

Integrable Kondo impurities in one-dimensional extended Hubbard models

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 (Received 15 June 1999)

Three kinds of integrable Kondo problems in one-dimensional extended Hubbard models are studied by means of the boundary graded quantum inverse scattering method. The boundary K matrices depending on the local moments of the impurities are presented as a nontrivial realization of the graded reflection equation algebras acting in a $(2s_\alpha + 1)$ -dimensional impurity Hilbert space. Furthermore, these models are solved using the algebraic Bethe ansatz method, and the Bethe ansatz equations are obtained.

I. INTRODUCTION

The study of integrable models of correlated electrons with open boundary conditions has been the subject of considerable attention.¹⁻⁹ Recently it has become apparent that for models of open chains it is possible to obtain integrable impurity boundary conditions as operators that need not be expressed in terms of the (super)symmetry of the bulk model. A very important application of this procedure is in the context of Kondo, i.e., spin, impurities in models of correlated electrons. For the case of the supersymmetric t - J model boundary spin- $\frac{1}{2}$ impurities were introduced in Ref. 10 and the resulting model was solved by means of the coordinate Bethe ansatz method.

A reformulation of this model in the context of the quantum inverse scattering method (QISM) was given in Ref. 11, demonstrating that the model could be obtained via a family of commuting transfer matrices and thus establishing integrability. Central to this approach is the representations of the reflection equation algebras originally introduced by Sklyanin.¹² Such a solution guarantees that boundary terms may be applied to any model whose bulk integrability is associated with a solution of the Yang-Baxter equation. An interesting observation made in Ref. 11 was that the necessary solution of the reflection equation was not regular in the sense that it is not obtained by ‘‘dressing’’; i.e., it cannot be factorized into a product of local monodromy matrices and a c -number matrix.

By utilizing the underlying algebraic structure it was subsequently shown in Ref. 13 that more general classes of integrable t - J models with Kondo impurities exist. These were derived from both $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(3)$ invariant solutions of the Yang-Baxter equation, and the solution of the reflection equation was extended to accommodate arbitrary spin s impurities situated on the boundaries. Again, the new solutions of the reflection equation are not regular. Moreover, it was also demonstrated in Ref. 13 that the algebraic Bethe ansatz is applicable for these models and explicit solutions were given.

Recently, the work of Frahm and Slavnov¹⁴ has provided a representation-theoretic explanation for the existence of these nonregular solutions of the reflection equation. In essence, such solutions are obtained by suitable projection onto a subspace of the impurity Hilbert space for a regular solution. A consequence of this projection method is that the

remaining (super)symmetry in the new boundary operator on the impurity site corresponds to a subalgebra of the (super)symmetry of the original regular solution. As examples, this was illustrated in Ref. 14 for the case of $\mathfrak{gl}(m)$ impurities coupled to an open $\mathfrak{gl}(n)$ invariant chain for $m < n$ and a reproduction of the integrable t - J model with Kondo impurities given in Ref. 13.

It is immediately evident in view of these results that integrable spin impurities, being characterized by the simplest Lie algebra $\mathfrak{su}(2)$, can be readily obtained from regular solutions coming from the larger (super)symmetry associated with the model in the bulk. In particular, it is possible to obtain integrable boundary Kondo impurity models associated with the Lie algebra $\mathfrak{gl}(4)$ and superalgebras $\mathfrak{gl}(3|1)$ and $\mathfrak{gl}(2|2)$, which we investigate here. In each case, the bulk Hamiltonian can be expressed in the form of an extended Hubbard model and thus is worthy of investigation in terms of the physical properties that are exhibited. The bulk Hamiltonian associated with the $\mathfrak{gl}(2|2)$ solution is well known from previous work of Essler *et al.*¹⁵ For the cases of $\mathfrak{gl}(4)$ and $\mathfrak{gl}(3|1)$, although the quantum R matrices are well known in the literature,¹⁶⁻¹⁸ a realization of the associated Hamiltonians in terms of Fermi operators appears lacking. Here we present for these two cases the bulk Hamiltonians, which are new models for integrable correlated electron systems.

In the next section we introduce the three forms of extended Hubbard models with integrable boundary Kondo impurities. Following this we undertake an algebraic Bethe ansatz approach to solve each case. In the last section we conclude with some final remarks.

II. INTEGRABLE NON- c -NUMBER BOUNDARY K MATRICES AND KONDO IMPURITIES IN ONE-DIMENSIONAL EXTENDED HUBBARD MODELS

Let $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ denote fermionic creation and annihilation operators for spin σ at site j , which satisfy the anticommutation relations $\{c_{i,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, \dots, L$ and $\sigma, \tau = \uparrow, \downarrow$. We consider the following Hamiltonian, which describes two impurities coupled to the supersymmetric extended Hubbard open chain of Essler *et al.*,¹⁵

$$\begin{aligned}
H = & - \sum_{j=1, \sigma}^{L-1} (c_{j, \sigma}^\dagger c_{j+1, \sigma} + \text{H.c.}) (1 - n_{j, -\sigma} - n_{j+1, -\sigma}) \\
& - \sum_{j=1}^{L-1} (c_{j, \uparrow}^\dagger c_{j, \downarrow}^\dagger c_{j+1, \downarrow} c_{j+1, \uparrow} + \text{H.c.}) \\
& + 2 \sum_{j=1}^{L-1} (\mathbf{S}_j \cdot \mathbf{S}_{j+1} - \frac{1}{4} n_j n_{j+1}) + J_a \mathbf{S}_1 \cdot \mathbf{S}_a + V_a n_1 \\
& + U_a n_{1\uparrow} n_{1\downarrow} + J_b \mathbf{S}_L \cdot \mathbf{S}_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow}, \quad (1)
\end{aligned}$$

where J_α, V_α , and U_α ($\alpha = a, b$) are the Kondo coupling constants, the impurity scalar potentials, and the boundary Hubbard-like interaction constants, respectively; \mathbf{S} is the vector spin operator for the conduction electrons; \mathbf{S}_α ($\alpha = a, b$) are the local moments with spin $\frac{1}{2}$ located at the left and right ends of the system, respectively; $n_{j\sigma}$ is the number density operator $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$, $n_j = n_{j\uparrow} + n_{j\downarrow}$.

The supersymmetry algebra underlying the bulk Hamiltonian of this model is $\text{gl}(2|2)$. It is quite interesting to note that although the introduction of the impurities spoils the supersymmetry, there still remains $\text{u}(2) \otimes \text{u}(2)$ symmetry in the Hamiltonian (1) whose representation contains the spin and η -pairing realizations. As a result, one may add some terms like $U \sum_{j=1}^L n_{j\uparrow} n_{j\downarrow}$, $\mu \sum_{j=1}^L n_j$, and $h \sum_{j=1}^L (n_{j\uparrow} - n_{j\downarrow})$ to the Hamiltonian (1), without spoiling the integrability. Below we will establish the quantum integrability of the Hamiltonian (1) for a special choice of the model parameters J_α , V_α , and U_α :

$$\begin{aligned}
J_\alpha = & - \frac{2}{c_\alpha (c_\alpha + 2s_\alpha + 1)}, \quad V_\alpha = - \frac{c_\alpha^2 + 2c_\alpha s_\alpha - s_\alpha}{c_\alpha (c_\alpha + 2s_\alpha + 1)}, \\
U_\alpha = & - \frac{2s_\alpha - c_\alpha^2 - c_\alpha (2s_\alpha - 1)}{c_\alpha (c_\alpha + 2s_\alpha + 1)}. \quad (2)
\end{aligned}$$

This is achieved by showing that it can be derived from the (graded) boundary quantum inverse scattering method.^{5,8} Here we emphasize that a special case of this model, corresponding to $s_\alpha = \frac{1}{2}$, has been studied in Ref. 19.

The second choice of couplings that leads to an integrable model is given by

$$\begin{aligned}
H = & - \sum_{j=1, \sigma}^{L-1} (c_{j, \sigma}^\dagger c_{j+1, \sigma} + \text{H.c.}) (1 - n_{j, -\sigma} - n_{j+1, -\sigma}) \\
& - \sum_{j=1}^{L-1} (c_{j, \uparrow}^\dagger c_{j, \downarrow}^\dagger c_{j+1, \downarrow} c_{j+1, \uparrow} + \text{H.c.}) \\
& - 2 \sum_{j=1}^{L-1} (\mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{3}{4} n_j n_{j+1}) \\
& - 2 \sum_{j=1}^{L-1} n_{j, \downarrow} n_{j, \uparrow} (n_{j+1, \downarrow} n_{j+1, \uparrow} - n_{j+1}) \\
& - 2 \sum_{j=1}^{L-1} n_{j+1, \downarrow} n_{j+1, \uparrow} (n_{j, \downarrow} n_{j, \uparrow} - n_j) + J_a \mathbf{S}_1 \cdot \mathbf{S}_a + V_a n_1 \\
& + U_a n_{1\uparrow} n_{1\downarrow} + J_b \mathbf{S}_L \cdot \mathbf{S}_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow}. \quad (3)
\end{aligned}$$

In this case we can introduce integrable Kondo impurities on the boundary by choosing

$$\begin{aligned}
J_\alpha = & \frac{8}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}, \\
V_\alpha = & - \frac{4c_\alpha^2 + 4c_\alpha - 4s_\alpha(s_\alpha + 1) - 3}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}, \quad (4) \\
U_\alpha = & \frac{4c_\alpha^2 + 8c_\alpha - 4s_\alpha(s_\alpha + 1) - 5}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}.
\end{aligned}$$

A third choice of couplings that leads to an integrable model is

$$\begin{aligned}
H = & - \sum_{j=1, \sigma}^{L-1} (c_{j, \sigma}^\dagger c_{j+1, \sigma} + \text{H.c.}) (1 - n_{j, -\sigma} - n_{j+1, -\sigma}) \\
& - \sum_{j=1}^{L-1} (c_{j, \uparrow}^\dagger c_{j, \downarrow}^\dagger c_{j+1, \downarrow} c_{j+1, \uparrow} + \text{H.c.}) \\
& - 2 \sum_{j=1}^{L-1} (\mathbf{S}_j \cdot \mathbf{S}_{j+1} - \frac{1}{4} n_j n_{j+1}) \\
& - 2 \sum_{j=1}^{L-1} n_{j, \downarrow} n_{j, \uparrow} n_{j+1, \downarrow} n_{j+1, \uparrow} + J_a \mathbf{S}_1 \cdot \mathbf{S}_a + V_a n_1 \\
& + U_a n_{1\uparrow} n_{1\downarrow} + J_b \mathbf{S}_L \cdot \mathbf{S}_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow}, \quad (5)
\end{aligned}$$

where integrable Kondo impurities on the boundary are obtained by the choice

$$\begin{aligned}
J_\alpha = & \frac{8}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}, \\
V_\alpha = & \frac{(2c_\alpha^2 - 1)^2 - 4s_\alpha(s_\alpha + 1)}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}, \quad (6) \\
U_\alpha = & - \frac{4(c_\alpha^2 - 1)^2 - (2s_\alpha + 1)^2}{(2c_\alpha + 2s_\alpha + 1)(2c_\alpha - 2s_\alpha - 1)}.
\end{aligned}$$

