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DETERMINING WHEN THE ABSOLUTE STATE COMPLEXITY OF A HERMITIAN CODE ACHIEVES ITS DLP BOUND[*]

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Abstract. Let g be the genus of the Hermitian function field $H/\mathbb{F}_{q^2}$ and let $C_{\mathcal{L}}(D, mQ_\infty)$ be a typical Hermitian code of length n. In [*Des. Codes Cryptogr.*, to appear], we determined the dimension/length profile (DLP) lower bound on the state complexity of $C_{\mathcal{L}}(D, mQ_\infty)$. Here we determine when this lower bound is tight and when it is not.

For $m \leq \frac{n-2}{2}$ or $m \geq \frac{n-2}{2} + 2g$, the DLP lower bounds reach Wolf's upper bound on state complexity and thus are trivially tight. We begin by showing that for about half of the remaining values of m the DLP bounds cannot be tight. In these cases, we give a lower bound on the absolute state complexity of $C_{\mathcal{L}}(D, mQ_\infty)$, which improves the DLP lower bound.

Next we give a "good" coordinate order for $C_{\mathcal{L}}(D, mQ_\infty)$. With this good order, the state complexity of $C_{\mathcal{L}}(D, mQ_\infty)$ achieves its DLP bound (whenever this is possible). This coordinate order also provides an upper bound on the absolute state complexity of $C_{\mathcal{L}}(D, mQ_\infty)$ (for those values of m for which the DLP bounds cannot be tight). Our bounds on absolute state complexity do not meet for some of these values of m, and this leaves open the question whether our coordinate order is best possible in these cases.

A straightforward application of these results is that if $C_{\mathcal{L}}(D, mQ_\infty)$ is self-dual, then its state complexity (with respect to the lexicographic coordinate order) achieves its DLP bound of $\frac{n}{2} - \frac{q^2}{4}$, and, in particular, so does its absolute state complexity.

Key words. Hermitian code, state complexity, dimension/length profile bound

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1. Introduction.

Let C be a linear code of length n. Many soft-decision decoding algorithms for C (such as the Viterbi algorithm and lower complexity derivatives of it) take place along a minimal trellis for C. The complexity of trellis decoding algorithms can be measured by various trellis complexities. The most common one is the state complexity $\mathrm{s}(C)$ of C, which varies with the coordinate order of C. Since the number of operations required for Viterbi decoding of C is proportional to $\mathrm{s}(C)$, it is desirable that $\mathrm{s}(C)$ be small. A classical upper bound for $\mathrm{s}(C)$ is the Wolf bound $\mathrm{W}(C) = \min\{\dim(C), n - \dim(C)\}$ [9]. It is well known that if C is a Reed–Solomon code, then $\mathrm{s}(C) = \mathrm{W}(C)$.

Let $[C]$ denote the set of codes equivalent to C by a change of coordinate order. We write $s[C]$ for the minimum of $\mathrm{s}(C)$ over all coordinate orders of C and call it the *absolute state complexity* of C. (We note that state-complexity notation and terminology varies in the literature. For example, state complexity is called minimal trellis size in [2]; absolute state complexity is called absolute minimal trellis size in [2] and minimal state complexity in [13].) Finding a coordinate order of C that achieves $s[C]$ is called the "art of trellis decoding" in [10] since exhaustive computation of $\mathrm{s}(C)$

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over all possible coordinate orders of C is infeasible, even for quite short codes. An important step towards attaining this goal is determining good lower bounds on $s[C]$.

The dimension/length profile (DLP) of C is a deep property which is equivalent to the generalized weight hierarchy (GWH) of C. (For a survey of GWH, see [15].) The DLP of C is independent of the coordinate order of C and provides a natural lower bound $\nabla(C)$ for $s[C]$. For example, if C is a Reed–Solomon code, then $\nabla(C) = \mathrm{W}(C)$ [9], so that $s[C]$ is as bad as possible and uninteresting. However, determining when $\nabla(C) = s[C]$ is still important. An obvious and useful way of doing this is to find a coordinate order of C for which $\mathrm{s}(C) = \nabla(C)$. In particular, this provides one route to the art of trellis decoding. It is also important to develop methods for determining when $\nabla(C) < \mathrm{s}(C)$ and, in these cases, to improve on $\nabla(C)$.

Geometric Goppa codes generalize Reed–Solomon codes. Hermitian codes are widely studied geometric Goppa codes which are longer than Reed–Solomon codes and have very good parameters for their lengths. Let q be a fixed prime power, $n = q^3$, and $g = \binom{q}{2}$. For $m \in [0, n + 2g - 2]$, we write $C_{\mathcal{L}}(D, mQ_\infty)$ for a typical Hermitian code of length n defined over $\mathbb{F}_{q^2}$. In [5], we determined $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$ using some of the GWH of Hermitian codes obtained in [11, 16]. (The complete GWH of Hermitian codes has subsequently appeared in [1].) From [5], we have $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)) = \mathrm{W}(C_{\mathcal{L}}(D, mQ_\infty))$ for $m < \frac{n-1}{2}$ or $m > \frac{n-3}{2} + 2g$, so we restrict ourselves to the interesting Hermitian codes, i.e., to $C_{\mathcal{L}}(D, mQ_\infty)$ with $m \in I(n, g) = [\frac{n-1}{2}, \frac{n-3}{2} + 2g]$.

Here we determine precisely when $\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = \mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty))$. In the process, we exhibit a good coordinate order which often gives $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)) < \mathrm{W}(C_{\mathcal{L}}(D, mQ_\infty))$. We also improve on the DLP bound (when it is strictly less than the state complexity).

"Points of gain and fall" were introduced in [3, 4, 6, 7] to help determine the state complexity of certain generalizations of Reed–Muller codes. For these codes, the points of gain and fall had particularly nice characterizations. For Hermitian codes, however, their characterization is not quite as nice, and so our approach is slightly different. We describe a coordinate order giving $C_m \in [C_{\mathcal{L}}(D, mQ_\infty)]$ and characterize the points of gain and fall of C_m. We also characterize these points of gain and fall in terms of runs. This has the advantage of greatly reducing (from n to $q + 1$) the number of trellis depths needed to find $\mathrm{s}(C_m)$.

The paper is arranged as follows. Section 2 contains terminology, notation, and some previous results that will be used throughout the paper. The paper proper begins with section 3. Here we show that, for $m \in I(n, g)$, just under half of the Hermitian codes cannot attain their DLP bound. In these cases we give an improvement of the DLP bound, written $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_\infty))$.

The main goal of section 4 is to characterize the points of gain and fall of C_m in runs. In section 5 we determine $\mathrm{s}(C_m)$ using section 4. We show that $\mathrm{s}(C_m) = \nabla(C_m)$ for just over half the $m \in I(n, g)$. Thus we have determined precisely when the DLP bound for Hermitian codes is tight. Furthermore, $\mathrm{s}(C_m) = \nabla^{\imath}(C_m)$ for around a further quarter (respectively, $1/q$) of $m \in I(n, g)$ when q is odd (respectively, even).

In conclusion, we have found $s[C_m]$ for three quarters (respectively, one half) of the $m \in I(n, g)$ when q is odd (respectively, even). For the remaining $m \in I(n, g)$, we do not know a better coordinate order (than that described in section 4) nor a better bound (than that given in section 3). Thus, although we have reduced the possible range of $s[C_m]$, some of its actual values remain open. Finally, our method of characterizing points of gain and fall is essentially the same as the one used to determine $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$ in [5] and may be able to be used quite generally in determining DLP bounds and state complexity.

The state complexity of Hermitian codes has also been studied in [13]. For a stronger version of [13, Proposition 1] (an application of Clifford's theorem), see [5, Proposition 3.4]. Also, Example 5.11 below generalizes the main result of [13] to arbitrary self-dual Hermitian codes. An initial account of some of these results appeared in [8].

2. Terminology, notation, and background.

State complexity. Let C be a linear code of length n and $0 \leq i \leq n$. The state space dimension of C at depth i is

$$\mathrm{s}_i(C) = \dim(C) - \dim(C_{i,-}) - \dim(C_{i,+}), \tag{1}$$

where $C_{i,-} = \{c \in C : \ c_{i+1} = \cdots = c_n = 0\}$ and $C_{i,+} = \{c \in C : \ c_1 = \cdots = c_i = 0\}$. The *state complexity of* C is $\mathrm{s}(C) = \max\{\mathrm{s}_i(C) : 0 \leq i \leq n\}$. It is well known that $\mathrm{s}(C^{\perp}) = \mathrm{s}(C)$. A simple upper bound on $\mathrm{s}(C)$ (and hence on $s[C]$) is the Wolf bound $\mathrm{W}(C) = \min\{\dim(C), n - \dim(C)\}$. We write $[C]$ for the set of codes equivalent to C by a change of coordinate order; i.e., $C' \in [C]$ if and only if there exists a permutation $(l_1, \ldots, l_n)$ of $(1, \ldots, n)$ such that $C' = \{(c_{l_1}, \ldots, c_{l_n}) : (c_1, \ldots, c_n) \in C\}$. Then we define the *absolute state complexity of* C to be

$$s[C] = \min\{\mathrm{s}(C') : C' \in [C]\}.$$

The DLP of C is $(k_0(C), \ldots, k_n(C))$, where $k_i(C) = \max\{\dim(C_J) : |J| = i\}$. Clearly, $\dim(C_{i,-}) \leq k_i(C)$ and $\dim(C_{i,+}) \leq k_{n-i}(C)$, so that $\mathrm{s}_i(C) \geq \dim(C) - k_i(C) - k_{n-i}(C)$. The *DLP bound on* $\mathrm{s}_i(C)$ is

$$\nabla_i(C) = \dim(C) - k_i(C) - k_{n-i}(C),$$

and the *DLP bound on* $\mathrm{s}(C)$ is $\nabla(C) = \max\{\nabla_i(C) : 0 \leq i \leq n\}$. *We will use DLP bound to mean* $\nabla(C)$ *for some* C. It is well known that $\nabla(C^{\perp}) = \nabla(C)$. Since $\nabla(C)$ is independent of the coordinate order of C, $\nabla(C) \leq s[C]$. If $s[C] = \nabla(C)$, we say that C is *DLP-tight*; e.g., if $\nabla(C) = \mathrm{W}(C)$, then C is DLP-tight.

Hermitian codes. Our terminology and notation for Hermitian codes for the most part follow [14]. We write $H/\mathbb{F}_{q^2}$ for the Hermitian function field. Thus $H = \mathbb{F}_{q^2}[x, y]$, where x is transcendental over $\mathbb{F}_{q^2}$ and $y^q + y = x^{q+1}$ is the minimal polynomial of y over $\mathbb{F}_{q^2}[x]$. The genus of $H/\mathbb{F}_{q^2}$ is $g = \binom{q}{2} > 0$. We write $\mathbb{P}_H$ for the set of places of $H/\mathbb{F}_{q^2}$ and $\mathcal{D}_H$ for the divisor group of $H/\mathbb{F}_{q^2}$. For $Q \in \mathbb{P}_H$ and $z \in H/\mathbb{F}_{q^2}$, we write $v_Q(z)$ for the valuation of z at Q. Thus $v_Q(z) < 0$ if and only if Q is a pole of z and $v_Q(z) > 0$ if and only if Q is a zero of z. Also, $(z) \in \mathcal{D}_H$ is given by $(z) = \sum_{Q \in \mathbb{P}_H} v_Q(z) Q$ and for $A \in \mathcal{D}_F$, $\mathcal{L}(A) = \{z \in H/\mathbb{F}_{q^2} : (z) \geq -A\} \cup \{0\}$.

There are $q^3 + 1$ places of degree one in $\mathbb{P}_H$. One of these is the place at infinity, which we denote Q_∞. We denote the others as $Q_1, \ldots, Q_{q^3}$. *For the rest of the paper, unless otherwise stated,* $n = q^3$. We put $D = \sum_{j=1}^{n} Q_j$. For an integer m, $\mathcal{L}(mQ_\infty) = \{z \in H/\mathbb{F}_{q^2} : (z) \geq -mQ_\infty\} \cup \{0\}$. The Hermitian codes over $\mathbb{F}_{q^2}$ are $C_{\mathcal{L}}(D, mQ_\infty) = \{z(Q_{l_1}), \ldots, z(Q_{l_n}) : z \in \mathcal{L}(mQ_\infty)\}$ for some permutation $(l_1, \ldots, l_n)$ of $(1, \ldots, n)$. Strictly speaking, the code $C(D, mQ_\infty)$ depends on the permutation $(l_1, \ldots, l_n)$ of $(1, \ldots, n)$ and may be better denoted $C_{\mathcal{L}}(Q_{l_1}, \ldots, Q_{l_n}; mQ_\infty)$. However, this notation is cumbersome and $C_{\mathcal{L}}(D, mQ_\infty)$ is standard. *Unless otherwise stated, when we write* $C_{\mathcal{L}}(D, mQ_\infty)$ *we have some fixed but arbitrary coordinate order in mind.*

From the usual expression for the dimension of geometric Goppa codes,

$$\dim(C_{\mathcal{L}}(D, mQ_\infty)) = \dim(mQ_\infty) - \dim(mQ_\infty - D).$$

When m is understood, $k = \dim(C_{\mathcal{L}}(D, mQ_\infty))$ *unless stated otherwise.* The *abundance* of $C_{\mathcal{L}}(D, mQ_\infty)$ is $\dim(mQ_\infty - D)$. For $m < n$, the abundance is 0 and the code is nonabundant. For $m < 0$, $C_{\mathcal{L}}(D, mQ_\infty) = \{0\}$ and for $m > n + 2g - 2$, $C_{\mathcal{L}}(D, mQ_\infty) = \mathbb{F}_{q^2}^n$, so we restrict our attention to $m \in [0, n + 2g - 2]$. With $m^\perp = n + 2g - 2 - m$, the dual of $C_{\mathcal{L}}(D, mQ_\infty)$ is $C_{\mathcal{L}}(D, m^\perp Q_\infty)$.

Let $\Pi : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ be the pole number sequence of Q_∞. Also, for $i, j \in \mathbb{Z}$ we put $[i, j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$ and $[i, \infty) = \{k \in \mathbb{Z} : k \geq i\}$. Thus $\Pi[1, \infty)$ is the set of pole numbers of Q_∞, $\Pi(r)$ is the rth pole number, and $\Pi^{-1}[R_1, R_2] = \{r : R_1 \leq \Pi(r) \leq R_2\}$. We note that $\Pi^{-1}[0, R] = \{r : \Pi(r) \leq R\}$ and $\Pi^{-1}[R_1, R_2] = \Pi^{-1}[0, R_2] \setminus \Pi^{-1}[0, R_1 - 1]$. From [14, Proposition VI.4.1] we deduce that

$$\Pi[1, \infty) = \{iq + j : 0 \leq i \leq q - 2, 0 \leq j \leq i\} \cup [2g, \infty). \tag{2}$$

We note that, for $m < n$, $\dim(mQ_\infty - D) = 0$ and $k = \dim(mQ_\infty) = |\Pi^{-1}[0, m]|$.

State complexity of Hermitian codes. For $0 \leq i \leq n$ we put $D_{i,-} = \sum_{j=1}^{i} Q_{l_j}$ and $D_{i,+} = \sum_{l=i+1}^{n} Q_{l_j}$ (where $(l_1, \ldots, l_n)$ is a fixed but arbitrary permutation of $(1, \ldots, n)$). We deduce that $\mathrm{s}_i(C_{\mathcal{L}}(D, mQ_\infty)) = k - \dim(mQ_\infty - D_{i,-}) - \dim(mQ_\infty - D_{i,+}) + 2\dim(mQ_\infty - D)$. In particular, for $m < n$,

$$\mathrm{s}_i(C_{\mathcal{L}}(D, mQ_\infty)) = k - \dim(mQ_\infty - D_{i,-}) - \dim(mQ_\infty - D_{i,+}). \tag{3}$$

These identities yield $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty) = \mathrm{W}(C_{\mathcal{L}}(D, mQ_\infty))$ for $m \in [0, \frac{n-2}{2}] \cup [\frac{n-2}{2} + 2g, n + 2g - 2]$. Thus we will almost exclusively be interested in $m \in I(n, g) = [\frac{n-1}{2}, \frac{n-3}{2} + 2g]$. In fact, since $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$ if and only if $m^\perp \in [\frac{n-1}{2}, \frac{n-3}{2} + g]$, we will often restrict our attention to $m \in [\frac{n-1}{2}, \frac{n-3}{2} + g]$, deducing results for $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$ from $\mathrm{s}(C^\perp) = \mathrm{s}(C)$ and $\nabla(C^\perp) = \nabla(C)$.

It is convenient to put $J(n, g) = [\frac{n-1}{2}, \frac{n-2}{2} + g]$. Using results of [11, 16], [5, Proposition 5.1] shows that for $m \in I(n, g)$,

$$\nabla_i(C_{\mathcal{L}}(D, mQ_\infty)) = k - |\Pi^{-1}[0, m - i]| - |\Pi^{-1}[0, m + i - n]|, \tag{4}$$

which is used to prove the following theorem.

