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Casimir invariants and characteristic identities for $gl(\infty)$

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A full set of (higher-order) Casimir invariants for the Lie algebra $gl(\infty)$ is constructed and shown to be well defined in the category O_{FS} generated by the highest weight (unitarizable) irreducible representations with only a finite number of nonzero weight components. Moreover, the eigenvalues of these Casimir invariants are determined explicitly in terms of the highest weight. Characteristic identities satisfied by certain (infinite) matrices with entries from $gl(\infty)$ are also determined and generalize those previously obtained for $gl(n)$ by Bracken and Green [A. J. Bracken and H. S. Green, J. Math. Phys. **12**, 2099 (1971); H. S. Green, *ibid*. **12**, 2106 (1971)]. © *1997 American Institute of Physics.* [S0022-2488(97)02508-5]

I. INTRODUCTION

In recent years infinite-dimensional Lie algebras have become a subject of interest in both mathematics and physics (see Refs. 1 and 2 and the references therein). We mention as an example, related to the topic of the present article, that the Lie algebra $gl(\infty)$ and its completion and central extension a_∞ play an important role in the theory of soliton equations,[3,4] string theory, two-dimensional statistical models, etc.[5] In addition, these algebras provide an example of Kac–Moody Lie algebras of an infinite type.[1,6]

In this paper, we derive a full set of Casimir invariants for the infinite-dimensional general linear Lie algebra $gl(\infty)$, corresponding to the following matrix realization (see the notation at the end of the Introduction):

$$gl(\infty)=\{x=(a_{ij})|i,j\in\mathbf{N},\ \textit{all but a finite number of } a_{ij}\in\mathbf{C} \textit{ are zero}\}. \tag{1}$$

Characteristic identities satisfied by certain infinite matrices with entries from $gl(\infty)$ are also determined and generalize those obtained by Bracken and Green[7,8] for $gl(n)$. Such identities are of interest and have found applications to state labeling problems[9] and to the determination of Racah–Wigner coefficients.[10]

A basis for the Lie algebra $gl(\infty)$ is given by the Weyl generators e_{ij}, $i,j\in\mathbf{N}$, satisfying the commutation relations:

$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj}. \tag{2}$$

The category O generated by highest weight irreducible $gl(\infty)$ modules, corresponding to the ''Borel'' subalgebra,

$$N_+=\text{lin. env.}\{e_{ij}|i<j\in\mathbf{N}\}, \tag{3}$$

has been constructed in Ref. 11. By definition, each $gl(\infty)$ module $V\in O$ contains a unique (up to a multiplicative constant) vector v_Λ, the highest weight vector, with the properties

$$N_+v_\Lambda=0,\quad e_{ii}v_\Lambda=\Lambda_i v_\Lambda,\quad \forall i\in\mathbf{N}. \tag{4}$$

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The highest weight $\Lambda\equiv(\Lambda_1,\Lambda_2,\Lambda_3,\ldots)$ of $V\in O$ uniquely labels the module, $V\equiv V(\Lambda)$. Moreover, all unitarizable irreducible highest weight $gl(\infty)$ modules $V(\Lambda)$, corresponding to the natural conjugation operation: $(e_{ij})^{\dagger}=e_{ji}$, $\forall i,j\in\mathbf{N}$, have been determined.[11] The module $V(\Lambda)\in O$ carries a unitarizable representation of $gl(\infty)$ if and only if

$$\Lambda_i-\Lambda_j\in\mathbf{Z}_+,\quad \forall i<j\in\mathbf{N},\quad \Lambda_i\in\mathbf{R},\quad \forall i\in\mathbf{N}. \tag{5}$$

In the paper we will consider the category $O_{FS}\subset O$, of modules generated by all unitarizable irreducible $gl(\infty)$ modules with a finite number of nonzero highest weight components Λ_i. These are modules $V(\Lambda)$ with highest weights,

$$\Lambda\equiv(\Lambda_1,\Lambda_2,\ldots,\Lambda_k,0,\ldots)\equiv(\Lambda_1,\Lambda_2,\ldots,\Lambda_k,\dot{0}). \tag{6}$$

The paper is organized as follows. In Sec. II we give some useful results on the representations of $gl(\infty)$ with a finite number of nonzero components of the highest weight. In Sec. III we construct a full set of convergent Casimir invariants on each module $V(\Lambda)$. Section IV is devoted to the computation of the eigenvalues of these Casimir invariants for all modules from the subcategory O_{FS}. In Sec. V we present a derivation of the polynomial identities satisfied by certain matrices with entries from $gl(\infty)$, which generalize those obtained previously for $gl(n)$.