Let us recall that the Hamiltonian of the one-dimensional (1D) supersymmetric extended Hubbard model with periodic boundary conditions commutes with the transfer matrix, which is the supertrace of the monodromy matrix $T(u)$:

$$T(u) = R_{0L}(u) \cdots R_{01}(u). \quad (7)$$

Here the quantum R matrix $R(u)$ comes from the fundamental representation of $\text{gl}(2|2)$ and takes the form

$$R(u) = \begin{pmatrix} u-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+2 \end{pmatrix}. \quad (8)$$

It should be noted that the supertrace is carried out for the auxiliary superspace V . The elements of the supermatrix $T(u)$ are the generators of an associative superalgebra \mathcal{A} defined by the relations

$$R_{12}(u_1 - u_2)T_1(u_1)T_2(u_2) = T_2(u_2)T_1(u_1)R_{12}(u_1 - u_2), \quad (9)$$

where $X_1 \equiv X \otimes 1$, $X_2 \equiv 1 \otimes X$ for any supermatrix $X \in \text{End}(V)$. For later use, we list some useful properties enjoyed by the R matrix: (i) unitarity: $R_{12}(u)R_{21}(-u) = \rho(u)$; (ii) crossing unitarity: $R_{12}^{st_2}(-u+1)R_{21}^{st_2}(u) = \tilde{\rho}(u)$ with $\rho(u), \tilde{\rho}(u)$ being some scalar functions.

In order to describe integrable models on open chains, we introduce two associative superalgebras \mathcal{T}_- and \mathcal{T}_+ defined by the R matrix $R(u_1 - u_2)$ and the relations^{5,8}

$$\begin{aligned} R_{12}(u_1 - u_2)\mathcal{T}_{1,-}(u_1)R_{21}(u_1 + u_2)\mathcal{T}_{2,-}(u_2) \\ = \mathcal{T}_{2,-}(u_2)R_{12}(u_1 + u_2)\mathcal{T}_{1,-}(u_1)R_{21}(u_1 - u_2), \end{aligned} \quad (10)$$

$$\begin{aligned} R_{21}^{st_1 ist_2}(-u_1 + u_2)\mathcal{T}_{1,+}^{st_1}(u_1)\{[R_{21}^{st_1}(u_1 + u_2)]^{-1}\}^{ist_2}\mathcal{T}_{2,+}^{ist_2}(u_2) \\ = \mathcal{T}_{2,+}^{ist_2}(u_2)\{[R_{12}^{ist_2}(u_1 + u_2)]^{-1}\}^{st_1} \\ \times \mathcal{T}_{1,+}^{st_1}(u_1)R_{12}^{st_1 ist_2}(-u_1 + u_2), \end{aligned} \quad (11)$$

respectively. Here the supertransposition st_α ($\alpha=1,2$) is only carried out in the α th factor superspace of $V \otimes V$, whereas ist_α denotes the inverse operation of st_α . By modifying Sklyanin's arguments,¹² one may show that the quantities $\tau(u)$ given by $\tau(u) = \text{str}[\mathcal{T}_+(u)\mathcal{T}_-(u)]$ constitute a commutative family, i.e., $[\tau(u_1), \tau(u_2)] = 0$.

One can obtain a class of realizations of the superalgebras \mathcal{T}_+ and \mathcal{T}_- by choosing $\mathcal{T}_\pm(u)$ to be the form

$$\mathcal{T}_-(u) = T_-(u)\tilde{\mathcal{T}}_-(u)T_-^{-1}(-u),$$

$$\mathcal{T}_+^{st}(u) = T_+^{st}(u)\tilde{\mathcal{T}}_+^{st}(u)[T_+^{-1}(-u)]^{st} \quad (12)$$

with

$$T_-(u) = R_{0M}(u) \cdots R_{01}(u),$$

$$T_+(u) = R_{0L}(u) \cdots R_{0,M+1}(u), \quad \tilde{\mathcal{T}}_\pm(u) = K_\pm(u), \quad (13)$$

where $K_\pm(u)$, called boundary K matrices, are representations of \mathcal{T}_\pm in some representation superspace.

We now solve Eqs. (10) and (11) for $K_-(u)$ and $K_+(u)$. For the quantum R matrix (8), one may check that the matrix $K_-(u)$ given by

$$K_-(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A_-(u) & B_-(u) \\ 0 & 0 & C_-(u) & D_-(u) \end{pmatrix}, \quad (14)$$

where

$$\begin{aligned} A_-(u) &= -\frac{u^2 + 2u - 4c_a^2 - 4c_a(2s_a + 1) + 4u\mathbf{S}_a^z}{(u - 2c_a)(u - 2c_a - 4s_a - 2)}, \\ B_-(u) &= -\frac{4u\mathbf{S}_a^-}{(u - 2c_a)(u - 2c_a - 4s_a - 2)}, \\ C_-(u) &= -\frac{4u\mathbf{S}_a^+}{(u - 2c_a)(u - 2c_a - 4s_a - 2)}, \\ D_-(u) &= -\frac{u^2 + 2u - 4c_a^2 - 4c_a(2s_a + 1) - 4u\mathbf{S}_a^z}{(u - 2c_a)(u - 2c_a - 4s_a - 2)}, \end{aligned} \quad (15)$$

Again we solve Eqs. (10) and (11) for $K_-(u)$ and $K_+(u)$. For Eq. (27) we obtain Eqs. (14) and (22), and

$$T_+^{st}(u) = JT_-(-u+2), \quad J = \text{diag}(1, -1, 1, 1), \quad (28)$$

giving

$$K_+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & A_+(u) & B_+(u) \\ 0 & 0 & C_+(u) & D_+(u) \end{pmatrix} \quad (29)$$

with

$$\begin{aligned} A_+(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4s_a(s_b + 1) + 1 - 4(u-2)\mathbf{S}_b^z}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \\ B_+(u) &= \frac{4(u-2)\mathbf{S}_b^-}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \\ C_+(u) &= \frac{4(u-2)\mathbf{S}_b^+}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}, \\ D_+(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4s_a(s_b + 1) + 1 + 4(u-2)\mathbf{S}_b^z}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}. \end{aligned} \quad (30)$$

The Hamiltonian (5) is related to the logarithmic derivative of the transfer matrix $\tau(u)$ with respect to the spectral parameter u at $u=0$ (up to an additive chemical potential term)

$$H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} K'_{1,-}(0) + \frac{\text{str}_0(K_{0,+}^{(0)} H_{L0})}{\text{str}_0 K_{0,+}(0)}, \quad (31)$$

with

$$h = -\frac{1}{2} \frac{d}{du} PR(u).$$

For this case we obtain Eq. (5) subject to the constraints (6).

III. THE BETHE ANSATZ SOLUTIONS

Having established the quantum integrability of the models, let us now diagonalize the Hamiltonians by means of the algebraic Bethe ansatz method.^{12,20} For the first case (1), introduce the ‘‘doubled’’ monodromy matrix $U(u)$:

$$\begin{aligned} U(u) &= T(u)K_-(u)\tilde{T}(u) \\ &\equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_1(u) & \mathcal{D}_{11}(u) & \mathcal{D}_{12}(u) & \mathcal{D}_{13}(u) \\ \mathcal{C}_2(u) & \mathcal{D}_{21}(u) & \mathcal{D}_{22}(u) & \mathcal{D}_{23}(u) \\ \mathcal{C}_3(u) & \mathcal{D}_{31}(u) & \mathcal{D}_{32}(u) & \mathcal{D}_{33}(u) \end{pmatrix}, \end{aligned} \quad (32)$$

where $\tilde{T}(u) = T^{-1}(-u)$. Substituting into the reflection equation (10) we may draw the following commutation relations:

$$\begin{aligned} \check{D}_{bd}(u_1)\mathcal{B}_c(u_2) &= \frac{(u_1 - u_2 - 2)(u_1 + u_2 - 4)}{(u_1 - u_2)(u_1 + u_2 - 2)} r(u_1 + u_2 - 2)_{gh}^{eb} \\ &\quad \times r(u_1 - u_2)_{cd}^{ih} \mathcal{B}_e(u_2) \check{D}_{gi}(u_1) \\ &\quad - \frac{2(u_1 - 2)u_2}{(u_1 + u_2 - 2)(u_1 - 1)(u_2 - 1)} \\ &\quad \times r(2u_1 - 2)_{cd}^{gb} \mathcal{B}_g(u_1) \mathcal{A}(u_2) \\ &\quad + \frac{2(u_1 - 2)}{(u_1 - u_2)(u_1 - 1)} r(2u_1 - 2)_{id}^{gb} \\ &\quad \times \mathcal{B}_g(u_1) \check{D}_{ic}(u_2), \end{aligned} \quad (33)$$

$$\begin{aligned} \mathcal{A}(u_1)\mathcal{B}_\beta(u_2) &= \frac{(u_1 - u_2 + 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 - 2)} \mathcal{B}_\beta(u_2) \mathcal{A}(u_1) \\ &\quad - \frac{2(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 - 2)} \mathcal{B}_\beta(u_1) \mathcal{A}(u_2) \\ &\quad + \frac{2}{u_1 + u_2 - 2} \left[\mathcal{B}_\alpha(u_1) \left(\check{D}_{\alpha\beta}(u_2) \right. \right. \\ &\quad \left. \left. - \frac{1}{u_2 - 1} \delta_{\alpha\beta} \mathcal{A}(u_2) \right) \right]. \end{aligned} \quad (34)$$

