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Similitude applied to centrifugal scaling of unsaturated flow

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Abstract. Centrifuge experiments modeling single-phase flow in prototype porous media typically use the same porous medium and permeant. Then, well-known scaling laws are used to transfer the results to the prototype. More general scaling laws that relax these restrictions are presented. For permeants that are immiscible with an accompanying gas phase, model-prototype (i.e., centrifuge model experiment–target system) scaling is demonstrated. Scaling is shown to be feasible for Miller-similar (or geometrically similar) media. Scalings are presented for a more general class, Lisle-similar media, based on the equivalence mapping of Richards' equation onto itself. Whereas model-prototype scaling of Miller-similar media can be realized easily for arbitrary boundary conditions, Lisle-similarity in a finite length medium generally, but not always, involves a mapping to a moving boundary problem. An exception occurs for redistribution in Lisle-similar porous media, which is shown to map to spatially fixed boundary conditions. Complete model-prototype scalings for this example are derived.

1. Introduction

Experiments carried out using a geotechnical centrifuge generally fall into two (not independent) classes. First, there are those that are designed to investigate processes directly and, second, those that aim to produce data that can be used to predict behavior of a process that occurs naturally at an unwieldy spatial scale or at an impracticably large timescale. In the latter class, which is the main concern here, centrifuge modeling invariably involves questions of scaling of the results of laboratory experiments (the model scale) to the problem under investigation (the prototype scale). In addition, centrifuge experiments offer a convenient method for estimating soil hydraulic properties or checking those that might have been determined by different experimental methods.

Flow and transport (both of chemicals and energy) in porous media occur in a wide variety of applications, ranging from industrial to environmental. Centrifuge modeling efforts pertaining to the latter have gathered pace over the past decade or so. Such efforts have provided valuable data for testing and furthering the development of theoretical and numerical modeling of environmental problems [e.g., *Alemi et al.*, 1976; *Celorie et al.*, 1989b; *Nimmo*, 1990; *Cooke and Mitchell*, 1991; *Culligan et al.*, 1996; *Nakajima et al.*, 1998; *Gamerdinger and Kaplan*, 2000; *Poulose et al.*, 2000]. As a result, centrifugal modeling has established itself as a significant contributor to the emerging discipline of geoenvironmental engineering.

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Prototype-model scaling has received much attention previously [e.g., *Taylor*, 1995]. Several authors [e.g., *Cargill and Ko*, 1983; *Bear et al.*, 1984; *Goodings*, 1984; *Hensley and Schofield*, 1987; *Arulanandan et al.*, 1988; *Celorie et al.*, 1989a; *Li et al.*, 1993; *Mitchell*, 1994; *Culligan-Hensley and Savvidou*, 1995; *Culligan et al.*, 1997; *Savvidou et al.*, 1997; *Griffioen and Barry*, 1997, 1999; *Culligan and Barry*, 1998] have presented scaling analyses specifically for porous media flow and/or transport processes. For example, the scaling is straightforward for flow in saturated porous media that undergo negligible consolidation during the centrifuge experiment. Darcy's law, describing the seepage of a liquid through a porous medium, is

$$q = -K\nabla\phi, \qquad (1)$$

where q is the Darcy flux, K is the hydraulic conductivity, and ϕ is the head. If the centrifugal acceleration is Ng (where $N > 1$), then the (centrifugal model) experimental results correspond to a prototype (the target for which data is desired) that is N times larger than the sample, with a Darcy flux that is N times smaller. The liquid travels through the model porous medium in a time that is N^2 times smaller than that in the prototype. This N^2 decrease in timescale in the experiment timescale relative to the target system timescale is, traditionally, a key motivation for geoenvironmental centrifuge experimentation.

Any prototype will include a set (or sets) of boundary conditions for which results are desired. Of course, these conditions must be scaled analogously to the scaling applied to the soil and permeant.

Typically, scaling of geotechnical centrifuge results involves consideration of appropriate dimensionless numbers characterizing the processes under consideration. This approach was taken by *Arulanandan et al.* [1988] in their analysis of contaminant transport. Another general technique is based on the governing equations describing such processes. Below, the focus is on scaling of the governing equation for unsaturated flow in soil. However, as mentioned above, it should be borne in mind that the problem specification is not complete nor is the scaling analysis necessarily accurate without consideration of the initial and boundary conditions.

Table 1. Standard Scale Factors for Centrifuge Modeling Based on the Assumption That the Same Soil and Permeant are Used in Both Model and Prototype[a]

Parameter	Symbol	Prototype-Model Ratio
Acceleration	g	$1/N$
Length	z	N
Pressure	p	1
Temperature	T	1
Contaminant mass fraction	w	1
Hydraulic conductivity	K	$1/N$
Permeability	k	1
Fluid density	ρ	1
Fluid viscosity	μ	1
Hydraulic head	h	N
Capillary rise height	h_c	N
Porosity	n	1
Fluid flux	q	$1/N$
Time	t	N^2
Mechanical dispersion coefficient	D_m	1
Linear decay rate	λ_d	$1/N^2$

[a]See *Cargill and Ko* [1983] and *Culligan-Hensley and Savvidou* [1995].

