PHYSICAL REVIEW LETTERS

VOLUME 78 | 9 JUNE 1997 | NUMBER 23

Ideal Soliton Environment Using Parametric Band Gaps

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(Received 9 October 1996)

Simultaneous solitary wave solutions for laser propagation in nonlinear parametric media with up to (3 + 1) dimensions are proved to exist. The combination of the large dispersion of a Bragg grating and the strong nonlinearity of $\chi^{(2)}$ optical material results in stable behavior with short interaction distances and low power requirements. The solutions are obtained by using the effective mass approximation to reduce the coupled propagation equations to those describing a dispersive parametric nonlinear waveguide, and are verified by solving the complete set of coupled band-gap equations numerically. [S0031-9007(97)03325-5]

PACS numbers: 03.40.Kf, 42.70.Qs

Simultaneous solitary-wave ("simultons") solutions in dispersive parametric waveguides, involving a wave copropagating with its second harmonic, were first found theoretically by Karamzin and Sukhorukov [1,2]. Recently, new solutions of different types were found by many researchers [3–11]. Parametric simultons have now been experimentally observed using continuous wave propagation [12], in $\chi^{(2)}$ media, but time-dependent simultons have yet to be generated experimentally.

This is due to a number of material requirements, especially that of group-velocity matching, and the requirement of having dispersions of identical sign in both the signal and its harmonic. In addition, nonlinear crystals have a relatively small dispersion, which usually results in long formation distances that are easily achieved only in optical fibers (which normally have a $\chi^{(3)}$ rather than a $\chi^{(2)}$ nonlinearity). Despite this, there are clear advantages to the parametric medium for soliton formation. The nonlinear phase shift is much larger at low intensities for parametric nonlinear materials, since it scales as E^2, not E^3. Furthermore, temporal $\chi^{(2)}$ solitary waves are known to exist in higher dimensions (provided the dispersion is anomalous), unlike the bright nonlinear Schrödinger equation solitons of a $\chi^{(3)}$ medium, which are always unstable in higher dimensions.

Bragg grating optical materials have a strong dispersion when the wavelength is near the band gap. This characteristic has been used to theoretically propose techniques of second-harmonic phase matching [13], which have been observed experimentally in periodic multilayer GaAs structures [14]. Extending this concept to doubly periodic Bragg gratings allows a simultaneous band gap to open up at both fundamental and second-harmonic frequencies. We show in this Letter that the novel double band-gap structure is an ideal candidate for the formation of $\chi^{(2)}$ simultons with short interaction distances, by transforming the coupled equations to an exactly soluble form.

Using a Bragg grating also helps solve other problems that occur with conventional parametric solitons. Group-velocity matching is no longer necessary with band gaps: solitons can form even at low or zero velocity in the laboratory frame. In addition, we will show that it is always possible to choose branches of the dispersion relation that give anomalous dispersion at both wavelengths, thus allowing higher-dimensional solitons to form. We have verified the stability of these in the (1 + 2) dimensional case, using a direct numerical simulation of the complete set of equations. By comparison, gap solitons of $\chi^{(3)}$ media are known to occur both theoretically and experimentally [15,16]. However, due to the small nonlinear coupling, formation of these single-wavelength solitons requires very large powers, and cannot occur in higher transverse dimensions.

Consider a Bragg grating structure with $n_j(z) = \bar{n}_j + \Delta n_j(z)$, where $\bar{n}_j$ is the spatial average of $n_j(z)$ (the refractive index at frequency $j\omega_1$), where $j = 1, 2$. Here

$\Delta n_j(z)$ is a periodic function with period d. We can expand $\Delta n_j(z)$ in a Fourier series, with

$$\Delta n_j(z) = \sum_m \Delta n_{jm} \cos(2mk_1 z), \qquad (1)$$

given that Δn_{jm} is a real coefficient, and $k_1 = \pi/d = 2\pi\bar{n}_1/\lambda_1$, where λ_1 is the free-space wavelength of the fundamental field. More general types of grating can be treated, but this is sufficient to illustrate our main point.