THEOREM 2.1 (see [5, Theorem 5.5]). *For $m \in J(n, g)$, write $n - 2m + 4g + q - 2 = uq + v$, where $0 \leq v \leq q - 1$. Then $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$ is attained at $m - 2g + 1 + \lfloor \frac{u}{2} \rfloor q$ and equals*

$$k - \binom{q - \lfloor \frac{u}{2} \rfloor}{2} - \binom{q - \lceil \frac{u}{2} \rceil}{2} - \min\left\{q - \left\lceil \frac{u}{2} \right\rceil, q - v\right\}.$$

If $C_{\mathcal{L}}(D, mQ_\infty)$ is DLP-tight, then we just say m *is DLP-tight.*

3. When the DLP bound is not tight.

Let $m \in [0, \frac{n-2}{2}] \cup [\frac{n-2}{2} + 2g, n + 2g - 2]$. Then by [5, Proposition 4.3, Example 4.9], we have $\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = \mathrm{W}(C_{\mathcal{L}}(D, mQ_\infty))$ and so

$$\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = s[C_{\mathcal{L}}(D, mQ_\infty)] = \mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)),$$

where $C_{\mathcal{L}}(D, mQ_\infty)$ can have any coordinate order. Such m are therefore DLP-tight, and we are reduced to determining which $m \in I(n, g)$ are DLP-tight. We note that $\frac{n-3}{3} + 2g < n$, so that the codes that we are interested in are nonabundant.

Table 1
Table of new notation.

m	integer
q	fixed prime power
$m^{\perp}$	$n+2g-2-m$
q_2	q mod 2
$I(n,g)$	$[\frac{n-1}{2}, \frac{n-3}{2}+2g]$
$J(n,g)$	$[\frac{n-1}{2}, \frac{n-2}{2}+g]$
M	$m-\frac{q^2-q_2}{2}q$ if $m \in J(n,g)$
$M^{\bullet}, M^{\circ}$	$M = M^{\bullet}(q+1)+M^{\circ}$ where $0 \leq M^{\circ} \leq q$
$\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$	Improved DLP bound for $m \in I(n,g)$ Definition 3.11
$\Delta(m)$	$\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty})) - \nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$ (Theorem 3.9 and Corollary 3.10)
$\mathbb{P}_H^1$	Finite places of degree one in $\mathbb{P}_H$
α_{ab}	Elements of $\mathbb{F}_{q^2}$ such that $\alpha_{ab}^{q+1} = a \in \mathbb{F}_q$
β_{ac}	Elements of $\mathbb{F}_{q^2}$ such that $\beta_{ac}^{q} + \beta_{ac} = a \in \mathbb{F}_q$
$Q_{a,b,c} = Q_{\alpha_{ab}, \beta_{ac}}$	Element of $\mathbb{P}_H^1$ such that $x(Q_{a,b,c}) = \alpha_{ab}$ and $y(Q_{a,b,c}) = \beta_{ac}$
C_m	Element of $[C_{\mathcal{L}}(D, mQ_{\infty})]$ with coordinate order given in section 4
$P_{\text{gain}}(m), P_{\text{fall}}(m)$	Sets of points of gain and fall of C_m
$P_{\text{gain}}^{i,-}(m), P_{\text{fall}}^{i,-}(m)$	$\vert P_{\text{gain}}(m) \cap [1,i]\vert$ and $\vert P_{\text{fall}}(m) \cap [1,i]\vert$
Λ	$\Lambda : [0,\infty) \times [0,q-1] \longrightarrow [0,\infty)$ given by $\Lambda(j,l) = jq + l(q+1)$
ζ_{gain}	$0, \frac{q-q_2}{2}, q$ depending on M° (defined before Proposition 4.8)
ζ_{fall}	$0, \frac{q+q_2}{2}, q$ depending on M° (defined before Proposition 4.8)
$\theta_{\text{gain}}, \theta_{\text{fall}}$	$M^{\bullet} + M^{\circ} - \zeta_{\text{gain}}$ and $M^{\bullet} + M^{\circ} - \zeta_{\text{fall}}$
ζ_{norm}	$(\zeta_{\text{gain}} + \zeta_{\text{fall}})/2$
η	$2q - 2M^{\bullet} + q_2 - \zeta_{\text{norm}} - 3$

In this section we determine the $m \in I(n,g)$ which are not DLP-tight, i.e, with $s[C_{\mathcal{L}}(D, mQ_{\infty})] > \nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$. The coordinate order of $C_{\mathcal{L}}(D, mQ_{\infty})$ is arbitrary, so it suffices to show that $\mathrm{s}(C_{\mathcal{L}}(D, mQ_{\infty})) > \nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$.

Our approach has three steps.

(i) We prove the key lemma, Lemma 3.2, and indicate how this can be used to show that m is not DLP-tight (Example 3.3).

(ii) We prove a generalization of the key lemma (Lemma 3.4) and an application of Proposition 3.5. We indicate how this can be used to improve on the DLP bound by more than one (Example 3.6).

(iii) We prove an application of Proposition 3.5 to improve the DLP bound for $m \in I(n,g)$, Theorem 3.9, and Corollary 3.10.

We conclude section 3 with a table of the improved DLP bound for small values of q (2) and an analysis of the proportion of those $m \in I(n,g)$ for which our bound is strictly better than the DLP bound (Proposition 3.12).

The key lemma. We begin with a clarification of (3) and (4).

Lemma 3.1. *For $0 \leq i \leq n$ and $m \in I(n,g)$,*

$$\dim(mQ_{\infty} - D_{i,-}) \leq |\Pi^{-1}[0, m-i]| \textit{ and } \dim(mQ_{\infty} - D_{i,+}) \leq |\Pi^{-1}[0, m+i-n]| \tag{5}$$

and $\mathrm{s}_i(C_{\mathcal{L}}(D, mQ_{\infty})) = \nabla_i(C_{\mathcal{L}}(D, mQ_{\infty}))$ if and only if there is equality in both.

Proof. The first part follows from [5, Lemma 4.1] and the fact that the gonality sequence of $H/\mathbb{F}_{q^2}$ equals the pole number sequence of Q_{∞} by [12, Corollary 2.4]. The second part then follows from (3) and (4). □

We note that Lemma 3.1 implies that a coordinate order is inefficient, in the sense of [9], if and only if there exists an i, $0 \leq i \leq n$, such that $|\Pi^{-1}[0, m-i]| >$

$\dim(mQ_\infty - D_{i,-})$ or $|\Pi^{-1}[0, m+i-n]| > \dim(mQ_\infty - D_{i,+})$. To show the stronger result that $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)) > \nabla(C_{\mathcal{L}}(D, mQ_\infty))$, we require a stronger condition on i, namely, that it satisfies

$$\begin{aligned}|\Pi^{-1}[0, m-i]| - \dim(mQ_\infty - D_{i,-}) + |\Pi^{-1}[0, m+i-n]| - \dim(mQ_\infty - D_{i,+})\\ > \nabla(C_{\mathcal{L}}(D, mQ_\infty)) - \nabla_i(C_{\mathcal{L}}(D, mQ_\infty)),\end{aligned}$$

so that $\mathrm{s}_i(C_{\mathcal{L}}(D, mQ_\infty)) > \nabla(C_{\mathcal{L}}(D, mQ_\infty))$.

This stronger condition is clearly more likely to hold if $\nabla_i(C_{\mathcal{L}}(D, mQ_\infty))$ attains or is close to attaining $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$. For now, we concentrate on determining when the equalities in (5) cannot hold. For these equalities to hold, $\dim(mQ_\infty - D_{i,-})$ and $\dim(mQ_\infty - D_{i,+})$ must change with $|\Pi^{-1}[0, m-i]|$ and $|\Pi^{-1}[0, m+i-n]|$, respectively. We shall see that it is possible that both $|\Pi^{-1}[0, m-i]| = |\Pi^{-1}[0, m-(i-1)]| - 1$ and $|\Pi^{-1}[0, m+i-n]| = |\Pi^{-1}[0, m+(i-1)-n]| + 1$ (i.e., it is possible that both $m-i+1$ and $m+i-n$ are pole numbers of Q_∞).

Lemma 3.2. *For $m \le \frac{n-2}{2} + g$, it is not possible that* $\dim(mQ_\infty - D_{i,-}) = \dim(mQ_\infty - D_{i-1,-}) - 1$ *and* $\dim(mQ_\infty - D_{i,+}) = \dim(mQ_\infty - D_{i-1,+}) + 1$.

Proof. We assume that $\dim(mQ_\infty - D_{i,-}) = \dim(mQ_\infty - D_{i-1,-}) - 1$ and $\dim(mQ_\infty - D_{i,+}) = \dim(mQ_\infty - D_{i-1,+}) + 1$ and derive a contradiction. Suppose we have $z_1, z_2 \in H/\mathbb{F}_{q^2}$ such that (i) $(z_1) \ge -mQ_\infty + D_{i-1,-}$, $v_{Q_{l_i}}(z_1) = 0$ and (ii) $(z_2) \ge -mQ_\infty + D_{i,+}$, $v_{Q_{l_i}}(z_2) = 0$. Thus $(z_1 z_2) \ge -2mQ_\infty + D - Q_{l_i}$ and $v_{Q_{l_i}}(z_1 z_2) = 0$. Now $nQ_\infty - D$ is a principal divisor of $H/\mathbb{F}_{q^2}$ (e.g., as in the proof of [14, Proposition VII.4.2]), say $nQ_\infty - D = (z_3)$. Thus $(z_1 z_2 z_3) \ge (n-2m)Q_\infty - Q_{l_i}$ and $v_{Q_{l_i}}(z_1 z_2 z_3) = -1$. Hence $z_1 z_2 z_3 \in \mathcal{L}((2m-n)Q_\infty + Q_{l_i}) \setminus \mathcal{L}((2m-n)Q_\infty)$ so that by [14, Lemma I.4.8]

$$\dim((2m-n)Q_\infty + Q_{l_i}) = \dim((2m-n)Q_\infty) + 1. \tag{6}$$

Now $(2g-2)Q_\infty$ is a canonical divisor of $H/\mathbb{F}_{q^2}$ (e.g., by [14, Lemma VI.4.4] or because $2g-2$ is the gth pole number of Q_∞ and [14, Proposition I.6.2]). Thus $\dim((2m-n)Q_\infty + Q_{l_i}) = 2m - n + 2 - g + \dim((2g-2-2m+n)Q_\infty - Q_{l_i})$ by the Riemann–Roch theorem, so from (6), $\dim((2g-2-2m+n)Q_\infty - Q_{l_i}) = \dim((2m-n)Q_\infty) - 2m + n + g - 1$. Again, by the Riemann–Roch theorem, $\dim((2g-2-2m+n)Q_\infty) = g - 1 - 2m + n + \dim((2m-n)Q_\infty)$, so that

$$\dim((2g-2-2m+n)Q_\infty - Q_{l_i}) = \dim((2g-2-2m+n)Q_\infty),$$

and hence $\mathcal{L}((2g-2-2m+n)Q_\infty - Q_{l_i}) = \mathcal{L}((2g-2-2m+n)Q_\infty)$. However, for $2g-2-2m+n \ge 0$, i.e., for $m \le \frac{n-2}{2} + g$, $\mathbb{F}_{q^2} \subseteq \mathcal{L}((2g-2-2m+n)Q_\infty) \setminus \mathcal{L}((2g-2-2m+n)Q_\infty - Q_{l_i})$, giving the required contradiction. □

Example 3.3. *Let $q = 3$. We show $m = 13$ is not DLP-tight. From* (2), *we have* $\Pi[1, 11] = \{0, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13\}$, *so that $k = 11$. From Theorem* 2.1, $\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = 10$. *Now, from* (4),

$$\nabla_{13}(C_{\mathcal{L}}(D, mQ_\infty)) = k - |\Pi^{-1}[0,0]| - |\Pi^{-1}[0,-1]| = 10 = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$$

and, similarly, $\nabla_{14}(C_{\mathcal{L}}(D, mQ_\infty)) = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$. *Thus,* $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)) = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$ *implies that* $\mathrm{s}_i(C_{\mathcal{L}}(D, mQ_\infty)) = \nabla_i(C_{\mathcal{L}}(D, mQ_\infty))$ *for $i = 13, 14$. Lemma* 3.1 *then implies that* $\dim(mQ_\infty - D_{13,-}) = |\Pi^{-1}[0,0]| = 1$, $\dim(mQ_\infty - D_{13,+}) = 0$, $\dim(mQ_\infty - D_{14,-}) = 0$, *and* $\dim(mQ_\infty - D_{14,+}) = 1$, *which contradicts Lemma* 3.2. *Therefore* $\mathrm{s}(C_{\mathcal{L}}(D, mQ_\infty)) > \nabla(C_{\mathcal{L}}(D, mQ_\infty))$ *and since the coordinate* *order of $C_{\mathcal{L}}(D, mQ_\infty)$ is arbitrary, m is not DLP-tight. We will see in section* 5 *that* 14 *and* 15 *are DLP-tight.*

Generalization of the key lemma. Since $\dim(mQ_\infty - D_{i-1,-}) \leq \dim(mQ_\infty - D_{i,-}) + 1$ and $\dim(mQ_\infty - D_{i,+}) \leq \dim(mQ_\infty - D_{i-1,+}) + 1$ by [14, Lemma I.4.8], Lemma 3.2 can be restated as follows: for $m \leq \frac{n-2}{2} + g$, either $\dim(mQ_\infty - D_{i-1,-}) \leq \dim(mQ_\infty - D_{i,-})$ or $\dim(mQ_\infty - D_{i,+}) \leq \dim(mQ_\infty - D_{i-1,+})$. This generalizes as the following lemma.

LEMMA 3.4. *For $m \leq \frac{n-2}{2} + g$ and $0 < t \leq i \leq n$,* (i) $\dim(mQ_\infty - D_{i-t,-}) \leq \dim(mQ_\infty - D_{i,-}) + \lfloor \frac{t}{2} \rfloor$ *or* (ii) $\dim(mQ_\infty - D_{i,+}) \leq \dim(mQ_\infty - D_{i-t,+}) + \lfloor \frac{t}{2} \rfloor$.

Proof. Suppose that $\dim(mQ_\infty - D_{i,-}) < \dim(mQ_\infty - D_{i-t,-}) - \lfloor \frac{t}{2} \rfloor$ and $\dim(mQ_\infty - D_{i,+}) > \dim(mQ_\infty - D_{i-t,+}) + \lfloor \frac{t}{2} \rfloor$. Therefore there are $r, s > \lfloor \frac{t}{2} \rfloor$ and $\{i_1, \dots, i_r\}, \{j_1, \dots, j_s\} \subseteq \{i-t+1, \dots, i\}$ such that $\dim(mQ_\infty - D_{i_k,-}) = \dim(mQ_\infty - D_{i_k-1,-}) - 1$ for $1 \leq k \leq r$ and $\dim(mQ_\infty - D_{j_k,+}) = \dim(mQ_\infty - D_{j_k-1,+}) + 1$ for $1 \leq k \leq s$. However, $r + s > t$ so that, since $|\{i-t+1, \dots, i\}| = t$, $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} \neq \emptyset$, contradicting Lemma 3.2. □

The following application of Lemmas 3.1 and 3.4 is a straightforward consequence of (3), (4).

PROPOSITION 3.5. *For $m \in J(n,g)$ and $0 < t \leq i \leq n$,*

$$s[C_{\mathcal{L}}(D, mQ_\infty)] \geq \nabla_i(C_{\mathcal{L}}(D, mQ_\infty)) + |\Pi^{-1}[m+i-n-t+1, m+i-n]| - \left\lfloor \frac{t}{2} \right\rfloor.$$

EXAMPLE 3.6. *Let $q = 7$ and $m = 186$. Then $s[C_{\mathcal{L}}(D, mQ_\infty)] \geq \nabla(C_{\mathcal{L}}(D, mQ_\infty))$ $+ 2 = 159$. We have $k = 166$ (e.g., by the Riemann–Roch theorem). From* (2)*, the first few pole numbers of Q_∞ are $\Pi[1,6] = \{0,7,8,14,15,16\}$. From Theorem* 2.1*, we have $\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = 157$. For $i = 173$, $\Pi^{-1}[0, m-i] = \{0,7,8\}$ and $\Pi^{-1}[0, m+i-n] = \{0,7,8,14,15,16\}$, so that, from* (4)*, $\nabla_i(C_{\mathcal{L}}(D, mQ_\infty)) = 157 = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$. Also, with $t = 3$, we have $\Pi^{-1}(m+i-n-t) = \{0,7,8\}$. Thus Proposition* 3.5 *gives*

$$s[C_{\mathcal{L}}(D, mQ_\infty)] \geq \nabla_i(C_{\mathcal{L}}(D, mQ_\infty)) + 2 = \nabla(C_{\mathcal{L}}(D, mQ_\infty)) + 2 = 159.$$

We shall see in section 5 *that $s[C_{\mathcal{L}}(D, mQ_\infty)] = 159$.*

Improvement on the DLP bound. We show how Proposition 3.5 can be used to improve on the DLP bound generally. First, we introduce some useful notation: $q_2 = 0$ if q is even and $q_2 = 1$ if q is odd. For a fixed $m \in J(n,g)$, we put $M = m - \frac{q^2-q_2}{2} q$ and write $M = M^\bullet(q+1) + M^\circ$, where $0 \leq M^\circ \leq q$. We easily deduce the following lemma.

LEMMA 3.7. (i) *For q odd, $0 \leq M^\bullet \leq \frac{q-3}{2}$ and if $M^\bullet = 0$, then $M^\circ \geq \frac{q-1}{2}$;* (ii) *for q even, $0 \leq M^\bullet \leq \frac{q-2}{2}$ and if $M^\bullet = \frac{q-2}{2}$, then $M^\circ = 0$.*

We begin by reinterpreting Theorem 2.1 in terms of $M^\bullet$ and M°.