A powerful means of analyzing the scaling of physical laws is inspectional analysis, whereby a given equation is mapped to a nondimensional form of the same equation while undergoing changes to the independent variables [*Ruark*, 1935; *Gukhman*, 1965; *Kline*, 1965; *Zierep*, 1971; *Tillotson and Nielsen*, 1984; *Li et al.*, 1994]. Inspectional analysis relies on the invariance of the physical law under changes of scale [*Birkhoff*, 1960]. The technique is most relevant when the mathematical description of the process is well developed, such that the theoretical model captures the pertinent physical features. Although one can obtain the same results using dimensional analysis, the physical insight often required in dimensional analysis is, at least to a known extent, already included in the governing equation [*Focken*, 1953]. In cases where a theoretical model is not available or is not sufficiently mature, centrifugal modeling often involves "modeling of models" [*Schofield*, 1980], thereby providing data at different acceleration levels so that scaling laws can be inferred directly from the measured data. *Culligan-Hensley and Savvidou* [1995], who considered single-phase fluid flow in a saturated medium along with associated chemical and energy transport, presented typical results. Assuming that the centrifuge model uses the prototype's soil and fluid and identical concentration and temperature boundary conditions, the scalings presented in Table 1 can be obtained easily by inspection. Several authors [e.g., *Arulanandan et al.*, 1988; *Celorie et al.*, 1989a] have presented similar scaling results.

The use of scaling of the governing equation for single-phase flow in porous media as a means to provide scaling rules for centrifugal modeling is investigated here. The technique of inspectional analysis is exploited to derive scaling rules for flow that involves different fluids in the model and prototype. For example, in environmental applications, movement of petroleum products to groundwater is a common problem. In the case of single-phase fluid flow, inspectional analysis reveals in a straightforward manner the centrifugal scaling rules that allow a different fluid to be used, while still allowing direct scaling of the experimental (model) results to the prototype scale. Such scaling allows a benign liquid to be used in a centrifuge experiment, at the same time permitting predictions to be made for a hazardous contaminant. Furthermore, geometric similarity of porous media allows modeling of different porous media on the basis of an experiment on any one of the same class.

Inspectional analysis, although a valuable technique, is limited in that it involves linear scalings only. These act to stretch or compress time, position, etc., such that the product of the scaling is a linearly distorted version of the starting point. More general (e.g., nonlinear) scalings are sometimes available [e.g., *Nielsen et al.*, 1998]. Richards' equation possesses a rich class of mappings that reduces to geometric scaling as a special case. This class will be examined in some detail.

2. Scaling of Richards' Equation

Scaling, using the term somewhat loosely, occurs almost as a matter of course in engineering and science. For example, it is typical to set the start of an experiment to time $t = 0$. That is, the data measurement times are translated such that they are offset from this time. Or data collected in a Lagrangian coordinate system are converted to an Eulerian coordinate system. In this section, various formal scaling approaches are applied to Richards' equation.

2.1. Inspectional Analysis of Richards' Equation

Richards' equation [*Richards*, 1931] describes water movement in unsaturated porous media. It is given by

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{dK(\theta)}{d\theta}\frac{\partial \theta}{\partial z}, \tag{2}$$

where θ is the volumetric moisture content, z is (positive downward) position, K is the hydraulic conductivity, and D is the soil water diffusivity. Inspectional analyses of Richards' equation have appeared previously [e.g., *Sposito and Jury*, 1985; *Youngs*, 1990]. Here use is made of such analyses to provide scaling rules for centrifuge-based modeling.

Inherent in (2) are the assumptions that the water is at a constant temperature, the porous medium is homogeneous and rigid, and the air in the soil is subjected to minimal pressure gradients. We consider first the problem of scaling unsaturated flow in a porous medium using a centrifuge model with different porous media and permeants.

Equation (2) is written in its usual form using the moisture content θ as the dependent variable. It can be written in three dimensions for an arbitrary liquid phase (in the presence of a gas phase such as air) in mixed form with both the volumetric moisture content and liquid pressure p as the dependent variables:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot \left(\frac{k}{\mu}\nabla p\right) - f\frac{\rho}{\mu}\frac{\partial k}{\partial z}, \tag{3}$$

where k is the permeability, ρ is the liquid density, μ is its viscosity, and f is the magnitude of the acceleration in the vertical direction (equal to g for the prototype). Note that in (3), k is taken as a function of p. Since $p \equiv p(\theta)$, k can be considered also as a function of θ. In what follows, the dependence of k on p or θ will be noted if necessary, unless obvious from the context. The equivalence between (2) and (3) follows from noting:

$$K = \frac{k\rho f}{\mu} \tag{4}$$

$$D \frac{\partial \theta}{\partial z} = K \frac{d\psi}{d\theta} \frac{\partial \theta}{\partial z} = \frac{k}{\mu} \frac{\partial p}{\partial z}, \tag{5}$$

where

$$\psi = \frac{p}{\rho f} \tag{6}$$

is the pressure head of the interstitial liquid. In this notation, Darcy's law (1) is written

$$\mathbf{q} = -k \frac{\rho f}{\mu} \nabla \left(z + \frac{p}{\rho f} \right). \tag{7}$$