Throughout the paper we use the following notation:

irrep(s)—irreducible representation(s);
lin. env. $\{X\}$-the linear envelope of X;
C—the complex numbers;
R—the real numbers;
$\mathbf{Z}_+$—all non-negative integers;
N—all positive integers;
$U(A)$—the universal enveloping algebra of A.

II. PRELIMINARIES

Denote by H the Cartan subalgebra of $gl(\infty)$. The space H^* dual to H is described by the forms ε_i, $i\in\mathbf{N}$, where $\varepsilon_i:x\rightarrow a_{ii}$, and x is given by (1) only for diagonal x. Let (,) be the bilinear form on H^* defined by $(\epsilon_i,\epsilon_j)=\delta_{ij}$. For a weight $\mu=\Sigma_{i=1}^{\infty}\mu_i\varepsilon_i\in H^*$ with μ_i being complex numbers we write $\mu\equiv(\mu_1,\mu_2,\ldots,\mu_n,\ldots)$. The roots $\varepsilon_i\rightarrow\varepsilon_j$ $(i\neq j)$ of $gl(\infty)$ are the nonzero weights of the adjoint representation. The positive roots are given by the set

$$\Phi^+=\{\varepsilon_i-\varepsilon_j|1\leqslant i<j\in\mathbf{N}\}. \tag{7}$$

Define

$$\rho=\frac{1}{2}\sum_{i=1}^{\infty}(1-2i)\epsilon_i. \tag{8}$$

Let D_n be the set of $gl(\infty)$ weights:

$$D_n=\{\nu|\nu=(\nu_1,\ldots,\nu_n,\dot{0}),\quad \nu_i\in\mathbf{Z}_+,\quad i=1,2,\ldots,n-1,\ \ \nu_n\in\mathbf{N}\}, \tag{9}$$

and let $D_n^+\subset D_n$ be the subset of dominant weights in D_n:

$$D_n^+=\{\nu|\nu\in D_n,(\nu,\varepsilon_i-\varepsilon_{i+1})\in\mathbf{Z}_+,\quad \forall i\in\mathbf{N}\}. \tag{10}$$

Denote

$$D_{FS}^{+}\equiv\cup_{n=1}^{\infty}D_n^{+}, \quad D_{FS}\equiv\cup_{n=1}^{\infty}D_n. \tag{11}$$

Note the following.

(1) The irreducible $gl(\infty)$ modules $V(\Lambda)$ with highest weights $\Lambda\in D_k^{+}\subset D_{FS}^{+}$, corresponding to the natural conjugation operation, generate the subcategory $O_{FS}\subset O$ of unitarizable $gl(\infty)$ modules (6);

(2) Each module $V(\Lambda)$ gives rise to a unitarizable module for the canonical subalgebra $gl(n)\subset gl(\infty)$ with generators e_{ij}, $i,j=1,\ldots,n$. In general, $V(\Lambda)$ is a reducible $gl(n)$ module; more precisely, it is a completely reducible $gl(n)$ module;

(3) If ν is a weight in $V(\Lambda)$, then $\nu\in D_n$, for some $n\in\mathbf{Z}_+$.

Let Λ_n be the projection of the $gl(\infty)$ highest weight $\Lambda\in D_k^{+}$ onto the weight space of $gl(n)$ so that, for $n>k$,

$$\Lambda_n=(\Lambda_1,\ldots,\Lambda_k,0,\ldots,0_n)=(\Lambda_1,\ldots,\Lambda_k,\dot{0}_{n-k}). \tag{12}$$

Theorem 1: *(i) The $gl(n)$ module $V_n(\Lambda)\subset V(\Lambda)$, $\Lambda\in D_k^{+}$, cyclically generated by the highest weight vector $v_\Lambda^{+}\in V(\Lambda)$, is irreducible with highest weight Λ_n.*

(ii) If $v\in V(\Lambda)$ is a weight vector of weight $\nu\in D_n$, then $v\in V_n(\Lambda)$.

Proof: (i) The cyclic $gl(n)$ module $V_n(\Lambda)$ generated by v_Λ^{+} is well known to be indecomposable (see, for instance, Ref. 12). The result then follows from the complete reducibility of $V(\Lambda)$ considered as a $gl(n)$ module.

