

## Quantum slow motion

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(Received 15 May 2000; published 18 January 2001)

We investigate the center-of-mass motion of cold atoms in a standing amplitude modulated laser field. We use a simple model to explain the momentum distribution of the atoms after any distinct number of modulation cycles. The atoms starting near a classical phase-space resonance move slower than we would expect classically. We explain this by showing that for a wave packet on the classical resonances we can replace the complicated dynamics in the quantum Liouville equation in phase space by its classical dynamics with a modified potential.

DOI: 10.1103/PhysRevA.63.023413

PACS number(s): 42.50.Vk, 03.75.Be, 05.45.-a

To have an intuitive picture of the quantum-mechanical dynamics of a wave packet we are usually confined to the semiclassical regime, that is, to orbits with action large compared to Planck's constant [1,2], or to special systems like the harmonic oscillator, where the quantum evolution equations in phase space are identical to the classical ones [3]. In this paper we analyze the center-of-mass motion of cold atoms in an amplitude-modulated standing laser field in the limit of large detuning. In this limit we can describe the dynamics by a sinusoidally modulated cosine potential.

In terms of this physical system we propose a scheme which enables us to describe a wave packet, localized near a resonance of a classical mixed phase space, by classical dynamics in a modified potential. We apply the theory of Henriksen *et al.* [4] to replace the potential in the high-order quantum Liouville equation by an effective potential in such a way that we obtain a classical Liouville equation. Hence we describe the quantum motion as a classical motion in this modified potential. We are then able to characterize the quantum effect by comparing the modified dynamics with the dynamics in the original potential. This method is applicable well beyond the semiclassical regime for many different potentials.

Usually quantum effects on wave packets express themselves in the revival and fractional revival properties [5] or in the occurrence of tunneling phenomena [6]. Both take place on a comparatively long-time scale so that we intuitively do not expect quantum effects to be visible on a short-time scale. We disprove this intuitive assumption in our model where we use the center-of-mass motion of cold atoms in a standing amplitude-modulated laser field. Here we demonstrate that the momentum distribution after each cycle of the modulation is peaked at smaller momenta than we would expect classically. This shows that the atoms are traveling slower than we would expect from classical simulations and we can give a very simple explanation of this "quantum slow motion" phenomenon.

We investigate a cloud of two-level atoms situated in a standing laser field with a periodic modulated amplitude. This system has been the subject of several experiments [7,8]. The Hamiltonian of the center-of-mass motion in the limit of large detuning is [9]

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon \cos t)\cos q, \quad (1)$$

where  $p$  and  $q$  denote scaled dimensionless momentum and position,  $t$  time, and  $\kappa$  and  $\epsilon$  are the parameters defining the depth of the standing wave and the strength of the amplitude modulation, respectively. Note that  $p$  and  $q$  fulfill the commutator relation  $[p, q] = i\bar{\hbar}$ , where  $\bar{\hbar}$  is a scaled Planck's constant that is in some sense a measure for the "quantum mechanicality" of the problem since it defines the size of a minimum uncertainty wave packet in relationship to the resonances [10].

In Fig. 1 (left) we show as an example the classical stroboscopic phase-space portrait [11] for  $\epsilon = 0.2$  with  $\kappa = 1.2$ . This choice of parameters is able to show classical stable period-one resonances after each modulation period symmetrically situated along the momentum axes. Until recently all published experimental results did not show any indication of the peaks corresponding to period-one resonances. With increasing laser intensities experimentalists at the University of Queensland [8] were just recently able to show the experimental indication of atoms loaded in a classical period-one phase-space resonance. As these experiments will be improved we are convinced that the quantum effect we are describing in this paper will be observed.

The specific phase-space structure allows a quantum-mechanical wave packet, situated initially near one of these resonances, to coherently tunnel to the other resonance. This can be described as Rabi oscillations between two Floquet states of the problem. The tunneling takes place on a long-time scale in terms of cycles of the modulation.<sup>1</sup> One of the ultimate goals of the experiments is to load *one* resonance in order to observe quantum tunneling, which is modeled in this paper.

We simulate the tunneling dynamics by starting each realization with a minimum uncertainty wave packet that may be squeezed [12], centered on the classical resonance. We then simulate the full quantum-mechanical dynamics by applying a split operator algorithm with adapted time-step size [13] in the context of a standard quantum Monte Carlo integration scheme to include stimulated and spontaneous tran-

<sup>1</sup>Note that this tunneling cannot be understood in terms of the presence of a potential barrier as it is present in several publications [6] regarding tunneling in mixed systems.