A straightforward approach is to rewrite (3) in dimensionless (superscript pound symbol) form:

$$\frac{\partial \theta^{\#}}{\partial t^{\#}} = \nabla^{\#} \cdot (k^{\#} \nabla^{\#} p^{\#}) - f^{\#} \frac{\partial k^{\#}}{\partial z^{\#}}, \tag{8}$$

using the scalings [e.g., *Youngs*, 1990; cf. *Reichardt et al.*, 1972]

$$k^{\#}(p^{\#}) = \frac{k(p)}{l^2}, \tag{9}$$

$$p^{\#} = \frac{l}{\sigma} p, \tag{10}$$

$$t^{\#} = \left(\frac{\sigma l}{\mu L^2 \Delta \theta} \right) t, \tag{11}$$

$$z^{\#} = \frac{z}{L}, \tag{12}$$

$$\nabla^{\#} = L \nabla, \tag{13}$$

$$f^{\#} = \left(\frac{\rho l L}{\sigma} \right) f, \tag{14}$$

$$\theta^{\#} = \frac{\theta - \theta_r}{\Delta \theta}, \tag{15}$$

where l is a microscopic length scale, L is a macroscopic length, $\Delta \theta = \theta_s - \theta_r$, θ_r is the residual moisture content, θ_s is the saturated moisture content, and σ is the surface tension. Observe in (14) that three length scales are present: the microscopic (pore size) length scale l, the macroscopic length scale L, and the capillary length scale $(\sigma/\rho f)^{1/2}$. In a centrifuge model experiment, L is adjusted to account for changes in the acceleration level and the other two length scales. Observe that the scaling results presented in this paper would apply also if σ was factored by the appropriate contact angle. Whether or not the contact angle should be included is a question best addressed by analyses of experimental data.

Equation (9) makes explicit the relationship between the permeability and pressure. Since, as already stated following (3), the pressure and moisture content are related through the soil moisture characteristic curve [e.g., *Rose*, 1966], (9) could equally well be written with moisture content as the independent variable. Observe that the scaling of θ, given in (15), is unnecessary, as θ is already dimensionless. It is used to scale $\theta^{\#}$ so that it varies between predefined limits, in this case 0 and 1. Note also that the liquid pressure is usually offset such that $p = 0$ at atmospheric pressure, $p \geq 0$ if the medium is saturated, and $p < 0$ if it is unsaturated.

For any given soil the scalings listed in (9)–(15) can be carried out. However, the presence of two porous medium length scales is not strictly necessary mathematically, although physically speaking, in any circumstance, one can envisage that at least two length scales will always be present in dealing with a porous medium: a microscopic length scale (e.g., the grain size) and a macroscopic length scale (e.g., the sample size). The microscopic length scale l is significant in the case of two (or more) soil samples. If the scaling in (9) can be achieved by varying only the value of l between the samples, then the porous media are geometrically similar [e.g., *Sposito and Jury*, 1990]. Furthermore, if the moisture content distributions in each sample are identical and pressure scaling shown in (10) also holds, then this class of porous media is called Miller-similar [*Miller and Miller*, 1955a, 1955b, 1956; *Miller*, 1980]. Analyses of porous media that exhibit Miller similitude are considerably simplified since simple scaling rules relate the moisture movement for all members of the class. Miller similitude has been used to relate water movement through different (nongeometrically similar) soils [*Warrick*, 1990]. However, in such soils it is common to relax the requirement that $\theta^{\#} = \theta$ (i.e., all soils in the class have the same θ_r and θ_s), a relationship that holds for geometrically similar media [e.g., *Sposito and Jury*, 1985]. This equality is not necessary to put (3) into the dimensionless form of (8) and, more importantly, is not a useful assumption in practice. The moisture content scaling in (15) is almost always invoked in data analysis [e.g., *Warrick et al.*, 1977; *Nielsen et al.*, 1998].

The microscopic and macroscopic length scales present in (9)–(14) can be defined in various ways, as already indicated. For example, *Youngs* [1990] used the length scaling choices:

$$l = \sqrt{k(\theta_s)} \tag{16}$$

$$L = \frac{\sigma}{\rho f l} \tag{17}$$

to coalesce numerous data sets on both infiltration and redistribution of water in soil. Equation (16) has the feature that it scales the maximum value of $k^{\#}$ to unity. Also, since the permeability at the residual saturation θ_r is 0, the range $0 \leq k^{\#} \leq 1$ results. Equation (17), which follows from (14), was used to define a macroscopic length scale. It is pertinent that the soils tested [*Youngs and Price*, 1981; *Youngs*, 1983] were not a priori geometrically similar, suggesting, along with the above-mentioned work of Warrick, that the scaling given by (9)–(17) might extend beyond the strict confines of Miller similitude and have more widespread application.