(ii) Let $v\in V(\Lambda)$ have weight $\nu\in D_n$. From the Poincaré–Birkhoff–Witt theorem we may write

$$v=pv_\Lambda^{+}, \quad p\in U(N_-), \tag{13}$$

with N_- the subalgebra of $gl(\infty)$ generated by all negative root vectors,

$$N_-=\text{lin. env.}\{e_{ij}|\ \ i>j\in\mathbf{N}\}. \tag{14}$$

The weight $\nu\in H^*$ has the form

$$\nu=\Lambda-\sum_{i=1}^{\infty} m_i(\varepsilon_i-\varepsilon_{i+1}), \tag{15}$$

and $m_i=0$ for all but a finite number of i. Since $\nu\in D_n$, $m_i=0$ for $i>n$, so that

$$\nu=\Lambda-\sum_{i=1}^{n} m_i(\varepsilon_i-\varepsilon_{i+1}). \tag{16}$$

In view of the linear independence of the simple roots $\varepsilon_i-\varepsilon_{i+1}$, (16) implies that

$$p\in U(N_-)\cap U[gl(n)]. \tag{17}$$

Therefore v is a vector from the $gl(n)$ module $V_n(\Lambda)$, $v\in V_n(\Lambda)$. □

Consider the $gl(\infty)$ modules $V(\Lambda)$ and $V(\mu)$, with highest weights $\Lambda\in D_k^{+}$ and $\mu\in D_l^{+}$, respectively. Take the tensor product of them,

$$V(\Lambda)\otimes V(\mu), \tag{18}$$

and suppose that v_ν^{+} is a $gl(\infty)$ highest weight vector in (18). Then for some n, $\nu\in D_n^{+}$ so that v_ν^{+} is a linear combination of vectors of the form

$$v \otimes w, \tag{19}$$

where v and w have weights in D_n. *Theorem 1* then implies that $v \in V_n(\Lambda)$, $w \in V_n(\mu)$. Therefore

$$v_\nu^+ \in V_n(\Lambda) \otimes V_n(\mu). \tag{20}$$

Since Λ has k and μ has l nonzero components, then ν can have at most $k+l$ nonzero components, so that $n \leqslant k+l$. Hence w.l.o.g. we may take $n=k+l$. Thus, if v_ν^+ is a $gl(\infty)$ highest weight vector in (18) then

$$v_\nu^+ \in V_n(\Lambda) \otimes V_n(\mu), \quad n=k+l, \tag{21}$$

is a $gl(n)$ highest weight vector. Conversely, given a $gl(n)$ highest weight vector,

$$v_\nu^+ \in V_n(\Lambda) \otimes V_n(\mu), \quad n=k+l,$$

we have

$$e_{ij} v_\nu^+ = 0, \quad \forall i<j=1,\ldots,n,$$

while

$$e_{ij} v_\nu^+ = 0, \quad \forall j>n,$$

since all weights in $V(\Lambda)$ and $V(\mu)$ have entries in $\mathbf{Z}_+$. Therefore v_ν^+ must be a $gl(\infty)$ highest weight vector. $V_n(\Lambda)$ and $V_n(\mu)$ are $gl(n)$ irreducible modules with highest weights Λ_n and μ_n, respectively. For their tensor product decomposition we write

$$V_n(\Lambda) \otimes V_n(\mu) \equiv V(\Lambda_n) \otimes V(\mu_n) = \oplus_\nu m_\nu V(\nu_n) \equiv \oplus_\nu m_\nu V_n(\nu), \tag{22}$$

where $\nu \equiv (\nu_n, \dot{0})$.

Hence we have proved the following.

Theorem 2: *The irreducible* $gl(n)$ *module decomposition,*

$$V_n(\Lambda) \otimes V_n(\mu) = \oplus_\nu m_\nu V_n(\nu), \tag{23}$$

implies the $gl(\infty)$ *irreducible module decomposition*

$$V(\Lambda) \otimes V(\mu) = \oplus_\nu m_\nu V(\nu), \tag{24}$$

where $\Lambda \in D_k^+$, $\mu \in D_l^+$, $n=k+l$.