LEMMA 3.8. *For $m \in J(n,g)$, the DLP bound is attained at*

$$\begin{array}{ll} m + 1 - M^\bullet q & \text{if } 0 \leq M^\circ \leq \frac{q-2}{2} - M^\bullet, \\ m + 1 - (M^\bullet + 1 - q_2) q & \text{if } \frac{q-1}{2} - M^\bullet \leq M^\circ \leq q - M^\bullet - 1, \\ m + 1 - (M^\bullet + 1) q & \text{if } q - M^\bullet \leq M^\circ \leq q. \end{array}$$

Proof. If u, v are defined as in Theorem 2.1, then

$$(u,v) = \begin{cases} (2q - 2 - 2M^\bullet + q_2, q - 2M^\bullet - 2M^\circ - 2) & \text{if } 0 \leq M^\circ \leq \frac{q-2}{2} - M^\bullet, \\ (2q - 3 - 2M^\bullet + q_2, 2q - 2M^\bullet - 2M^\circ - 2) & \text{if } \frac{q-1}{2} - M^\bullet \leq M^\circ \leq q - M^\bullet - 1, \\ (2q - 4 - 2M^\bullet + q_2, 3q - 2M^\bullet - 2M^\circ - 2) & \text{if } q - M^\bullet \leq M^\circ \leq q. \end{cases}$$

The result now follows from the fact that the DLP bound is attained at $m-2g+1+\lfloor\frac{u}{2}\rfloor q$. □

Next we give our improvement on the DLP bounds for $m \in J(n,g)$. The size of the improvement is given by

$$\Delta(m) = \begin{cases} 1+M^\bullet+M^\circ-\frac{q-q_2}{2} & \text{if } \frac{q-q_2}{2}-M^\bullet \le M^\circ \le \frac{q-M^\bullet-1}{2}, \\ \frac{q+q_2}{2}-M^\circ & \text{if } \frac{q-M^\bullet}{2} \le M^\circ \le \frac{q-2+q_2}{2}, \\ 1+M^\bullet+M^\circ-q & \text{if } q-M^\bullet \le M^\circ \le q-\frac{M^\bullet+1}{2}, \\ 1+q-q_2-M^\circ & \text{if } q-\frac{M^\bullet}{2} \le M^\circ \le q-q_2, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\Delta(m)>0$ if and only if $\frac{q-q_2}{2}-M^\bullet \le M^\circ \le \frac{q-2+q_2}{2}$ or $q-M^\bullet \le M^\circ \le q-q_2$.

Theorem 3.9. *For $m \in J(n,g)$, $s[C_{\mathcal{L}}(D,mQ_\infty)] \ge \nabla(C_{\mathcal{L}}(D,mQ_\infty))+\Delta(m)$.*

Proof. First assume that $\frac{q-q_2}{2}-M^\bullet \le M^\circ \le \frac{q-2+q_2}{2}$. From Lemma 3.8, $\nabla(C_{\mathcal{L}}(D,mQ_\infty))$ is attained at $i=m+1-(M^\bullet+1-q_2)q$. We take $i=m+1-(M^\bullet+1-q_2)q$ and $t=2M^\bullet+2M^\circ+1-q+q_2$ in Proposition 3.5. Now $m+i-t-n=M^\bullet q-q_2$. We have two subcases.

(a) For $\frac{q-q_2}{2}-M^\bullet \le M^\circ \le \frac{q-M^\bullet-1}{2}$ we have $0<t\le M^\bullet+q_2$. Now, from (2), $M^\bullet q,\ldots,M^\bullet q+M^\bullet \in \Pi[1,\infty)$, so that $|\Pi^{-1}[m+i-n-t+1,m+i-n]|=t$, and Proposition 3.5 gives

$$s[C_{\mathcal{L}}(D,mQ_\infty)]-\nabla(C_{\mathcal{L}}(D,mQ_\infty)) \ge \left\lceil\frac{t}{2}\right\rceil = 1+M^\bullet+M^\circ-\frac{q-q_2}{2}.$$

(b) For $\frac{q-M^\bullet}{2} \le M^\circ \le \frac{q-2+q_2}{2}$ we have $M^\bullet+q_2+1 \le t \le 2M^\bullet+2q_2-1 \le q-1$. From (2), $M^\bullet q+M^\bullet+1,\ldots,M^\bullet q+q-1 \notin \Pi(\mathbb{N})$ since $M^\bullet \le q-2$, so that $|\Pi^{-1}[m+i-n-t+1,m+i-n]|=M^\bullet+q_2$, and Proposition 3.5 gives

$$s[C_{\mathcal{L}}(D,mQ_\infty)]-\nabla(C_{\mathcal{L}}(D,mQ_\infty)) \ge M^\bullet+q_2-\left(M^\bullet+M^\circ-\frac{q-q_2}{2}\right) = \frac{q+q_2}{2}-M^\circ.$$

Now suppose that $q-M^\bullet \le M^\circ \le q-q_2$. From Lemma 3.8, $\nabla(C_{\mathcal{L}}(D,mQ_\infty))$ is attained at $m+1-(M^\bullet+1)q$. We take $i=m+1-(M^\bullet+1)q$ and $t=2M^\bullet+2M^\circ-2q+2-q_2$ in Proposition 3.5. Now $m+i-t-n=(M^\bullet+1-q_2)q-(1-q_2)$ and again we have two subcases.

(a) For $q-M^\bullet \le M^\circ \le q-\frac{M^\bullet+1}{2}$ we have $0<t\le M^\bullet+1-q_2$. From (2), $(M^\bullet+1-q_2)q,\ldots,(M^\bullet+1-q_2)q+(M^\bullet+1-q_2) \in \Pi[1,\infty)$, so that $|\Pi^{-1}[m+i-n-t+1,m+i-n]|=t$, and Proposition 3.5 gives $s[C_{\mathcal{L}}(D,mQ_\infty)]-\nabla(C_{\mathcal{L}}(D,mQ_\infty)) \ge \lceil\frac{t}{2}\rceil = 1+M^\bullet+M^\circ-q$.

(b) For $q-\frac{M^\bullet}{2} \le M^\circ \le q-q_2$ we have $M^\bullet+2-q_2 \le t \le 2M^\bullet+2-3q_2 \le q-q_2$. From (2), $(M^\bullet+1-q_2)q+(M^\bullet+2-q_2),\ldots,(M^\bullet+1-q_2)+(q-1) \notin \Pi[1,\infty)$, since $M^\bullet+1-q_2 \le q-2$, so that $|\Pi^{-1}[m+i-n-t+1,m+i-n]|=M^\bullet+2-2q_2$, so that from Proposition 3.5,

$$\begin{aligned} s[C_{\mathcal{L}}(D,mQ_\infty)]-\nabla(C_{\mathcal{L}}(D,mQ_\infty)) &\ge M^\bullet+2-2q_2-(M^\bullet+M^\circ-q+1-q_2) \\ &= 1+q-q_2-M^\circ. \qquad \square \end{aligned}$$

For $m \in [\frac{n-1}{2}+g,\frac{n-3}{2}+2g]$ we put $\Delta(m)=\Delta(m^\perp)\ge 0$.

Corollary 3.10. *For $m \in I(n,g)$, $s[C_{\mathcal{L}}(D,mQ_\infty)] \ge \nabla(C_{\mathcal{L}}(D,mQ_\infty))+\Delta(m)$.*

Proof. The proof is an easy consequence of Theorem 3.9, $\nabla(C) = \nabla(C^{\perp})$, and the definition of $\Delta(m)$. □

Definition 3.11. *For $m \in I(n, g)$, we put $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty})) = \nabla(C_{\mathcal{L}}(D, mQ_{\infty})) + \Delta(m)$.*

We note that for $m \in I(n, g)$,

$$\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty})) = \nabla^{\imath}(C_{\mathcal{L}}(D, m^{\perp}Q_{\infty})). \tag{7}$$

In Table 2 we have written $\nabla^{\imath}(m)$ for $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$, and the DLP bound is calculated using Theorem 2.1. The bold face entries are those for which $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty})) > \nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$. (The values of $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$ for $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$ can of course be deduced from (7).)

Table 2
$\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$ for $q \in \{2, 3, 4, 5, 7, 8\}$ and $m \in J(n, g)$.

q															
2	m	4													
	$\nabla^{\imath}(m)$	3													
3	m	13	14	15											
	$\nabla^{\imath}(m)$	**11**	11	11											
4	m	32	33	34	35	36	37								
	$\nabla^{\imath}(m)$	26	27	27	28	**28**	28								
5	m	62	63	64	65	66	67	68	69	70	71				
	$\nabla^{\imath}(m)$	**53**	53	54	54	55	**56**	**56**	56	**56**	56				
7	m	171	172	173	174	175	176	177	178	179	180				
	$\nabla^{\imath}(m)$	**151**	151	152	153	153	154	155	**156**	**156**	156				
	m	181	182	183	184	185	186	187	188	189	190	191			
	$\nabla^{\imath}(m)$	157	**157**	157	158	**159**	**159**	**159**	159	**159**	**159**	159			
8	m	256	257	258	259	260	261	262	263	264	265	266	267	268	269
	$\nabla^{\imath}(m)$	228	229	230	231	231	232	233	234	**234**	234	235	236	**236**	236
	m	270	271	272	273	274	275	276	277	278	279	280	281	282	283
	$\nabla^{\imath}(m)$	237	238	**238**	**238**	238	239	**239**	**239**	239	240	**240**	**240**	**240**	240

We conclude this section by calculating the proportion of $m \in I(n, g)$ for which $\Delta(m) > 0$.

Proposition 3.12.

$$|\Delta^{-1}(0, \infty)|/|I(n, g)| = \begin{cases} \frac{1}{2} - \frac{1}{2q} & \textit{if } q \textit{ is odd,} \\ \frac{1}{2} - \frac{3q-5}{2(q^2-q-1)} & \textit{if } q \textit{ is even.} \end{cases}$$

Proof. We note first that $|I(n, g)| = 2g + q_2 - 1$. Recall from the definition of $\Delta(m)$ that

$$\Delta^{-1}(0, \infty) \cap \left\{\frac{n-1}{2}, \ldots, \frac{n-2}{2} + g\right\} = \left\{m : \frac{q - q_2}{2} - M^{\bullet} \leq M^{\circ} \leq \frac{q - 2 + q_2}{2} \text{ or } q - M^{\bullet} \leq M^{\circ} \leq q - q_2\right\}.$$

Next we note that $|\Delta^{-1}(0, \infty)| = 2\left|\Delta^{-1}(0, \infty) \cap J(n, g)\right|$. This follows from the definition of $\Delta(m)$ for $\frac{n-1}{2} + g \leq m \leq \frac{n-3}{2} + 2g$ when q is odd and from $\frac{n-2}{2} + g \notin \Delta^{-1}(0, \infty)$ when q is even. Now, fixing $0 \leq M^{\bullet} \leq \frac{q-3}{2}$, we have

$$\left|\left\{M^{\circ} : \frac{q - q_2}{2} - M^{\bullet} \leq M^{\circ} \leq \frac{q - 2 + q_2}{2} \text{ or } q - M^{\bullet} \leq M^{\circ} \leq q - q_2\right\}\right| = 2M^{\bullet} + 1.$$

We note that the restriction $M^\circ \geq \frac{q-1}{2}$ for q odd and $M^\bullet = 0$ from Lemma 3.7 does not affect this. We also note that for q even and $M^\bullet = \frac{q-2}{2}$, the restriction of $M^\circ = 0$ in Lemma 3.7 gives $|\{M^\circ : 1 \leq M^\circ \leq \frac{q-2}{2} \text{ or } q - \frac{q-2}{2} \leq M^\circ \leq q\}| = 0$.

Thus the result follows from

$$|\Delta^{-1}(0,\infty)| = \begin{cases} 2\sum_{M^\bullet=0}^{\frac{q-3}{2}}(2M^\bullet+1) = (q-1)+4\binom{\frac{q-1}{2}}{2} = \frac{(q-1)^2}{2} \\ \qquad \text{if } q \text{ is odd,} \\ 2\sum_{M^\bullet=0}^{\frac{q-4}{2}}(2M^\bullet+1) = (q-2)+4\binom{\frac{q-2}{2}}{2} = \frac{(q-2)^2}{2} \\ \qquad \text{if } q \text{ is even.} \end{cases} \qquad \square$$

Thus, for large q at least, $\nabla^\imath(C_\mathcal{L}(D, mQ_\infty))$ improves on $\nabla(C_\mathcal{L}(D, mQ_\infty))$ for just under half the $m \in I(n,g)$. We shall see in section 5 that m is DLP-tight when $\nabla^\imath(C_\mathcal{L}(D, mQ_\infty))$ fails to improve on $\nabla(C_\mathcal{L}(D, mQ_\infty))$.

4. A good coordinate order.

We describe a "good" coordinate order for Hermitian codes, denoting the code in $[C_\mathcal{L}(D, mQ_\infty)]$ with this coordinate order by C_m. After recalling the notions of points of gain and fall for a linear code, we give the most natural description of the points of gain and fall of C_m in Propositions 4.2 and 4.4. We conclude by characterizing the points of gain and fall of C_m as "runs" in Theorem 4.10 (which we will use in section 5 to derive a formula for $\mathrm{s}(C_m)$).

The good coordinate order. As noted at the beginning of section 3, for $m \leq \frac{n-2}{2}$ or $m \geq \frac{n-2}{2} + 2g$, all coordinate orders of $C_\mathcal{L}(D, mQ_\infty)$ are equally bad with regard to state complexity. Thus we are interested in $m \in I(n,g)$.

Recall that $H/\mathbb{F}_{q^2}$ has $n+1$ places of degree one, namely Q_∞, and the finite places of degree one, $Q_1, \ldots, Q_n$. We put $\mathbb{P}_H^1 = \{Q_1, \ldots, Q_n\}$. Now

$$C_\mathcal{L}(D, mQ_\infty) = \{(z(Q_{l_1}), \ldots, z(Q_{l_n})) : z \in \mathcal{L}(mQ_\infty)\}$$

for some fixed but arbitrary ordering $(Q_{l_1}, \ldots, Q_{l_n})$ of $\mathbb{P}_H^1$. Thus the order of $\mathbb{P}_H^1$ determines the coordinate order of $C_\mathcal{L}(D, mQ_\infty)$. As in [14], for each $(\alpha, \beta) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ such that $\beta^q + \beta = \alpha^{q+1}$, there exists a unique $Q_{\alpha\beta} \in \mathbb{P}_H^1$ such that $x(Q_{\alpha\beta}) = \alpha$ and $y(Q_{\alpha\beta}) = \beta$.

We now describe an order of $\mathbb{P}_H^1$ giving $C_m \in [C_\mathcal{L}(D, mQ_\infty)]$. First we relabel the elements of $\mathbb{P}_H^1$ as $Q_{a,b,c}$ for certain integers a, b, c. We write $\{0, 1, \ldots, q-1\}$ for $\mathbb{F}_q$, where $0 = 0_{\mathbb{F}_q}$. Now for each $a \in \mathbb{F}_q \setminus \{0\}$ there exist $\beta_{a0}, \ldots, \beta_{a,q-1} \in \mathbb{F}_{q^2}$ and $\alpha_{a0}, \ldots, \alpha_{aq} \in \mathbb{F}_{q^2}$ such that $\beta_{ac}^q + \beta_{ac} = \alpha_{ab}^{q+1} = a$ for $0 \leq c \leq q-1$ and $0 \leq b \leq q$. Thus for each $a \in \mathbb{F}_q \setminus \{0\}$, $0 \leq c \leq q-1$ and $0 \leq b \leq q$, there exists $Q_{a,b,c} \in \mathbb{P}_H^1$ such that $x(Q_{a,b,c}) = \alpha_{a,b}$ and $y(Q_{a,b,c}) = \beta_{a,c}$, giving $q^3 - q$ elements of $\mathbb{P}_H^1$.

For $a = 0$ there exist $\beta_{00}, \ldots, \beta_{0q}$ and $\alpha_{00} = 0$ such that $\beta_{0c}^q + \beta_{0c} = \alpha_{00}^{q+1} = 0$ for $0 \leq c \leq q-1$. Thus the remaining q elements of $\mathbb{P}_H^1$, which we write as $Q_{0,0,c}$ for $0 \leq c \leq q-1$, are such that $x(Q_{0,0,c}) = 0$ and $y(Q_{0,0,c}) = \beta_{0,c}$. We note that $Q_{a,b,c} = Q_{\alpha_{ab},\beta_{ac}}$.

When a, b, or c takes any of its possible values we write $Q_{*,b,c}$, $Q_{a,*,c}$, or $Q_{a,b,*}$. Note that for $a = 0$ we have $b = 0$ and for $1 \leq a \leq q-1$ we have $0 \leq b \leq q$. Thus there are q places of the form $Q_{0,*,*}$ and for $1 \leq a \leq q-1$ there are $q^2 - 1$ places of the form $Q_{a,*,*}$.

We first describe the ordering of $\mathbb{P}_H^1$ giving $C_m \in [C_\mathcal{L}(D, mQ_\infty)]$ for $m \in J(n,g)$. This uses lexicographic order of t-tuples of integers: $(i_1, \ldots, i_t) < (j_1, \ldots, j_t)$ if and only if there exists u such that $i_1 = j_1, \dots, i_{u-1} = j_{u-1}$ and $i_u < j_u$. For $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$ or $q - \frac{M^\bullet}{2} \le M^\circ \le q$, C_m is defined by simply using the order

$$\text{O1:}\quad Q_{a,b,c} < Q_{a',b',c'} \text{ if } (a,b,c) < (a',b',c')$$

of $\mathbb{P}^1_H$. For $\frac{q-M^\bullet-1}{2} \le M^\circ \le q - \frac{M^\bullet+1}{2}$, C_m is defined by the "Order O2" of $\mathbb{P}^1_H$: partition $\mathbb{P}^1_H$ into the following three sets:

$$\begin{aligned} \mathbb{P}^1_1 &= \{Q_{1,*,c} : 0 \le c \le \tfrac{q-q_2}{2} - 1\}, \\ \mathbb{P}^1_2 &= \{Q_{a,*,*} : a \ne 1\}, \\ \mathbb{P}^1_3 &= \{Q_{1,*,c} : \tfrac{q-q_2}{2} \le c \le q-1\}. \end{aligned} \tag{8}$$

Then Order O2 of $\mathbb{P}^1_H$ is given by putting $P^1_1 < P^1_2 < P^1_3$, ordering $\mathbb{P}^1_1$ and $\mathbb{P}^1_3$ by $Q_{1,b,c} < Q_{1,b',c'}$ if $(c,b) < (c',b')$, and ordering $\mathbb{P}^1_2$ by $Q_{a,b,c} < Q_{a',b',c'}$ if $(a,b,c) < (a',b',c')$.