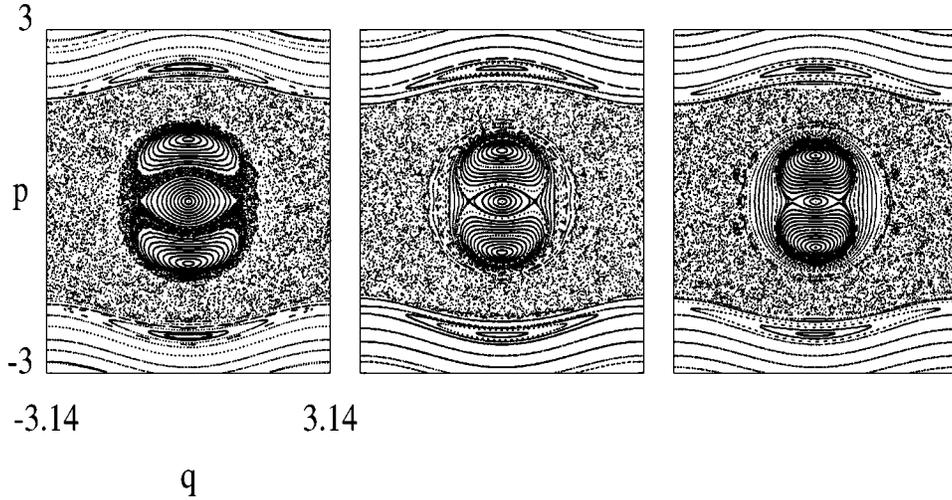


FIG. 1. Left: Stroboscopic phase-space portrait of the classical motion described by the Hamiltonian Eq. (1) for  $\kappa=1.2$  and  $\epsilon=0.2$ . Middle and right: Stroboscopic phase-space portraits of the corresponding effective potentials Eq. (7) with  $\tilde{\kappa}=0.25$  and  $\tilde{\kappa}=0.35$ . The scales are identical to the left figure.

sitions [9].<sup>2</sup> We calculate the mean momenta and the corresponding variance from the momentum distribution taken after each cycle of the modulation at  $t=2n\pi$ . In Fig. 2 (full line) we show the result of this simulation for  $\epsilon=0.2$ ,  $\kappa=1.2$ , and  $\tilde{\kappa}=0.25$ . Related to recent experiments [8] we used the parameters for rubidium to obtain a realistic scenario. We plot the mean momentum after each cycle of the modulation of the standing wave against the number of cycles. As expected and clearly indicated by the decreasing variance, we observe coherent tunneling of the mean momentum from the location of the resonance at approximately  $p=1$  to the corresponding resonance at  $p=-1$ .

However, there are additional oscillations that might lead to the conclusion that the wave packet is not sitting precisely on the classical period-one fixed point but is indeed circulating around an alternative stable point in phase space. It seems like the wave packet, centered on the classical resonance, is not appropriately centered on the “true” resonance but sitting beside it. Therefore the mean momentum at each kick strongly oscillates around its mean motion. This leads us to the conclusion that if we would move the initial wave packet onto this alternative stable point and start the simulation of the dynamics from there, we expect the oscillations to vanish. This is exactly what we see in Fig. 2 (dashed line). The oscillations are strongly compressed. Hence for the dynamics of the wave packet obviously not the classical resonance is important but a modified resonance, shifted towards slower momentum. This indicates that any tunneling experiment depends very sensitively on the initial conditions and it will be very helpful for the experiment, if all parameters and initial conditions are well investigated. The present paper

shows that in order to observe tunneling, one has to start with a cloud of atoms sitting on the modified resonance rather than the classical one. Depending on the value of  $\tilde{\kappa}$  this can be very significant, because for large values of  $\tilde{\kappa}$  the modified resonance can be shifted by a such a large amount that tunneling is observable when the wave packet starts on the modified resonance, but it is not observable when it starts on the classical resonance.

