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Level-one highest weight representation of $U_q[\mathrm{sl}(\hat{N}|1)]$ and Bosonization of the multicomponent Super $t-J$ model

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We study the level-one irreducible highest weight representations of the quantum affine superalgebra $U_q[\mathrm{sl}(\hat{N}|1)]$, and calculate their characters and supercharacters. We obtain bosonized q-vertex operators acting on the irreducible $U_q[\mathrm{sl}(\hat{N}|1)]$ modules and derive the exchange relations satisfied by the vertex operators. We give the bosonization of the multicomponent super $t-J$ model by using the bosonized vertex operators.
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I. INTRODUCTION

The purpose of this paper is twofold. One is to study irreducible highest weight representations and q-vertex operators[1] of the quantum affine superalgebra $U_q[\mathrm{sl}(\hat{N}|1)]$, $N>2$. Another one is to apply these results to bosonize the multicomponent super $t-J$ model on an infinite lattice.

We shall adapt the bosonization technique initiated in Refs. 2 and 3, which turns out to be very powerful in constructing highest weight representations and q-vertex operators. Recently, free bosonic realizations of the level-one representations and ''elementary'' q-vertex operators have been obtained for $U_q[\mathrm{sl}(\hat{M}|N)]$, $M\neq N$[4] and $U_q[\mathrm{gl}(\hat{N}|N)]$.[5] However, these free boson representations are not irreducible in general. Moreover, the elementary q-vertex operators obtained in Refs. 4 and 5 were determined solely from their commutation relations with the bosonized Drinfeld generators[6] of the relevant algebras, and thus one can ask on which representations these bosonized q-vertex operators act. To construct irreducible highest weight representations and q-vertex operators acting on them, we need to study in details the structure of the bosonic Fock space generated by the free boson fields. This has been done for $U_q[\mathrm{sl}(\hat{2}|1)]$[4,7] and $U_q[\mathrm{gl}(\hat{N}|N)]$, $N\leqslant 2$.[8] In this paper we treat the $U_q[\mathrm{sl}(\hat{N}|1)]$ $(N>2)$ case.

Irreducible highest weight representations and bosonized q-vertex operators acting on them play an essential role in the algebraic analysis method of lattice integrable models, which was invented by the Kyoto group and collaborators.[9,10] In this approach, the following assumption is the vital key.

$$\text{``the physical space of states of the model''}=\bigoplus_{\alpha,\alpha'} V(\lambda_\alpha)\otimes V(\lambda_{\alpha'})^{*S}, \tag{I.1}$$

where $V(\lambda_\alpha)$ is the level-one irreducible highest weight module of the underlying quantum affine algebras and $V(\lambda_\alpha)^{*S}$ is the dual module of $V(\lambda_\alpha)$. By this method, various integrable models have been analyzed such as the higher spin XXZ chains,[11–13] the higher rank cases,[14,15] the twisted $A_2^{(2)}$ case,[16] and the face type statistical models.[17,18]

Spin chain models with quantum superalgebra symmetries have been the focus of recent studies in the context of strongly correlated fermion systems.[19–23] It is natural to generalize the algebraic analysis method to treat super spin chains on an infinite lattice. In Ref. 7, the q-deformed supersymmetric $t-J$ model which has $U_q'[\mathrm{sl}(\hat{2}|1)]$ as its non-abelian symmetry has been analyzed. However, the super case is fundamentally different from the nonsuper case. Unlike the latter, $U_q[\mathrm{sl}(\hat{2}|1)]$ has infinite number of level-one irreducible highest weight representations and the bosonized q-vertex operators act in all of them. This leads to[7] the assumption that for the q-deformed supersymmetric $t-J$ model α, α' in (I.1) take infinite number of integer values.

In this paper we extend the work[7] to treat the multicomponent $t-J$ model with $U_q'[\mathrm{sl}(\hat{N}|1)]$ $(N>2)$ symmetry. As we shall see, the level-one irreducible highest weight representations of $U_q[\mathrm{sl}(\hat{N}|1)]$ $(N>2)$ have similar structures as the $N=2$ case. So we shall make the assumption that the physical space of states of the multicomponent $t-J$ model on an infinite lattice is of the form (I.1) with α, α' being any integers.