For $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$, the coordinate order of C_m is defined to be that of $C_{m^\perp}$.

From now on, Q_i denotes the ith element of $\mathbb{P}^1_H$ ordered as above. Thus

$$C_m = \{(z(Q_1), \dots, z(Q_n)) : z \in \mathcal{L}(mQ_\infty)\}.$$

The points of gain and fall of C_m. Points of gain and fall were introduced in [3, 6]. For this paragraph, C is a length n linear code with dimension k. We note that $\dim(C_{i,-})$ (as defined in section 2) increases in unit steps from 0 to k and $\dim(C_{i,+})$ decreases in unit steps from k to 0 as i increases from 0 to n. If $0 \le i \le n$, then

- i is a point of gain of C if $\dim(C_{i,+}) = \dim(C_{i,+}) - 1$ and
- i is a point if fall of C if $\dim(C_{i,-}) = \dim(C_{i,-}) + 1$.

These definitions are motivated by (1). We note that there are k points of gain and k points of fall. Points of gain and fall describe the local behavior of a minimal trellis [6], and being able to give a succinct characterization of them for particular families of codes has been useful in calculating formulae for their state complexity; see, e.g., [3, 6]. The same proves to be the case here. We note that, as in [6], i is a point of gain of C_m if and only if i is the "initial point" of a codeword of C_m, i.e., if and only if there exists $z \in \mathcal{L}(mQ_\infty)$ such that

$$z(Q_1) = \cdots = z(Q_{i-1}) = 0 \text{ and } z(Q_i) \ne 0.$$

Similarly, i is a point of fall of C_m if and only if i is the "final point" of a codeword of C_m, i.e., if and only if there exists $z \in \mathcal{L}(mQ_\infty)$ such that

$$z(Q_i) \ne 0 \text{ and } z(Q_{i+1}) = \cdots = z(Q_n) = 0.$$

We write $P_{\text{gain}}(C)$ and $P_{\text{fall}}(C)$ for the sets of points of gain and fall of C. With $P^{i,-}_{\text{gain}}(C) = |P_{\text{gain}}(C) \cap [1,i]|$ and $P^{i,-}_{\text{fall}}(C) = |P_{\text{fall}}(C) \cap [1,i]|$ we have

$$\mathrm{s}_i(C) = P^{i,-}_{\text{gain}}(C) - P^{i,-}_{\text{fall}}(C). \tag{9}$$

We also write $P_{\text{gain}}(m) := P_{\text{gain}}(C_m)$ and $P_{\text{fall}}(m) := P_{\text{fall}}(C_m)$. We will need a function Λ closely related to Π. Define $\Lambda : [0,\infty) \times [0, q-1] \longrightarrow [0,\infty)$ by

$$\Lambda(j,l) = jq + l(q+1).$$

We have $\Pi[1,\infty) = \mathrm{Im}(\Lambda)$ from [14]. We note that

$$\Lambda^{-1}[0,m] = \{(j,l) \in \mathbb{Z}\times\mathbb{Z} : j \ge 0, 0 \le l \le q-1, jq + l(q+1) \le m\}$$

and for $m < n$, $k = \dim(C_m) = |\Lambda^{-1}[0,m]|$ [14, Proposition VII.4.3]. For $0 \le a \le q-1$, we put

$$A(a) = \begin{cases} \{\alpha_{00}\} & \text{for } a = 0, \\ \{\alpha_{ab} : 0 \le b \le q\} & \text{for } 1 \le a \le q-1 \end{cases}$$

and $B(a) = \{\beta_{ac} : 0 \le c \le q-1\}$. Thus $\mathbb{P}_H^1 = \{Q_{\alpha\beta} : 0 \le a \le q-1, \alpha \in A(a), \beta \in B(a)\}$. We also put $A = \bigcup_{a=0}^{q-1} A(a)$ and $B = \bigcup_{a=0}^{q-1} B(a)$. We will determine the initial and final points of certain $z \in H/\mathbb{F}_{q^2}$ of the form

$$z = (x-\alpha_0)\cdots(x-\alpha_{l-1})(y-\beta_0)\cdots(y-\beta_{j-1}),$$

where $\alpha_0,\ldots,\alpha_{l-1} \in A$ and $\beta_0,\ldots,\beta_{j-1} \in B$. Note that $(x-\alpha_{ab})(Q_{a',b',*}) = 0$ if and only if $a = a'$, $b = b'$ and $(y-\beta_{ac})(Q_{a',*,c'}) = 0$ if and only if $a = a'$, $c = c'$. Of course, we are interested in when $(z(Q_1),\ldots,z(Q_n)) \in C_m$, i.e., when $z \in \mathcal{L}(mQ_\infty)$.

LEMMA 4.1. *If $(j,l) \in \Lambda^{-1}[0,m]$, $\alpha_0,\ldots,\alpha_{j-1} \in A$ and $\beta_0,\ldots,\beta_{l-1} \in B$, then*

$$(x-\alpha_0)\cdots(x-\alpha_{j-1})(y-\beta_0)\cdots(y-\beta_{l-1}) \in \mathcal{L}(mQ_\infty).$$

Proof. We put $z_{jl} = (x-\alpha_0)\cdots(x-\alpha_{j-1})(y-\beta_0)\cdots(y-\beta_{l-1}) \in \mathcal{L}(mQ_\infty)$. Using the facts that (i) $v_{Q_\infty}(x) = -q$ and $v_{Q_\infty}(y) = -(q+1)$, (ii) for $Q \in \mathbb{P}_H \setminus \{Q_\infty\}$, $v_Q(x) \ge 0$ and $v_Q(y) \ge 0$, and (iii) for $\alpha \in \mathbb{F}_{q^2}$ and $Q \in \mathbb{P}_H$, $v_Q(\alpha) = 0$, we get $v_{Q_\infty}(z_{jl}) = -\Lambda(j,l)$ and $v_Q(z_{jl}) \ge 0$ for all $Q \in \mathbb{P}_H \setminus \{Q_\infty\}$. Hence $(j,l) \in \Lambda^{-1}[0,m]$ implies that $z_{jl} \in \mathcal{L}(mQ_\infty)$. □

PROPOSITION 4.2 (O1 ordering of $\mathbb{P}_H^1$). *For $m \in J(n,g)$ with $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$ or $q - \frac{M^\bullet}{2} \le M^\circ \le q$,*

1. *$P_{\text{gain}}(m) = \{jq + l + 1 : (j,l) \in \Lambda^{-1}[0,m]\}$ and*
2. *$P_{\text{fall}}(m) = \{n - jq - l : (j,l) \in \Lambda^{-1}[0,m]\} = n - P_{\text{gain}}(m) + 1$.*

Proof. We order the set A by $\alpha_{ab} < \alpha_{a'b'}$ if and only if $(a,b) < (a',b')$. Thus $\alpha_{ab} < \alpha_{a'b'}$ if and only if $Q_{a,b,*} < Q_{a',b',*}$. For $0 \le d \le q^2-1$, we write α_d for the $(d+1)$st element of A. Thus $\alpha_0 = \alpha_{00}, \alpha_1 = \alpha_{10}, \ldots, \alpha_{q+1} = \alpha_{1q}, \ldots, \alpha_{q^2-1} = \alpha_{q-1,q}$. For $0 \le d \le q^2-1$, we define $a(d)$ by $\alpha_{a(d)b} = \alpha_d$ for some b. Thus $a(0) = 0, a(1) = \cdots = a(q+1) = 1, \ldots, a(q^2-q-1) = \cdots = a(q^2-1) = q-1$. Then for $1 \le i \le q^3$, writing $i - 1 = dq + c$, where $0 \le d \le q^2-1$ and $0 \le c \le q-1$, we have

$$Q_i = Q_{\alpha_d,\beta_{a(d)c}}.$$

Thus

$$(x-\alpha_d)(Q_i) = 0 \text{ if and only if } dq+1 \le i \le (d+1)q$$

and

$$(y-\beta_{ac})(Q_i) = 0 \text{ if and only if } i = dq+c+1, \text{ where } a(d) = a.$$

We begin with $P_{\text{gain}}(m)$. For $(j,l) \in \Lambda^{-1}[0,m]$ we put

$$u_j^{\text{gain}} = (x-\alpha_0)\cdots(x-\alpha_{j-1}),\ v_{jl}^{\text{gain}} = (y-\beta_{a(j),0})\cdots(y-\beta_{a(j),l-1}),\ z_{jl}^{\text{gain}} = u_j^{\text{gain}} v_{jl}^{\text{gain}}.$$

We note that $jq \le \Lambda(j,l) \le m \le \frac{n-2}{2} + g = \frac{q^3+q^2-q-2}{2}$, which implies that $j < \frac{q^2+q-1}{2} \le q^2$, so that u_j^{gain}, v_{jl}^{gain}, and z_{jl}^{gain} are well-defined for all $(j,l) \in \Lambda^{-1}[0,m]$. Now $u_j^{\text{gain}}(Q_i) = 0$ if and only if $1 \le i \le jq$, $v_{jl}^{\text{gain}}(Q_i) = 0$ for $jq+1 \le i \le jq+l$, and $v_{jl}^{\text{gain}}(Q_{jq+l+1}) \ne 0$. Hence the initial point of z_{jl}^{gain} is $jq+l+1$ so that $jq+l+1 \in P_{\text{gain}}(m)$. Also, by Lemma 4.1, $z_{jl}^{\text{gain}} \in \mathcal{L}(mQ_\infty)$. Finally, each $(j,l) \in \Lambda^{-1}[0,m]$ gives a different point of gain of C_m and, since $|\Lambda^{-1}[0,m]| = k$, these are all the points of gain, and similarly for points of fall. □

We use Proposition 4.2 to determine $\mathrm{s}(C_m)$ for $q = 2$ and $m \in [\frac{n-1}{2}, \frac{n-3}{2} + g]$. To do this we use (9) and so we put

$$P_{\text{gain}}^{i,-}(m) := P_{\text{gain}}^{i,-}(C_m) \text{ and } P_{\text{fall}}^{i,-}(m) := P_{\text{fall}}^{i,-}(C_m).$$

Example 4.3. *If $q = 2$, then $P_{\text{gain}}(4) = [1,3] \cup \{5\}$ and $P_{\text{fall}}(4) = \{4\} \cup [6,8]$, giving $\mathrm{s}(C_4) = 3$. (Thus C_4 is our first example of a geometric Goppa code with $\mathrm{s}(C_4) < \mathrm{W}(C_4)$. Also, $\mathrm{s}(C_4) = \nabla(C_4)$, where the latter is given by Theorem 2.1.)*

Proof. The coordinate order of C_4 is $Q_{0,0,0} < Q_{0,0,1} < Q_{1,0,0} < Q_{1,0,1} < Q_{1,1,0} < Q_{1,1,1} < Q_{1,2,0} < Q_{1,2,1}$. In the notation of Proposition 4.2, we have $\alpha_0 = \alpha_{0,0}$, $\alpha_1 = \alpha_{1,0}$, $\alpha_2 = \alpha_{1,1}$, $\alpha_3 = \alpha_{1,2}$. Thus $a(0) = 0$ and $a(1) = a(2) = a(3) = 1$. Also $\Lambda^{-1}[0,4] = \{(0,0),(1,0),(0,1),(2,0)\}$, and $k = 4$.

Now $P_{\text{gain}}(4)$ is the set of initial points of z_{jl}^{gain}, where $(j,l) \in \Lambda^{-1}[0,4]$. These are given in the table below. The third column in the table gives the "initial place," i.e., the $Q_{a,b,c}$, such that $Q_{a,b,c} = Q_i$, where i is the initial point.

(j,l)	z_{jl}^{gain}	Initial place	Initial point
$(0,0)$	1	$Q_{0,0,0}$	1
$(1,0)$	$(x-\alpha_0)$	$Q_{1,0,0}$	3
$(0,1)$	$(y-\beta_{0,0})$	$Q_{0,0,1}$	2
$(2,0)$	$(x-\alpha_0)(x-\alpha_1)$	$Q_{1,1,0}$	5

Thus $P_{\text{gain}}(4) = [1,3] \cup \{5\}$. Also $P_{\text{fall}}(4)$ is given by the final points of z_{jl}^{fall} such that $(j,l) \in \Lambda^{-1}[0,4]$, as follows.

(j,l)	z_{jl}^{fall}	Final place	Final point
$(0,0)$	1	$Q_{1,2,1}$	8
$(1,0)$	$(x-\alpha_3)$	$Q_{1,1,1}$	6
$(0,1)$	$(y-\beta_{1,1})$	$Q_{1,2,0}$	7
$(2,0)$	$(x-\alpha_2)(x-\alpha_3)$	$Q_{1,0,1}$	4

Thus $P_{\text{fall}}(4) = \{4\} \cup [6,8]$. Hence, using (9) we have

i	0	1	2	3	4	5	6	7	8
$P_{\text{gain}}^{i,-}(4)$	0	1	2	3	3	4	4	4	4
$P_{\text{fall}}^{i,-}(4)$	0	0	0	0	1	1	2	3	4
$\mathrm{s}_i(C_4)$	0	1	2	3	2	3	2	1	0

giving $\mathrm{s}(C_4) = 3$. □

For $m \in J(n,g)$ such that $\frac{q-M^\bullet-1}{2} \le M^\circ \le q - \frac{M^\bullet+1}{2}$ we put

$$\zeta_{\text{gain}} = \frac{q-q_2}{2} \text{ and } \zeta_{\text{fall}} = \frac{q+q_2}{2}.$$

Proposition 4.4 (O2 ordering of $\mathbb{P}^1_H$). *For $m \in J(n,g)$ with $\frac{q-M^\bullet-1}{2} \le M^\circ \le q - \frac{M^\bullet+1}{2}$, $P_{\text{gain}}(m) = P^1_{\text{gain}}(m) \cup P^2_{\text{gain}}(m)$ and $P_{\text{fall}}(m) = P^1_{\text{fall}}(m) \cup P^2_{\text{fall}}(m)$, where*

$$P^1_{\text{gain}}(m) = \{l(q+1)+j+1 : (j,l) \in \Lambda^{-1}[0,m], 0 \le j \le q, 0 \le l \le \zeta_{\text{gain}}-1\},$$
$$P^2_{\text{gain}}(m) = \{\zeta_{\text{gain}}(q+1)+jq+l+1 : (j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{gain}}(q+1)]\},$$
$$P^1_{\text{fall}}(m) = \{n-l(q+1)-j : (j,l) \in \Lambda^{-1}[0,m], 0 \le j \le q, 0 \le l \le \zeta_{\text{fall}}-1\},$$
$$P^2_{\text{fall}}(m) = \{n-\zeta_{\text{fall}}(q+1)-jq-l : (j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{fall}}(q+1)]\}.$$

Proof. We recall that P^1_1, P^1_2, and P^1_3 were defined in (8). We note that

- for $1 \le i \le \zeta_{\text{gain}}(q+1)$, $Q_i \in \mathbb{P}^1_1$, so that writing $i-1 = c(q+1)+b$, where $0 \le c \le \zeta_{\text{gain}}-1$ and $0 \le b \le q$, $Q_i = Q_{1,b,c}$;
- for $\zeta_{\text{gain}}(q+1)+1 \le i \le \zeta_{\text{gain}}(q+1)+q^3-q^2-q$, $Q_i \in \mathbb{P}^1_2$; and
- for $\zeta_{\text{gain}}(q+1)+q^3-q^2-q \le i \le q^3$, $Q_i \in \mathbb{P}^1_3$.

We begin by showing that $P^1_{\text{gain}}(m) \subseteq P_{\text{gain}}(m)$. For $(j,l) \in \Lambda^{-1}[0,m]$ such that $0 \le j \le q$ and $0 \le l \le \zeta_{\text{gain}}-1$ we exhibit an element of $\mathcal{L}(mQ_\infty)$ with initial point $l(q+1)+j+1$. Put

$$u^{\text{gain}}_j = (x-\alpha_{1,0})\cdots(x-\alpha_{1,j-1}),\ v^{\text{gain}}_l = (y-\beta_{1,0})\cdots(y-\beta_{1,l-1}),\ z^{\text{gain}}_{jl} = u^{\text{gain}}_j v^{\text{gain}}_l.$$

Thus $v^{\text{gain}}_l(Q_{a,*,c}) = 0$ if and only if $a=1$, and $0 \le c \le l-1$ and $u^{\text{gain}}(Q_{a,b,*}) = 0$ if and only if $a = 1$, and $0 \le b \le j-1$. Therefore $v^{\text{gain}}_l(Q_i) = 0$ if and only if $1 \le i \le l(q+1)$, $u^{\text{gain}}_j(Q_i) = 0$ for $l(q+1)+1 \le i \le l(q+1)+j$ (taking $c = l \le \zeta_{\text{gain}}$), and $u^{\text{gain}}_j(Q_{l(q+1)+j+1}) \ne 0$ (taking $c = l$ and $b = j \le q$). Hence the initial point of z^{gain}_{jl} is $l(q+1)+j+1$. Also, from Lemma 4.1, $z^{\text{gain}}_{jl} \in \mathcal{L}(mQ_\infty)$, so that $P^1_{\text{gain}}(m) \subseteq P_{\text{gain}}(m)$.