III. CONSTRUCTION OF CASIMIR INVARIANTS

An obvious invariant for $gl(\infty)$ is the first-order invariant,

$$I_1 = \sum_{i=1}^{\infty} e_{ii}. \tag{25}$$

However, it is not clear how to construct appropriate higher-order invariants for $gl(\infty)$. Let us therefore consider the second-order invariant $I_2^{(n)}$ of $gl(n)$:

$$I_2^{(n)}=\sum_{i,j=1}^{n} e_{ij}e_{ji}=\sum_{i=1}^{n}\sum_{j<i=1}^{n} e_{ij}e_{ji}+\sum_{i=1}^{n}\sum_{j>i=1}^{n} e_{ij}e_{ji}+\sum_{i=1}^{n} e_{ii}^2$$
$$=2\sum_{i=1}^{n}\sum_{j<i=1}^{n} e_{ij}e_{ji}+\sum_{i=1}^{n}\sum_{j>i=1}^{n} (e_{ii}-e_{jj})+\sum_{i=1}^{n} e_{ii}^2$$
$$=2\sum_{i=1}^{n}\sum_{j<i=1}^{n} e_{ij}e_{ji}+\sum_{i=1}^{n} (n+1-2i)e_{ii}+\sum_{i=1}^{n} e_{ii}^2$$
$$=2\sum_{i=1}^{n}\sum_{j<i=1}^{n} e_{ij}e_{ji}+\sum_{i=1}^{n} e_{ii}(e_{ii}+1-2i)+nI_1^{(n)}, \tag{26}$$

where $I_1^{(n)}\equiv\Sigma_{i=1}^{n}e_{ii}$ is the first-order invariant of $gl(n)$. Due to the last term in (26) the $gl(n)$ second-order invariant diverges as $n\to\infty$. Eliminating the last term in (26) (the rest of the expression is also an invariant) and taking the limit $n\to\infty$, one obtains the following quadratic Casimir for $gl(\infty)$:

$$I_2=2\sum_{i=1}^{\infty}\sum_{j<i}^{\infty} e_{ij}e_{ji}+\sum_{i=1}^{\infty} e_{ii}(e_{ii}+1-2i), \tag{27}$$

which is convergent [see formula (36)] on the category O_{FS} of irreps considered. On $V(\Lambda)$, $\Lambda \in D_k^+$, I_2 takes the constant value

$$\chi_\Lambda(I_2)=\sum_{i=1}^{k} \Lambda_i(\Lambda_i+1-2i)=(\Lambda,\Lambda+2\rho). \tag{28}$$

This construction suggests how to proceed to the higher-order invariants of $gl(\infty)$.

To begin with we introduce the characteristic matrix,

$$A_i^j=e_{ji}. \tag{29}$$

This matrix, in fact, arises naturally in the context of characteristic identities, to be discussed in Sec. V. Powers of the matrix A are defined recursively by

$$(A^m)_i^j=\sum_{k=1}^{\infty} A_i^k(A^{m-1})_k^j, \quad [(A^0)_i^j\equiv\delta_{ij}]. \tag{30}$$

Using induction and the $gl(\infty)$ commutation relations (2) one obtains the following.

Proposition 1:

$$[e_{kl},(A^m)_i^j]=\delta_{jl}(A^m)_i^k-\delta_{ik}(A^m)_l^j. \qquad \square \tag{31}$$

Therefore the matrix traces,

$$\mathrm{tr}(A^m)\equiv\sum_{i=1}^{\infty} (A^m)_i^i, \tag{32}$$

are formally Casimir invariants. They are, however, divergent except for $m=1$, in which case we obtain the first-order invariant (25). The purpose of the present investigation is to construct a full set of Casimir invariants that are well defined and convergent on the category O_{FS}.

The following is the main result of the paper.

Theorem 3: *The Casimir invariants defined recursively by*

$$I_1=\sum_{i=1}^{\infty} A_i^i=\text{tr}(A);$$

$$I_m=\sum_{i=1}^{\infty} [(A^m)_i^i-I_{m-1}]=\text{tr}[A^m-I_{m-1}], \tag{33}$$

form a full set of convergent Casimir invariants on each module, $V(\Lambda)\in O_{FS}$. □

Observe first that the I_m so defined (33) are indeed Casimir invariants (see *Proposition 1*). It remains to prove that they are convergent on the category O_{FS}. We will do this by induction. It is constructive to consider first the case $m=2$:

$$I_2\equiv\sum_{j=1}^{\infty} [(A^2)_j^j-I_1]=\sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty} e_{ij}e_{ji}-I_1\right]=\sum_{j=1}^{\infty}\left[\sum_{i>j}^{\infty} e_{ij}e_{ji}+\sum_{i<j}^{\infty} e_{ij}e_{ji}+e_{jj}^2-I_1\right]$$

$$=\sum_{j=1}^{\infty}\left[2\sum_{i>j}^{\infty} e_{ij}e_{ji}+\sum_{i<j}^{\infty}(e_{ii}-e_{jj})+e_{jj}^2-I_1\right]=\sum_{j=1}^{\infty}\left[2\sum_{i>j}^{\infty} e_{ij}e_{ji}+e_{jj}(e_{jj}-j+1)+\sum_{i<j}^{\infty} e_{ii}-I_1\right]$$