Here

$$\mathcal{D}_{bd}(u) = \check{D}_{bd}(u) - \frac{1}{u-1} \delta_{bd} \mathcal{A}(u)$$

and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

$$r_{11}^{11}(u) = 1, \quad r_{22}^{22}(u) = r_{33}^{33}(u) = -\frac{u+2}{u-2},$$

$$r_{12}^{12}(u) = r_{13}^{13}(u) = r_{21}^{21}(u) = r_{31}^{31}(u) = r_{23}^{23}(u) = r_{32}^{32}(u) = -\frac{2}{u-2},$$

$$r_{21}^{12}(u) = r_{12}^{21}(u) = r_{31}^{13}(u) = r_{13}^{31}(u) = \frac{u}{u-2},$$

$$r_{32}^{23}(u) = r_{23}^{32}(u) = -\frac{u}{u-2}. \quad (35)$$

Next choose Bethe state $|\Omega\rangle$ of the form

$$|\Omega\rangle = \mathcal{B}_{i_1}(u_1) \cdots \mathcal{B}_{i_N}(u_N) |\bar{\Psi}\rangle F^{i_1 \cdots i_N}, \quad (36)$$

with $|\bar{\Psi}\rangle$ being the pseudovacuum. Acting the transfer matrix $\tau(u)$ on the state $|\Omega\rangle$ we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$ with the eigenvalue

$$\begin{aligned} \Lambda(u) &= \frac{u}{u-1} \frac{(c_b - \frac{1}{2}u)}{(c_b - \frac{1}{2}u - 1)} \frac{(c_b - \frac{1}{2}u + 2s_b + 1)}{(c_b - \frac{1}{2}u + 2s_b)} \\ &\times \prod_{j=1}^N \frac{(u+u_j)(u-u_j+2)}{(u-u_j)(u+u_j-2)} + \frac{u}{u-1} \left(\frac{u}{u-2}\right)^{2L} \\ &\times \prod_{j=1}^N \frac{(u-u_j-2)(u+u_j-4)}{(u-u_j)(u+u_j-2)} \Lambda^{(1)}(u; \{u_i\}), \end{aligned} \quad (37)$$

provided the parameters $\{u_j\}$ satisfy

$$\begin{aligned} \frac{u_j}{u_j-2} \frac{(c_b - \frac{1}{2}u_j)}{(c_b - \frac{1}{2}u_j - 1)} \frac{(c_b - \frac{1}{2}u_j + 2s_b + 1)}{(c_b - \frac{1}{2}u_j + 2s_b)} \left(\frac{u_j-2}{u_j}\right)^{2L} \\ = \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j - u_i - 2)}{(u_j - u_i + 2)} \frac{(u_j + u_i - 4)}{(u_j + u_i)} \Lambda^{(1)}(u_j; \{u_{ij}\}). \end{aligned} \quad (38)$$

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem that arises out of the r matrices from the first term in the right-hand side of Eq. (33) with the reduced boundary K matrices $K_{\pm}^{(1)}(u)$:

$$K_{-}^{(1)}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{-}^{(1)}(u) & B_{-}^{(1)}(u) \\ 0 & C_{-}^{(1)}(u) & D_{-}^{(1)}(u) \end{pmatrix}, \quad (39)$$

where

$$\begin{aligned} A_{-}^{(1)}(u) &= -\frac{u^2 - 4c_a^2 - 8s_a c_a + 4s_a + 4(u-1)\mathbf{S}_a^z}{(u-2c_a)(u-2c_a-4s_a-2)}, \\ B_{-}^{(1)}(u) &= -\frac{4(u-1)\mathbf{S}_a^-}{(u-2c_a)(u-2c_a-4s_a-2)}, \\ C_{-}^{(1)}(u) &= -\frac{4(u-1)\mathbf{S}_a^+}{(u-2c_a)(u-2c_a-4s_a-2)}, \\ D_{-}^{(1)}(u) &= -\frac{u^2 - 4c_a^2 - 8s_a c_a + 4s_a - 4(u-1)\mathbf{S}_a^z}{(u-2c_a)(u-2c_a-4s_a-2)} \end{aligned} \quad (40)$$

and

$$K_{+}^{(1)}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{+}^{(1)}(u) & B_{+}^{(1)}(u) \\ 0 & C_{+}^{(1)}(u) & D_{+}^{(1)}(u) \end{pmatrix}, \quad (41)$$

where

$$\begin{aligned} A_{+}^{(1)}(u) &= -\frac{u^2 - 2u - 4c_b^2 - 4c_b(2s_b - 1) + 8s_b + 4u\mathbf{S}_b^z}{(u-2c_b+2)(u-2c_b-4s_b)}, \\ B_{+}^{(1)}(u) &= -\frac{4u\mathbf{S}_b^-}{(u-2c_b+2)(u-2c_b-4s_b)}, \end{aligned}$$

$$\begin{aligned} C_{+}^{(1)}(u) &= -\frac{4u\mathbf{S}_b^+}{(u-2c_b+2)(u-2c_b-4s_b)}, \\ D_{+}^{(1)}(u) &= -\frac{u^2 - 2u - 4c_b^2 - 4c_b(2s_b - 1) + 8s_b - 4u\mathbf{S}_b^z}{(u-2c_b+2)(u-2c_b-4s_b)}. \end{aligned} \quad (42)$$

Here $K_{-}^{(1)}(u)$, the boundary K matrix after the first nesting, follows from the relation

$$\begin{aligned} \check{D}_{dd}(u)|\Psi\rangle &\equiv \frac{u}{u-1} K_{dd}^{(1)}(u)|\Psi\rangle \\ &= \left(K_{-}(u)_{dd} + \frac{1}{u-1}\right) \left(\frac{u}{u-2}\right)^{2L} |\Psi\rangle, \\ \check{D}_{db}(u)|\Psi\rangle &\equiv \frac{u}{u-1} K_{db}^{(1)}(u)|\Psi\rangle = K_{-}(u)_{db} \left(\frac{u}{u-2}\right)^{2L} |\Psi\rangle. \end{aligned} \quad (43)$$

Indeed, applying the monodromy matrix $T(u)$ and its ‘‘adjoint’’ $\tilde{T}(u)$ to the pseudovacuum, we have

$$\begin{aligned} T_{11}(u)|\Psi\rangle &= |\Psi\rangle, \quad T_{dd}(u)|\Psi\rangle = \left(\frac{u}{u-2}\right)^L |\Psi\rangle, \\ T_{1d}(u)|\Psi\rangle &\neq 0, \quad T_{db}(u)|\Psi\rangle = 0, \quad T_{d1}(u)|\Psi\rangle = 0, \\ \tilde{T}_{11}(u)|\Psi\rangle &= |\Psi\rangle, \quad \tilde{T}_{dd}(u)|\Psi\rangle = \left(\frac{u}{u-2}\right)^L |\Psi\rangle, \end{aligned} \quad (44)$$

$\tilde{T}_{1d}(u)|\Psi\rangle \neq 0$, $\tilde{T}_{db}(u)|\Psi\rangle = 0$, $\tilde{T}_{d1}(u)|\Psi\rangle = 0$, where $d \neq b$, $d, b = 2, 3, 4$. Then we have

$$\begin{aligned} \mathcal{A}(u)|\Psi\rangle &= |\Psi\rangle, \\ \mathcal{B}_d(u)|\Psi\rangle &\neq 0, \quad \mathcal{C}_d(u)|\Psi\rangle = 0, \\ \mathcal{D}_{db}(u)|\Psi\rangle &= \left(\frac{u}{u-2}\right)^{2L} K_{-}(u)_{db} |\Psi\rangle, \\ \mathcal{D}_{dd}(u)|\Psi\rangle &= \left(\frac{u}{u-2}\right)^{2L} \left(K_{-}(u)_{dd} + \frac{1}{u-1}\right) |\Psi\rangle - \frac{1}{u-1} |\Psi\rangle. \end{aligned} \quad (45)$$

$$\begin{aligned} &(u-1)T_{21}(u)\tilde{T}_{12}(u) - T_{22}(u)\tilde{T}_{22}(u) - T_{23}(u)\tilde{T}_{32}(u) \\ &\quad - T_{24}(u)\tilde{T}_{42}(u) \\ &= -\tilde{T}_{11}(u)T_{11}(u) + (u-1)\tilde{T}_{12}(u)T_{21}(u) \\ &\quad - \tilde{T}_{13}(u)T_{31}(u) - \tilde{T}_{14}(u)T_{41}(u), \\ &(u-1)T_{21}(u)\tilde{T}_{13}(u) - T_{22}(u)\tilde{T}_{23}(u) - T_{23}(u)\tilde{T}_{33}(u) \\ &\quad - T_{24}(u)\tilde{T}_{43}(u) = u\tilde{T}_{13}(u)T_{21}(u), \\ &(u-1)T_{21}(u)\tilde{T}_{14}(u) - T_{22}(u)\tilde{T}_{24}(u) - T_{23}(u)\tilde{T}_{34}(u) \\ &\quad - T_{24}(u)\tilde{T}_{44}(u) = u\tilde{T}_{14}(u)T_{21}(u), \end{aligned}$$

$$T_{31}(u)\tilde{T}_{12}(u) - (u-1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u) \\ + T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{22}(u)T_{32}(u),$$

$$T_{31}(u)\tilde{T}_{13}(u) - (u-1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) \\ + T_{34}(u)\tilde{T}_{43}(u) \\ = \tilde{T}_{21}(u)T_{12}(u) + \tilde{T}_{22}(u)T_{22}(u) + (u+1)\tilde{T}_{23}(u)T_{32}(u) \\ + \tilde{T}_{24}(u)T_{42}(u),$$

$$T_{31}(u)\tilde{T}_{14}(u) - (u-1)T_{32}(u)\tilde{T}_{24}(u) + T_{33}(u)\tilde{T}_{34}(u) \\ + T_{34}(u)\tilde{T}_{44}(u) = u\tilde{T}_{24}(u)T_{32}(u)$$

$$T_{41}(u)\tilde{T}_{12}(u) + T_{42}(u)\tilde{T}_{22}(u) + (u+1)T_{43}(u)\tilde{T}_{32}(u) \\ + T_{44}(u)\tilde{T}_{42}(u) \\ = u\tilde{T}_{32}(u)T_{43}(u)$$

$$T_{41}(u)\tilde{T}_{13}(u) + T_{42}(u)\tilde{T}_{23}(u) + (u+1)T_{43}(u)\tilde{T}_{33}(u) \\ + T_{44}(u)\tilde{T}_{43}(u) \\ = u\tilde{T}_{33}(u)T_{43}(u),$$

$$T_{41}(u)\tilde{T}_{14}(u) + T_{42}(u)\tilde{T}_{24}(u) + (u+1)T_{43}(u)\tilde{T}_{34}(u) \\ + T_{44}(u)\tilde{T}_{44}(u) \\ = \tilde{T}_{31}(u)T_{13}(u) + \tilde{T}_{32}(u)T_{23}(u) \\ + \tilde{T}_{33}(u)T_{33}(u) + (u+1)\tilde{T}_{34}(u)T_{43}(u), \quad (46)$$