Next we show that $P^2_{\text{gain}}(m) \subseteq P_{\text{gain}}(m)$. We order $A \setminus A(1)$ by $\alpha_{ab} < \alpha_{a'b'}$ if and only if $(a,b) < (a',b')$ and write α_d for the $(d+1)$st element of $A \setminus A(1)$, where $0 \le d \le q^2-q-2$. (This is different from the labelling in the proof of Proposition 4.2 since we do not include $A(1)$ in the relabelling.) We define $a(d)$ by $\alpha_{a(d)b} = \alpha_d$ for some b. Then, for $\zeta_{\text{gain}}(q+1)+1 \le i \le \zeta_{\text{gain}}(q+1)+q^3-q^2-q$, writing $i-1 = \zeta_{\text{gain}}(q+1)+dq+c$, where $0 \le d \le q^2-q-2$ and $0 \le c \le q-1$, we have $Q_i = Q_{\alpha_d,\beta_{a(d)c}}$. We put $w^{\text{gain}} = (y-\beta_{1,0})\cdots(y-\beta_{1,\zeta_{\text{gain}}-1})$. For $(j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{gain}}(q+1)]$, set

$$(u^{\text{gain}})'_j = (x-\alpha_0)\cdots(x-\alpha_{j-1}),\ (v^{\text{gain}})'_{jl} = (y-\beta_{a(j),0})\cdots(y-\beta_{a(j),l-1}),$$
$$z^{\text{gain}}_{j,l+\zeta_{\text{gain}}} = w^{\text{gain}}(u^{\text{gain}})'_j(v^{\text{gain}})'_{jl}.$$

We note that $jq \le \Lambda(j,l) \le m-\zeta_{\text{gain}}(q+1) \le \frac{q^3-q-1}{2}$, which implies that $j \le q^2-q-2$. Thus $(u^{\text{gain}})'_j$, $(v^{\text{gain}})'_{jl}$, and $z^{\text{gain}}_{j,l+\zeta_{\text{gain}}}$ are well-defined for all $(j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{gain}}(q+1)]$. Now $w^{\text{gain}}(Q_i) = 0$ if and only if $1 \le i \le \zeta_{\text{gain}}(q+1)$. Also $(u^{\text{gain}})'_j(Q_{\alpha_d\beta_{a(d)c}}) = 0$ if and only if $0 \le d \le j-1$ and $0 \le c \le q-1$, and $(v^{\text{gain}})'_{jl}(Q_{\alpha_d\beta_{a(d)c}}) = 0$ if and only if $a(d) = a(j)$ and $0 \le c \le l-1$. Thus $(u^{\text{gain}})'_j(Q_i) = 0$ if and only if $\zeta_{\text{gain}}(q+1)+1 \le i \le \zeta_{\text{gain}}(q+1)+jq$, $(v^{\text{gain}})'_{jl}(Q_i) = 0$ for $\zeta_{\text{gain}}(q+1)+jq+1 \le i \le \zeta_{\text{gain}}(q+1)+jq+l$ and $(v^{\text{gain}})'_{jl}(Q_{\zeta_{\text{gain}}(q+1)+jq+l+1}) \ne 0$. Therefore the initial point of $z^{\text{gain}}_{j,l+\zeta_{\text{gain}}}$ is $\zeta_{\text{gain}}(q+1)+jq+l+1$. Also, by Lemma 4.1, $w^{\text{gain}} \in \mathcal{L}(\zeta_{\text{gain}}(q+1)Q_\infty)$ and $(z^{\text{gain}}_{j,l+\zeta_{\text{gain}}}/w) \in \mathcal{L}((m-\zeta_{\text{gain}}(q+1))Q_\infty)$. Hence $z^{\text{gain}}_{j,l+\zeta_{\text{gain}}} \in \mathcal{L}(mQ_\infty)$, $P^2_{\text{gain}} \subseteq P_{\text{gain}}(m)$, and $P^1_{\text{gain}}(m) \cup P^2_{\text{gain}}(m) \subseteq P_{\text{gain}}(m)$.

Therefore it remains to show that $|P^1_{\text{gain}}(m) \cup P^2_{\text{gain}}(m)| = k$. To do this we exhibit a bijection $\Lambda^{-1}[0,m] \to P^1_{\text{gain}}(m) \cup P^2_{\text{gain}}(m)$. First, for $(j,l) \in \Lambda^{-1}[0,m]$ we map (j,l) to $l(q+1)+j+1 \in P^1_{\text{gain}}(m)$ if $0 \le j \le q$ and $0 \le l \le \zeta_{\text{gain}} - 1$. Now we are left with defining a bijection F

$$\{(j,l) \in \Lambda^{-1}[0,m] : 0 \le j \le q, \zeta_{\text{gain}} \le l \le q-1 \text{ or } j \ge q+1\} \to P^2_{\text{gain}}(m) \text{ by}$$

$$F(j,l) = \begin{cases} \zeta_{\text{gain}}(q+1) + jq + (l - \zeta_{\text{gain}}) + 1 & \text{if } \zeta_{\text{gain}} \le l \le q-1, \\ \zeta_{\text{gain}}(q+1) + (j-q-1)q + (l + \frac{q+q_2}{2}) + 1 & \text{if } 0 \le l \le \zeta_{\text{gain}} - 1. \end{cases}$$

It is easy to check that F maps into $P^2_{\text{gain}}(m)$ and F is one-to-one since for $\zeta_{\text{gain}} \le l \le q-1$ and $0 \le l' \le \zeta_{\text{gain}} - 1$, $0 \le l - \zeta_{\text{gain}} \le \frac{q+q_2}{2} - 1 < \frac{q+q_2}{2} \le l' + \frac{q+q_2}{2} \le q-1$. Finally we prove F is onto. For $i \in P^2_{\text{gain}}(m)$, such that $i = \zeta_{\text{gain}}(q+1) + jq + l + 1$ for $(j,l) \in \Lambda^{-1}[0, m - \zeta_{\text{gain}}(q+1)]$, we put

$$(j',l') = \begin{cases} (j, l + \zeta_{\text{gain}}) & \text{if } 0 \le l \le \frac{q+q_2}{2} - 1, \\ (j+q+1, l - \frac{q+q_2}{2}) & \text{if } \frac{q+q_2}{2} \le l \le q-1. \end{cases}$$

It is straightforward to see that (i) $(j',l') \in \Lambda^{-1}[0,m]$; (ii) if $j' \le q$, then $\zeta_{\text{gain}} \le l' \le q-1$; and (iii) $F((j',l')) = i$. This completes the proof for $P_{\text{gain}}(m)$, and similarly for the points of fall. □

EXAMPLE 4.5. *If* $q = 3$, *then* $P_{\text{gain}}(13) = [1,9] \cup \{11,14\}$ *and* $P_{\text{fall}}(13) = \{16\} \cup [18,27]$, *giving* $\mathrm{s}(C_{13}) = \mathrm{W}(C_{13}) = \nabla^\imath(C_{\mathcal{L}}(D, 13Q_\infty)) = 11$ *using Theorem* 3.9, *but* $\mathrm{s}(C_{13}) = \nabla(C_{13}) + 1$.

Proof. The coordinate order of C_{13} is

$$\begin{gathered} Q_{1,0,0} < Q_{1,1,0} < Q_{1,2,0} < Q_{1,3,0} < Q_{0,0,0} < Q_{0,0,1} < Q_{0,0,2} < Q_{2,0,0} < Q_{2,0,1} \\ < Q_{2,0,2} < Q_{2,1,0} < Q_{2,1,1} < Q_{2,1,2} < Q_{2,2,0} < Q_{2,2,1} < Q_{2,2,2} < Q_{2,3,0} < Q_{2,3,1} \\ < Q_{2,3,2} < Q_{1,0,1} < Q_{1,1,1} < Q_{1,2,1} < Q_{1,3,1} < Q_{1,0,2} < Q_{1,1,2} < Q_{1,2,2} < Q_{1,3,2}. \end{gathered}$$

We use the notation of the proof of Proposition 4.4. We note that $\zeta_{\text{gain}} = 1$. Thus for $0 \le j \le q$ and $0 \le l \le \zeta_{\text{gain}} - 1$, $jq + l(q+1) \le 9 \le 13$, so that $(j,l) \in \Lambda^{-1}[0,13]$. Thus $P^1_{\text{gain}}(13)$ is the set of initial points of $z^{\text{gain}}_{j,0}$ for $0 \le j \le 3$, which are as follows.

(j,l)	z^{gain}_{jl}	Initial place	Initial point
$(0,0)$	1	$Q_{1,0,0}$	1
$(1,0)$	$(x-\alpha_{1,0})$	$Q_{1,1,0}$	2
$(2,0)$	$(x-\alpha_{1,0})(x-\alpha_{1,1})$	$Q_{1,2,0}$	3
$(3,0)$	$(x-\alpha_{1,0})(x-\alpha_{1,1})(x-\alpha_{1,2})$	$Q_{1,3,0}$	4

Thus $P^1_{\text{gain}}(13) = [1,4]$. Next we consider $P^2_{\text{gain}}(13)$. Now we have

$$\alpha_0 = \alpha_{0,0},\ \alpha_1 = \alpha_{2,0},\ \alpha_2 = \alpha_{2,1},\ \alpha_3 = \alpha_{2,2},\ \alpha_4 = \alpha_{2,3}$$

so that $a(0) = 0$, $a(1) = a(2) = a(3) = a(4) = 2$. Then $P^2_{\text{gain}}(13)$ is the set of initial points of $z^{\text{gain}}_{j,l+\zeta_{\text{gain}}}$ such that $(j,l) \in \Lambda^{-1}[0, 13 - \zeta_{\text{gain}}(q+1)] = \Lambda^{-1}[0,9]$ and

$$\Lambda^{-1}[0,9] = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0)\},$$

giving the following.

$(j, l+1)$	$z^{\text{gain}}_{j,l+1}$	Initial place	Initial point
$(0,1)$	$(y-\beta_{1,0})$	$Q_{0,0,0}$	5
$(1,1)$	$(y-\beta_{1,0})(x-\alpha_0)$	$Q_{2,0,0}$	8
$(0,2)$	$(y-\beta_{1,0})(y-\beta_{0,0})$	$Q_{0,0,1}$	6
$(2,1)$	$(y-\beta_{1,0})(x-\alpha_0)(x-\alpha_1)$	$Q_{2,1,0}$	11
$(1,2)$	$(y-\beta_{1,0})(x-\alpha_0)(y-\beta_{2,0})$	$Q_{2,0,1}$	9
$(0,3)$	$(y-\beta_{1,0)}(y-\beta_{0,0})(y-\beta_{0,1})$	$Q_{0,0,2}$	7
$(3,1)$	$(y-\beta_{1,0})(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)$	$Q_{2,2,0}$	14

Thus $P^2_{\text{gain}}(13) = \{5,8,6,11,9,7,14\} = [5,9]\cup\{11,14\}$ and $P_{\text{gain}}(13) = [1,9]\cup\{11,14\}$, and similarly for $P_{\text{fall}}(13)$. We have $P_{\text{gain}}(13) < P_{\text{fall}}(13)$ and so $\mathrm{s}(C_{13}) = 11$. □

From Propositions 4.2 and 4.4 we have that, if (i) $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$ or (ii) $q - \frac{M^\bullet}{2} \le M^\circ \le q$ or (iii) $\zeta_{\text{gain}} = \zeta_{\text{fall}}$ and $\frac{q-M^\bullet-1}{2} \le M^\circ \le q - \frac{M^\bullet+1}{2}$, then $P_{\text{fall}}(m) = n - P_{\text{gain}}(m) + 1$. In these cases the following useful property holds.

Remark 4.6. *For a length n code C, if $P_{\text{fall}}(C) = n - P_{\text{gain}}(C) + 1$, then $\mathrm{s}_{n-i}(C) = \mathrm{s}_i(C)$ for $0 \le i \le n$. In particular, for $m \in J(n,g)$, if* (i) *q is odd and $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$ or $q - \frac{M^\bullet}{2} \le M^\circ \le q$ or* (ii) *q is even and $0 \le M^\circ \le q$, then $\mathrm{s}_i(C_m) = \mathrm{s}_{n-i}(C_m)$ for $0 \le i \le n$. The same holds for $m \in [\frac{n-1}{2}+g, \frac{n-3}{2}+2g]$ if $m^\perp$ satisfies* (i) *or* (ii).

Proof. The proof is similar to that of [6, Proposition 2.5] and in fact can be modified to hold for branch complexity as in [6, Proposition 2.5]. We put $P^{i,+}_{\text{gain}}(C) = |P_{\text{gain}}(C) \cap [i+1,n]|$ and $P^{i,+}_{\text{fall}}(C) = |P_{\text{fall}}(C) \cap [i+1,n]|$. Of course, with $k = \dim(C)$,

$$P^{i,+}_{\text{gain}}(C) = k - P^{i,-}_{\text{gain}}(C) \text{ and } P^{i,+}_{\text{fall}}(C) = k - P^{i,-}_{\text{fall}}(C)$$

for any linear code C. The condition $P_{\text{fall}}(C) = n - P_{\text{gain}}(C) + 1$ also implies that

$$P^{i,-}_{\text{gain}}(C) = P^{n-i,+}_{\text{fall}}(C) \text{ and } P^{i,-}_{\text{fall}}(C) = P^{i,+}_{\text{gain}}(C).$$

Thus, from (9), we have

$$\begin{aligned}\mathrm{s}_i(C) &= P^{n-i,+}_{\text{fall}}(C) - P^{i,+}_{\text{gain}}(C)\\ &= (k - P^{n-i,-}_{\text{fall}}(C)) - (k - P^{n-i,-}_{\text{gain}}(C)) = \mathrm{s}_{n-i}(C).\end{aligned}$$ □

Remark 4.7. *If $C \in [C_{\mathcal{L}}(D, 14Q_\infty)]$ is ordered by $O1$, then as in the proof of Proposition* 4.2 *$P_{\text{gain}}(C) = [1,11] \cup \{13\}$ and $P_{\text{fall}}(C) = \{15\} \cup [17,27]$, so that* $\mathrm{s}(C) = 12$. *However, if C is ordered by $O2$, $P_{\text{gain}}(14) = [1,9] \cup [11,12] \cup \{14\}$ and $P_{\text{fall}}(14) = \{13,16\} \cup [18,27]$, giving* $\mathrm{s}(C_{14}) = \mathrm{W}(C_{14}) - 1 = \nabla(C_{14}) = 11$. *Thus $O2$ is strictly better than $O1$ for m=14.*

If $C \in [C_{\mathcal{L}}(D, 15Q_\infty)]$ is ordered by $O1$, then as in the proof of Proposition 4.2*, $P_{\text{gain}}(15) = [1,11] \cup \{13,16\}$ and $P_{\text{fall}}(15) = \{12,15\} \cup [17,27]$, giving* $\mathrm{s}(C_{15}) = \nabla(C_{15}) = \mathrm{W}(C_{15}) - 2 = 11$. *However, if C is ordered by $O2$, we get $P_{\text{gain}}(15) = [1,12] \cup \{14\}$ and $P_{\text{fall}}(15) = \{13\} \cup [18,27]$, giving* $\mathrm{s}(C_{15}) = 12$. *Thus $O1$ is strictly better than $O2$ for $m = 15$.*

To summarize, for $q = 2,3$ and $m \in J(n,g) \subseteq I(n,g)$, $\mathrm{s}(C_m) = \nabla^{\iota}(C_{\mathcal{L}}(D, mQ_\infty))$. Thus, in these cases $\mathrm{s}(C_m) = s[C_{\mathcal{L}}(D, mQ_\infty)]$, and the coordinate order for C_m is optimal with regard to $\mathrm{s}(C_m)$. In fact, except for $q = 3$ and $m \in \{11,18\}$, $\mathrm{s}(C_m) = \nabla(C_m) < \mathrm{W}(C_m)$.

Another characterization of the points of gain and fall of C_m. We now characterize $P_{\text{gain}}(m)$ and $P_{\text{fall}}(m)$ as runs, i.e., as sequences of noncontiguous intervals of integers. This is useful since $\mathrm{s}(C_m)$ must be attained at the end of a run of points of gain. Thus to determine $\mathrm{s}(C_m)$, we need only to find the maximum of $\mathrm{s}_i(C_m)$ over those i that end a run of points of gain, i.e. over those i such that $i \in P_{\text{gain}}(m)$ and $i+1 \notin P_{\text{gain}}(m)$.