$$=\sum_{j=1}^{\infty}\left[2\sum_{i>j}^{\infty} e_{ij}e_{ji}+e_{jj}(e_{jj}-j)-\sum_{i>j}^{\infty} e_{ii}\right]=2\sum_{j=1}^{\infty}\sum_{i>j}^{\infty} e_{ij}e_{ji}+\sum_{j=1}^{\infty} e_{jj}(e_{jj}-2j+1), \tag{34}$$

which agrees with the definition (27).

Now let $v\in V(\Lambda)$, $\Lambda\in D_k^+$, be an arbitrary weight vector. Then the weight of v has the form.

$$\nu=(\nu_1,\nu_2,\ldots,\nu_r,\dot{0}), \tag{35}$$

so that $\Sigma_{i=1}^r\nu_i=\Sigma_{i=1}^k\Lambda_i=\chi_\Lambda(I_1)$. Note that

$$A_i^j v=e_{ji}v=0,\quad \forall i>r, \tag{36}$$

and that the second-order invariant I_2 is convergent on each $V(\Lambda)\in O_{FS}$ [cf. formula (27)].

Applying *Proposition 1* and (36) for $i>r$, one obtains

$$(A^m)_i^i v=\sum_{j=1}^{\infty} A_i^j(A^{m-1})_j^i v=\sum_{j=1}^{\infty} e_{ji}(A^{m-1})_j^i v=\sum_{j=1}^{\infty}\{[(A^{m-1})_j^j-(A^{m-1})_i^i]v+(A^{m-1})_j^i e_{ji}v\}$$

$$=\sum_{j=1}^{\infty}[(A^{m-1})_j^j-(A^{m-1})_i^i]v. \tag{37}$$

In particular, for the case $m=2$ we have

$$(A^2)_i^i v=\sum_{j=1}^{\infty}[A_j^j-A_i^i]v=\sum_{j=1}^{\infty} e_{jj}v=I_1v,\quad \forall i>r, \tag{38}$$

so that

$$((A^2)_i^i-I_1)v=0,\quad \forall i>r, \tag{39}$$

which is another proof for the convergence of I_2. More generally, we have the following.

Proposition 2: For any weight vector $v\in V(\Lambda)$, *and* $m\in\mathbf{N}$ *there exist* $r\in\mathbf{N}$ *such that*

$$((A^m)^i_i - I_{m-1})v = 0, \quad \forall i > r. \tag{40}$$

Proof: We proceed by induction and assume v has weight ν as in (35). Formula (40) is valid for $m=2$ (39). Assuming the result is true for a given m, i.e.

$$(A^m)^i_i v = I_{m-1} v, \quad \forall i > r,$$

we have [see (37)]

$$(A^{m+1})^i_i v = \sum_{j=1}^{\infty} [(A^m)^j_j - (A^m)^i_i] v = \sum_{j=1}^{\infty} [(A^m)^j_j - I_{m-1}] v = I_m v, \quad \forall i > r, \tag{41}$$

which proves (40). □

I_m (33) is convergent on each $V(\Lambda)$ for $m=2$. Assume it is convergent and well defined on $V(\Lambda)$ for a given m. Then, with v as in (40), we have

$$I_{m+1} v \equiv \sum_{i=1}^{\infty} [(A^{m+1})^i_i - I_m] v = \sum_{i=1}^{r} [(A^{m+1})^i_i - I_m] v = \sum_{i=1}^{r} (A^{m+1})^i_i v - r I_m v, \tag{42}$$

so that I_{m+1} is convergent and well defined on $V(\Lambda)$.

This completes the (inductive) proof of *Theorem 3*.

In the next section we will obtain an explicit eigenvalue formula for these invariants.

IV. EIGENVALUE FORMULA FOR CASIMIR INVARIANTS

In this section we apply our previous results to evaluate the spectrum of the invariants (33).