Given the validity of (8), (9)–(15) must be satisfied for the prototype and centrifuge model, where these are identified by the subscripts p and m, respectively. On the assumption that Miller-similar porous media are to be modeled the key relationship concerns f, the acceleration. In the prototype, $f_p = g$, whereas for the model, $f_m = Ng = Nf_p$. If the same soil and permeant are used in each case, then l, k, ρ, σ, and μ are identical for each, and the appropriate scaling relationships in Table 1 are immediately apparent. More generally, if these conditions are relaxed, then (14) shows that the model macroscopic length is given by

$$L_m = \frac{\rho_p}{\rho_m} \frac{l_p}{l_m} \frac{\sigma_m}{\sigma_p} \frac{L_p}{N}. \tag{18}$$

Equation (18) shows that the model length is related to the imposed centrifugal acceleration and, since different permeants and Miller-similar media are involved, the characteristics of the porous medium and liquid. The timescaling follows directly from (11) and (18):

$$t_m = \left(\frac{\theta_{sm} - \theta_{rm}}{\theta_{sp} - \theta_{rp}}\right)\left(\frac{\rho_p}{\rho_m}\right)^2\left(\frac{l_p}{l_m}\right)^3 \frac{\sigma_m}{\sigma_p}\frac{\mu_m}{\mu_p}\frac{t_p}{N^2}. \quad (19)$$

In media that are geometrically similar, the first factor on the right-hand side of (19) is unity. The term is kept for cases where this situation does not hold. The liquid pressure scales according to (10), or

$$p_m = \frac{l_p}{l_m}\frac{\sigma_m}{\sigma_p} p_p. \quad (20)$$

In summary, (18)–(20) show directly the effects on the model spatial and temporal scaling of (1) centrifugal acceleration, (2) changes of porous media type, and (3) use of a different permeant. For point 2, strictly speaking, a generalized Miller-similar class of porous media is considered, while for point 3, there is an immiscible two-fluid combination (one liquid and one gaseous), e.g., water-air, oil-air, etc. Scaling of boundary conditions is simply a matter of applying the appropriate mappings.

2.2. Equivalence Transformation for Richards' Equation

Equivalence transformations include the inspectional analysis scalings as a special case.

2.2.1. Lisle's equivalence transformations. The scaling transformations, (9)–(15), leading from (3) to (8) are all stretchings. The only exception is (15), which involves an offset also. Given that (2) is well established as being a good approximation to the process of single-phase liquid movement in soil, a more general approach to the model-prototype scaling problem consists of examining the conditions under which (2) can be mapped into other equations of exactly the same form. For example, using superscript asterisks to denote the mapping of (2), the goal is to define the conditions under which it maps to:

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\partial}{\partial z^*}\left[D^*(\theta^*)\frac{\partial \theta^*}{\partial z^*}\right] - \frac{dK^*}{d\theta^*}\frac{\partial \theta^*}{\partial z^*}. \quad (21)$$

The problem of mapping from (2) to (21) amounts to finding the appropriate set of equivalence transformations [*Ovsiannikov*, 1982]. This set, given by *Lisle* [1992], consists of the Galileian transformation, scalings, and translations. Significantly, a more general set of transformations results when (2) is written in potential form as defined by the following two equations:

$$\frac{\partial I}{\partial z} = -\theta, \quad (22)$$

$$\frac{\partial I}{\partial t} = q, \quad (23)$$

where

$$q = K(\theta) - D(\theta)\frac{\partial \theta}{\partial z} \quad (24)$$

is the Darcy flux and $I(z, t)$ is the cumulative volume of water that has passed location z at time t. For (22)–(24) the equivalence group is [*Lisle*, 1992]

$$I^* = \frac{\lambda}{\zeta}(\alpha I - \beta z) - \vartheta t - I_0, \quad (25)$$

$$z^* = \frac{\lambda}{\zeta}(\delta z - \gamma I) + \nu t + z_0, \quad (26)$$

$$t^* = \frac{\lambda^2}{\zeta} t + t_0, \quad (27)$$

$$\theta^* = \frac{\alpha\theta + \beta}{\gamma\theta + \delta}, \quad (28)$$

$$q^* = \frac{\lambda q + \zeta(\alpha\nu - \gamma\vartheta)\theta + \zeta(\beta\nu - \delta\vartheta)}{\lambda^2(\gamma\theta + \delta)}, \quad (29)$$

$$K^* = \frac{\lambda K + \zeta(\alpha\nu - \gamma\vartheta)\theta + \zeta(\beta\nu - \delta\vartheta)}{\lambda^2(\gamma\theta + \delta)}, \quad (30)$$

$$D^* = \frac{(\gamma\theta + \delta)^2}{\zeta} D. \quad (31)$$

From (30) and (31) the transformed diffusivity and conductivity functions are given, respectively, by

$$K(\theta) = \lambda(\gamma\theta + \delta)K^*\left(\frac{\alpha\theta + \beta}{\gamma\theta + \delta}\right) + \frac{\zeta}{\lambda}[\theta(\gamma\vartheta - \alpha\nu) + \delta\vartheta - \beta\nu] \quad (32)$$

$$D(\theta) = \frac{\zeta}{(\gamma\theta + \delta)^2} D^*\left(\frac{\alpha\theta + \beta}{\gamma\theta + \delta}\right). \quad (33)$$

In (25)–(33) the various nonasterisked Greek parameters are constants whose dimensions depend on the meanings assigned to the asterisked variables, with $\lambda > 0$, $\zeta > 0$, and [*Lisle*, 1992],

$$\alpha = \frac{1 + \beta\gamma}{\delta}, \quad (34)$$

while the constants with a subscript 0 are offsets. Several (groups of) parameters have clear physical meanings. For example, in (28), two of the three independent parameters are used to scale moisture content.