We begin by combining Propositions 4.2 and 4.4 for a common development of the cases (i) $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$ or $q-\frac{M^\bullet}{2} \le M^\circ \le q$ and (ii) $\frac{q-M^\bullet-1}{2} \le M^\circ \le q-\frac{M^\bullet+1}{2}$. First, we extend the definitions of ζ_{gain} and ζ_{fall} as follows:

$$\zeta_{\text{gain}} = \begin{cases} 0 & \text{for } 0 \le M^\circ \le \frac{q-M^\bullet-2}{2}, \\ \frac{q-q_2}{2} & \text{for } \frac{q-M^\bullet-1}{2} \le M^\circ \le q-\frac{M^\bullet+1}{2}, \\ q & \text{for } q-\frac{M^\bullet}{2} \le M^\circ \le q \end{cases}$$

and

$$\zeta_{\text{fall}} = \begin{cases} 0 & \text{for } 0 \le M^\circ \le \frac{q-M^\bullet-2}{2}, \\ \frac{q+q_2}{2} & \text{for } \frac{q-M^\bullet-1}{2} \le M^\circ \le q-\frac{M^\bullet+1}{2}, \\ q & \text{for } q-\frac{M^\bullet}{2} \le M^\circ \le q. \end{cases}$$

Proposition 4.8. *For $m \in J(n,g)$, $P_{\text{gain}}(m) = P^1_{\text{gain}}(m) \cup P^2_{\text{gain}}(m)$ and $P_{\text{fall}}(m) = P^1_{\text{fall}}(m) \cup P^2_{\text{fall}}(m)$, where*

$$\begin{aligned} P^1_{\text{gain}}(m) &= \{l(q+1)+j+1 : (j,l) \in \Lambda^{-1}[0,m], 0 \le j \le q, 0 \le l \le \zeta_{\text{gain}}-1\}, \\ P^2_{\text{gain}}(m) &= \{\zeta_{\text{gain}}(q+1)+jq+l+1 : (j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{gain}}(q+1)]\}, \\ P^1_{\text{fall}}(m) &= \{n-l(q+1)-j : (j,l) \in \Lambda^{-1}[0,m], 0 \le j \le q, 0 \le l \le \zeta_{\text{fall}}-1\}, \\ P^2_{\text{fall}}(m) &= \{n-\zeta_{\text{fall}}(q+1)-jq-l : (j,l) \in \Lambda^{-1}[0,m-\zeta_{\text{fall}}(q+1)]\}. \end{aligned}$$

Proof. From the examples above and Remark 4.7, we can assume that $q \ge 4$. For $\frac{q-M^\bullet-2}{2} \le M^\circ \le q-\frac{M^\bullet+1}{2}$, the result is just a restatement of Proposition 4.4. Also, for $0 \le M^\circ \le \frac{q-M^\bullet-1}{2}$, the result states that $P_{\text{gain}}(m) = P^2_{\text{gain}}(m) = \{jq+l+1 : (j,l) \in \Lambda^{-1}[0,m]\}$ and $P_{\text{fall}}(m) = P^2_{\text{fall}}(m) = \{n-jq-l : (j,l) \in \Lambda^{-1}[0,m]\}$, in agreement with Proposition 4.2. Therefore we are reduced to m such that $q-\frac{M^\bullet}{2} \le M^\circ \le q$ for which $\zeta_{\text{gain}} = \zeta_{\text{fall}} = q$. Rewriting $j'q+l'+1$ as $q(q+1)+(j'-q-1)q+l'+1$ and $q(q+1)+jq+l+1$ as $(j+q+1)q+l+1$, we see that $P^1_{\text{gain}}(m) = \{j'q+l'+1 : (j',l') \in \Lambda^{-1}[0,m], 0 \le j' \le q\}$.

We claim that $P^2_{\text{gain}}(m) = \{j'q+l'+1 : (j',l') \in \Lambda^{-1}[0,m], j' \ge q+1\}$. First, if $0 \le j \le q$ and $0 \le l \le q-1$, then $(j,l) \in \Lambda^{-1}[0,m]$ since $q \ge 4$. Thus we need to show that

$$\{j'q+l'+1 : 0 \le j' \le q, 0 \le l' \le q-1\} = \{l(q+1)+j+1 : 0 \le j \le q, 0 \le l \le q-1\}.$$

If k is in the left-hand side, $k = j'q+l'+1$ for some $0 \le j' \le q$ and $0 \le l' \le q-1$. Put

$$(j,l) = \begin{cases} (l'-j'+q+1, j'-1) & \text{if } 0 \le l' < j', \\ (l'-j', j') & \text{if } j' \le l' \le q-1. \end{cases}$$

In either case, $0 \le j \le q$, $0 \le l \le q-1$ and $l(q+1)+j+1 = j'q+l'+1 = k$, so that k is in the right-hand side. The reverse inclusion is similar. The result now follows from Proposition 4.2 since for $q - \frac{M^\bullet}{2} \leq M^\circ \leq q$, $P^1_{\mathrm{fall}}(m) = n - P^1_{\mathrm{gain}}(m) + 1$ and $P^2_{\mathrm{fall}}(m) = n - P^2_{\mathrm{fall}}(m) + 1$. □

LEMMA 4.9. *If* $\theta_{\mathrm{gain}} = M^\bullet + M^\circ - \zeta_{\mathrm{gain}}$ *and* $\theta_{\mathrm{fall}} = M^\bullet + M^\circ - \zeta_{\mathrm{fall}}$*, then* $0 \leq \theta_{\mathrm{gain}} \leq q - 2$ *and* $-1 \leq \theta_{\mathrm{fall}} \leq q - 2$*.*

Proof. The proof is straightforward using Lemma 3.7. □

THEOREM 4.10. *For* $m \in J(n, g)$*,*

1. $P_{\mathrm{gain}}(m)$ *is the union of*
   (a) $[1, m - 2g - \theta_{\mathrm{gain}}]$*;*
   (b) $\{m - 2g - \theta_{\mathrm{gain}} + eq + f + 1 : 0 \leq e \leq q - 2 - \theta_{\mathrm{gain}}, 0 \leq f \leq q - 2 - e\}$*; and*
   (c) $\{m - 2g - \theta_{\mathrm{gain}} + eq + f + 1 : q - 1 - \theta_{\mathrm{gain}} \leq e \leq q - 1, 0 \leq f \leq q - 1 - e\}$*;*
   and
2. $P_{\mathrm{fall}}(m)$ *is the union of*
   (a) $[n - m + 2g + \theta_{\mathrm{fall}} + 1, n]$*;*
   (b) $\{n - m + 2g + \theta_{\mathrm{fall}} - eq - f : 0 \leq e \leq q - 2 - \theta_{\mathrm{fall}}, 0 \leq f \leq q - 2 - e\}$*; and*
   (c) $\{n - m + 2g + \theta_{\mathrm{fall}} - eq - f : q - 1 - \theta_{\mathrm{fall}} \leq e \leq q - 1, 0 \leq f \leq q - 1 - e\}$*.*

Proof. As in the proof of Proposition 4.8, we assume that $q \geq 4$. We will use the fact that

$$m - 2g - \theta_{\mathrm{gain}} = \zeta_{\mathrm{gain}}(q + 1) + \left(\frac{q^2 - q_2}{2} + M^\bullet - q + 1 - \zeta_{\mathrm{gain}}\right) q. \tag{10}$$

For convenience we put $R^1_{\mathrm{gain}}(m) = [1, m - 2g - \theta_{\mathrm{gain}}]$, $R^2_{\mathrm{gain}}(m) = \{m - 2g - \theta_{\mathrm{gain}} + eq + f + 1 : 0 \leq e \leq q - 2 - \theta_{\mathrm{gain}}, 0 \leq f \leq q - 2 - e\}$, and $R^3_{\mathrm{gain}}(m) = \{m - 2g - \theta_{\mathrm{gain}} + eq + f + 1 : q - 1 - \theta_{\mathrm{gain}} \leq e \leq q - 1, 0 \leq f \leq q - 1 - e\}$.

We show that $R^1_{\mathrm{gain}}(m) \subseteq P_{\mathrm{gain}}(m)$ in two steps. First we note that $P^1_{\mathrm{gain}}(m) = [1, \zeta_{\mathrm{gain}}(q + 1)]$, since for $q \geq 4$, $0 \leq j \leq q$ and $0 \leq l \leq \zeta_{\mathrm{gain}} - 1 \leq q - 1$, $\Lambda(j, l) \leq 2q^2 - 1 < \frac{n-1}{2} \leq m$. Next we show that $[\zeta_{\mathrm{gain}}(q + 1) + 1, m - 2g - \theta_{\mathrm{gain}}] \subseteq P^2_{\mathrm{gain}}(m)$. Now from (10) we have that for $\zeta_{\mathrm{gain}}(q + 1) + 1 \leq k \leq m - 2g - \theta_{\mathrm{gain}}$,

$$k = \zeta_{\mathrm{gain}}(q + 1) + jq + l + 1 \text{ for some } 0 \leq j \leq \left(\frac{q^2 - q_2}{2} + M^\bullet - q - \zeta_{\mathrm{gain}}\right) \\ \text{and } 0 \leq l \leq q - 1.$$

Also, if $0 \leq j \leq \left(\frac{q^2 - q_2}{2} + M^\bullet - q - \zeta_{\mathrm{gain}}\right)$ and $0 \leq l \leq q - 1$, then, again using (10),

$$\Lambda(j, l) \leq \left(\frac{q^2 - q_2}{2} + M^\bullet - q - \zeta_{\mathrm{gain}}\right) q + (q - 1)(q + 1) = m - \theta_{\mathrm{gain}} - \zeta_{\mathrm{gain}}(q + 1) - 1,$$

so that $(j, l) \in \Lambda^{-1}[0, m - \zeta_{\mathrm{gain}}(q + 1)]$. Thus $k \in P^2_{\mathrm{gain}}(m)$. Next we show that $R^2_{\mathrm{gain}}(m) \cup R^3_{\mathrm{gain}}(m) \subseteq P^2_{\mathrm{gain}}(m)$. Take $k = m - 2g - \theta_{\mathrm{gain}} + eq + f + 1$. Then, from (10), $k = \zeta_{\mathrm{gain}}(q+1) + jq + l + 1$, where $j = (\frac{q^2 - q_2}{2} + M^\bullet - q + 1 - \zeta_{\mathrm{gain}} + e)$ and $l = f$. Also, again using (10), $\Lambda(j, l) = m - 2g - \theta_{\mathrm{gain}} - \zeta_{\mathrm{gain}}(q + 1) + (e + f)q + f$. Thus $k \in P^2_{\mathrm{gain}}(m)$ if $(e + f)q + f \leq 2g + \theta_{\mathrm{gain}}$. If $0 \leq e \leq q - 2 - \theta_{\mathrm{gain}}$ and $0 \leq f \leq q - 2 - e$, then $(e + f)q \leq (q - 2)q = 2g - q$ and $f \leq q - 2$, so that $R^2_{\mathrm{gain}}(m) \subseteq P^2_{\mathrm{gain}}(m)$. If $q - 1 - \theta_{\mathrm{gain}} \leq e \leq q - 1$ and $0 \leq f \leq q - 1 - e$, then $(e + f)q \leq 2g$ and $f \leq \theta_{\mathrm{gain}}$, so that $R^3_{\mathrm{gain}}(m) \subseteq P^2_{\mathrm{gain}}(m)$.

Thus $\bigcup_{i=1}^3 R^i_{\mathrm{gain}}(m) \subseteq P_{\mathrm{gain}}(m)$, and it suffices to show that $|\bigcup_{i=1}^3 R^i_{\mathrm{gain}}(m)| = |P_{\mathrm{gain}}|$. We recall that $|P_{\mathrm{gain}}| = \dim(C_m)$ and since $2g - 2 < m < n$, $\dim(C_m) =$

$m-g+1$. Also,

$$\left|\bigcup_{i=1}^{3} R^i_{\text{gain}}(m)\right| = (m-2g-\theta_{\text{gain}})+\sum_{e=0}^{q-1}(q-1-e)+(\theta_{\text{gain}}+1) = m-2g+1+\sum_{e=0}^{q-1} e = m-g+1.$$

The proof for $P_{\text{fall}}(m)$ is similar and we omit the details. □

5. When the DLP bound is tight.

Here we use Theorem 4.10 to determine $\text{s}(C_m)$. We know (from Corollary 3.10 and Proposition 3.12) that $s[C_{\mathcal{L}}(D, mQ_\infty)] > \nabla(C_m)$ for just under half of the m in the range $I(n,g)$. We show that for the remaining m in this range, $\text{s}(C_m) = \nabla(C_m)$. As a consequence, we have determined $s[C_{\mathcal{L}}(D, mQ_\infty)]$ and a coordinate order that achieves $s[C_{\mathcal{L}}(D, mQ_\infty)]$ for such m. For those m with $\text{s}(C_m) > \nabla(C_{\mathcal{L}}(D, mQ_\infty))$ we compare the upper bound, $\text{s}(C_m)$, on $s[C_{\mathcal{L}}(D, m_\infty)]$ with the lower bound $\nabla^\imath[(C_{\mathcal{L}}(D, mQ_\infty)]$ given in Corollary 3.10. When q is odd, these bounds meet for over three quarters of those m in $I(n,g)$, but when q is even, the bounds meet for only a little over one half of those m in $I(n,g)$.

Determining $\mathbf{s(C_m)}$. As discussed in section 4, it suffices to find the maximum of $\text{s}_i(C_m)$ over those i such that $i \in P_{\text{gain}}(m)$ and $i+1 \notin P_{\text{gain}}(m)$. From Theorem 4.10, there are only $q+1$ such i. Thus concentrating on these i is significantly simpler. Therefore we calculate $\text{s}_i(C_m)$ for these $q+1$ values of i (in Proposition 5.5) by determining $P^{i,-}_{\text{gain}}(m)$ and $P^{i,-}_{\text{fall}}(m)$ (in Lemmas 5.1 and 5.4). We determine which of these i gives the largest $\text{s}_i(C_m)$ (in Lemma 5.6). This enables us to write down $\text{s}(C_m)$ (in Theorem 5.7).

Early on we introduce a variable $\eta = \eta(m)$ which plays a crucial role in the proofs and statements of many of the results, and we end with a table of $\text{s}(C_m)$ for $m \in [\frac{n-1}{2}, \frac{n-3}{2}+g]$ when $q \in \{2,3,4,5,7,8\}$.

We begin by determining $\text{s}(C_m)$ for $m \in J(n,g)$. We note first that $\theta_{\text{gain}} = M^\bullet + M^\circ - \zeta_{\text{gain}}$ and $\theta_{\text{fall}} = M^\bullet + M^\circ - \zeta_{\text{fall}}$, where ζ_{gain} and ζ_{fall} were defined just before Proposition 4.8.

As noted above, $\text{s}_i(C_m) = \text{s}(C_m)$ for some i such that $i \in P_{\text{gain}}(m)$ and $i+1 \notin P_{\text{gain}}(m)$. From Theorem 4.10 such i are either (i) of the form $m-2g-\theta_{\text{gain}}+eq+(q-1-e)$ for some $-1 \le e \le q-2-\theta_{\text{gain}}$ or (ii) of the form $m-2g+\theta_{\text{gain}}+eq+(q-e)$ for some $q-1-\theta_{\text{gain}} \le e \le q-1$. Thus putting

$$i_e = \begin{cases} m-2g-\theta_{\text{gain}}+eq+(q-1-e) & \text{for } -1 \le e \le q-2-\theta_{\text{gain}}, \\ m-2g-\theta_{\text{gain}}+eq+(q-e) & \text{for } q-1-\theta_{\text{gain}} \le e \le q-1, \end{cases}$$

we have

$$\text{s}(C_m) = \max\{\text{s}_{i_e}(C_m) : -1 \le e \le q-1\}. \tag{11}$$

From (9), $\text{s}_{i_e}(C_m) = P^{i_e,-}_{\text{gain}}(m) - P^{i_e,+}_{\text{gain}}(m)$, so we wish to determine $P^{i_e,-}_{\text{gain}}(m)$ and $P^{i_e,-}_{\text{fall}}(m)$ for $-1 \le e \le q-1$. The first of these is straightforward.

LEMMA 5.1. *For $m \in J(n,g)$,*

$$P^{i_e,-}_{\text{gain}}(m) = \begin{cases} k-\binom{q-e}{2}+(q-2-\theta_{\text{gain}}-e) & \textit{for } -1 \le e \le q-2-\theta_{\text{gain}}, \\ k-\binom{q-e}{2} & \textit{for } q-1-\theta_{\text{gain}} \le e \le q-1. \end{cases}$$

Proof. Since $2g-2 < m < n$ we have $k = m-g+1$. For $-1 \le e \le q-2-\theta_{\text{gain}}$, Theorem 4.10 gives

$$P^{i_e,-}_{\text{gain}}(m) = m-2g-\theta_{\text{gain}}+\sum_{\nu=0}^{e}(q-1-\nu) = k-g-(\theta_{\text{gain}}+1)+\sum_{\nu=q-1-e}^{q-1}\nu.$$

The first case follows since $\sum_{\nu=q-1-e}^{q-1}\nu = g-\binom{q-1-e}{2}$ and $\binom{q-1-e}{2}=\binom{q-e}{2}-(q-1-e)$. In the second case,

$$P_{\text{gain}}^{i_e,-}(m) = m-g+1-\binom{q-e}{2}-(q-2-\theta_{\text{gain}}-e)-(e-(q-2-\theta_{\text{gain}})). \qquad \square$$

For $P_{\text{fall}}^{i_e,-}(m)$ it is convenient to introduce some more notation. For fixed m we put

$$\zeta_{\text{norm}} = \frac{\zeta_{\text{gain}}+\zeta_{\text{fall}}}{q}.$$

Thus ζ_{norm} is 0, 1, or 2, depending on whether $0 \le M^\circ \le \frac{q-M^\bullet-2}{2}$, $\frac{q-M^\bullet-1}{2} \le M^\circ \le q-\frac{M^\bullet+1}{2}$, or $q-\frac{M^\bullet}{2} \le M^\circ \le q$. Also, we put

$$\eta = 2q-2M^\bullet+q_2-\zeta_{\text{norm}}-3.$$

In Lemma 5.4 and Proposition 5.5 we will see a symmetry between the roles of e in $P_{\text{gain}}^{i_e,-}(m)$ and $\eta-e$ in $P_{\text{fall}}^{i_e,-}(m)$. We will see in Lemma 5.6 that $\mathrm{s}_{i_e}(C_m)$ is maximized near $\frac{\eta}{2}$, and hence η appears naturally in our formula for $\mathrm{s}(C_m)$.