Let $v \in V(\Lambda)$, be an arbitrary vector of weight $\nu = (\nu_1, \ldots, \nu_r, \dot{0})$. Then, keeping in mind *Proposition 1*, the fact that $(A^{m-1})^j_k$ has weight $\varepsilon_j - \varepsilon_k$ under the adjoint representation of $gl(\infty)$ and that all vectors of $V(\Lambda)$ have weight components in $\mathbf{Z}_+$, we must have for $j \leqslant r$,

$$(A^{m-1})^j_k v = 0, \quad \forall k > r. \tag{43}$$

Therefore

$$(A^m)^j_i v = \sum_{k=1}^{\infty} A^k_i (A^{m-1})^j_k v = \sum_{k=1}^{r} A^k_i (A^{m-1})^j_k v. \tag{44}$$

Proceeding recursively, we may therefore write

$$(A^m)^j_i v = (\overline{A^m})^j_i v, \quad \forall i,j = 1,\ldots,r, \tag{45}$$

where $(\overline{A})^j_i = e_{ji}$, $\forall i,j = 1,\ldots,r$, is the $gl(r)$ characteristic matrix, and the powers of the matrix $\overline{A}$ are defined by (30) with $i,j,k = 1,\ldots,r$ and $\overline{A}$ instead of A. It follows then that the formula (42) can be written as

$$I_m v = \sum_{i=1}^{r} [(\overline{A^m})^i_i - I_{m-1}] v = [I_m^{(r)} - r I_{m-1}] v, \tag{46}$$

with

$$I_m^{(r)} = \sum_{i=1}^{r} (\overline{A^m})^i_i, \tag{47}$$

being the mth-order invariant of $gl(r)$. Formula (46) is valid $\forall m\in\mathbf{N}$, which gives a recursion relation for the I_m with the initial condition

$$I_1 v=\chi_\Lambda(I_1)v. \tag{48}$$

In particular, it follows from (46) that the invariants I_m are certainly convergent on all weight vectors $v\in V(\Lambda)$.

To determine the eigenvalues of I_m let $v=v_\Lambda^+$ be the highest weight vector of the unitarizable module $V(\Lambda)$ and let

$$\Lambda=(\overline{\Lambda},\dot{0})\in D_k^+,\quad \overline{\Lambda}\equiv(\Lambda_1,\ldots,\Lambda_k). \tag{49}$$

Then for the eigenvalues of the I_m one obtains the recursion relation [see (46)]

$$\chi_\Lambda(I_m)=\chi_{\overline{\Lambda}}(I_m^{(k)})-k_{\chi_\Lambda}(I_{m-1}),\quad \chi_\Lambda(I_1)=\sum_{i=1}^{k}\Lambda_i, \tag{50}$$

where $\chi_{\overline{\Lambda}}(I_m^{(k)})$ is the eigenvalue of the mth-order invariant (47) of $gl(k)$ on the irreducible $gl(k)$ module with highest weight $\overline{\Lambda}$; the latter is given explicitly by[13]

$$\chi_{\overline{\Lambda}}(I_m^{(k)})=\sum_{i=1}^{k}\alpha_i^m\prod_{j\neq i=1}^{k}\left(\frac{\alpha_i-\alpha_j+1}{\alpha_i-\alpha_j}\right), \tag{51}$$

where

$$\alpha_i=\Lambda_i+1-i.$$

We thereby obtain for the eigenvalues of the Casimir invariants I_m,

$$\chi_\Lambda(I_m)=\sum_{i=1}^{k}P_m(\alpha_i)\prod_{j\neq i=1}^{k}\left(\frac{\alpha_i-\alpha_j+1}{\alpha_i-\alpha_j}\right), \tag{52}$$

for suitable polynomials $P_m(x)$, which, from Eq. (50), satisfy the recursion relation

$$P_m(x)=x^m-kP_{m-1}(x),\quad P_1(x)=x. \tag{53}$$

In particular,

$$P_2(x)=x^2-kx=x\,\frac{x^2-k^2}{x+k}; \tag{54a}$$

$$P_3(x)=x^3-k(x^2-kx)=x\,\frac{x^3+k^3}{x+k}, \tag{54b}$$

and more generally, it is easily established by induction that

$$P_m(x)=x\,\frac{x^m-(-1)^m k^m}{x+k}. \tag{55}$$

Thus we have the following.