Assuming a quasimonochromatic electric field in a second-harmonic generation process, we can write solutions to Maxwell's equation as

$$\mathbf{E} = \mathbf{e}_1[\mathcal{E}_{1+}(z,t)e^{-i(\omega_1 t - k_1 z)} + \mathcal{E}_{1-}(z,t)e^{-i(\omega_1 t + k_1 z)}] + \mathbf{e}_2[\mathcal{E}_{2+}(z,t)e^{-2i(\omega_1 t - k_1 z)} + \mathcal{E}_{2-}(z,t)e^{-2i(\omega_1 t + k_1 z)}] + \text{c.c.},$$

where $\mathbf{e}_n$ are the polarizations, and the sign $\pm$ represents right or left propagation.

Substituting the above ansatz into Maxwell's equation and assuming slowly evolving envelopes, we obtain the following coupled first order equations:

$$\begin{aligned}
&i\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial z}\right)\mathcal{E}_{1+} + \kappa_1\mathcal{E}_{1-} + \chi_E\mathcal{E}_{1+}^*\mathcal{E}_{2+} = 0,\\
&i\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial z}\right)\mathcal{E}_{1-} + \kappa_1\mathcal{E}_{1+} + \chi_E\mathcal{E}_{1-}^*\mathcal{E}_{2-} = 0,\\
&i\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial z} - i\delta k_2\right)\mathcal{E}_{2+} + \kappa_2\mathcal{E}_{2-} + \chi_E\mathcal{E}_{1+}^2 = 0,\\
&i\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial z} - i\delta k_2\right)\mathcal{E}_{2-} + \kappa_2\mathcal{E}_{2+} + \chi_E\mathcal{E}_{1-}^2 = 0.
\end{aligned} \qquad (2)$$

Here $\chi_E = \chi^{(2)}k_1/\bar{n}_1^2$, $\kappa_j = \pi\Delta n_{jj}/\lambda_j$, $\tau = vt$, and $\delta k_2 = \omega_2(\bar{n}_2 - \bar{n}_1)/c$ represents the phase mismatch between the fundamental at frequency ω_1, and the second harmonic at frequency $\omega_2 = 2\omega_1$. We neglect dispersion of the medium, as it is usually much smaller than the gap dispersion. For simplicity, the two group velocities are taken equal to v, with no mismatch.

Following standard techniques [15], we introduce a vector for the right and left propagating fields:

$$\vec{\mathcal{E}}_j(z,t) = \begin{bmatrix} \mathcal{E}_{j+} \\ \mathcal{E}_{j-} \end{bmatrix}. \qquad (3)$$

Inserting the ansatz,

$$\vec{\mathcal{E}}_j(z,t) = \vec{f}_j(Q)e^{i(Qz - \Omega_j t)}, \qquad j = 1,2, \qquad (4)$$

into the linear part of the above equations, one obtains two sets of normalized eigenvectors, corresponding to linear propagation above and below the band gap:

$$\vec{f}_j^s(Q) = \frac{(\kappa_j, Q - s\sqrt{\kappa_j^2 + Q^2})^T}{\sqrt{2(\kappa_j^2 + Q^2 - sQ\sqrt{\kappa_j^2 + Q^2})}}, \qquad (5)$$

where the eigenvalues corresponding to $s = \pm 1$ are

$$\Omega_j^s(Q) = v(s\sqrt{Q_j^2 + \kappa_j^2} - \delta k_j), \qquad j = 1,2. \qquad (6)$$

Next, we introduce normalized mode functions:

$$u_{jQ}^s(z) = f_{j+}^s(Q)e^{jik_1 z} + f_{j-}^s(Q)e^{-jik_1 z}, \qquad j = 1,2. \qquad (7)$$

For example, when $\kappa_j > 0$, the mode functions stand for symmetric ($s = -1$, lower branch) and antisymmetric ($s = 1$, upper branch) standing waves at $Q = 0$. Near $Q = 0$, these modes satisfy the orthogonality conditions

$$(\bar{n}_j^2 L)^{-1}\int n_j^2(z)\,[u_{j0}^s(z)u_{j'0}^{s'}(z)]^* dz = \delta_{ss'}\delta_{jj'}, \qquad (8)$$

where L is the quantization length.