which come from a variant of the (graded) Yang-Baxter algebra (9) with the R matrix (8),

$$T_1(u)R(2u)\tilde{T}_2(u) = \tilde{T}_2(u)R(2u)T_1(u). \quad (47)$$

Noticing the change $u \rightarrow u-1$ with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra the reduced transfer matrix $\tau^{(1)}(u)$ may be recognized as that for the inhomogeneous supersymmetric t - J open chain interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. 13. The final result is

$$\Lambda^{(1)}(u; \{u_j\}) = \frac{u}{u-2} \frac{(c_b - \frac{1}{2}u)}{(c_b - \frac{1}{2}u + 2s_b)} \frac{(c_b - \frac{1}{2}u + 2s_b + 1)}{(c_b - \frac{1}{2}u - 1)} \\ \times \prod_{\alpha=1}^{M_1} \frac{(u - v_\alpha + 2)(u + v_\alpha - 2)}{(u - v_\alpha)(u + v_\alpha - 4)} \\ - \frac{u-1}{u-2} \prod_{j=1}^N \frac{(u - u_j)(u + u_j - 2)}{(u - u_j - 2)(u + u_j - 4)} \\ \times \prod_{\alpha=1}^{M_1} \frac{(u - v_\alpha + 2)(u + v_\alpha - 2)}{(u - v_\alpha)(u + v_\alpha - 4)} \\ \times \Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) \quad (48)$$

provided the parameters $\{v_\alpha\}$ satisfy

$$\frac{v_\alpha}{v_\alpha - 1} \frac{(c_b - \frac{1}{2}v_\alpha)(c_b - \frac{1}{2}v_\alpha + 2s_b + 1)}{(c_b - \frac{1}{2}v_\alpha + 2s_b)(c_b - \frac{1}{2}v_\alpha - 1)} \\ \times \prod_{j=1}^N \frac{(v_\alpha - u_j - 2)(v_\alpha + u_j - 4)}{(v_\alpha - u_j)(v_\alpha + u_j - 2)} \\ = -\Lambda^{(2)}(v_\alpha; \{u_j\}, \{v_\beta\}). \quad (49)$$

Here $\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})$ is the eigenvalue of the transfer matrix $\tau^{(2)}(u)$ for the M_1 -site inhomogeneous XXX open chain interacting with the Kondo impurities of arbitrary spins,

$$\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) = - \frac{(c_b - \frac{1}{2}u)}{(c_b - \frac{1}{2}u + 2s_b)} \frac{(c_b - \frac{1}{2}u + 2s_b + 1)}{(c_b - \frac{1}{2}u - 1)} \prod_{\gamma=a,b} \frac{c_\gamma + \frac{1}{2}u + 2s_\gamma - 1}{c_\gamma - \frac{1}{2}u + 2s_\gamma + 1} \left\{ \frac{u}{u-1} \prod_{\beta=1}^{M_2} \frac{(u - w_\beta - 3)(u + w_\beta - 3)}{(u - w_\beta - 1)(u + w_\beta - 1)} \right. \\ + \frac{u-2}{u-1} \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}u - 1)}{(c_\gamma - \frac{1}{2}u)} \frac{(c_\gamma - \frac{1}{2}u + 2s_\gamma)}{(c_\gamma + \frac{1}{2}u + 2s_\gamma - 1)} \prod_{\alpha=1}^{M_1} \frac{(u - v_\alpha)(u + v_\alpha - 4)}{(u - v_\alpha + 2)(u + v_\alpha - 2)} \\ \left. \times \prod_{\beta=1}^{M_2} \frac{(u - w_\beta + 1)(u + w_\beta + 1)}{(u - w_\beta - 1)(u + w_\beta - 1)} \right\}, \quad (50)$$

provided the parameters $\{w_\beta\}$ satisfy

$$\prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}w_\beta - \frac{1}{2})c_\gamma - \frac{1}{2}w_\beta + 2s_\gamma - \frac{1}{2}}{(c_\gamma - \frac{1}{2}w_\beta - \frac{1}{2})(c_\gamma + \frac{1}{2}w_\beta + 2s_\gamma - \frac{1}{2})} \prod_{\alpha=1}^{M_1} \frac{(w_\beta - v_\alpha + 1)(w_\beta + v_\alpha - 3)}{(w_\beta - v_\alpha + 3)(w_\beta + v_\alpha - 1)} = \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 2)}{(w_\beta - w_\delta + 2)(w_\beta + w_\delta + 2)}. \quad (51)$$

After a shift of the parameters $u_j \rightarrow u_j + 1, v_\alpha \rightarrow v_\alpha + 2$, the Bethe ansatz equations (38), (49), and (51) may be rewritten as follows:

$$\begin{aligned}
& \left(\frac{u_j-1}{u_j+1} \right)^{2L} \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j-u_i+2)(u_j+u_i+2)}{(u_j-u_i-2)(u_j+u_i-2)} \\
&= \prod_{\alpha=1}^{M_1} \frac{(u_j-v_\alpha+1)(u_j+v_\alpha+1)}{(u_j-v_\alpha-1)(u_j+v_\alpha-1)}, \\
& \prod_{\gamma=a,b} \frac{c_\gamma+\frac{1}{2}v_\alpha+2s_\gamma}{c_\gamma-\frac{1}{2}v_\alpha+2s_\gamma} \prod_{j=1}^N \frac{(v_\alpha-u_j+1)(v_\alpha+u_j+1)}{(v_\alpha-u_j-1)(v_\alpha+u_j-1)} \\
&= \prod_{\beta=1}^{M_2} \frac{(v_\alpha-w_\beta+1)(v_\alpha+w_\beta+1)}{(v_\alpha-w_\beta-1)(v_\alpha+w_\beta-1)}, \\
& \prod_{\gamma=a,b} \frac{(c_\gamma+\frac{1}{2}w_\beta-\frac{1}{2})(c_\gamma-\frac{1}{2}w_\beta+2s_\gamma-\frac{1}{2})}{(c_\gamma-\frac{1}{2}w_\beta-\frac{1}{2})(c_\gamma+\frac{1}{2}w_\beta+2s_\gamma-\frac{1}{2})} \\
& \quad \times \prod_{\alpha=1}^{M_1} \frac{(w_\beta-v_\alpha-1)(w_\beta+v_\alpha-1)}{(w_\beta-v_\alpha+1)(w_\beta+v_\alpha+1)} \\
&= \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta-w_\delta-2)(w_\beta+w_\delta-2)}{(w_\beta-w_\delta+2)(w_\beta+w_\delta+2)}, \quad (52)
\end{aligned}$$

with the corresponding energy eigenvalue E of the model

$$E = - \sum_{j=1}^N \frac{4}{u_j^2-1}. \quad (53)$$

We now perform the algebraic Bethe ansatz method^{12,20} procedure for the second couplings (3). We introduce the ‘‘doubled’’ monodromy matrix $U(u)$, as in Eq. (32). Substituting in the reflection equation (10) we may draw commutation relations Eqs. (33) and (34). Here

$$\mathcal{D}_{bd}(u) = \check{D}_{bd}(u) - \frac{1}{u-1} \delta_{bd} \mathcal{A}(u)$$

and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

$$r_{11}^{11}(u) = r_{22}^{22}(u) = r_{33}^{33}(u) = 1,$$

$$r_{12}^{12}(u) = r_{13}^{13}(u) = r_{21}^{21}(u) = r_{31}^{31}(u) = r_{23}^{23}(u) = r_{32}^{32}(u) = -\frac{2}{u-2},$$

$$r_{21}^{12}(u) = r_{12}^{21}(u) = r_{31}^{13}(u) = r_{13}^{31}(u) = r_{32}^{23}(u) = r_{23}^{32}(u) = \frac{u}{u-2}. \quad (54)$$

Choosing the Bethe state $|\Omega\rangle$ as Eq. (36) with $|\Psi\rangle$ being the pseudovacuum, and acting the transfer matrix $\tau(u)$ on the state $|\Omega\rangle$, we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$, with the eigenvalue

$$\begin{aligned}
\Lambda(u) &= \frac{u-4}{u-1} \frac{(c_b+\frac{1}{2}u+s_b+\frac{1}{2})(c_b+\frac{1}{2}u-s_b-\frac{1}{2})}{(c_b+\frac{1}{2}u+s_b-\frac{1}{2})(c_b+\frac{1}{2}u-s_b-\frac{3}{2})} \\
& \quad \times \prod_{j=1}^N \frac{(u+u_j)(u-u_j+2)}{(u-u_j)(u+u_j-2)} \\
& \quad + \frac{u}{u-1} \left(\frac{u}{u-2} \right)^{2L} \prod_{j=1}^N \frac{(u-u_j-2)(u+u_j-4)}{(u-u_j)(u+u_j-2)} \\
& \quad \times \Lambda^{(1)}(u; \{u_i\}), \quad (55)
\end{aligned}$$

provided the parameters $\{u_j\}$ satisfy

$$\begin{aligned}
& \frac{u_j-4}{u_j-2} \frac{(c_b+\frac{1}{2}u_j+s_b+\frac{1}{2})(c_b+\frac{1}{2}u_j-s_b-\frac{1}{2})}{(c_b+\frac{1}{2}u_j+s_b-\frac{1}{2})(c_b+\frac{1}{2}u_j-s_b-\frac{3}{2})} \left(\frac{u_j-2}{u_j} \right)^{2L} \\
&= \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j-u_i-2)(u_j+u_i-4)}{(u_j-u_i+2)(u_j+u_i)} \Lambda^{(1)}(u_j; \{u_i\}). \quad (56)
\end{aligned}$$