How can we explain this effect? To give an explanation we first recall that a wave packet localized near a classical resonance has been shown [11] to remain localized without changing its shape, at least for a long time. Therefore we may assume that a minimum uncertainty wave packet sitting near a classical resonance will remain unchanged in shape for several cycles. This is the main assumption we need to apply to the theory of Henriksen *et al.* [4] where the effect of quantum mechanics on a wave packet is described as classical motion, that is, as motion following the classical Liouville equations in phase space, but in a modified potential.

The convenient quantum-mechanical phase-space representation is the Wigner function  $W(q,p,t)$ , because it has the correct quantum-mechanical marginal distributions. Since in the experiments we are seeking to describe the momentum distribution and the position distribution of the center-of-mass motion, this property of the Wigner function allows us to compare the marginals directly with the measured distributions. The phase-space dynamics of the Wigner function is given by [14,4]

$$\begin{aligned} \frac{\partial W}{\partial t} = & -p \frac{\partial W}{\partial q} \\ & + \frac{i}{\tilde{\kappa}} \left[ \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( \frac{\tilde{\kappa}}{2i} \right)^{\nu} \frac{\partial^{\nu} V(q,t)}{\partial q^{\nu}} \frac{\partial^{\nu} W}{\partial p^{\nu}} \right. \\ & \left. - \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( -\frac{\tilde{\kappa}}{2i} \right)^{\nu} \frac{\partial^{\nu} V(q,t)}{\partial q^{\nu}} \frac{\partial^{\nu} W}{\partial p^{\nu}} \right], \end{aligned} \quad (2)$$

<sup>2</sup>Note that stimulated and spontaneous transitions may have some influence on the results since we are using realistic experimental parameters. In order to control and simulate these processes we have to apply the Monte Carlo method to the full Hamiltonian given in [9] rather than the simplified one of Eq. (1).

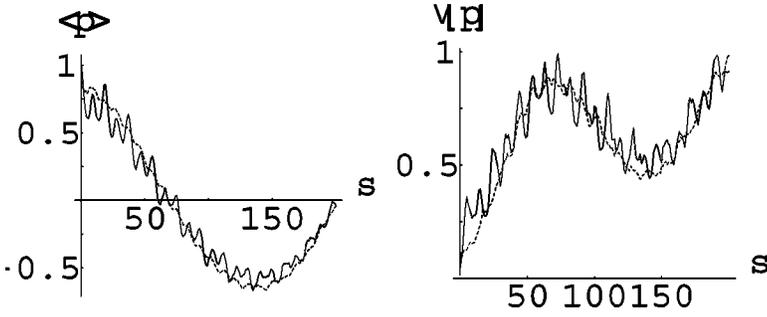


FIG. 2. Left: Mean momentum  $\langle p \rangle$  of the quantum-mechanical simulation of the dynamics of two wave packets dependent on the number of cycles  $s$ . The first (straight line) is initially sitting on the classical resonance ( $p_m = 1.03$ ), the second (dashed line) on the modified one at  $p_m = 0.84$ . Here the parameters are  $\bar{\kappa} = 0.25, \kappa = 1.2$ , and  $\epsilon = 0.2$ . Right: Corresponding variance  $V[p]$ .

where  $V(q, t) = \kappa(1 - \epsilon \cos t) \cos q$  denotes the potential. This representation is convenient for our further analysis and corresponds to the well-known one given by Wigner where only one sum over odd derivatives occurs. We can formally replace the infinite sum by defining an effective potential  $V_{\text{eff}}$  by

$$\frac{\partial V_{\text{eff}}}{\partial q} = \frac{i}{\bar{\kappa}} \left[ \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( \frac{\bar{\kappa}}{2i} \right)^{\nu} \frac{\partial^{\nu} V(q, t)}{\partial q^{\nu}} \frac{\partial^{\nu} W}{\partial p^{\nu}} - \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( -\frac{\bar{\kappa}}{2i} \right)^{\nu} \frac{\partial^{\nu} V(q, t)}{\partial q^{\nu}} \frac{\partial^{\nu} W}{\partial p^{\nu}} \right] \bigg/ \frac{\partial W}{\partial p}. \quad (3)$$

E.g., for an exponential potential  $V'(q) \equiv \exp(iq)$  this simplifies to [4]

$$\frac{\partial V'_{\text{eff}}}{\partial q} = \frac{\partial V'}{\partial q} \frac{1}{\bar{\kappa}} \left[ W \left( q, p + \frac{\bar{\kappa}}{2}, t \right) - W \left( q, p - \frac{\bar{\kappa}}{2}, t \right) \right] \bigg/ \frac{\partial W(q, p, t)}{\partial p}. \quad (4)$$

With this effective potential Eq. (2) is replaced by the first-order equation

$$\frac{\partial W}{\partial t} = -p \frac{\partial W}{\partial q} + \frac{\partial V_{\text{eff}}}{\partial q} \frac{\partial W}{\partial p} \quad (5)$$

which is identical to the classical Liouville equation describing the classical dynamics in the modified potential  $V_{\text{eff}}$ . In this sense the action of quantum mechanics can be described by the classical motion in a modified potential.