Neglecting dispersion, the Hamiltonian in a nonlinear medium [17] can be written as

$$\begin{aligned}
H = &\int \sum_j \left(\frac{1}{\mu}\mathcal{B}_j^*\mathcal{B}_j + \frac{1}{\epsilon_0 n^2(z)}\mathcal{D}_j^*\mathcal{D}_j\right)A\,dz\\
&+ \int \frac{\eta^{(2)}(z)}{\epsilon_0}(\mathcal{D}_1^{*2}\mathcal{D}_2 + \mathcal{D}_1^2\mathcal{D}_2^*)A\,dz,
\end{aligned} \qquad (9)$$

where $\eta^{(2)}(z) = -\chi^{(2)}/[\epsilon_0 n^6(t)]$ [18].

In terms of the symmetric or antisymmetric modes, the electric displacement field can be expanded as

$$\begin{aligned}
D = &\sum_{s=\pm1}\sum_Q n_1^2(z)\sqrt{\frac{\hbar\omega_1^s\epsilon_0}{2\bar{n}_1^2 LA}}\,\hat{a}_{1Q}^s u_{1Q}^s(z)e^{i(Qz - \omega_0 t)}\\
&+ \sum_{s=\pm1}\sum_Q n_2^2(z)\sqrt{\frac{\hbar\omega_2^s\epsilon_0}{2\bar{n}_2^2 LA}}\,\hat{a}_{2Q}^s u_{2Q}^s(z)e^{i(Qz - 2\omega_0 t)}\\
&+ \text{c.c.}
\end{aligned} \qquad (10)$$

Substituting Eq. (10) into the above Hamiltonian, we find that the linear part can be written in terms of photon annihilation and creation operators as

$$H_0 = \sum_s \sum_j \sum_Q \hbar\omega_{jQ}^s \hat{a}_{jQ}^{\dagger s}\hat{a}_{jQ}^s. \qquad (11)$$

We are interested in photon properties near the center of the band-gap region in momentum space, where $Q \ll \kappa_j$. Therefore, the well-known effective-mass approximation (EMA) in solid state physics gives

$$\hbar\omega_{jQ}^s = \hbar\omega_j^s + \frac{\hbar^2 Q^2}{2m_j^s}, \qquad j = 1,2, \qquad (12)$$

where the effective mass is $m_j^s = s\hbar|\kappa_j|/v$, and the mode frequency at the gap center is

$$\omega_j^s = j\omega_1 + \Omega_j^s(0) = j\omega_1 + (s|\kappa_j| - \delta k_j)v. \qquad (13)$$

It is convenient to work in the coordinate representation. Taking a Fourier transform of the annihilation operators

$\hat{a}_j^s$, we obtain the following envelope field operators for the photon field:

$$\psi_j^s(z) = L^{-1/2} \sum_Q \hat{a}_{jQ}^s e^{iQz}, \qquad j = 1,2. \tag{14}$$

The Hamiltonian can therefore be expressed in terms of these field operators, provided that $|Q_j| \ll |\kappa_j|$:

$$\frac{\hat{H}}{\hbar} = \int \sum_s \sum_j \left(\frac{\hbar}{2m_j^s} \partial_z \psi_j^{s\dagger} \partial_z^s \psi_j + \omega_j^s \psi_j^{s\dagger} \psi_j^s \right) dz - \sum_{\vec{s}} \int \frac{\chi(\vec{s})}{2} (\psi_2^{s_2\dagger} \psi_1^{s_1} \psi_1^{s_1'} + \psi_2^{s_2} \psi_1^{s_1\dagger} \psi_1^{s_1'\dagger})\, dz\,, \tag{15}$$

where the nonlinear coupling is

$$\chi(\vec{s}) = \frac{\chi^{(2)}}{2\bar{n}^3} \sqrt{\frac{\omega_1^3 \hbar}{2\epsilon_0 A}} [\mathrm{sgn}(\kappa_2) - s_1 s_1' s_2]. \tag{16}$$