Theorem 4: *The eigenvalues of the Casimir invariants* I_m (33), *on the irreducible unitarizable* $gl(\infty)$ *module* $V(\Lambda)$, $\Lambda\in D_k^+$ *are given by*

$$\chi_\Lambda(I_m)=\sum_{i=1}^{k}\alpha_i\left(\frac{\alpha_i^m+(-1)^{m+1}k^m}{\alpha_i+k}\right)\prod_{j\neq i}^{k}\left(\frac{\alpha_i-\alpha_j+1}{\alpha_i-\alpha_j}\right),\quad \text{where}\quad \alpha_i=\Lambda_i+1-i. \tag{56}$$

□

V. POLYNOMIAL IDENTITIES

Let Δ be the comultiplication on the enveloping algebra $U[gl(\infty)]$ of $gl(\infty)$ $[\Delta(e_{ij})=e_{ij}\otimes 1+1\otimes e_{ij}$, $i,j\in\mathbf{N}$, with 1 being the unit in $U[gl(\infty)]]$. Applying Δ to the second-order Casimir invariant (27) of $gl(\infty)$, we obtain

$$\Delta(I_2)=I_2\otimes 1+1\otimes I_2+2\sum_{i,j=1}^{\infty}e_{ij}\otimes e_{ji}. \tag{57}$$

Therefore

$$\sum_{i,j=1}^{\infty}e_{ij}\otimes e_{ji}=\frac{1}{2}[\Delta(I_2)-I_2\otimes 1-1\otimes I_2]. \tag{58}$$

Denote by π_{ε_1} the irrep of $gl(\infty)$ afforded by $V(\varepsilon_1)$. The weight spectrum for the vector module $V(\varepsilon_1)$ consists of all weights ε_i, $i=1,2,\ldots$, each occurring exactly once. Denote by E_{ij}, $i,j\in\mathbf{N}$ the generators on this space,

$$\pi_{\varepsilon_1}(e_{ij})=E_{ij}, \tag{59}$$

with E_{ij} an elementary matrix.

As for the algebra $gl(n)$, we introduce the characteristic matrix

$$A=\sum_{i,j=1}^{\infty}\pi_{\varepsilon_1}(e_{ij})e_{ji}=\sum_{i,j=1}^{\infty}E_{ij}e_{ji}=\frac{1}{2}(\pi_{\varepsilon_1}\otimes 1)[\Delta(I_2)-I_2\otimes 1-1\otimes I_2]. \tag{60}$$

Therefore A is the infinite matrix introduced in Sec. III [see (29)] and the entries of the matrix powers A^m are given recursively by (30). We will show that the characteristic matrix satisfies a polynomial identity acting on the $gl(\infty)$ module $V(\Lambda)$, $\Lambda\in D_k^+$. Let π_Λ be the representation afforded by $V(\Lambda)$. From Eq. (60) acting on $V(\Lambda)$ we may interpret A as an invariant operator on the tensor product module $V(\varepsilon_1)\otimes V(\Lambda)$:

$$A\equiv\tfrac{1}{2}(\pi_{\varepsilon_1}\otimes\pi_\Lambda)[\Delta(I_2)-I_2\otimes 1-1\otimes I_2]. \tag{61}$$

From *Theorem 2*, we have, for the tensor product decomposition,

$$V(\varepsilon_1)\otimes V(\Lambda)=\oplus_{i=1}^{k+1}{}'V(\Lambda+\varepsilon_i), \tag{62}$$

where the prime signifies that it is necessary to retain only those summands for which $\Lambda+\varepsilon_i\in D_{FS}^+$. Therefore on each $gl(\infty)$ module $V(\Lambda+\varepsilon_i)$ in (62), A takes the eigenvalue

$$\begin{aligned}\tfrac{1}{2}[\chi_{\Lambda+\varepsilon_i}(I_2)-\chi_{\varepsilon_1}(I_2)-\chi_\Lambda(I_2)]&=\tfrac{1}{2}[(\Lambda+\varepsilon_i,\Lambda+\varepsilon_i+2\rho)-(\varepsilon_1,\varepsilon_1+2\rho)-(\Lambda,\Lambda+2\rho)]\\&=\Lambda_i+1-i\end{aligned} \tag{63}$$

(see *Theorem 4*). Thus we have the following theorem.

Theorem 5: *On each* $gl(\infty)$ *module* $V(\Lambda)$, $\Lambda\in D_k^+$ *the characteristic matrix satisfies the polynomial identity*

$$\prod_{i=1}^{k+1}(A-\alpha_i)=0, \tag{64}$$

with $\alpha_i=\Lambda_i+1-i$ *the characteristic roots.* □

The characteristic identities (64) are the $gl(\infty)$ counterpart of the polynomial identities encountered for $gl(n)$ by Bracken and Green[7,8] (more precisely their adjoint identities). It is worth noting, in view of the decomposition (62), that these identities may frequently be reduced. Some reduced identities are indicated below for certain choices $\Lambda\in D_{FS}^+$ of the $gl(\infty)$ highest weight:

$$\Lambda=(\dot{1}_k,\dot{0}):\ (A-1)(A+k)=0; \tag{65a}$$

$$\Lambda=(k,\dot{0}):\ (A+1)(A-k)=0; \tag{65b}$$

$$\Lambda=(\dot{p}_k,\dot{q}_l,\dot{0}):\ (A-p)(A+k-q)(A+k+l)=0,\quad p<q. \tag{65c}$$

Note: Sometimes the characteristic and reduced identities are the same; for instance, in (65b) the reduced identity coincides with the characteristic identity. This is in stark contrast to the characteristic identities for $gl(n)$.