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem, that arises out of the r matrices from the first term in the right-hand side of Eq. (58), with the reduced boundary K matrices $K_{\pm}^{(1)}(u)$ from Eq. (39) where

$$\begin{aligned}
A_{-}^{(1)}(u) &= - \frac{u^2-4c_a^2-4c_a+4s_a(s_a+1)+3-4(u-1)\mathbf{S}_a^z}{(u+2c_a-2s_a-1)(u+2c_a+2s_a+1)}, \\
B_{-}^{(1)}(u) &= \frac{4(u-1)\mathbf{S}_a^-}{(u+2c_a-2s_a-1)(u+2c_a+2s_a+1)}, \quad (57) \\
C_{-}^{(1)}(u) &= \frac{4(u-1)\mathbf{S}_a^+}{(u+2c_a-2s_a-1)(u+2c_a+2s_a+1)}, \\
D_{-}^{(1)}(u) &= - \frac{u^2-4c_a^2-4c_a+4s_a(s_a+1)+3+4(u-1)\mathbf{S}_a^z}{(u+2c_a-2s_a-1)(u+2c_a+2s_a+1)}
\end{aligned}$$

and Eq. (41) where

$$\begin{aligned}
A_{+}^{(1)}(u) &= \frac{u^2-6u-4c_b^2-8c_b+4s_b(s_b+1)+5-4(u-4)\mathbf{S}_b^z}{(u+2c_b-2s_b-3)(u+2c_b+2s_b-1)} \\
B_{+}^{(1)}(u) &= - \frac{4(u-4)\mathbf{S}_b^-}{(u+2c_b-2s_b-3)(u+2c_b+2s_b-1)}, \\
C_{+}^{(1)}(u) &= - \frac{4(u-4)\mathbf{S}_b^+}{(u+2c_b-2s_b-3)(u+2c_b+2s_b-1)},
\end{aligned}$$

$$D_+^{(1)}(u) = \frac{u^2 - 6u - 4c_b^2 - 8c_b + 4s_b(s_b + 1) + 5 + 4(u-4)\mathbf{S}_b^z}{(u + 2c_b - 2s_b - 3)(u + 2c_b + 2s_b - 1)}. \quad (58)$$

Here $K_-^{(1)}(u)$, the boundary K matrices after the first nesting, follows from relations (43). Indeed, applying the monodromy matrix $T(u)$ and its ‘‘adjoint’’ $\tilde{T}(u)$ to the pseudo-vacuum, we have Eqs. (44) and (45)

$$\begin{aligned} (u-1)T_{21}(u)\tilde{T}_{12}(u) - T_{22}(u)\tilde{T}_{22}(u) - T_{23}(u)\tilde{T}_{32}(u) \\ - T_{24}(u)\tilde{T}_{42}(u) \\ = -\tilde{T}_{11}(u)T_{11}(u) + (u-1)\tilde{T}_{12}(u)T_{21}(u) \\ - \tilde{T}_{13}(u)T_{31}(u) - \tilde{T}_{14}(u)T_{41}(u), \end{aligned}$$

$$\begin{aligned} (u-1)T_{21}(u)\tilde{T}_{13}(u) - T_{22}(u)\tilde{T}_{23}(u) - T_{23}(u)\tilde{T}_{33}(u) \\ - T_{24}(u)\tilde{T}_{43}(u) = u\tilde{T}_{13}(u)T_{21}(u) \end{aligned}$$

$$\begin{aligned} (u-1)T_{21}(u)\tilde{T}_{14}(u) - T_{22}(u)\tilde{T}_{24}(u) - T_{23}(u)\tilde{T}_{34}(u) \\ - T_{24}(u)\tilde{T}_{44}(u) = u\tilde{T}_{14}(u)T_{21}(u) \end{aligned}$$

$$\begin{aligned} T_{31}(u)\tilde{T}_{12}(u) - (u-1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u) \\ + T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{22}(u)T_{32}(u) \end{aligned}$$

$$\begin{aligned} T_{31}(u)\tilde{T}_{13}(u) - (u-1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) \\ + T_{34}(u)\tilde{T}_{43}(u) \\ = \tilde{T}_{21}(u)T_{12}(u) + \tilde{T}_{22}(u)T_{22}(u) \\ - (u-1)\tilde{T}_{23}(u)T_{32}(u) + \tilde{T}_{24}(u)T_{42}(u), \end{aligned}$$

$$\begin{aligned} T_{31}(u)\tilde{T}_{14}(u) - (u-1)T_{32}(u)\tilde{T}_{24}(u) + T_{33}(u)\tilde{T}_{34}(u) \\ + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u) \end{aligned}$$

$$\begin{aligned} T_{41}(u)\tilde{T}_{12}(u) + T_{42}(u)\tilde{T}_{22}(u) - (u-1)T_{43}(u)\tilde{T}_{32}(u) \\ + T_{44}(u)\tilde{T}_{42}(u) = u\tilde{T}_{32}(u)T_{43}(u) \end{aligned}$$

$$\begin{aligned} T_{41}(u)\tilde{T}_{13}(u) + T_{42}(u)\tilde{T}_{23}(u) - (u-1)T_{43}(u)\tilde{T}_{33}(u) \\ + T_{44}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{33}(u)T_{43}(u), \end{aligned}$$

$$\begin{aligned} T_{41}(u)\tilde{T}_{14}(u) + T_{42}(u)\tilde{T}_{24}(u) - (u-1)T_{43}(u)\tilde{T}_{34}(u) \\ + T_{44}(u)\tilde{T}_{44}(u) \\ = \tilde{T}_{31}(u)T_{13}(u) + \tilde{T}_{32}(u)T_{23}(u) + \tilde{T}_{33}(u)T_{33}(u) \\ - (u-1)\tilde{T}_{34}(u)T_{43}(u), \end{aligned} \quad (59)$$

which come from a variant of the (graded) Yang-Baxter algebra (9) with the R matrix (21), as in Eq. (47).

Noticing the change $u \rightarrow u-1$ with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix $\tau^{(1)}(u)$ may be recognized as that for the inhomogeneous $\text{su}(3)$ t - J open chain interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. 13. The final result is

$$\begin{aligned} \Lambda^{(1)}(u; \{u_j\}) &= \frac{u-4}{u-2} \frac{(c_b + \frac{1}{2}u + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}u + s_b - \frac{1}{2})} \frac{(c_b + \frac{1}{2}u - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}u - s_b - \frac{3}{2})} \\ &\times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha+2)(u+v_\alpha-2)}{(u-v_\alpha)(u+v_\alpha-4)} \\ &- \frac{u-1}{u-2} \prod_{j=1}^N \frac{(u-u_j)(u+u_j-2)}{(u-u_j-2)(u+u_j-4)} \\ &\times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha-2)(u+v_\alpha-6)}{(u-v_\alpha)(u+v_\alpha-4)} \\ &\times \Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}), \end{aligned} \quad (60)$$

provided the parameters $\{v_\alpha\}$ satisfy

$$\begin{aligned} \frac{v_\alpha-4}{v_\alpha-3} \frac{(c_b + \frac{1}{2}v_\alpha + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}v_\alpha + s_b - \frac{1}{2})} \frac{(c_b + \frac{1}{2}v_\alpha - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}v_\alpha - s_b - \frac{3}{2})} \prod_{j=1}^N \frac{(v_\alpha-u_j-2)(v_\alpha+u_j-4)}{(v_\alpha-u_j)(v_\alpha+u_j-2)} \\ \times \prod_{\substack{\xi=1 \\ \xi \neq \alpha}}^{M_1} \frac{(v_\alpha-v_\xi+2)(v_\alpha+v_\xi-2)}{(v_\alpha-v_\xi-2)(v_\alpha+v_\xi-6)} = -\Lambda^{(2)}(v_\alpha; \{u_i\}, \{v_\beta\}). \end{aligned} \quad (61)$$

Here $\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})$ is the eigenvalue of the transfer matrix $\tau^{(2)}(u)$ for the M_1 -site inhomogeneous XXX open chain interacting with the Kondo impurities of arbitrary spins,

$$\begin{aligned} \Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) = & - \frac{(c_b + \frac{1}{2}u + s_b + \frac{1}{2})(c_b + \frac{1}{2}u - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}u + s_b - \frac{1}{2})(c_b + \frac{1}{2}u - s_b - \frac{3}{2})} \prod_{\gamma=a,b} \frac{c_\gamma - \frac{1}{2}u + s_\gamma + \frac{5}{2}}{c_\gamma + \frac{1}{2}u + s_\gamma + \frac{1}{2}} \\ & \times \left\{ \frac{u-4}{u-3} \prod_{\beta=1}^{M_2} \frac{(u-w_\beta+2)(u+w_\beta-4)}{(u-w_\beta)(u+w_\beta-6)} + \frac{u-2}{u-3} \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}u + s_\gamma - \frac{1}{2})(c_\gamma - \frac{1}{2}u - s_\gamma + \frac{5}{2})}{(c_\gamma + \frac{1}{2}u - s_\gamma - \frac{1}{2})(c_\gamma - \frac{1}{2}u + s_\gamma + \frac{5}{2})} \right. \\ & \left. \times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha)(u+v_\alpha-4)}{(u-v_\alpha-2)(u+v_\alpha-6)} \prod_{\beta=1}^{M_2} \frac{(u-w_\beta-2)(u+w_\beta-8)}{(u-w_\beta)(u+w_\beta-6)} \right\}, \end{aligned} \tag{62}$$

provided the parameters $\{w_\beta\}$ satisfy

$$\begin{aligned} & \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}w_\beta + s_\gamma - \frac{1}{2})(c_\gamma - \frac{1}{2}w_\beta - s_\gamma + \frac{5}{2})}{(c_\gamma - \frac{1}{2}w_\beta + s_\gamma + \frac{5}{2})(c_\gamma + \frac{1}{2}w_\beta - s_\gamma - \frac{1}{2})} \\ & \times \prod_{\alpha=1}^{M_1} \frac{(w_\beta - v_\alpha)(w_\beta + v_\alpha - 4)}{(w_\beta - v_\alpha - 2)(w_\beta + v_\alpha - 6)} \\ & = \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta - 4)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 8)}. \end{aligned} \tag{63}$$

After a shift of the parameters $u_j \rightarrow u_j + 1, v_\alpha \rightarrow v_\alpha + 2, w_\beta \rightarrow w_\beta + 3$, the Bethe ansatz equations (56), (61), and (63) may be rewritten as follows:

$$\begin{aligned} & \left(\frac{u_j - 1}{u_j + 1} \right)^{2L} \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j - u_i + 2)(u_j + u_i + 2)}{(u_j - u_i - 2)(u_j + u_i - 2)} \\ & = \prod_{\alpha=1}^{M_1} \frac{(u_j - v_\alpha + 1)(u_j + v_\alpha + 1)}{(u_j - v_\alpha - 1)(u_j + v_\alpha - 1)}, \\ & \prod_{\gamma=a,b} \frac{c_\gamma + \frac{1}{2}v_\alpha + s_\gamma + \frac{3}{2}}{c_\gamma - \frac{1}{2}v_\alpha + s_\gamma + \frac{3}{2}} < \prod_{j=1}^N \frac{(v_\alpha - u_j - 1)(v_\alpha + u_j - 1)}{(v_\alpha - u_j + 1)(v_\alpha + u_j + 1)} \\ & = \prod_{\beta=1}^{M_2} \frac{(v_\alpha - w_\beta + 1)(v_\alpha + w_\beta + 1)}{(v_\alpha - w_\beta - 1)(v_\alpha + w_\beta - 1)}, \\ & \times \prod_{\substack{\zeta=1 \\ \zeta \neq \alpha}}^{M_1} \frac{(v_\alpha - v_\zeta - 2)(v_\alpha + v_\zeta - 2)}{(v_\alpha - w_\zeta + 2)(v_\alpha + v_\zeta + 2)} \\ & \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}w_\beta + s_\gamma + 1)(c_\gamma - \frac{1}{2}w_\beta - s_\gamma + 1)}{(c_\gamma - \frac{1}{2}w_\beta + s_\gamma + 1)(c_\gamma + \frac{1}{2}w_\beta - s_\gamma + 1)} \\ & \times \prod_{\alpha=1}^{M_1} \frac{(w_\beta - v_\alpha + 1)(w_\beta + v_\alpha + 1)}{(w_\beta - v_\alpha - 1)(w_\beta + v_\alpha - 1)} \\ & = \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta + 2)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 2)}, \end{aligned} \tag{64}$$

with the corresponding energy eigenvalue E of the model in Eq. (53).