Lemma 5.2. $q-1 \le \eta \le 2q-3$.

Proof. First, it follows from Lemma 3.7 that

$$(12)\quad \eta \ge \left\{ \begin{array}{ll} 2q-(q-3)+1-2-3 = q-1 & \text{if } q \text{ is odd,} \\ 2q-(q-4)-2-3 = q-1 & \text{if } q \text{ is even and } M^\circ > 0, \\ 2q-(q-2)-3 = q-1 & \text{if } q \text{ is even and } M^\circ = 0 \end{array} \right\} = q-1.$$

Next, clearly $\eta \le 2q-2$, with equality only if $M^\bullet = \zeta_{\text{norm}} = 0$ and $q_2 = 1$. However, from Lemma 3.7, if $M^\bullet = 0$ and q is odd, then $M^\circ \ge \frac{q-1}{2}$ so that $\zeta_{\text{norm}} \ge 1$. $\square$

Now, in order to use Theorem 4.10 to calculate $P_{\text{fall}}^{i_e,-}(m)$, we need to write i_e as $n-m+2g+\theta_{\text{fall}}-e'q-f$ for some, preferably nonnegative, integer e' and $0 \le f \le q-1$. We could then determine an expression for $P_{\text{fall}}^{i_e,-}(m)$ in terms of e' and f in a similar way to the proof of Lemma 5.1, except that f would add complications. This would give us an expression for $\mathrm{s}_{i_e}(C_m)$ in terms of e, e', and f. To maximize this over $-1 \le e \le q-1$ we would need to relate e' and f to e. Fortunately, these relationships are reasonably simple.

Lemma 5.3. *Let $m \in I(n,g)$ and $-1 \le e \le q-1$. If we write*

$$i_e = n-m+2g+\theta_{\text{fall}}-e'q-f \text{ for some } 0 \le f \le q-1,$$

then $e' = \eta - e$ and

$$f = \left\{ \begin{array}{ll} e+1 & \text{for } -1 \le e \le q-2-\theta_{\text{gain}}, \\ e & \text{for } q-1-\theta_{\text{gain}} \le e \le q-1. \end{array} \right.$$

In particular, $e' \ge 0$. Also, if $e \le \eta-q+1+\theta_{\text{fall}}$, then $q-\eta+e-f \le 0$.

Proof. For $-1 \le e \le q-2-\theta_{\text{gain}}$, we have $(e+e')q+(q-1-e+f) = n-2m+4g+\theta_{\text{gain}}+\theta_{\text{fall}}$. Now $2m = n-q_2q+2M^\bullet(q+1)+2M^\circ$, $4g = 2q^2-2q$, and $\theta_{\text{gain}}+\theta_{\text{fall}} = 2M^\bullet+2M^\circ-\zeta_{\text{norm}}q$, giving $(e+e')q+(q-1-e+f) = (2q-2M^\bullet+q_2-\zeta_{\text{norm}})q$ which implies that $f = e+1$ (since $q-1-e > 0$ from Lemma 4.9) and $e' = \eta-e$. Similarly, for $q-1-\theta_{\text{gain}} \le e \le q-1$ we get $(e+e')q+(q-e+f) = (\eta+1)q$, giving $f = e$ and $e' = \eta - e$.

For the second part we have $\eta \geq q-1$ (from Lemma 5.2) and $f \geq e$ (from the first part). Thus $q-\eta+e-f \leq 1$ with equality only if $\eta = q-1$ and $f = e$. We show that, for $e \leq \eta - q + 1 + \theta_{\text{fall}}$, it is not possible that $\eta = q-1$ and $f = e$. First, $f = e$ implies that $e \geq q - 1 - \theta_{\text{gain}}$. Also, $\eta = q-1$ and $e \leq \eta - q + 1 + \theta_{\text{fall}}$ imply that $e \leq \theta_{\text{fall}}$. Thus $q - 1 - \theta_{\text{gain}} \leq e \leq \theta_{\text{fall}}$ so that, adding θ_{gain} to both sides,

$$2M^{\bullet} + 2M^{\circ} - q\zeta_{\text{norm}} \geq q - 1. \tag{13}$$

Now, as in (12), $\eta = q-1$ implies that either (i) $\zeta_{\text{norm}} = 2$ and $M^{\bullet} \leq \frac{q-3}{2}$ or (ii) $M^{\bullet} = \frac{q-2}{2}$ and $M^{\circ} = 0$. Each of these clearly contradicts (13). □

LEMMA 5.4. *For $m \in J(n,g)$,*

$$P_{\text{fall}}^{i_e,-}(m) = \begin{cases} \binom{q-\eta+e}{2} & \text{for } -1 \leq e \leq \eta - q + 1 + \theta_{\text{fall}}, \\ \binom{q-\eta+e}{2} - (q - 2 - \theta_{\text{fall}} - \eta + e) & \text{for } \eta - q + 2 + \theta_{\text{fall}} \leq e \leq q - 1. \end{cases}$$

Proof. We write $i_e = n - m + 2g + \theta_{\text{fall}} - e'q - f$ if $0 \leq f \leq q-1$, as in Lemma 5.3, and work from Theorem 4.10. First, if $e' \geq q$, i.e., if $e \leq \eta - q$, then $P_{\text{fall}}^{i_e,-}(m) = 0$. We note also that, for $e \leq \eta - q$, $\binom{q-\eta+e}{2} = 0$. Next, if $q - 1 - \theta_{\text{fall}} \leq e' \leq q-1$, i.e., if $\eta - q + 1 \leq e \leq \eta - q + 1 + \theta_{\text{fall}}$, then

$$\begin{aligned} P_{\text{fall}}^{i_e,-}(m) &= \sum_{\nu=1}^{q-1-e'} \nu + \max\{0, q - e' - f\} \\ &= \binom{q-\eta+e}{2} + \max\{0, q - \eta + e - f\} = \binom{q-\eta+e}{2}, \end{aligned}$$

the last equality following from the second part of Lemma 5.3. Finally (since $e' \geq 0$ by Lemma 5.3), if $0 \leq e' \leq q - 2 - \theta_{\text{fall}}$, i.e., if $\eta - q + 2 + \theta_{\text{fall}} \leq e \leq q - 1$ (since $\eta \geq q-1$), then

$$\begin{aligned} P_{\text{fall}}^{i_e,-}(m) &= \sum_{\nu=1}^{q-1-e'} \nu - (q - 2 - \theta_{\text{fall}} - e') + \max\{0, q - e' - 1 - f\} \\ &= \binom{q-\eta+e}{2} - (q - 2 - \theta_{\text{fall}} - \eta + e) + \max\{0, q - \eta + e - f - 1\} \end{aligned}$$

and $q - \eta + e - f - 1 \leq 0$ since $\eta \geq q-1$ and $f \geq e$, by Lemma 5.3. □

We use the convention that, for $b \geq 0$, $\binom{a}{b} = 0$ if $a < b$. In particular,

$$\binom{a}{1} = \begin{cases} a & \text{for } a \geq 0, \\ 0 & \text{for } a \leq 0, \end{cases} \quad \binom{a}{0} = \begin{cases} 1 & \text{for } a \geq 0, \\ 0 & \text{for } a < 0, \end{cases} \quad \binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1},$$

where $b \geq 1$. Lemmas 5.1 and 5.4, together with (9), give the following proposition.

PROPOSITION 5.5. *For $m \in J(n,g)$,*

$$\mathrm{s}_{i_e}(C_m) = k - \binom{q-e}{2} - \binom{q-\eta+e}{2} + \binom{q-2-\theta_{\text{gain}}-e}{1} + \binom{q-2-\theta_{\text{fall}}-\eta+e}{1}.$$

Now we determine for which e, $-1 \leq e \leq q-1$, $\mathrm{s}_{i_e}(C_m)$ is maximized.

LEMMA 5.6. *For $m \in J(n,g)$, $\mathrm{s}_{i_e}(C_m)$ is maximized*

1. *at $e = \lfloor \frac{\eta}{2} \rfloor$ if $\eta \leq 2q - 6 - 2\theta_{\text{fall}}$ or*
2. *at $e = \lceil \frac{\eta}{2} \rceil$ if $\eta \geq 2q - 5 - 2\theta_{\text{fall}}$.*

Proof. From Proposition 5.5, with

$$\sigma(e) = \binom{q-e}{2} + \binom{q-\eta+e}{2} - \binom{q-2-\theta_{\text{gain}}-e}{1} - \binom{q-2-\theta_{\text{fall}}-\eta+e}{1},$$

we have $\mathrm{s}_{i_e}(C_m) = k - \sigma(e)$ and maximizing $\mathrm{s}_{i_e}(C_m)$ is equivalent to minimizing $\sigma(e)$ over $-1 \le e \le q-1$. Now, for $0 \le e \le q-1$,

$$\begin{aligned}\sigma(e) - \sigma(e-1) = &-\binom{q-e}{1} + \binom{q-\eta+e-1}{1} + \binom{q-2-\theta_{\text{gain}}-e}{0} \\ &- \binom{q-3-\theta_{\text{fall}}-\eta+e}{0}.\end{aligned}$$

Thus, since $0 \le \binom{q-2-\theta_{\text{gain}}-e}{0} \le 1$, we have

$$\begin{array}{lll} & e-q \le \sigma(e)-\sigma(e-1) \le e-q+1 & \text{for } 0 \le e \le \eta-q+1, \\ (14) & 2e-\eta-1 \le \sigma(e)-\sigma(e-1) \le 2e-\eta & \text{for } \eta-q+2 \le e \le \eta-q+2+\theta_{\text{fall}}, \\ & 2e-\eta-2 \le \sigma(e)-\sigma(e-1) \le 2e-\eta-1 & \text{for } \eta-q+3+\theta_{\text{fall}} \le e \le q-1. \end{array}$$

First, for $0 \le e \le \eta-q+1$, (14) implies that $\sigma(e)-\sigma(e-1) \le \eta-2q+2 \le 0$, so that $\sigma(e)$ is minimized over $-1 \le e \le \eta-q+1$ at $e = \eta-q+1$. Thus it is sufficient to determine where $\sigma(e)$ is minimized over $\eta-q+1 \le e \le q-1$. We note that, since $\eta \le 2q-3$ (Lemma 5.2),

$$\eta - q + 1 \le \left\lfloor \frac{\eta}{2} \right\rfloor \le \left\lceil \frac{\eta}{2} \right\rceil \le q-1.$$

Now, for $\eta-q+2 \le e \le \eta-q+2+\theta_{\text{fall}}$, (14) implies that if $e \le \lfloor \frac{\eta}{2} \rfloor$, then $\sigma(e)-\sigma(e-1) \le 0$ and if $e \ge \lfloor \frac{\eta}{2} \rfloor + 1 \ge \frac{\eta+1}{2}$, then $\sigma(e)-\sigma(e-1) \ge 0$. Similarly, for $\eta-q+3+\theta_{\text{fall}} \le e \le q-1$, (14) implies that if $e \le \lfloor \frac{\eta+1}{2} \rfloor = \lceil \frac{\eta}{2} \rceil$, then $\sigma(e) \le \sigma(e-1)$ and if $e \ge \lceil \frac{\eta}{2} \rceil + 1$, then $\sigma(e) \ge \sigma(e-1)$. Thus

1. if $\lceil \frac{\eta}{2} \rceil \le \eta-q+2+\theta_{\text{fall}}$, then $\sigma(e)$ is minimized over $\eta-q+1 \le e \le q-1$ at $e = \lfloor \frac{\eta}{2} \rfloor$ and
2. if $\lfloor \frac{\eta}{2} \rfloor \ge \eta-q+3+\theta_{\text{fall}}$, then $\sigma(e)$ is minimized over $\eta-q+1 \le e \le q-1$ at $e = \lceil \frac{\eta}{2} \rceil$.

This leaves the case $\lfloor \frac{\eta}{2} \rfloor = \lceil \frac{\eta}{2} \rceil - 1 = \eta-q+2+\theta_{\text{fall}}$, i.e., $\eta = 2q-5-2\theta_{\text{fall}}$. In this case, the above analysis implies that $\sigma(e)$ is minimized at either $\lfloor \frac{\eta}{2} \rfloor = \eta-q+2+\theta_{\text{fall}} = q-3-\theta_{\text{fall}}$ or $\lceil \frac{\eta}{2} \rceil = \eta-q+3+\theta_{\text{fall}} = q-2-\theta_{\text{fall}}$. Also, we have

$$\sigma(q-2-\theta_{\text{fall}}) - \sigma(q-3-\theta_{\text{fall}}) = -(\theta_{\text{fall}}+2) + (\theta_{\text{fall}}+2) + \binom{\theta_{\text{fall}}-\theta_{\text{gain}}}{0} - 1 \le 0,$$

so that $\sigma(e)$ is minimized at $\lceil \frac{\eta}{2} \rceil$.

Finally, we note that if $\eta \ge 2q-3-2\theta_{\text{fall}}$, then $-\eta \le -2q+3+2\theta_{\text{fall}}$, so that adding $2\eta+1$ to both sides and dividing by 2 implies

$$\left\lceil \frac{\eta}{2} \right\rceil \le \frac{\eta+1}{2} \le \eta - q + 2 + \theta_{\text{fall}},$$

and we are in case 1 above. Also, if $\eta = 2q-4-2\theta_{\text{fall}}$ we have $\lceil \frac{\eta}{2} \rceil = \eta-q+2+\theta_{\text{fall}}$, and again we are in case 1. Similarly for $\eta \le 2q-6-2\theta_{\text{fall}}$ we are in case 2 above. □

Proposition 5.5 and Lemma 5.6 give us the following theorem.

THEOREM 5.7. *For $m \in J(n,g)$,*

$$\mathrm{s}(C_m) = \begin{cases} k - \binom{q-\lfloor \frac{\eta}{2} \rfloor}{2} - \binom{q-\lceil \frac{\eta}{2} \rceil}{2} + (2q - 4 - \theta_{\mathrm{fall}} - \theta_{\mathrm{gain}} - \eta) & \textit{for } \eta \le 2q - 6 - 2\theta_{\mathrm{fall}}, \\ k - \binom{q-\lfloor \frac{\eta}{2} \rfloor}{2} - \binom{q-\lceil \frac{\eta}{2} \rceil}{2} + 1 & \textit{for } \eta = 2q - 5 - 2\theta_{\mathrm{fall}}, \\ k - \binom{q-\lfloor \frac{\eta}{2} \rfloor}{2} - \binom{q-\lceil \frac{\eta}{2} \rceil}{2} & \textit{for } \eta \ge 2q - 4 - 2\theta_{\mathrm{fall}}. \end{cases}$$

Proof. The result follows since

1. for $\eta \le 2q - 6 - 2\theta_{\mathrm{fall}}$, $q - 2 - \theta_{\mathrm{gain}} - \lfloor \frac{\eta}{2} \rfloor \ge 0$ and $q - 2 - \theta_{\mathrm{fall}} - \lceil \frac{\eta}{2} \rceil \ge 1$;
2. for $\eta = 2q - 5 - 2\theta_{\mathrm{fall}}$, $q - 2 - \theta_{\mathrm{gain}} - \lceil \frac{\eta}{2} \rceil \le 0$ and $q - 2 - \theta_{\mathrm{fall}} - \lfloor \frac{\eta}{2} \rfloor = 1$; and
3. for $\eta \ge 2q - 4 - 2\theta_{\mathrm{gain}}$, $q - 2 - \theta_{\mathrm{gain}} - \lceil \frac{\eta}{2} \rceil \le 0$ and $q - 2 - \theta_{\mathrm{fall}} - \lfloor \frac{\eta}{2} \rfloor \le 0$.

For example, $\eta \le 2q - 6 - 2\theta_{\mathrm{fall}}$ implies that $\lfloor \frac{\eta}{2} \rfloor \le q - 3 - \theta_{\mathrm{fall}}$, so that

$$q - 2 - \theta_{\mathrm{gain}} - \left\lfloor \frac{\eta}{2} \right\rfloor \ge 1 + \theta_{\mathrm{fall}} - \theta_{\mathrm{gain}} = 1 + \zeta_{\mathrm{gain}} - \zeta_{\mathrm{fall}} \ge 0.$$

The other equalities and inequalities follow similarly. □

Of course, Theorem 5.7 essentially gives the values of $\mathrm{s}(C_m)$ for $I(n,g)$ since $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$ implies $m^{\perp} \in J(n,g)$ and $\mathrm{s}(C_m) = \mathrm{s}(C_{m^{\perp}})$.

Table 3 gives $\mathrm{s}(C_m)$ for $q \in \{2,3,4,5,7,8\}$ and $m \in J(n,g)$. Comparing these values of $\mathrm{s}(C_m)$ with the values of $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$ given in Table 2 we have $\mathrm{s}(C_m) = \nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$ except for $q = 5$ and $m = 70$, $q = 7$ and $m \in \{182, 189, 190\}$, and $q = 8$ and $m \in \{268, 272, 276, 277, 280, 281\}$. In particular, $\mathrm{s}(C_m)$ achieves the DLP bound for C_m for $q \in \{2,3,4,5,7,8\}$ and $m \in I(n,g)$ when this is not excluded by Corollary 3.10, i.e., whenever the entry for m or $m^{\perp}$ in Table 2 is not in boldface.