This paper is organized as follows. After presenting some necessary preliminaries, in Sec. III we construct the level-one irreducible highest weight representations of $U_q[\mathrm{sl}(\hat{N}|1)]$ and calculate their (super)characters by means of the BRST resolution. In Sec. IV, we compute the exchange relations of the q-vertex operators and show that they form the graded Faddeev–Zamolodchikov algebra. In Sec. V, we consider the application of these results to the multicomponent super $t-J$ model on an infinite lattice. Generalizing the Kyoto group's work,[9] we give the bosonization of this model using the bosonized vertex operators of $U_q[\mathrm{sl}(\hat{N}|1)]$. Finally, we compute the one-point correlation functions of the local operators and give an integral expression of the correlation functions.

II. PRELIMINARIES

A. Quantum affine superalgebra $U_q[\mathrm{sl}(\hat{N}|1)]$

Let us introduce orthonormal basis $\{\epsilon_i'|i=1,2,...,N+1\}$ with the bilinear form $(\epsilon_i',\epsilon_j')=\nu_i\delta_{ij}$, where $\nu_i=1$ for $i\neq N+1$ and $\nu_{N+1}=-1$. The classical fundamental weights are defined by $\overline{\Lambda}_i=\Sigma_{j=1}^{i}\epsilon_j$ $(i=1,2,...,N)$, with $\epsilon_i=\epsilon_i'-[\nu_i/(N-1)]\Sigma_{j=1}^{N+1}\epsilon_j'$. Introduce the affine weight Λ_0 and the null root δ having $(\Lambda_0,\epsilon_i')=(\delta,\epsilon_i')=0$ for $i=1,2,...,N+1$ and $(\Lambda_0,\Lambda_0)=(\delta,\delta)=0$, $(\Lambda_0,\delta)=1$. The affine simple roots and fundamental weights are given by

$$\alpha_i=\nu_i\epsilon_i'-\nu_{i+1}\epsilon_{i+1}',\quad i=1,2,...,N,\quad \alpha_0=\delta-\sum_{i=1}^{N}\alpha_i,$$
$$\Lambda_0=\Lambda_0,\quad \Lambda_i=\Lambda_0+\overline{\Lambda}_i,\quad i=1,2,...,N. \tag{II.1}$$

The Cartan matrix of the affine superalgebra $\mathrm{sl}(\hat{N}|1)$ reads as

$$(a_{ij})=\begin{pmatrix} 0 & -1 & & & & 1 & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ 1 & & & & -1 & 0 \end{pmatrix}\quad (i,j=0,1,2,...,N). \tag{II.2}$$

The quantum affine superalgebra $U_q[\mathrm{sl}(\hat{N}|1)]$ is a q-analog of the universal enveloping algebra of $\mathrm{sl}(\hat{N}|1)$ generated by the Chevalley generators $\{e_i,f_i,q^{h_i},d|i=0,1,2,...,N\}$, where d is the usual derivation operator. The Z_2-grading of the generators are $[e_0]=[f_0]=[e_N]=[f_N]=1$ and zero otherwise. The defining relations are

$$[h_i,h_j]=0,\quad h_i d=d h_i,\quad [d,e_i]=\delta_{i,0}e_i,\quad [d,f_i]=-\delta_{i,0}f_i,$$

$$q^{h_i}e_j q^{-h_i}=q^{a_{ij}}e_j,\quad q^{h_i}f_j q^{-h_i}=q^{-a_{ij}}f_j,\quad [e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}},$$

$$[e_i,e_j]=[f_i,f_j]=0,\quad \text{for } a_{ij}=0,$$

$$[e_j,[e_j,e_i]_{q^{-1}}]_q=0,\quad [f_j,[f_j,f_i]_{q^{-1}}]_q=0\quad \text{for } |a_{ij}|=1,\quad j\neq 0,\ N.$$

Here and throughout, $[a,b]_x\equiv ab-(-1)^{[a][b]}xba$ and $[a,b]\equiv[a,b]_1$. We do not write down the extra q-Serre relations which can be obtained by using Yamane's Dynkin diagram procedure.[24]