We now perform the algebraic Bethe ansatz method^{12,20} procedure for the third couplings (5). We introduce the ‘‘doubled’’ monodromy matrix $U(u)$, as in Eq. (32) where $\tilde{T}(u) = T^{-1}(-u)$. Substituting into the reflection equation (10), we may draw the following commutation relations,

$$\begin{aligned} \check{D}_{bd}(u_1)\mathcal{B}_c(u_2) = & \frac{(u_1 - u_2 - 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)} r(u_1 + u_2 + 2)_{gh}^{eb} \\ & \times r(u_1 - u_2)_{cd}^{ih} \mathcal{B}_e(u_2) \check{D}_{gi}(u_1) \\ & - \frac{2u_1 u_2}{(u_1 + u_2 + 2)(u_1 + 1)(u_2 + 1)} \\ & \times r(2u_1 + 2)_{cd}^{gb} \mathcal{B}_g(u_1) \mathcal{A}(u_2) \\ & + \frac{2u_1}{(u_1 - u_2)(u_1 + 1)} \\ & \times r(2u_1 + 2)_{id}^{gb} \mathcal{B}_g(u_1) \check{D}_{ic}(u_2), \end{aligned} \tag{65}$$

$$\begin{aligned} \mathcal{A}(u_1)\mathcal{B}_\beta(u_2) = & \frac{(u_1 - u_2 - 2)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)} \mathcal{B}_\beta(u_2) \mathcal{A}(u_1) \\ & - \frac{2(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 2)} \mathcal{B}_\beta(u_1) \mathcal{A}(u_2) \\ & - \frac{2}{u_1 + u_2 + 2} \left[\mathcal{B}_\alpha(u_1) \left(\check{D}_{\alpha\beta}(u_2) \right. \right. \\ & \left. \left. - \frac{1}{u_2 - 1} \delta_{\alpha\beta} \mathcal{A}(u_2) \right) \right]. \end{aligned} \tag{66}$$

Here

$$\mathcal{D}_{bd}(u) = \check{D}_{bd}(u) + \frac{1}{u+1} \delta_{bd} \mathcal{A}(u)$$

and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form, of Eq. (54). Choosing the Bethe state $|\Omega\rangle$ as Eq. (36) with $|\Psi\rangle$ being the pseudo-vacuum, and acting the transfer matrix $\tau(u)$ on the state $|\Omega\rangle$, we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$, with the eigenvalue

$$\begin{aligned} \Lambda(u) &= \frac{u-2}{u+1} \frac{(c_b + \frac{1}{2}u - s_b - \frac{1}{2})(c_b + \frac{1}{2}u + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}u + s_b - \frac{1}{2})(c_b + \frac{1}{2}u - s_b - \frac{3}{2})} \\ &\times \prod_{j=1}^N \frac{(u+u_j)(u-u_j-2)}{(u-u_j)(u+u_j+2)} + \frac{u}{u+1} \left(-\frac{u}{u+2}\right)^{2L} \\ &\times \prod_{j=1}^N \frac{(u+u_j)(u-u_j-2)}{(u-u_j)(u+u_j+2)} \Lambda^{(1)}(u; \{u_i\}), \end{aligned} \quad (67)$$

provided the parameters $\{u_j\}$ satisfy

$$\begin{aligned} &\frac{u_j-2}{u_j} \frac{(c_b + \frac{1}{2}u_j - s_b - \frac{1}{2})(c_b + \frac{1}{2}u_j + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}u_j + s_b - \frac{1}{2})(c_b + \frac{1}{2}u_j - s_b - \frac{3}{2})} \left(-\frac{u_j+2}{u_j}\right)^{2L} \\ &= \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j - u_i + 2)(u_j + u_i + 4)}{(u_j - u_i - 2)(u_j + u_i)} \Lambda^{(1)}(u_j; \{u_i\}). \end{aligned} \quad (68)$$

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem, which arises out of the r matrices from the first term in the right-hand side of Eq. (65), with the reduced boundary K matrices $K_{\pm}^{(1)}(u)$ from Eq. (39) where

$$\begin{aligned} A_{-}^{(1)}(u) &= -\frac{u^2 - 4c_a^2 + 4c_a + 4s_a(s_a + 1) - 1 - 4(u+1)\mathbf{S}_a^z}{(u+2c_a - 2s_a - 1)(u+2c_a + 2s_a + 1)}, \\ B_{-}^{(1)}(u) &= \frac{4(u+1)\mathbf{S}_a^-}{(u+2c_a - 2s_a - 1)(u+2c_a + 2s_a + 1)}, \\ C_{-}^{(1)}(u) &= \frac{4(u+1)\mathbf{S}_a^+}{(u+2c_a - 2s_a - 1)(u+2c_a + 2s_a + 1)}, \\ D_{-}^{(1)}(u) &= -\frac{u^2 - 4c_a^2 + 4c_a + 4s_a(s_a + 1) - 1 + 4(u+1)\mathbf{S}_a^z}{(u+2c_a - 2s_a - 1)(u+2c_a + 2s_a + 1)}, \end{aligned} \quad (69)$$

and

$$K_{+}^{(1)}(u) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & A_{+}^{(1)}(u) & B_{+}^{(1)}(u) \\ 0 & C_{+}^{(1)}(u) & D_{+}^{(1)}(u) \end{pmatrix}, \quad (70)$$

where

$$\begin{aligned} A_{+}^{(1)}(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4s_b(s_b + 1) + 1 - 4(u-2)\mathbf{S}_b^z}{(u+2c_b - 2s_b - 3)(u+2c_b + 2s_b - 1)}, \\ B_{+}^{(1)}(u) &= \frac{4(u-2)\mathbf{S}_b^-}{(u+2c_b - 2s_b - 3)(u+2c_b + 2s_b - 1)}, \\ C_{+}^{(1)}(u) &= \frac{4(u-2)\mathbf{S}_b^+}{(u+2c_b - 2s_b - 3)(u+2c_b + 2s_b - 1)}, \\ D_{+}^{(1)}(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4s_b(s_b + 1) + 1 + 4(u-2)\mathbf{S}_b^z}{(u+2c_b - 2s_b - 3)(u+2c_b + 2s_b - 1)}. \end{aligned} \quad (71)$$

Here $K_{\pm}^{(1)}(u)$, the boundary K matrices after the first nesting, follows from the relations

$$\begin{aligned} \check{D}_{dd}(u)|\Psi\rangle &\equiv \frac{u}{u+1} K_{dd}^{(1)}(u)|\Psi\rangle \\ &= \left(K_{-}(u)_{dd} - \frac{1}{u+1}\right) \left(-\frac{u}{u+2}\right)^{2L} |\Psi\rangle, \end{aligned} \quad (72)$$

$$\check{D}_{db}(u)|\Psi\rangle \equiv \frac{u}{u+1} K_{db}^{(1)}(u)|\Psi\rangle = K_{-}(u)_{db} \left(-\frac{u}{u+2}\right)^{2L} |\Psi\rangle.$$

Indeed, applying the monodromy matrix $T(u)$ and its ‘‘adjoint’’ $\tilde{T}(u)$ to the pseudovacuum, we have

$$\begin{aligned} T_{11}(u)|\Psi\rangle &= |\Psi\rangle, \quad T_{dd}(u)|\Psi\rangle = \left(-\frac{u}{u+2}\right)^L |\Psi\rangle, \\ T_{1d}(u)|\Psi\rangle &\neq 0, \quad T_{db}(u)|\Psi\rangle = 0, \quad T_{d1}(u)|\Psi\rangle = 0, \end{aligned} \quad (73)$$

$$\tilde{T}_{11}(u)|\Psi\rangle = |\Psi\rangle, \quad \tilde{T}_{dd}(u)|\Psi\rangle = \left(-\frac{u}{u+2}\right)^L |\Psi\rangle,$$

$$\tilde{T}_{1d}(u)|\Psi\rangle \neq 0, \quad \tilde{T}_{db}(u)|\Psi\rangle = 0, \quad \tilde{T}_{d1}(u)|\Psi\rangle = 0,$$

where $d \neq b$, $d, b = 2, 3, 4$. Then we have

$$\begin{aligned} \mathcal{A}(u)|\Psi\rangle &= |\Psi\rangle, \\ \mathcal{B}_d(u)|\Psi\rangle &\neq 0, \quad \mathcal{C}_d(u)|\Psi\rangle = 0, \\ \mathcal{D}_{db}(u)|\Psi\rangle &= \left(-\frac{u}{u+2}\right)^{2L} K_{-}(u)_{db} |\Psi\rangle, \\ \mathcal{D}_{dd}(u)|\Psi\rangle &= \left(-\frac{u}{u+2}\right)^{2L} \left(K_{-}(u)_{dd} - \frac{1}{u+1}\right) |\Psi\rangle \\ &\quad + \frac{1}{u+1} |\Psi\rangle. \end{aligned} \quad (74)$$

$$(u+1)T_{21}(u)\tilde{T}_{12}(u) + T_{22}(u)\tilde{T}_{22}(u) + T_{23}(u)\tilde{T}_{32}(u) + T_{24}(u)\tilde{T}_{42}(u)$$

$$= \tilde{T}_{11}(u)T_{11}(u) - (u-1)\tilde{T}_{12}(u)T_{21}(u)$$

$$+ \tilde{T}_{13}(u)T_{31}(u) + \tilde{T}_{14}(u)T_{41}(u),$$

$$(u+1)T_{21}(u)\tilde{T}_{13}(u) + T_{22}(u)\tilde{T}_{23}(u) + T_{23}(u)\tilde{T}_{33}(u)$$