Observe that the functional forms of the soil hydraulic properties, K and D in (32) and (33), are not identical to the forms K^* and D^*. That their relationship is known is all that concerns the scaling presented here since the objective is to map measurements from one system (the model) to another (the prototype). Investigations that identify the classes of precise forms for which K and D in (2) and (21) are identical (except for the asterisks) have been studied by several authors [e.g., *Sposito*, 1990, 1998; *Lisle*, 1992; *Yung et al.*, 1994; *Sophocleous*, 1996; *Vijayakumar*, 1997]. These forms are very important, e.g., for reduction of field data sets. However, these forms do not directly concern the present investigation.

Some of the parameters in (25)–(33) are related to familiar transformations, e.g., translations and stretchings. However, (28) is nonlinear except for $\gamma = 0$. Indeed, it is this parameter upon which the nonlinearity of the above set of transformations depends. The influence of γ will be evident below in section 2.2.2.3.

2.2.2. Interpretation of the set of equivalence transformations. It might appear, at first sight anyway, that numerous centrifugal scaling possibilities are available. In the following, several cases are considered, starting with the simplest scaling.

Table 2. Parameter Settings Such That the Equivalence Transformations in Section 2.2.1 Produce Different Transformation Classes

Parameter	Standard Scalings Based on Identical Porous Medium and Permeant in Model and Prototype[a]	Miller-Similar Porous Media and Different Permeants[b]	Lisle-Similar Porous Media and Different Permeants[c]	Transformation of *Rogers et al.* [1983][d]
λ	N	$\sigma l/L(\Delta\theta)^{1/2}$	$\sigma l/L\delta$	$\sigma l/L\delta$
ζ	1	$l\sigma$	$l\sigma$	$l\sigma$
β	0	$-\theta_r/(\Delta\theta)^{1/2}$	$-\theta_r[\gamma + (\gamma^2 + 4/\Delta\theta)^{1/2}]/2$	$-1/\gamma$
ϑ	0	0	0	0
I_0	0	0	0	0
δ	1	$(\Delta\theta)^{1/2}$	$\{\Delta\theta[(\gamma^2 + 4/\Delta\theta)^{1/2} - \gamma] - 2\gamma\theta_r\}/2$	$-1/\gamma - \gamma\theta_r$
γ	0	0	as appropriate for the porous medium	as appropriate for the porous medium, <0
ξ	NA[e]	NA[e]	δ/l^2	δ/l^2
ν	0	0	0	0
z_0	0	0	0	0
t_0	0	0	0	0

[a]See section 2.2.2.1 and Table 1.
[b]See section 2.2.2.2.
[c]See section 2.2.2.3.
[d]See section 3.
[e]NA is not applicable.

2.2.2.1. Identical prototype and model porous medium and permeant: Consider the practically important case where the goal is to scale the model centrifuge results directly to a prototype, when the prototype's porous medium is used in the model. In (25)–(33), then, it is necessary to maintain the same moisture content θ and diffusivity D in the model and prototype. As well, translations and moving coordinate systems are not relevant in that case. Removing such constants results in the usual scaling:

$$z^* = \lambda z, \tag{35}$$

$$t^* = \lambda^2 t, \tag{36}$$

$$q^* = \frac{q}{\lambda}, \tag{37}$$

$$K^* = \frac{K}{\lambda}. \tag{38}$$

In (35)–(38), take the left sides as the prototype and the right sides as the centrifugal model. For instance, in one dimension, Darcy's law (equation (1)) applied to water flow states

$$q = -K\frac{\partial\phi}{\partial z}, \tag{39}$$

where $\phi = z + p/\rho_w g$ is the hydraulic head of the interstitial water and $K = k\rho_w g/\mu_w$ is its hydraulic conductivity. In the centrifuge model (system without the asterisk) the body force is increased to N times the force due to gravitational acceleration, such that $K = NK^*$. In (35)–(38), then, it follows that $\lambda = N$, noting that because the hydraulic gradient is identical in the model and prototype, the soil water pressure in each is identical. Thus, in Table 2 (standard scalings column) the familiar scalings contained in Table 1 appear.

2.2.2.2. Miller similitude, geometrically scaled porous media and different permeants: The mapping from (2) to (21) applies also when the parameters appearing in each take on different forms. For example, the scaling that leads from (3) to (8) is contained in (26)–(31). Because (3) and (8) are not identical, neither will (2) and (21) be, at least if they are to reproduce (3) and (8). Observe that (8) and (21) will have the same form only if $f^{\#} = f^*\rho$ and $t^{\#} = t^*/\mu$ (all other asterisks and pound symbol variables are identical). Again, various superfluous translations are ignored in (26)–(31). Then, the reduction of (21) to (8) occurs for the parameter values contained in Table 2 (Miller-similar porous media and different permeants column).