The nonlinear part of the Hamiltonian, Eq. (15), vanishes if the total coupling between gaps is antisymmetric. We consider only cases with nonvanishing coupling, and $s_1 = s_1'$, so that $s_2 = -\mathrm{sgn}(\kappa_2)$. These possible couplings between gaps are illustrated in Fig. 1.

The Hamiltonian approach therefore affords a physically intuitive understanding of the coupling processes. Not only does the use of the gap modes eliminates linear cross couplings, but it also introduces a powerful symmetry principle; the second harmonic that is coupled must have the same type of symmetry as the product of the two subharmonic modes. Because of this, the use of gap modes permits great simplifications even in this nonlinear problem. Either quantum soliton or classical soliton behavior can result; the classical solutions would, of course, be approximately valid only at large photon number. However, this technique does permit us to obtain novel solutions of some experimental interest.

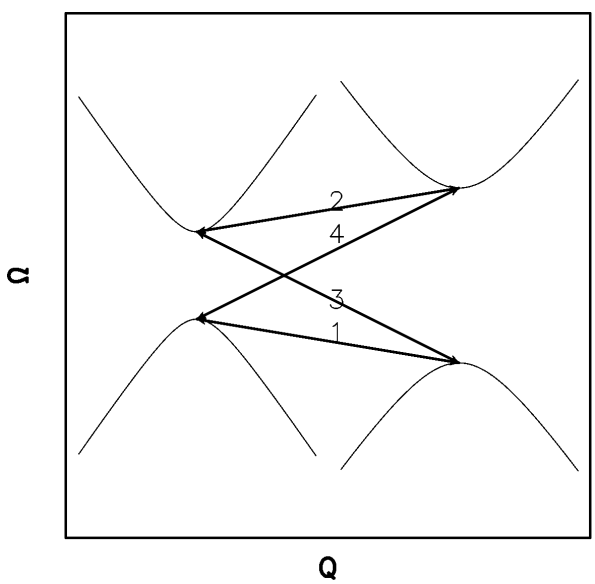

FIG. 1. Possible nonlinear couplings between gaps. Couplings 1 and 2 generate bright simultons while couplings 3 and 4 generate dark simultons.

We can now apply the known topological properties [19] of the parametric soliton equations to the band-gap case. From the above Hamiltonian, we derive classical equations for two coupled waves with symmetries s_1 and s_2:

$$\begin{aligned} \frac{\partial \psi_1^{s_1}}{\partial t} &= i \frac{\hbar}{2m_1^{s_1}} \frac{\partial^2 \psi_1^{s_1}}{\partial z^2} - i\omega_1^{s_1} \psi_1^{s_1} + i\chi(\vec{s}) \psi_2^{s_1} \psi_1^{\dagger s_1}, \\ \frac{\partial \psi_2^{s_2}}{\partial t} &= i \frac{\hbar}{2m_2^{s_2}} \frac{\partial^2 \psi_2^{s_2}}{\partial z^2} - i\omega_2^{s_2} \psi_2^{s_2} + i\frac{\chi(\vec{s})}{2} \psi_1^{2s_1}. \end{aligned} \tag{17}$$

Transforming the new coupled equations by $\tilde{\psi}_j = \psi_j e^{ij\omega_1^{s_1} t}$, a pair of equations is obtained that is identical to the usual description of a nonlinear dispersive parametric waveguide, with dispersions $\omega_j'' = \hbar/m_j^{s_j}$, and an effective phase mismatch of

$$\beta = (2\omega_1^{s_1} - \omega_2^{s_2}) = (\delta k_2 + 2s_1|\kappa_1| - s_2|\kappa_2|)v\,. \tag{18}$$

From previously known results [19], Eqs. (17) support dark (i.e., topological) solitary waves if $s_1 = -s_2$. Also, bright type solitary waves can occur, if the dispersions have identical sign, i.e., $s_1 = s_2 = s = -\mathrm{sgn}(\kappa_2)$. We will focus on this case in what follows.