TABLE 3
$\mathrm{s}(C_m)$ *for* $q \in \{2,3,4,5,7,8\}$ *and* $m \in J(n,g)$.

q															
2	m	4													
	$\mathrm{s}(C_m)$	3													
3	m	13	14	15											
	$\mathrm{s}(C_m)$	11	11	11											
4	m	32	33	34	35	36	37								
	$\mathrm{s}(C_m)$	26	27	27	28	28	28								
5	m	62	63	64	65	66	67	68	69	70	71				
	$\mathrm{s}(C_m)$	53	53	54	54	55	56	56	56	57	56				
7	m	171	172	173	174	175	176	177	178	179	180				
	$\mathrm{s}(C_m)$	151	151	152	153	153	154	155	156	156	156				
	m	181	182	183	184	185	186	187	188	189	190	191			
	$\mathrm{s}(C_m)$	157	158	157	158	159	159	159	159	160	160	159			
8	m	256	257	258	259	260	261	262	263	264	265	266	267	268	269
	$\mathrm{s}(C_m)$	228	229	230	231	231	232	233	234	234	234	235	236	237	236
	m	270	271	272	273	274	275	276	277	278	279	280	281	282	283
	$\mathrm{s}(C_m)$	237	238	239	238	238	239	240	240	239	240	241	241	240	240

Comparing $\mathrm{s}(C_m)$ with $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$. We start by reinterpreting $\nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$ in terms of η in Theorem 5.8. We use this to calculate (in Proposition 5.9) and hence to show (in Corollary 5.10) that $\mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_{\infty}))$ whenever this is not excluded by Corollary 3.10 . This means that $\mathrm{s}(C_m)$ achieves the DLP bound for C_m for just over half of those m in the range $[\frac{n-1}{2}, \frac{n-3}{3} + 2g]$. We then compare $\mathrm{s}(C_m)$ with $\nabla^{\imath}(C_{\mathcal{L}}(D, mQ_{\infty}))$ in Table 4 and see that $\mathrm{s}(C_m)$ achieves the bound

$\nabla^\imath(C_\mathcal{L}(D, mQ_\infty))$ for approximately a further quarter of those m in $[\frac{n-1}{2}, \frac{n-3}{3} + 2g]$ if q is odd but only for about a further $1/q$ of those m in $[\frac{n-1}{2}, \frac{n-3}{3} + 2g]$ if q is even.

Previously, we partitioned $J(n, g)$ into three subintervals, according to whether $0 \leq M^\circ \leq \frac{q-M^\bullet-2}{2}$, $\frac{q-M^\bullet-1}{2} \leq M^\circ \leq q - \frac{M^\bullet+1}{2}$, or $q - \frac{M^\bullet}{2} \leq M^\circ \leq q$. Now we consider a finer partition and say that $m \in J(n, g)$ satisfies (A), (B), (C), (D), or (E) according to whether (A) $0 \leq M^\circ \leq \frac{q-2}{2} - M^\bullet$, (B) $\frac{q-1}{2} - M^\bullet \leq M^\circ \leq \frac{q-M^\bullet-2}{2}$, (C) $\frac{q-M^\bullet-1}{2} \leq M^\circ \leq q - M^\bullet - 1$, (D) $q - M^\bullet \leq M^\circ \leq q - \frac{M^\bullet+1}{2}$, or (E) $q - \frac{M^\bullet}{2} \leq M^\circ \leq q$.

We compare $\mathrm{s}(C_m)$ with $\nabla^\imath(C_\mathcal{L}(D, mQ_\infty))$ by reinterpreting Theorems 3.9 and 5.7 using (A)–(E).

THEOREM 5.8. *If $m \in J(n, g)$, then*

$$\nabla(C_\mathcal{L}(D, mQ_\infty)) = \begin{cases} k - \binom{q-\lfloor\frac{\eta}{2}\rfloor}{2} - \binom{q-\lceil\frac{\eta}{2}\rceil}{2} & \textit{if } m \textit{ satisfies (A),(C),(E),} \\ k - \binom{q-\lfloor\frac{\eta}{2}\rfloor}{2} - \binom{q-\lceil\frac{\eta}{2}\rceil}{2} - (\theta_{\text{fall}} + \theta_{\text{gain}} - q + 2) & \textit{otherwise.} \end{cases}$$

Proof. Take u and v as in the statement of Theorem 2.1. It is straightforward to show, using the characterization of (u, v) given in the proof of Lemma 3.8, that if m satisfies (A), (C), or (E), then $\eta = u - 1$ and $v = q - \theta_{\text{gain}} - \theta_{\text{fall}} - 2$ and if m satisfies (B) or (D), then $\eta = u$ and $v = 2q - \theta_{\text{gain}} - \theta_{\text{fall}} - 2$. Thus Theorem 2.1 implies that, for m satisfying (A), (C), or (E),

$$\begin{aligned}\nabla(C_\mathcal{L}(D, mQ_\infty)) = k - \binom{q - \lfloor\frac{\eta+1}{2}\rfloor}{2} - \binom{q - \lceil\frac{\eta+1}{2}\rceil}{2} \\ - \min\left\{q - \left\lceil\frac{\eta+1}{2}\right\rceil, \theta_{\text{gain}} + \theta_{\text{fall}} + 2\right\}\end{aligned}$$

and for m satisfying (B) or (D),

$$\nabla(C_\mathcal{L}(D, mQ_\infty)) = k - \binom{q - \lfloor\frac{\eta}{2}\rfloor}{2} - \binom{q - \lceil\frac{\eta}{2}\rceil}{2} - \min\left\{q - \left\lceil\frac{\eta}{2}\right\rceil, \theta_{\text{gain}} + \theta_{\text{fall}} + 2 - q\right\}.$$

First, for m satisfying (A), (C), or (E) we have (i) $\lceil\frac{\eta+1}{2}\rceil \geq q - M^\bullet - 1$ if $\zeta_{\text{norm}} \in \{0, 1\}$ or (ii) $\lceil\frac{\eta+1}{2}\rceil \geq q - M^\bullet - 2$ if $\zeta_{\text{norm}} = 2$. Also, $\theta_{\text{gain}} + \theta_{\text{fall}} + 2 = 2M^\bullet + 2M^\circ - \zeta_{\text{gain}} - \zeta_{\text{fall}} + 2$ and (i) for $\zeta_{\text{norm}} = 0$, $2M^\circ - \zeta_{\text{gain}} - \zeta_{\text{fall}} \geq 0$, (ii) for $\zeta_{\text{norm}} = 1$, $2M^\circ - \zeta_{\text{gain}} - \zeta_{\text{fall}} \geq (q - M^\bullet - 1) - q = -M^\bullet - 1$ or (iii) for $\zeta_{\text{norm}} = 2$, $2M^\circ - \zeta_{\text{gain}} - \zeta_{\text{fall}} \geq (2q - M^\bullet) - 2q = M^\bullet$. Thus, for m satisfying (A), (C), or (E), $\nabla(C_\mathcal{L}(D, mQ_\infty))$ is equal to

$$k - \binom{q - \lfloor\frac{\eta+1}{2}\rfloor}{2} - \binom{q - \lceil\frac{\eta+1}{2}\rceil}{2} - q - \left\lceil\frac{\eta+1}{2}\right\rceil = k - \binom{q - \lfloor\frac{\eta+1}{2}\rfloor}{2} - \binom{q - \lceil\frac{\eta+1}{2}\rceil + 1}{2}$$

as required. Similarly, for m satisfying (B) or (D) (so that $\zeta_{\text{norm}} \leq 1$) it is easy to see that $q - \lceil\frac{\eta}{2}\rceil \geq M^\bullet + 1$ and (by considering the cases that $\zeta_{\text{norm}} = 0$ and $\zeta_{\text{norm}} = 1$ separately) $\theta_{\text{gain}} + \theta_{\text{fall}} + 2 - q \leq M^\bullet + 1$. Thus, for m satisfying (B) or (D),

$$\nabla(C_\mathcal{L}(D, mQ_\infty)) = k - \binom{q - \lfloor\frac{\eta}{2}\rfloor}{2} - \binom{q - \lceil\frac{\eta}{2}\rceil}{2} - (\theta_{\text{gain}} + \theta_{\text{fall}} + 2 - q)$$

as required. □

Before comparing $\mathrm{s}(C_m)$ with $\nabla^\imath(C_\mathcal{L}(D, mQ_\infty))$, we compare it with $\nabla(C_\mathcal{L}(D, mQ_\infty))$. To do this we refine (A)–(E) as follows: if m satisfies (C), then we say that m satisfies (C1), (C2), or (C3) if (C1) $\frac{q-M^\bullet-1}{2} \le M^\circ \le \frac{q-2}{2}$, (C2) $M^\circ = \frac{q-1}{2}$, or (C3) $\frac{q}{2} \le M^\circ \le q - M^\bullet - 1$.

Proposition 5.9. *For $m \in J(n,g)$,*

$$\mathrm{s}(C_m) - \nabla(C_{\mathcal{L}}(D, mQ_\infty)) = \begin{cases} 0 & \textit{if } m \textit{ satisfies (A)}, \\ 2M^\bullet + 2M^\circ - q + 2 & \textit{if } m \textit{ satisfies (B)}, \\ q - 2M^\circ - q_2 & \textit{if } m \textit{ satisfies (C1)}, \\ 1 & \textit{if } m \textit{ satisfies (C2)}, \\ 0 & \textit{if } m \textit{ satisfies (C3)}, \\ 2M^\bullet + 2M^\circ - 2q + 2 & \textit{if } m \textit{ satisfies (D)}, \\ 2q - 2M^\circ + 1 - q_2 & \textit{if } m \textit{ satisfies (E)}. \end{cases}$$

Proof. Using $\eta = 2q - 2\theta_{\text{fall}} + 2M^\circ - 2\zeta_{\text{fall}} + q_2 - \zeta_{\text{norm}} - 3$, it is straightforward to see that if $M^\circ \le q - 1$ and

1. if m satisfies (A), (B), (D), or (C3), then $\eta \ge 2q - 2\theta_{\text{fall}} - 4$;
2. if m satisfies (C1) or (E), then $\eta \le 2q - 6 - \theta_{\text{fall}}$; or
3. if m satisfies (C2), then $\eta = 2q - 5 - \theta_{\text{fall}}$.

Also, if $m = q$ is odd, then $\eta = 2q - 2\theta_{\text{fall}} - 4$. Likewise, if $m = q$ is even, then $\eta = 2q - 2\theta_{\text{fall}} - 5$. The result then follows from Theorems 5.7 and 5.8 noting that, for cases (B) and (D), $\theta_{\text{gain}} + \theta_{\text{fall}} - q + 2 = 2M^\bullet + 2M^\circ - (\zeta_{\text{norm}} + 1)q + 2$ and for cases (C1) and (E) with $M^\circ \le q - 1$, $2q - 4 - \theta_{\text{fall}} - \theta_{\text{gain}} - \eta = \zeta_{\text{norm}} q - 2M^\circ - q_2 + (\zeta_{\text{norm}} - 1)$. □

It follows from Proposition 5.9 that $\mathrm{s}(C_m)$ achieves the DLP bound for C_m as often as this is possible. We state this as the following corollary.

Corollary 5.10. *For $m \in I(n,g)$, $\mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$ if and only if $\Delta(m) = 0$.*

Proof. Since for $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$, $\Delta(m) = \Delta(m^\perp)$, $\nabla(C_{\mathcal{L}}(D, mQ_\infty)) = \nabla(C_{\mathcal{L}}(D, m^\perp Q_\infty))$, and $\mathrm{s}(C_m) = \mathrm{s}(C_{m^\perp})$, it suffices to show the result for $m \in J(n,g)$. It follows from the definition of $\Delta(m)$ for such m that $\Delta(m) = 0$ if and only if (i) m satisfies (A) or (ii) m satisfies (C3) or (iii) $q_2 = 1$ and $M^\circ = q$. These are exactly the values of M° for which Proposition 5.9 gives $\mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$. □

Example 5.11. *If C_m is self-dual, then $\nabla(C_m) = \mathrm{s}(C_m) = \frac{n}{2} - \frac{q^2}{4}$, where C_m has the lexicographic coordinate order. In particular, $s[C_m] = \frac{n}{2} - \frac{q^2}{4}$.*

Proof. We know that q is a power of 2, $k = \frac{n}{2}$, and $m = \frac{n}{2} + g - 1 \in J(n,g) \subseteq I(n,g)$. From the definitions, $M^\bullet = \frac{q-2}{2}$ and $M^\circ = \zeta_{\text{norm}} = 0$. Also, $\nabla(C_m) = \frac{n}{2} - \frac{q^2}{4}$ by Theorem 5.8. The result now follows since $\Delta(m) = 0$. □

We remark that the main result of [13] is Example 5.11 with $q \ge 4$. Corollary 5.10 and Proposition 3.12 imply that $\nabla(C_m)$ is attained for just over half the $m \in I(n,g)$. Explicitly, the proportion of these m for which the DLP bound is attained is $\frac{1}{2} + \frac{1}{2q}$ for q odd and $\frac{1}{2} + \frac{3q-5}{2(q^2-q-1)}$ for q even. Of course Corollary 5.10 implies that if m satisfies (A), (C3) or $M^\circ = q$ is odd, then

$$s[C_{\mathcal{L}}(D, mQ_\infty)] = \nabla(C_{\mathcal{L}}(D, mQ_\infty)) = \mathrm{s}(C_m).$$

The increments on $s[C_{\mathcal{L}}(D, mQ_\infty)]$ given by Theorem 3.9 and Proposition 5.9 for all m in $J(n,g)$ (and hence implicitly also for $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$) are given in Table 4. The first entry is $\Delta(m)$ and the second is $\mathrm{s}(C_m) - \nabla(C_{\mathcal{L}}(D, mQ_\infty))$. Thus our lower bound for $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is $\nabla^\iota(C_{\mathcal{L}}(D, mQ_\infty)) = \nabla(C_{\mathcal{L}}(D, mQ_\infty)) + \Delta(m)$ and our upper bound for $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$ plus the second entry,

TABLE 4
Table of bounds on $s[C_{\mathcal{L}}(D, mQ_\infty)]$ for $m \in J(n,g)$.

m satisfies	$\Delta(m)$	$s(C_m) - \nabla$	Range
(A)	0	0	0
(B)	$M^\bullet + M^\circ + 1 - \frac{q-q_2}{2}$	$2M^\bullet + 2M^\circ + 2 - q$	$M^\bullet + M^\circ + 1 - \frac{q+q_2}{2}$
(C1)	$\frac{q+q_2}{2} - M^\circ$	$q - q_2 - 2M^\circ$	$\frac{q-q_2}{2} - M^\circ$
(C2)	1	1	0
(C3)	0	0	0
(D)	$M^\bullet + M^\circ + 1 - q$	$2M^\bullet + 2M^\circ + 2 - 2q$	$M^\bullet + M^\circ + 1 - q$
(E)	$q - M^\circ + 1 - q_2$	$2q - 2M^\circ + 1 - q_2$	$q - M^\circ$

i.e., $\mathrm{s}(C_m)$. The third entry in the table (the range of $s[C_{\mathcal{L}}(D, mQ_\infty)]$) is $\mathrm{s}(C_m) - \nabla^\iota(C_{\mathcal{L}}(D, mQ_\infty))$.

As well as those m for which $\mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_\infty))$, Table 4 also gives

$$\mathrm{s}(C_m) = \nabla^\iota(C_{\mathcal{L}}(D, mQ_\infty)) = s[C_{\mathcal{L}}(D, mQ_\infty)] \tag{15}$$

for those $m \in J(n,g)$ such that

$$\begin{array}{ll} \frac{q-1}{2} - M^\bullet \leq M^\circ \leq \frac{q-1}{2} & \text{if } q \text{ is odd,} \\ M^\circ = q & \text{if } q \text{ is even.} \end{array} \tag{16}$$

Hence (15) also holds for those $m \in [\frac{n-1}{2} + g, \frac{n-3}{2} + 2g]$ such that $m^\perp$ satisfies (16). In all these cases except $M^\bullet \geq 2$ and $M^\circ = \frac{q-3}{3}$ we have

$$s[C_{\mathcal{L}}(D, mQ_\infty)] = \mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_\infty)) + 1.$$

For $M^\bullet \geq 2$ and $M^\circ = \frac{q-3}{3}$ we have

$$s[C_{\mathcal{L}}(D, mQ_\infty)] = \mathrm{s}(C_m) = \nabla(C_{\mathcal{L}}(D, mQ_\infty)) + 2.$$

For q odd, this gives $\frac{q^2-1}{4}$ values of $m \in I(n,g)$ for which $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is determined but is strictly greater than $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$. Thus, for q odd, the total proportion of those m in $I(n,g)$ for which we have determined $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is

$$\frac{1}{2} + \frac{1}{2q} + \frac{q^2-1}{4(q^2-q)} = \frac{3(q+1)}{4q}.$$

For q even, it gives $q-2$ values of $m \in I(n,g)$ for which $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is determined but is strictly greater than $\nabla(C_{\mathcal{L}}(D, mQ_\infty))$. Thus, for q even, the total proportion of those $m \in I(n,g)$ for which we have determined $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is

$$\frac{1}{2} + \frac{3q-5}{2(q^2-q-1)} + \frac{q-2}{q^2-q-1} = \frac{1}{2} + \frac{5q-9}{2(q^2-q-1)}.$$

Thus we have determined $s[C_{\mathcal{L}}(D, mQ_\infty)]$ for over three quarters of those m in $I(n,g)$ when q is odd but only for something over one half of those m in $I(n,g)$ when q is even. For q odd, the first m for which $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is not determined is $q = 5$ and $m = 70$ (when it is either 56 or 57), and for q even, the first m for which $s[C_{\mathcal{L}}(D, mQ_\infty)]$ is not determined is $q = 8$ and $m = 268$ (when it is either 236 or 237).

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