2.2.2.3. Lisle equivalence class similarity: Now consider centrifugal scaling making use of the complete set of equations given by *Lisle* [1992]. The mapping from (3) to (8) written for the vertical direction alone is targeted. In this case, however, the previous results are extended by taking $\gamma \neq 0$ in (28). As mentioned above, Miller similitude has been shown to be useful in characterizing different soils. Thus it is useful to reduce the (appropriate) mappings in (26)–(31), if possible, to the special case of Miller similitude scalings in (9)–(15) for $\gamma = 0$ (Table 2).

2.2.2.4. Lisle equivalence class generalization of Miller similitude: We consider first the manner in which the Lisle-similar class generalizes the Miller similitude. The mapping between (I^*, z^*) and (I, z) in (25) and (26) involves both position and infiltration. To clarify matters, consider a special case for which simple results pertain, namely, redistribution in a finite soil profile, with no infiltration at the top or bottom of the profile. For this case, we note that (22) is replaced by

$$\frac{\partial I}{\partial z} = -(\theta - \theta_i), \tag{40}$$

where $\theta_i(z)$ is the initial moisture content in the soil profile. In this case, it can be shown that (25) and (26) become

$$I^* - \int_{z^*(z=0)}^{z^*} \theta_i^*(\overline{z^*})\, d\overline{z^*} = \frac{\lambda}{\zeta}\left\{\alpha\left[I - \int_0^z \theta_i(\bar{z})\, d\bar{z}\right] - \beta z\right\} - \vartheta t - I_0 \tag{41}$$

$$z^* = \frac{\lambda}{\zeta}\left\{\delta z - \gamma\left[I - \int_0^z \theta_i(\bar{z})\, d\bar{z}\right]\right\} + \nu t + z_0, \tag{42}$$

respectively.

It is convenient to take the view that the standard translations are necessary only for purposes such as changing coordinate systems and should not be considered further (this was done above as well). Thus set I_0, z_0, t_0, and ν to 0, as shown in Table 2 (Lisle-similar porous media and different permeants column). Additionally, the parameter ϑ in (41) is used to account for "base flow" within the flow domain, and so it can be set to 0 without affecting the present example.

The parameters β and δ in (28) are used to scale θ^* and so should not be considered as free parameters. The most convenient scaling is for $\theta = \theta_r$ and θ_s to correspond to $\theta^* = 0$ and 1, respectively. The actual values of β and δ will therefore depend on the choice of γ (a material property), as shown in Table 2. Note that, although there are two solution pairs for β and δ, only one is given. This restriction is a direct result of condition (46) below.

When the above considerations are taken into account, from (30) there results

$$\frac{f^\#}{f} = \frac{\rho k(\theta)}{\lambda k^*(\theta^*)(\gamma\theta + \delta)}. \tag{43}$$

Since the left-hand side of (43) is independent of the moisture content, the relationship between k and k^* is

$$k^*(\theta^*) = \frac{\xi k(\theta)}{\gamma\theta + \delta}, \tag{44}$$

where ξ is a constant of proportionality. Note that the left-hand side of (44) is dimensionless. In keeping with the spirit of the Miller-similitude approach of *Youngs* [1990], set $\xi = \delta/l^2$. Then, observe that (44) contains (9) as a special case. Moreover, the crucial acceleration relationship in (14) is honored, and the combination of (43) and (44) gives $\lambda = (\sigma l)/(\delta L)$. Note that it is desirable, although not necessary, to limit k^* to the range [0, 1]. This will occur if the definition of the microscopic length scale is modified from (16) to

$$l = \sqrt{\frac{\delta k(\theta_s)}{\delta + \gamma\theta_s}}. \tag{45}$$

The definition of l given here is but one choice, motivated by the data analysis of *Youngs* [1990]. Other choices might also be reasonable, e.g., the typical grain size or a length scale derived from the intrinsic sorptivity [e.g., *Philip*, 1969; *Haverkamp et al.*, 1998].

Because body force must be positive, the right-hand side of (43) is similarly constrained. Thus, the term in parentheses in the denominator must satisfy the condition

$$\delta > -\gamma\theta_r. \tag{46}$$

Now consider (5), defining the soil moisture diffusivity in terms of K and ψ. Using this relationship and the results obtained already in this section, it is possible to obtain from (31) the following expression relating p^* and p:

$$p^* = \frac{l}{\sigma}\int_0^p \left[1 + \frac{\gamma}{\delta}\,\theta(\bar{p})\right] d\bar{p}, \tag{47}$$

where, as above, $\zeta = l\sigma$ has been used to make the left-hand side dimensionless. As expected, (47) reduces to (10) for $\gamma = 0$.

Equations (28), (44), and (47) together generalize the class of Miller-similar porous media to a new class. Lisle-similar porous media. Clearly, if boundary conditions are not considered (which, of course, they must be in any application), unsaturated flow in any Miller-similar porous medium is contained within the Lisle-similar class. Put another way, Lisle similarity allows mapping of a larger class of porous media than is obtained for Miller similarity.