Assuming a Gaussian squeezed minimum uncertainty wave packet with time-dependent squeeze parameter  $\xi(t)$ , we take the Wigner function to be of the form

$$W(q, p, t) = \frac{1}{\pi \bar{\kappa}} \exp \left( -\frac{\xi}{\bar{\kappa}} (q - \langle q \rangle)^2 - \frac{1}{\bar{\kappa} \xi} (p - \langle p \rangle)^2 \right), \quad (6)$$

with the mean time-dependent momentum and position,  $\langle p \rangle(t)$  and  $\langle q \rangle(t)$ , generally chosen in such a way that the wave packet always stays centered on the resonance in order for the assumption of staying unchanged in shape to remain valid. Since our analysis is only valid in the vicinity of the resonance, with this specific choice the inequalities

$(q - \langle q \rangle)^2 \ll 1$  and  $(p - \langle p \rangle)^2 \ll 1$  hold true for all times. It is straightforward to find the corresponding expression of Eq. (4) for our cosine potential  $V(q)$  and to insert the Wigner function Eq. (6) to obtain the analytical expression

$$V_{\text{eff}}(q, t) = V(q, t) \exp \left( -\frac{\bar{\kappa}}{4} \frac{\sinh(p - \langle p \rangle)}{p - \langle p \rangle} \right) \quad (7)$$

for the effective potential. Since  $(p - \langle p \rangle)^2 \ll 1$  the sinh factor can, for the sake of qualitative discussion, be approximated by 1. That means the motion of the wave packet is locally described by the original potential compressed by a factor of  $\exp(-\bar{\kappa}/4)$ .

Technically speaking, this analysis is nothing other than a first-order iteration procedure since we substitute an estimated Wigner function to get more information out of the equations. However, in the case where a Gaussian wave packet proves to be stable the first-order iteration turns out to be sufficient. This is the case in the vicinity of the resonances. In all the other phase-space areas the Wigner function is known to change in time since wave packets spread. To make use of this method in all the other phase-space regions we could include more iteration steps to describe the spread of an initial wave packet. In that case the first-order iterative shown here is not applicable. Therefore our interpretation is only valid for the resonances.

In Fig. 1 (middle and right) we show for  $\bar{\kappa} = 0.25$  and  $\bar{\kappa} = 0.35$  classical stroboscopic phase-space portraits for the effective potential and compare them to the phase-space portrait of the original potential. Note that our approximation is only valid in the vicinity of the period-one resonances. However, since we are interested in exactly these regions of phase space this kind of representation gives an idea of what is going on, although the other phase-space regions are not represented correctly. The main conclusion regarding the resonances is that the central resonance at  $(q, p) = (0, 0)$  becomes smaller and the second-order resonances we are interested in are pushed towards smaller momenta  $p$  which corresponds exactly to the observation made in Fig. 2, where we obtained the best simulation for the tunneling phenomenon for initially situating the wave packet at the shifted resonance.

Equation (7) indicates that the effect scales with  $\bar{\kappa}$  which identifies that it is a purely quantum-mechanical effect. We can clearly see this property by comparing Fig. 1 (middle) and (right), where we can directly see the relocation of the classical resonance for two values of  $\bar{\kappa}$ . In Fig. 3 we simulate wave packets for different values of  $\bar{\kappa}$  for the first few

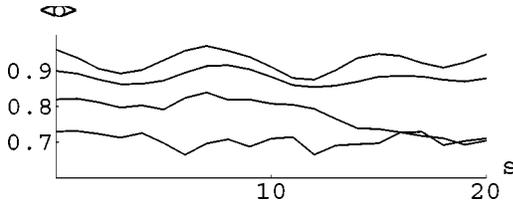


FIG. 3. Mean momenta  $\langle p \rangle$  of the quantum-mechanical simulation of the dynamics of several wave packets dependent on the number of cycles  $s$  with  $\kappa=1.2$  and  $\epsilon=0.2$ . Here  $\bar{\kappa}$  takes on the values 0.15, 0.2, 0.25, 0.3 (from top).