More generally, having in mind (58), introduce a characteristic matrix,

$$A_\Lambda=\sum_{i,j=1}^{\infty}\pi_\Lambda(e_{ij})e_{ji}=\frac{1}{2}(\pi_\Lambda\otimes 1)[\Delta(I_2)-I_2\otimes 1-1\otimes I_2], \tag{66}$$

corresponding to any irrep π_Λ of $gl(\infty)$ afforded by $V(\Lambda)$, $\Lambda\in D_k^+$. In a suitably chosen basis for $V(\Lambda)$, A_Λ is an infinite matrix with entries

$$(A_\Lambda)_\alpha^\beta=\sum_{i,j=1}^{\infty}\pi_\Lambda(e_{ij})_{\alpha\beta}e_{ji}. \tag{67}$$

Acting on an irreducible $gl(\infty)$ module $V(\mu)$, $\mu\in D_l^+$, A_Λ may be regarded as an invariant operator on the tensor product module $V(\Lambda)\otimes V(\mu)$:

$$A_\Lambda\equiv\tfrac{1}{2}(\pi_\Lambda\otimes\pi_\mu)[\Delta(I_2)-I_2\otimes 1-1\otimes I_2]. \tag{68}$$

Now applying *Theorem 2*, the decomposition of the tensor product space $V(\Lambda)\otimes V(\mu)$ is given by the $gl(k+l)$ branching rule,

$$V_n(\Lambda)\otimes V_n(\mu)=\oplus_\nu m_\nu V_n(\nu), \tag{69}$$

with $n=k+l$. Let $\{\lambda_n^i\}_{i=1}^d$ be the set of distinct weights in the $gl(n)$ module $V_n(\Lambda)$. Then the allowed highest weights ν_n occurring in the decomposition (69) are of the form $\nu_n=\mu_n+\lambda_n^i$, for some i. It follows that on $V(\nu)$, $\nu=(\nu_n,\dot{0})$, the matrix A_Λ takes the constant values

$$\alpha_{\Lambda,i}=\tfrac{1}{2}[\chi_{\mu+\lambda_i}(I_2)-\chi_\Lambda(I_2)-\chi_\mu(I_2)]=\tfrac{1}{2}[(\lambda_i,\lambda_i+2(\mu+\rho))-(\Lambda,\Lambda+2\rho)],\quad \lambda_i=(\lambda_n^i,\dot{0}), \tag{70}$$

which are the characteristic roots of the matrix A_Λ. Thus we have the following.

Theorem 6: *On the irreducible* $gl(\infty)$ *module* $V(\mu)$, $\mu \in D_{FS}^{+}$, *the characteristic matrix* A_{Λ} *satisfies the polynomial identity*

$$\prod_{i=1}^{d} (A_{\Lambda} - \alpha_{\Lambda,i}) = 0. \tag{71}$$

These identities are obvious generalizations of those of *Theorem 5* [see (64)]. Note, in this case, that Eq. (69) implies the reduced identity satisfied by the matrix A_{Λ} on the $gl(\infty)$ module $V(\mu)$, given by

$$\prod_{\nu} (A_{\Lambda} - \alpha_{\nu}) = 0, \tag{72}$$

where now

$$\alpha_{\nu} = \tfrac{1}{2}[(\nu, \nu + 2\rho) - (\Lambda, \Lambda + 2\rho) - (\mu, \mu + 2\rho)]. \tag{73}$$

Casimir invariants for the infinite-dimensional general linear Lie algebra have been obtained explicitly, and their eigenvalues on any irreducible highest weight unitarizable representation with a finite number of nonzero weight components computed. With the help of the second-order Casimir invariant, we have obtained characteristic identities for the Lie algebra $gl(\infty)$, which are a generalization of those for $gl(n)$.

It is well known that the invariants of finite-dimensional Lie algebras play an important role in their representation theory. However, for the infinite-dimensional Lie algebras, corresponding full sets of Casimir invariants have not yet been determined. The present paper is a step in solving this problem.

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