2.2.2.5. Application to redistribution: For the redistribution problem the boundary conditions are zero flux at $z = 0$ and L, or

$$I(0, t) = I(L, t) = 0. \tag{48}$$

Note that the boundaries are impermeable only to the liquid flow (the accompanying gas flow across those boundaries is unimpeded). Using these conditions, (26), with $\nu = 0$, shows that the end points of the spatial domain ($z = 0$ and L) map to fixed locations in the dimensionless variable z^*. Note that the amount of liquid in the porous medium is fixed in this problem. Initially, in an experiment the liquid is distributed according to

$$\theta(z, 0) = \theta_i(z), \tag{49}$$

where $\theta_r \le \theta_i \le \theta_s$, as indicated above in (40). The total amount in the profile, A, at any time during the experiment is:

$$A = \int_0^L \theta_i(z)\, dz = \int_0^L \theta(z, t)\, dz. \tag{50}$$

2.2.2.6. Scaling centrifuge model data: Now consider model-prototype scaling for a centrifuge experiment. Interpretation of centrifuge experimental data based on the complete set of equivalence transformations as presented here is slightly more complicated than either of the cases presented in sections 2.2.2.1 or 2.2.2.2. Since (14) is satisfied, then, as above, $Nf_p = f_m$, and the model-prototype macroscopic length scaling is therefore given by (18), with l defined by (45). The timescaling is deduced from (27) and the parameterization shown in Table 2:

$$t_m = \left(\frac{\delta_m}{\delta_p}\right)^2 \left(\frac{\rho_p}{\rho_m}\right)^2 \left(\frac{l_p}{l_m}\right)^3 \frac{\sigma_m}{\sigma_p}\frac{\mu_m}{\mu_p}\frac{t_p}{N^2}, \tag{51}$$

where δ_m and δ_p (Table 2, Lisle-similar column) are not written explicitly for compactness. Again, (51) reduces to (19) for $\gamma = 0$. The timescaling in (51) is slightly different to that in (19), the difference being the porous medium moisture content properties.

For a centrifugal model experiment the macroscopic length scaling is selected according to (18) and the timescaling defined by (51). It is now appropriate to address the scaling of the model data (collected at various locations in the model) to the prototype.

The scaling of z from the model to prototype depends on I, as shown in (41) and (42) and noted above. From the latter expression,

$$z_m + \frac{\gamma_m}{\delta_m}\left[I_m - \int_0^{z_m} \theta_{im}(\overline{z_m})\, d\overline{z_m}\right] = \frac{L_m}{L_p}\left\{z_p + \frac{\gamma_p}{\delta_p}\left[I_p - \int_0^{z_p} \theta_{ip}(\overline{z_p})\, d\overline{z_p}\right]\right\}, \tag{52}$$

where L_m/L_p is given by (18), and from (40),

$$I_m = \int_{z_m}^{L_m} [\theta_m(\bar{z}, t_m) - \theta_i(\bar{z})] \, d\bar{z}. \tag{53}$$

Here it has been assumed that moisture contents have been measured directly, or perhaps inferred from pressure measurements, with sufficient sampling density to make meaningful the integrals in (52) and (53). The infiltration mapping, (41), yields

$$\frac{(1+\beta_m\gamma_m)\left[I_m - \int_0^{z_m} \theta_{im}(\overline{z_m})\, d\overline{z_m}\right] - \delta_m\beta_m z_m}{\delta_m^2} = \frac{L_m}{L_p}\left\{\frac{(1+\beta_p\gamma_p)\left[I_p - \int_0^{z_p} \theta_{ip}(\overline{z_p})\, d\overline{z_p}\right] - \delta_p\beta_p z_p}{\delta_p^2}\right\}. \tag{54}$$

Equations (52) and (54) can be solved easily for z_p and I_p in terms of z_m and I_m and known characteristics of the porous medium and permeant. In other words, (52) and (54) can be used to transfer directly model results to prototype predictions.

Timescaling follows (51), while from (29) and (44) the flux scaling is

$$q_m = N q_p \frac{\rho_m}{\rho_p} \frac{k_m(\theta_m)}{k_p(\theta_p)}. \tag{55}$$

3. Discussion and Conclusions

The purpose of the present paper was to present scaling laws applicable for simulating a prototype single-phase porous media flow using a centrifugal model, where different permeants and porous media are used. Different permeants might be used for health and safety requirements. An advantage of similar media is that results for a single experiment extend directly to all members of the class upon application of appropriate scaling laws. This broadens significantly the application of centrifugal models in simulating movement of environmentally sensitive liquids in different unsaturated porous media.

Observe that the scaling in (25)–(31) shows that the flux q and hydraulic conductivity K scale similarly. If the same soil and permeant is used, then the ratio q/K should be identical in model and prototype. This reflects the fact that for unsaturated flow one must use an increase in body force to increase the flow rate in the model in order to maintain the same moisture content distribution in model and prototype. Increasing the flow rate by increasing the pressure gradient alone (e.g., in a $1g$ laboratory experiment) produces results that are not directly scalable to the prototype since moisture contents will not scale to the prototype. Another example where q/K must be maintained is for centrifugal modeling of unstable fingers [*Culligan et al.*, 1997; *Griffioen and Barry*, 1999].