cycles. We start each simulation with a minimum uncertainty wave packet in such a way that the oscillations in the evolution are suppressed most and observe that in correspondence to the modified potential the mean momenta and therefore the wave packets themselves are relocated towards smaller velocities with increasing  $\bar{\kappa}$ . Note that the curve corresponding to  $\bar{\kappa}=0.25$  is related to a situation where the conditions for tunneling are fulfilled; therefore, the mean momentum starts to decrease. This result has a considerable impact to current tunneling experiments: For the example shown in Fig. 3, for  $\bar{\kappa}=0.25$  the distribution of atoms would have to start at momenta down to about 0.82 instead of 1.03, where the classical resonance is situated in these units. For  $\bar{\kappa}=0.3$  it is even 0.72. We consider this effect significant for the experiments since the consequence of this shift is that if we would start on a classical resonance we would observe the oscillatory behavior shown in Fig. 2 (straight line) rather than the expected pure tunneling behavior (dotted line). If the deviation is too large, which is the case for even higher  $\bar{\kappa}$ , we expect to observe no tunneling at all, whereas with the correct initial conditions it can be observed. Note that these values of  $\bar{\kappa}$  are being realized in current experiments.

Furthermore, in a new generation of experiments it is intended to place an atomic distribution at any particular position in phase space. One target is to trap the atom at a resonance to observe the property of a cloud of atoms remaining in that resonance and to observe tunneling. Another target [7] is to investigate anomalous diffusion and Levy flights. We expect the effect described in this paper to have an impact on this problem as well.

In many other systems even larger values of  $\bar{\kappa}$  can be reached than those shown in this paper, e.g., in an optical hollow fiber. As Eq. (7) indicates, the size of the quantum slow motion effect depends strongly on  $\bar{\kappa}$  and we expect that related effects are even more dominant as higher values of  $\bar{\kappa}$  are realized. For more complicated systems like a Gaussian potential we expect to be able to extend our theory and there will be not only a shift but also squeezing and other related effects. Therefore we strongly believe that our predictions will have a strong impact on current and future experiments.

There is a second important consequence of this phenomenon in the scenario of present experiments [8] of investigating the short-time behavior of loading all the resonances from a spatially uniform distributed cloud of atoms. In order to effectively load the resonances we start with a phase shift of  $-\pi/2$ , that is to say, we now investigate the Hamiltonian

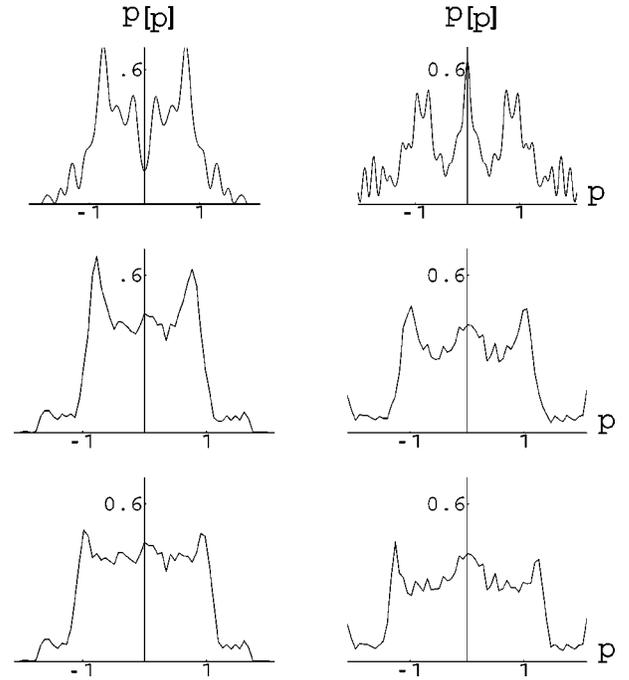


FIG. 4. Left: Quantum, modified classical, and purely classical simulation (from top) of momentum distributions  $P[p]$  of snapshots after 2.25 modulation cycles of the Hamiltonian equation (8) with  $\kappa=1.2$ ,  $\epsilon=0.2$ , and  $\bar{\kappa}=0.35$ . Right: The same simulations but for the unmodulated case  $\epsilon=0$ .