Soliton type solutions to (17) can be written as [19]

$$\begin{aligned} \psi_1 &= \pm|q| \sqrt{\left| \frac{\kappa_1}{\kappa_2} \right|} V_1(z/z_0) e^{i(q-\omega_1)t} / \chi(\vec{s})\,, \\ \psi_2 &= s_1 |q| V_2(z/z_0) e^{2i(q-\omega_1)t} / \chi(\vec{s})\,, \end{aligned} \tag{19}$$

where q is an arbitrary parameter describing the (inverse) soliton time scale, and the corresponding length scale is $z_0 = \sqrt{|v/(2q\kappa_1)|}$.

Assuming the condition $Q \ll \kappa_j$, we can expand the mode function $u_{jQ}^s(z)$ into a Taylor's series up to first order about $Q = 0$. Hence, the electric field at $t = 0$ can be expressed as

$$\mathcal{E}_j(z) = \sum_s \sqrt{\frac{\hbar \omega_j^s}{2\bar{n}_j^2 \epsilon_0 A}} \left(\psi_j^s(z) u_{j0}^s(z) - i \frac{\partial \psi_j^s(z)}{\partial z} \delta u_{j0}^s(z) \right). \tag{20}$$

Substituting Eq. (19) into the above result gives

$$\begin{aligned} \vec{\mathcal{E}}_1 &= \pm a_1 \left(V_1 \begin{bmatrix} \mathrm{sgn}(\kappa_1) \\ -s \end{bmatrix} - \frac{i}{2|\kappa_1|} \frac{dV_1}{dz} \begin{bmatrix} \mathrm{sgn}(\kappa_1) s \\ 1 \end{bmatrix} \right) \\ \vec{\mathcal{E}}_2 &= a_2 \left(sV_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{i}{2|\kappa_2|} \frac{dV_2}{dz} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \end{aligned} \tag{21}$$

Here $a_1 = \sqrt{|\kappa_1/2\kappa_2|}\,|q| e^{i(q-\omega_1)t}/(c\chi_E/\bar{n})$ and $a_2 = |q| e^{2i(q-\omega_1)t}/(c\chi_E/\bar{n})$.

It has been proved that there is a family of one parameter solitary wave solutions of Eq. (17) [19]. The parameter ρ in this case is

$$\rho = \left| \frac{\kappa_2}{\kappa_1} \right| \left| 2 - \frac{\beta}{q} \right|. \tag{22}$$

Provided that the effective mass approximation is valid, which means that $\kappa_i z_o \gg 1$, each of these known simultaneous solitary waves—simultons—generates a corresponding band-gap soliton, which has an additional phase modulation when compared with the usual solitons.

For example, in cases with $\rho = 1$, the corresponding solutions can be worked out for the case $\kappa_j > 0$, $s_1 = s_2 = s_3 = -1$, which involves coupling between lower branches. The solutions are

$$V_1(z) = \frac{3}{\sqrt{2}}\operatorname{sech}^2\left(\frac{z}{2z_0}\right), \quad V_2(z) = \frac{3}{2}\operatorname{sech}^2\left(\frac{z}{2z_0}\right). \tag{23}$$

Cases that involve cross couplings between the upper branch and the lower branch of different gaps result in dark simulton solutions. These are not available analytically, and therefore must be calculated numerically—as is also necessary for all cases with $\rho \neq 1$.