$$+ T_{24}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{13}(u)T_{21}(u),$$

$$(u+1)T_{21}(u)\tilde{T}_{14}(u) + T_{22}(u)\tilde{T}_{24}(u) + T_{23}(u)\tilde{T}_{34}(u)$$

$$+ T_{24}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{14}(u)T_{21}(u),$$

$$T_{31}(u)\tilde{T}_{12}(u) - (u-1)T_{32}(u)\tilde{T}_{22}(u) + T_{33}(u)\tilde{T}_{32}(u)$$

$$+ T_{34}(u)\tilde{T}_{42}(u) = -u\tilde{T}_{22}(u)T_{32}(u),$$

$$\begin{aligned}
& T_{31}(u)\tilde{T}_{13}(u) - (u-1)T_{32}(u)\tilde{T}_{23}(u) + T_{33}(u)\tilde{T}_{33}(u) \\
& \quad + T_{34}(u)\tilde{T}_{43}(u) \\
& = \tilde{T}_{21}(u)T_{12}(u) + \tilde{T}_{22}(u)T_{22}(u) \\
& \quad - (u-1)\tilde{T}_{23}(u)T_{32}(u) + \tilde{T}_{24}(u)T_{42}(u), \\
& T_{31}(u)\tilde{T}_{14}(u) - (u-1)T_{32}(u)\tilde{T}_{24}(u) + T_{33}(u)\tilde{T}_{34}(u) \\
& \quad + T_{34}(u)\tilde{T}_{44}(u) = -u\tilde{T}_{24}(u)T_{32}(u), \\
& T_{41}(u)\tilde{T}_{12}(u) + T_{42}(u)\tilde{T}_{22}(u) - (u-1)T_{43}(u)\tilde{T}_{32}(u) \\
& \quad + T_{44}(u)\tilde{T}_{42}(u) = u\tilde{T}_{32}(u)T_{43}(u), \\
& T_{41}(u)\tilde{T}_{13}(u) + T_{42}(u)\tilde{T}_{23}(u) - (u-1)T_{43}(u)\tilde{T}_{33}(u) \\
& \quad + T_{44}(u)\tilde{T}_{43}(u) = -u\tilde{T}_{33}(u)T_{43}(u), \\
& T_{41}(u)\tilde{T}_{14}(u) + T_{42}(u)\tilde{T}_{24}(u) - (u-1)T_{43}(u)\tilde{T}_{34}(u) \\
& \quad + T_{44}(u)\tilde{T}_{44}(u) \\
& = \tilde{T}_{31}(u)T_{13}(u) + \tilde{T}_{32}(u)T_{23}(u) \\
& \quad + \tilde{T}_{33}(u)T_{33}(u) - (u-1)\tilde{T}_{34}(u)T_{43}(u), \quad (75)
\end{aligned}$$

which come from a variant of the (graded) Yang-Baxter algebra (9) with the R matrix (27), as in Eq. (47).

Noticing the change $u \rightarrow u+1$ with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix $\tau^{(1)}(u)$ may be recognized as that for the inhomogeneous $\text{su}(3)$ t - J open chain

interacting with the Kondo impurities of arbitrary spins, which has been diagonalized in Ref. 13. The final result is

$$\begin{aligned}
\Lambda^{(1)}(u; \{u_j\}) &= \frac{u-2}{u} \frac{(c_b + \frac{1}{2}u - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}u + s_b - \frac{1}{2})} \frac{(c_b + \frac{1}{2}u + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}u - s_b - \frac{3}{2})} \\
&\quad \times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha+2)(u+v_\alpha+2)}{(u-v_\alpha)(u+v_\alpha)} \\
&\quad - \frac{u+1}{u} \prod_{j=1}^N \frac{(u-u_j)(u+u_j+2)}{(u-u_j-2)(u+u_j)} \\
&\quad \times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha-2)(u+v_\alpha-2)}{(u-v_\alpha)(u+v_\alpha)} \\
&\quad \times \Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}), \quad (76)
\end{aligned}$$

provided the parameters $\{v_\alpha\}$ satisfy

$$\begin{aligned}
& \frac{v_\alpha-2}{v_\alpha-1} \frac{(c_b + \frac{1}{2}v_\alpha - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}v_\alpha + s_b - \frac{1}{2})} \frac{(c_b + \frac{1}{2}v_\alpha + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}v_\alpha - s_b - \frac{3}{2})} \\
&\quad \times \prod_{j=1}^N \frac{(v_\alpha - u_j - 2)(v_\alpha + u_j)}{(v_\alpha - u_j)(v_\alpha + u_j + 2)} \\
&\quad \times \prod_{\substack{\xi=1 \\ \xi \neq \alpha}}^{M_1} \frac{(v_\alpha - v_\xi + 2)(v_\alpha + v_\xi + 2)}{(v_\alpha - v_\xi - 2)(v_\alpha + v_\xi - 2)} \\
&= -\Lambda^{(2)}(v_\alpha; \{u_i\}, \{v_\beta\}). \quad (77)
\end{aligned}$$

Here $\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\})$ is the eigenvalue of the transfer matrix $\tau^{(2)}(u)$ for the M_1 -site inhomogeneous XXX open chain interacting with the Kondo impurities of arbitrary spins,

$$\begin{aligned}
\Lambda^{(2)}(u; \{u_j\}, \{v_\alpha\}) &= - \frac{(c_b + \frac{1}{2}u - s_b - \frac{1}{2})}{(c_b + \frac{1}{2}u + s_b - \frac{1}{2})} \frac{(c_b + \frac{1}{2}u + s_b + \frac{1}{2})}{(c_b + \frac{1}{2}u - s_b - \frac{3}{2})} \prod_{\gamma=a,b} \frac{c_\gamma - \frac{1}{2}u + s_\gamma + \frac{1}{2}}{c_\gamma + \frac{1}{2}u + s_\gamma + \frac{1}{2}} \\
&\quad \times \left\{ \frac{u-2}{u-1} \prod_{\beta=1}^{M_2} \frac{(u-w_\beta+2)(u+w_\beta)}{(u-w_\beta)(u+w_\beta-2)} + \frac{u}{u-1} \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}u + s_\gamma - \frac{1}{2})}{(c_\gamma + \frac{1}{2}u - s_\gamma - \frac{1}{2})} \frac{(c_\gamma - \frac{1}{2}u - s_\gamma + \frac{1}{2})}{(c_\gamma - \frac{1}{2}u + s_\gamma + \frac{1}{2})} \right. \\
&\quad \left. \times \prod_{\alpha=1}^{M_1} \frac{(u-v_\alpha)(u+v_\alpha)}{(u-v_\alpha-2)(u+v_\alpha-2)} \prod_{\beta=1}^{M_2} \frac{(u-w_\beta-2)(u+w_\beta-4)}{(u-w_\beta)(u+w_\beta-2)} \right\}, \quad (78)
\end{aligned}$$

provided the parameters $\{w_\beta\}$ satisfy

$$\prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}w_\beta + s_\gamma - \frac{1}{2})(c_\gamma - \frac{1}{2}w_\beta - s_\gamma + \frac{1}{2})}{(c_\gamma - \frac{1}{2}w_\beta + s_\gamma + \frac{1}{2})(c_\gamma + \frac{1}{2}w_\beta - s_\gamma - \frac{1}{2})} \prod_{\alpha=1}^{M_1} \frac{(w_\beta - v_\alpha)(w_\beta + v_\alpha)}{(w_\beta - v_\alpha - 2)(w_\beta + v_\alpha - 2)} = \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 4)}. \quad (79)$$

After a shift of the parameters $u_j \rightarrow u_j - 1, w_\beta \rightarrow w_\beta + 1$, the Bethe ansatz equations (68), (77), and (79) may be rewritten as follows:

$$\left(\frac{u_j+1}{u_j-1} \right)^{2L} \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(u_j - u_i + 2)(u_j + u_i + 2)}{(u_j - u_i - 2)(u_j + u_i - 2)} = \prod_{\alpha=1}^{M_1} \frac{(u_j - v_\alpha + 1)(u_j + v_\alpha + 1)}{(u_j + v_\alpha - 1)(u_j - v_\alpha - 1)},$$

$$\begin{aligned}
& \prod_{\gamma=a,b} \frac{c_\gamma + \frac{1}{2}v_\alpha + s_\gamma + \frac{1}{2}}{c_\gamma - \frac{1}{2}v_\alpha + s_\gamma + \frac{1}{2}} \prod_{j=1}^N \frac{(v_\alpha - u_j - 1)(v_\alpha + u_j - 1)}{(v_\alpha - u_j + 1)(v_\alpha + u_j + 1)} \\
&= \prod_{\beta=1}^{M_2} \frac{(v_\alpha - w_\beta + 1)(v_\alpha + w_\beta + 1)}{(v_\alpha - w_\beta - 1)(v_\alpha + w_\beta - 1)} \\
&\times \prod_{\substack{\zeta=1 \\ \zeta \neq \alpha}}^{M_1} \frac{(v_\alpha - v_\zeta - 2)(v_\alpha + v_\zeta - 2)}{(v_\alpha - v_\zeta + 2)(v_\alpha + v_\zeta + 2)}, \\
& \prod_{\gamma=a,b} \frac{(c_\gamma + \frac{1}{2}w_\beta + s_\gamma)(c_\gamma - \frac{1}{2}w_\beta - s_\gamma)}{(c_\gamma - \frac{1}{2}w_\beta + s_\gamma)(c_\gamma + \frac{1}{2}w_\beta - s_\gamma)} \\
&\times \prod_{\alpha=1}^{M_1} \frac{(w_\beta - v_\alpha + 1)(w_\beta + v_\alpha + 1)}{(w_\beta - v_\alpha - 1)(w_\beta + v_\alpha - 1)} \\
&= \prod_{\substack{\delta=1 \\ \delta \neq \beta}}^{M_2} \frac{(w_\beta - w_\delta + 2)(w_\beta + w_\delta + 2)}{(w_\beta - w_\delta - 2)(w_\beta + w_\delta - 2)}, \quad (80)
\end{aligned}$$

with the corresponding energy eigenvalue E of the model in Eq. (53).

$$\tilde{L}(u) = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & u + 2c_a + 2s_a + 1 + 2\mathbf{S}^z & 2\mathbf{S}^- \\ 0 & 0 & 2\mathbf{S}^+ & u + 2c_a + 2s_a + 1 - 2\mathbf{S}^z \end{pmatrix}, \quad (82)$$

which constitutes a realization of the Yang-Baxter algebra (9) when ϵ tends to 0. The recent work of Frahm and Slavnov¹⁴ confirms the existence of such nonregular solutions by means of a projection method.