$$H(t) = \frac{p^2}{2} - \kappa(1 - 2\epsilon \sin t) \cos q \quad (8)$$

and take the snapshots at  $t = \pi/2 + 2n\pi$ . Then, in contrast to Fig. 1 the resonances are initially aligned on the  $q$  axes and are therefore covered best by the cloud of atoms. A classical picture of the dynamics suggests that as time goes by only those atoms initially sitting close to a resonance remain, whereas all the other atoms perform a nonlinear motion corresponding to the fact that they are sitting in a chaotic region [15] of phase space. Therefore we expect to observe after some time only the three peaks of loaded resonances. Since the assumption of a durable wave packet is only valid for a wave packet initially situated on a resonance and not for all the other wave packets, this motivates us to believe that the local relocation of the resonance described above only happens to those atoms trapped at the resonance. This should change the overall momentum distribution in comparison to a purely classical simulation.

In Fig. 4 (left) we compare the momentum distributions of snapshots at  $t = 9\pi/2$ , that is after 2.25 modulation cycles only, of three different simulations: a quantum-mechanical simulation (top), a modified classical simulation (middle), and a purely classical simulation (bottom). Note that this is a very short time compared to tunneling and revival experiments which are typically more than 100 modulation cycles. For the quantum simulation (top) we start with a large number of wave packets of the width of the distribution in momentum of the atom cloud. The width in position is chosen in order to have a minimum uncertainty wave packet. We

distribute them uniformly on the  $q$  axis, apply the Monte Carlo integration scheme to each of them and finally add up the contribution of each wave packet to get the whole momentum distribution. In the purely classical simulation (bottom) we simply take a cloud of point particles uniformly distributed in the  $q$  direction and Gaussian distributed in  $p$  direction. Then the individual motion of the atoms is treated classically by letting the atoms evolve following the classical Liouville dynamics, but we still have included stimulated and spontaneous transitions in a Monte Carlo integration scheme.

Note that the quantum peaks are shifted towards smaller momenta. This shift becomes larger with the scaled Planck's constant  $\hbar$  which is a further indication that this effect can be explained quantum mechanically as described above. To show that the occurrence of the effective potential may in principle be sufficient to explain this feature, we simulate this by applying the classical simulation again (middle), where we now change the trajectory according to the effective potential once we start on a resonance. This is a very simple approach which is certainly only useful to show qualitatively that our explanation is suitable to describe the quantum dynamics. But we note that this modified classical simulation indeed shows the essential features of the pure quantum simulations.

In Fig. 4 (right) we show the same simulations but without any modulation. Here the differences corresponding to the quantum-mechanical effect vanish and now more or less all three simulations show the same structure. This structure is due to classical transient effects, which appear in the first

few cycles and are closely related to the motion in the standing wave, since they are independent of the modulation. This transient is always there and interferes with the quantum-mechanical effect investigated in this paper. However, the quantum-mechanical effect is easy to identify since it vanishes for  $\epsilon=0$ . Therefore this effect is clearly related to the modulation and shows a quantum feature of the classical mixed phase space. Note that the fact that the effect is vanishing for  $\epsilon=0$  is consistent with our theory since in this case we face classical integrable motion. A wave packet in such a system is not stabilized but spreads and changes its shape and therefore the assumption for applying the theory of Henriksen *et al.* [4] is no longer valid.

To conclude, we have shown that we can use the property that wave packets stay localized on resonances of a classical mixed phase space to simplify the complicated quantum dynamics in phase space. In this case we can describe the quantum dynamics of the wave packet by the classical motion in a modified potential. This is not only valid for the cosine potential investigated, but also, as already mentioned in [4], for polynomial potentials of arbitrary high order and for other systems that have been topic of investigations of the relationship of classical chaotic motion and the corresponding quantum dynamics. For example, there is the atomic bouncer in an evanescent field [16],  $V(q,t)=\lambda q + \kappa(1 + \epsilon \cos t)\exp(-q)$ . This setup of evanescent light waves can be modified to get a Morse potential [17] which serves as an atomic trap. In these cases it is also very straightforward to find the modified potential and to come to similar conclusions as we did.

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