (39), the energy is given by

$$E_n = -(N/2 - n)U - \sum_{i=1}^n 2 \cos k_i. \tag{57}$$

Using the momenta  $k_i$  instead of the charge rapidity  $\tilde{v}_i$  via the relation (39) and making a scaling on the spin rapidity  $\tilde{\lambda}_j$  as  $\lambda_j = -(i/2)\tilde{\lambda}_j$ , then the Bethe equations (55) and (56) read

$$\begin{aligned} e^{ik_i} = & \prod_{l=1}^M \frac{(\sin k_i - \lambda_l - iU/4)}{(\sin k_i - \lambda_l + iU/4)}, \\ \prod_{i=1}^n \frac{(\sin k_i - \lambda_j - iU/4)}{(\sin k_i - \lambda_j + iU/4)} = & - \prod_{\substack{l=1, \\ l \neq j}}^M \frac{(\lambda_j - \lambda_l + iU/2)}{(\lambda_j - \lambda_l - iU/2)}, \tag{58} \end{aligned}$$

$$j = 1, \dots, M, \quad i = 1, \dots, n.$$

## V. CONCLUSIONS AND DISCUSSION

We have proposed an integrable variant of the Hubbard model with off-site Coulomb interaction. The integrability of the model was verified by showing that the quantum  $R$ -matrix satisfies the Yang–Baxter equation. It was argued that the model possess  $SO(4)$  symmetry, however, it contains a new representation of  $\eta$ -pairing  $SU(2)$  algebra and  $\zeta$ -pair spin  $SU(2)$  algebra. By means of the nested Bethe ansatz, we have presented the spectrum of the Hamiltonian, eigenvectors and the Bethe ansatz equations for the model with periodic boundary conditions. We found that the model exhibits a gauged  $r$ -matrix of the isotropic XXX model, which plays a crucial role in solving the model. Under periodic boundary conditions the alternative model and the standard Hubbard model share the same spectrum and Bethe ansatz equations. However, the new  $R$ -matrix we obtained permits different boundary conditions from that for the usual one.<sup>25,26</sup> This is meant that there does not exist simple transformation or gauge transformation between the new  $R$ -matrix and the original one. In turn, the differences in spectrum for the two models would be apparent in the case of open boundary conditions. We would like to remark that the 1D Hubbard model with long range Coulomb interaction, i.e.,  $U \sum_{j=1}^N (n_{j\uparrow} - \frac{1}{2})(n_{(j+r)\downarrow} - \frac{1}{2})$ ,  $r=1,2,\dots$ , instead of the on-site one in the standard Hubbard model would be also integrable. But this type of interaction would result in nondiagonal boundary scattering matrices which provide competing interaction terms in the Hamiltonian. This seems to open an opportunity to identify new boundary impurity effects<sup>27–29</sup> in a Luttinger liquid. An interesting problem is to identify the boost operator for the spectral parameter extension of this new model, which can iteratively generate all of the conserved currents, using the results of Ref. 14. We shall be focusing on these problems in the near future.

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