Finally, we would like to stress that here we have only considered the case of Kondo impurities in these extended Hubbard models that are based on the $\mathfrak{sl}(2)$ subalgebra of the bulk symmetry of the models. It is of course possible to consider other boundary impurities corresponding to different subalgebra embeddings such as $\mathfrak{sl}(1|1)$ for the $\mathfrak{gl}(2|2)$, $\mathfrak{gl}(3|1)$ cases or $\mathfrak{sl}(3)$ for the $\mathfrak{gl}(3|1)$, $\mathfrak{gl}(4)$ models and even $\mathfrak{gl}(2|1)$ for $\mathfrak{gl}(3|1)$, $\mathfrak{gl}(2|2)$. For the case of t - J models such other types of integrable boundary impurities have been studied in Ref. 21.

ACKNOWLEDGMENTS

H.-Q. Zhou and X.-Y. Ge were supported by the OPRS and UQPRS. H.-Q. Zhou was also supported by the NNSF of China. Jon Links and Mark D. Gould were supported by the Australian Research Council.

APPENDIX: DERIVATION OF THE NON- c -NUMBER BOUNDARY K MATRICES

In this appendix, we sketch the procedure of solving the (\mathbf{Z}_2 -graded) reflection equation for $K_-(u)$. To describe

IV. CONCLUSION

In conclusion, we have studied integrable Kondo problems describing two boundary impurities coupled to one-dimensional extended Hubbard open chains. The quantum integrability of these systems follows from the fact that the Hamiltonians in each case are derived from a one-parameter family of commuting transfer matrices. Moreover, the Bethe ansatz equations and expressions for the energies are derived by means of the algebraic Bethe ansatz approach. We would like to emphasize that the boundary K matrices found here are nonregular in that they cannot be factorized into the product of a c -number K matrix and the local monodromy matrices. However, similar to the cases discussed in Refs. 11 and 13, it is possible to introduce a singular local monodromy matrix $\tilde{L}(u)$ to express the boundary K matrix $K_-(u)$ as

$$K_-(u) = \tilde{L}(u)\tilde{L}^{-1}(-u), \quad (81)$$

where, for example, in the case of the superalgebra $\mathfrak{gl}(2|2)$ model,

integrable Kondo impurities coupled with the one-dimensional supersymmetric extended Hubbard model open chain, it is reasonable to assume that

$$K_-(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A(u) & B(u) \\ 0 & 0 & C(u) & D(u) \end{pmatrix}. \quad (A1)$$

Throughout, we have omitted all the subscripts for brevity, reflecting that the fermionic degrees of freedom do not occur, as it should be for a magnetic impurity. For the R matrix (8), one may get from the reflection equation (10) 54 functional equations, of which 14 are identities. After some algebraic analysis, together with the $\text{SU}(2)$ symmetry, we may assume that

$$\begin{aligned}
A(u) &= \alpha(u) + \beta(u)\mathbf{S}^z, & B(u) &= \beta(u)\mathbf{S}^-, \\
C(u) &= \beta(u)\mathbf{S}^+, & D(u) &= \alpha(u) - \beta(u)\mathbf{S}^z. \quad (A2)
\end{aligned}$$

There are 10 equations automatically satisfied and 10 same equations, leaving only 20 equations left to be solved:

$$\begin{aligned}
 A(u_1)B(u_2) + B(u_1)D(u_2) &= A(u_2)B(u_1) + B(u_2)D(u_1), \\
 C(u_1)A(u_2) + D(u_1)C(u_2) &= C(u_2)A(u_1) + D(u_2)C(u_1), \\
 u_-[A(u_1)B(u_2) + B(u_1)D(u_2)] &= u_+[B(u_1) - B(u_2)], \\
 u_-[A(u_2)B(u_1) + B(u_2)D(u_1)] &= u_+[B(u_1) - B(u_2)], \\
 u_-[C(u_1)A(u_2) + D(u_1)C(u_2)] &= u_+[C(u_1) - C(u_2)], \\
 u_-[C(u_2)A(u_1) + D(u_2)C(u_1)] &= u_+[C(u_1) - C(u_2)], \\
 u_-[A(u_1)A(u_2) + B(u_1)C(u_2) - 1] &= u_+[A(u_1) - A(u_2)], \\
 u_-[A(u_2)A(u_1) + B(u_2)C(u_1) - 1] &= u_+[A(u_1) - A(u_2)], \\
 u_-[C(u_1)B(u_2) + D(u_1)D(u_2) - 1] &= u_+[D(u_1) - D(u_2)], \\
 u_-[C(u_2)B(u_1) + D(u_2)D(u_1) - 1] &= u_+[D(u_1) - D(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_1)B(u_2) + B(u_1)D(u_2)] \\
 &= 2u_+[D(u_2)B(u_1) - D(u_1)B(u_2)] \\
 &\quad + u_+u_-[D(u_2)B(u_1) - B(u_1)D(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_2)B(u_1) + B(u_2)D(u_1)] \\
 &= 2u_+[B(u_1)A(u_2) - B(u_2)A(u_1)] \\
 &\quad + u_+u_-[B(u_1)A(u_2) - A(u_2)B(u_1)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[C(u_1)A(u_2) + D(u_1)C(u_2)] \\
 &= 2u_+[A(u_2)C(u_1) - A(u_1)C(u_2)] \\
 &\quad + u_+u_-[A(u_2)C(u_1) - C(u_1)A(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[C(u_2)A(u_1) + D(u_2)C(u_1)] \\
 &= 2u_+[C(u_1)D(u_2) - C(u_2)D(u_1)] \\
 &\quad + u_+u_-[C(u_1)D(u_2) - D(u_2)C(u_1)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_2)A(u_1) + B(u_2)C(u_1) \\
 - C(u_1)B(u_2) - D(u_1)D(u_2)] \\
 &= 2u_+[A(u_1)D(u_2) - A(u_2)D(u_1)] \\
 &\quad - u_+u_-[B(u_2)C(u_1) - C(u_1)B(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_1)A(u_2) + B(u_1)C(u_2) \\
 - C(u_2)B(u_1) - D(u_2)D(u_1)] \\
 &= 2u_+[D(u_2)A(u_1) - D(u_1)A(u_2)] \\
 &\quad - u_+u_-[B(u_1)C(u_2) - C(u_2)B(u_1)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_1)B(u_2) + B(u_1)D(u_2)] \\
 &\quad + u_+u_-[A(u_1)B(u_2) - B(u_2)A(u_1)] \\
 &= 2u_+[A(u_2)B(u_1) - A(u_1)B(u_2)] + 4[A(u_2)B(u_1) \\
 &\quad + B(u_2)D(u_1) - A(u_1)B(u_2) - B(u_1)D(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[A(u_2)B(u_1) + B(u_2)D(u_1)] \\
 &\quad + u_+u_-[B(u_2)D(u_1) - D(u_1)B(u_2)] \\
 &= 2u_+[B(u_1)D(u_2) - B(u_2)D(u_1)] + 4[A(u_1)B(u_2) \\
 &\quad + B(u_1)D(u_2) - A(u_2)B(u_1) - B(u_2)D(u_1)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[C(u_1)A(u_2) + D(u_1)C(u_2)] \\
 &\quad + u_+u_-[D(u_1)C(u_2) - C(u_2)D(u_1)] \\
 &= 2u_+[D(u_2)C(u_1) - D(u_1)C(u_2)] + 4[C(u_2)A(u_1) \\
 &\quad + D(u_2)C(u_1) - C(u_1)A(u_2) - D(u_1)C(u_2)],
 \end{aligned}$$

$$\begin{aligned}
 2u_-[C(u_2)A(u_1) + D(u_2)C(u_1)] \\
 &\quad + u_+u_-[C(u_2)A(u_1) - A(u_1)C(u_2)] \\
 &= 2u_+[C(u_1)A(u_2) - C(u_2)A(u_1)] + 4[C(u_1)A(u_2) \\
 &\quad + D(u_1)C(u_2) - C(u_2)A(u_1) - D(u_2)C(u_1)],
 \end{aligned} \tag{A3}$$

with $u_+ = u_1 + u_2, u_- = u_1 - u_2$. Substituting Eq. (A2) into these equations we find that all these equations are reduced to the following three equations:

$$\begin{aligned}
 u_+[\alpha(u_1) - \alpha(u_2)] \\
 &= u_-[-1 + \alpha(u_1)\alpha(u_2) + s(s+1)\beta(u_1)\beta(u_2)], \\
 u_+[\beta(u_1) - \beta(u_2)] \\
 &= u_-[\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1) - \beta(u_1)\beta(u_2), \\
 &\quad 2u_+[\alpha(u_2)\beta(u_1) - \alpha(u_1)\beta(u_2)] \\
 &= 2u_-[\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1)] \\
 &\quad - u_-[u_+ + 2]\beta(u_1)\beta(u_2)].
 \end{aligned} \tag{A4}$$

Taking the limit $u_1 \rightarrow u_2$, these equations become

$$\begin{aligned}
 \frac{d\alpha(u)}{du} &= \frac{1}{2u}[-1 + \alpha(u)^2 + s(s+1)\beta(u)^2], \\
 \frac{d\beta(u)}{du} &= \frac{1}{2u}[2\alpha(u)\beta(u) - \beta(u)^2], \\
 \alpha(u)\frac{d\beta(u)}{du} - \beta(u)\frac{d\alpha(u)}{du} \\
 &= \frac{1}{2u}[2\alpha(u)\beta(u) - (u+1)\beta(u)^2].
 \end{aligned} \tag{A5}$$

Solving the first two equations, we have

$$\begin{aligned}
 \alpha(u) &= \frac{(c_1c_2 - u^2)(2s+1) + (c_2 - c_1)u}{(2s+1)(c_1 - u)(c_2 - u)}, \\
 \beta(u) &= \frac{2(c_2 - c_1)u}{(2s+1)(c_1 - u)(c_2 - u)},
 \end{aligned} \tag{A6}$$

where c_1 and c_2 are integration constants. Substituting these results into the third equation in Eq. (A5), we may establish a relation between c_1 and c_2 : $c_2 = c_1 - 4s - 2$. This is just the non- c -number boundary K matrix (14) (after a redefini-

tion of the constant: $c_1 \rightarrow 2c + 4s + 2$). A similar construction also works for the quantum R matrix in Eqs. (21) and (27).

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