These solutions have many interesting properties, although space is too limited here to present all of them. An experimentally relevant point is that the solutions given already are completely stationary in the laboratory frame. This creates an unexpected problem: how can they be introduced into the band-gap material? In fact, this is easily solved. If the gap structure is fabricated with $m_2 = 2m_1$, there is a symmetry in the equations which allows for moving solutions with an identical form to the stationary ones. Thus, they can be generated at the boundary, and then move into the bulk medium.

The dispersive parametric equations we obtain also support higher-dimensional soliton solutions [6,19,20] in (2 + 1) and (3 + 1) dimensions. These correspond to striped or layered band-gap structures, respectively, and are described by adding a transverse Laplacian to each of the earlier propagation equations. Solutions of this type do not appear to exist in conventional ($\chi^{(3)}$) gap solitons. Our Hamiltonian mapping from band-gap to dispersive equations, therefore, proves that the parametric band-gap environment is able to support higher-dimensional solitons. Thus, it is possible to obtain parametric gap solitons in up to three spatial dimensions.

In order to demonstrate stability, we have numerically solved the original band-gap equations (2), using the effective mass approximation result as an initial condition [19], in (1 + 1) and (2 + 1) dimensions. To provide a suitable perturbation, the (small) imaginary part of the input solution was omitted. In both cases, this perturbation is quickly radiated, and a stable solitary wave is obtained. Our results in Fig. 2 show a (2 + 1) dimensional gap simulton propagation, with the complete set of four coupled partial differential equations. After a small initial oscillation, the simulton reaches a steady state, proving the stability of the soliton in higher dimensions.

In summary, a parametric band-gap waveguide can provide both large dispersion and large nonlinearity. As

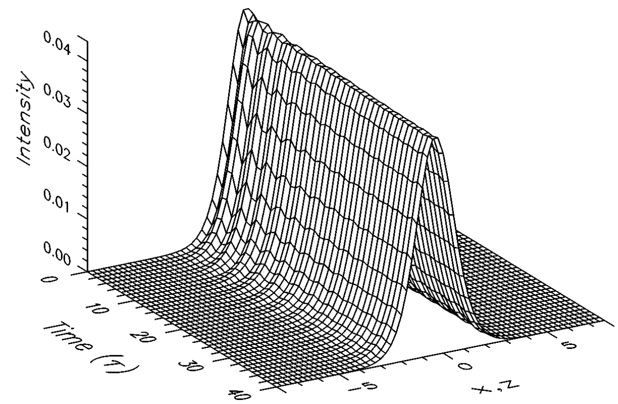

FIG. 2. First harmonic of a (2 + 1) dimensional gap simulton. Initial conditions were cylindrically symmetric in the reduced (EMA) coordinate system. The dimensionless parameters used for simplicity were $q = 1/8$, $-\kappa_2 = 2\kappa_1 = 8$, $z_0 = 1$, $\rho = 1$, $v = 1$, $\delta k_2 = 3/16$, and $\chi_E = 1$.

a numerical example, we consider a waveguide made of $LiNbO_3$. We use the following typical values: $\chi^{(2)} = 11.9$ pm/V [21], $\bar{n} = 2.5$, $\Delta n_{11} = \Delta n_{22} = 0.025$, and wavelength of the first harmonic $\lambda_1 = 1.06$ μm. This gives coupling parameters of $\kappa_1 \simeq 10^5$ m^{-1}, $\kappa_2 \simeq 2 \times 10^5$ m^{-1}, and $\chi_E \simeq 3 \times 10^{-5}$ V^{-1}. In order to satisfy the requirements of the effective mass approximation, the minimum soliton length possible is around $z_0 = 25$ μm. With this choice, the soliton period or reshaping time would be $q^{-1} \simeq 1$ ps. The corresponding energy is approximately $E \simeq 1$ nJ, for a waveguide area of $A = 5$ μm^2. This pulse energy is many orders of magnitude lower than the usual values for the corresponding $\chi^{(3)}$ gap solitons.

These characteristics of fast interaction times, low pulse energy, and stability in higher dimensions make the gap parametric system an ideal soliton environment for both fundamental physics and applications of solitons.

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