This investigation into porous media exhibiting Lisle similarity has shown that for $\gamma \neq 0$, boundary conditions need to be carefully considered so that results are appropriately mapped to members within the class. As an obvious case, it has been shown that redistribution problems for this class of porous media can be modeled, in principle, using a centrifuge. The imposition of impermeable boundaries is not quite as restrictive as might appear at first sight. For flow problems involving no interaction with the porous medium boundaries, the conditions applied at those boundaries are unimportant.

In general, the application of the scaling results presented here relies on knowledge of porous medium properties such as $k(\theta_s)$ and γ in the experimental and target soils. Recall that for $\gamma = 0$ the soils are taken to be Miller-similar, which is a strong assumption. Acquisition of knowledge of $k(\theta_s)$ might be regarded as routine; however, at this point, the physical meaning of the Lisle similarity parameter γ is unclear. Notwithstanding this, the theory presented here could be applied to centrifuge modeling data from different soils to ascertain whether the soils are Lisle-similar through determination of γ.

In contrast to Miller scaling the results obtained for Lisle-similar porous media do not apply to mixed saturated-unsaturated flow. Essentially, the reason for this is that the equivalence transformations apply to Richards' equation written in terms of moisture content (and hence apply only up to zero pressure) not pressure. We recall, in passing, that the derivation of the θ-based form of Richards' equation involves assumptions that inherently limit its application [e.g., *LaBolle and Clausnitzer*, 1999].

Rogers et al. [1983] and others [e.g., *Fokas and Yortsos*, 1982; *Rosen*, 1982; *Broadbridge and White*, 1988; *Sander et al.*, 1988; *White and Broadbridge*, 1988; *Warrick et al.*, 1990; *Kühnel et al.*, 1990; *Barry and Sander*, 1991; *Sander et al.*, 1991; *Parkin et al.*, 1995] used less general (although similar) forms than those given in (32) and (33) to linearize (21) and thereby derive exact solutions for the governing flow (2). In essence, their solutions are based on the special case of $\beta = -1/\gamma$ ($\gamma < 0$). Table 2 (transformation of *Rogers et al.* [1983] column) lists values for the other equivalence transformation parameters derived following the approach in section 2.2.2.3. Because β is fixed in this manner, normalization of θ to lie between given fixed limits is not possible. This means that porous media characterized by different values of γ do not map to the same limits, so it is not possible to match model and prototype boundary conditions. As noted by *Sposito* [1995], the Rogers et al. transformation breaks "the full space-time symmetry" of the Richards' equation mapping.

Flow of multiple-liquid phases in porous media is a strongly nonlinear process. Richards' equation models the simplest case of two-phase flow: that of a single liquid phase and a gas phase providing negligible resistance to flow. Its nonlinearity, fundamentally reflected in the marked dependence of pressure and permeability on saturation, underlies the scope for soil classifications such as that based on Lisle similarity. The question of whether more complicated multiphase flows are amenable to such classifications remains open.

Notation

- A total liquid redistributing in the porous medium $[L]$.
- D soil water diffusivity $[L^2\ T^{-1}]$.
- D_m mechanical dispersion coefficient $[L^2\ T^{-1}]$.
- f acceleration magnitude $[L\ T^{-2}]$.
- h_c capillary rise height $[L]$.
- g magnitude of gravitational acceleration $[L\ T^{-2}]$.
- h hydraulic head $[L]$.
- I cumulative infiltration $[L]$.
- k permeability $[L^2]$.
- K hydraulic conductivity $[L\ T^{-1}]$.
- l microscopic length scale $[L]$.

L macroscopic length scale $[L]$.
n porosity.
N ratio of centrifugal acceleration magnitude to g.
p soil water pressure $[M\ L^{-1}\ T^{-2}]$.
q Darcy flux $[L\ T^{-1}]$.
$\mathbf{q}$ Darcy flux vector $[L\ T^{-1}]$.
t time $[T]$.
T temperature.
w contaminant mass fraction.
z position $[L]$.
α $(1 + \beta\gamma)/\delta$.
β equivalence group constant.
γ equivalence group constant.
δ equivalence group constant.
$\Delta\theta$ $\theta_s - \theta_r$.
ζ equivalence group constant.
θ volumetric moisture content.
θ_i initial volumetric moisture content.
θ_r volumetric moisture content at residual saturation.
θ_s volumetric moisture content at saturation.
ν equivalence group constant.
λ equivalence group constant.
λ_d chemical decay rate $[T^{-1}]$.
μ viscosity $[M\ L^{-1}\ T^{-1}]$.
ξ constant of proportionality.
ρ fluid density $[M\ L^{-3}]$.
σ surface tension $[M\ T^{-2}]$.
ϕ piezometric head $[L]$.
ψ pressure head $[L]$.
ϑ equivalence group constant.
∇ del operator $[L^{-1}]$.

Subscripts

m model.
p prototype.
w water.
0 offset.

Overbar indicates variable of integration. A superscript pound symbol indicates dimensionless form. A superscript asterisk indicates transformed system.

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