$U_q[\mathrm{sl}(\hat{N}|1)]$ is a $\mathbf{Z}_2$-graded quasi-triangular Hopf algebra endowed with the following coproduct Δ, counit ϵ and antipode S:

$$\begin{gathered}\Delta(h_i)=h_i\otimes 1+1\otimes h_i,\quad \Delta(d)=d\otimes 1+1\otimes d,\\ \Delta(e_i)=e_i\otimes 1+q^{h_i}\otimes e_i,\quad \Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i,\\ \epsilon(e_i)=\epsilon(f_i)=\epsilon(h)=0,\\ S(e_i)=-q^{-h_i}e_i,\quad S(f_i)=-f_i q^{h_i},\quad S(h)=-h,\end{gathered}\tag{II.3}$$

where $i=0,1,...,N$. Notice that the antipode S is a $\mathbf{Z}_2$-graded algebra antihomomorphism. Namely, for any homogeneous elements $a,b\in U_q[\mathrm{sl}(\hat{N}|1)]$ $S(ab)=(-1)^{[a][b]}S(b)S(a)$, which extends to inhomogeneous elements through linearity. Moreover,

$$S^2(a)=q^{-2\rho}aq^{2\rho},\quad \forall a\in U_q[\mathrm{sl}(\hat{N}|1)],\tag{II.4}$$

where ρ is an element in the Cartan subalgebra such that $(\rho,\alpha_i)=(\alpha_i,\alpha_i)/2$ for any simple root α_i, $i=0,1,2,...,N$. Explicitly,

$$\rho=(N-1)d+\overline{\rho}=(N-1)d+\frac{1}{2}\sum_{k=1}^{N}(N-2k)\epsilon'_k-\frac{1}{2}N\epsilon'_{N+1},\tag{II.5}$$

which $\overline{\rho}$ is the half-sum of positive roots of $\mathrm{sl}(N|1)$. The multiplication rule on the tensor products is $\mathbf{Z}_2$ graded: $(a\otimes b)(a'\otimes b')=(-1)^{[b][a']}(aa'\otimes bb')$ for any homogeneous elements $a,b,a',b'\in U_q[\mathrm{sl}(\hat{N}|1)]$.

$U_q[\mathrm{sl}(\hat{N}|1)]$ can also be realized in terms of the Drinfeld generators[6] $\{X_m^{\pm,i},H_n^i,q^{\pm H_0^i},c,d|m\in\mathbf{Z},n\in\mathbf{Z}-\{0\},i=1,2,...,N\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators is given by $[X_m^{\pm,N}]=1$ for $m\in\mathbf{Z}$ and zero otherwise. The relations satisfied by the Drinfeld generators read[24,25]

$$\begin{gathered}[c,a]=[d,H_0^i]=[H_0^i,H_n^j]=0,\quad [d,H_n^i]=nH_n^i,\quad \forall a\in U_q[\mathrm{sl}(\hat{N}|1)],\\ [d,X_n^{\pm,i}]=nX_n^{\pm,i},\quad q^{H_0^j}X_n^{\pm,i}q^{-H_0^j}=q^{\pm a_{ij}}X_n^{\pm,i},\\ [H_n^i,H_m^j]=\delta_{n+m,0}\frac{[a_{ij}n]_q[nc]_q}{n},\quad [H_n^i,X_m^{\pm,j}]=\pm\frac{[a_{ij}n]_q}{n}X_{n+m}^{\pm j}q^{\pm|n|c/2},\\ [X_n^{+,i},X_m^{-j}]=\frac{\delta_{ij}}{q-q^{-1}}(q^{(c/2)(n-m)}\psi_{n+m}^{+,i}-q^{-(c/2)(n-m)}\psi_{n+m}^{-,i}),\\ [X_n^{\pm,i},X_m^{\pm,j}]=0\quad \text{for } a_{ij}=0,\end{gathered}\tag{II.6}$$

$$[X^{\pm,i}_{n+1},X^{\pm j}_{m}]_{q^{\pm a_{ij}}}-[X^{\pm,j}_{m+1},X^{\pm,i}_{n}]_{q^{\pm a_{ij}}}=0 \quad \text{for} \quad a_{ij}\neq 0,$$

$$\text{Sym}_{l,m}[X^{\pm,i}_{l},[X^{\pm,i}_{m},X^{\pm,j}_{n}]_{q^{-1}}]_{q}=0 \quad \text{for} \quad a_{ij}=0, \ \ i\neq N,$$

where $\Sigma_{n\in Z}\psi^{\pm,j}_{n}z^{-n}=q^{\pm H^{j}_{0}}\exp(\pm(q-q^{-1})\Sigma_{n>0}H^{j}_{\pm n}z^{\mp n})$, and the symbol $\text{Sym}_{k,l}$ means symmetrization with respect to k and l. We used the standard notation $[x]_{q}=(q^{x}-q^{-x})/(q-q^{-1})$. The Chevalley generators are related to the Drinfeld generators by the formulas:

$$h_{i}=H^{i}_{0}, \quad e_{i}=X^{+,i}_{0}, \quad f_{i}=X^{-,i}_{0}, \quad i=1,2,...,N, \quad h_{0}=c-\sum_{k=1}^{N}H^{k}_{0},$$

$$e_{0}=-[X^{-,N}_{0},[X^{-,N-1}_{0},\ldots,[X^{-,2}_{0},X^{-,1}_{1}]_{q^{-1}}\cdots]_{q^{-1}}q^{\Sigma^{N}_{k=1}H^{k}_{0}}, \tag{II.7}$$

$$f_{0}=q\sum_{k=1}^{N}H^{k}_{0}H^{k}_{0}[[\cdots[[X^{+,1}_{-1},X^{+,2}_{0},]_{q},\cdots,X^{+,N-1}_{0}]_{q},X^{+,N}_{0}]_{q}.$$

B. Free Bosonic realization of the quantum affine superalgebra $U_q[\widehat{sl(N|1)}]$ at level one

Introduce bosonic oscillators $\{a^{i}_{n},b_{n},c_{n},Q_{a^{i}},Q_{b},Q_{c}|n\in\mathbf{Z},i=1,2,...,N,\}$ which satisfy the commutation relations

$$[a^{i}_{n},a^{j}_{m}]=\delta_{n+m,0}\delta_{ij}\frac{[n]_{q}[n]_{q}}{n}, \quad [a^{i}_{0},Q_{a^{j}}]=\delta_{ij},$$

$$[b_{n},b_{m}]=-\delta_{n+m,0}\frac{[n]^{2}_{q}}{n}, \quad [b_{0},Q_{b}]=-1, \tag{II.8}$$

$$[c_{n},c_{m}]=\delta_{n+m,0}\frac{[n]^{2}_{q}}{n}, \quad [c_{0},Q_{c}]=1.$$

The remaining commutation relations are zero. Define $\{h^{i}_{m}|i=1,2,...,N,m\in\mathbf{Z}\}$:

$$h^{i}_{m}=a^{i}_{m}q^{-|m|/2}-a^{i+1}q^{|m|/2}, \quad Q_{h_{i}}=Q_{a^{i}}-Q_{a^{i+1}}, \quad i=1,2,...,N-1,$$
$$h^{N}_{m}=a^{N}_{m}q^{-|m|/2}+b_{m}q^{-|m|/2}, \quad Q_{h_{N}}=Q_{a^{N}}+Q_{b}. \tag{II.9}$$

Let us introduce the notation

$$h^{j}(z;\kappa)=Q_{h_{j}}+h^{j}_{0}\ln z-\sum_{n\neq 0}\frac{h^{j}_{n}}{[n]_{q}}q^{\kappa|n|}z^{-n}.$$

The bosonic fields $c(z;\beta)$, $b(z;\beta)$, and $h^{*}_{j}(z;\beta)$ are defined in the same way. Define the Drinfeld currents, $X^{\pm,i}(z)=\Sigma_{n\in\mathbf{Z}}X^{\pm,i}_{n}z^{-n-1}$, $i=1,2,...,N$, and the q-differential operator $\partial_{z}f(z)=[f(qz)-f(q^{-1}z)]/(q-q^{-1})z$. Then, the Drinfeld generators of $U_{q}[\widehat{sl(N|1)}]$ at level one can be realized by the free boson fields as[4]

$$c=1, \quad H^{i}_{m}=h^{i}_{m}, \quad X^{+,N}(z)=:e^{-h^{N}(z;-1/2)}e^{c(z;0)}:e^{-\sqrt{-1}\pi\sum_{i=1}^{N-1}a^{i}_{0}},$$

$$X^{-,N}(z)=:e^{-h^{N}(z;1/2)}\partial_{z}\{e^{-c(z;0)}\}:e^{\sqrt{-1}\pi\sum_{i=1}^{N-1}a^{i}_{0}}, \tag{II.10}$$

$$X^{\pm,i}(z)=\pm:e^{\pm h^i(z;\mp 1/2)}:e^{\pm\sqrt{-1}\pi a_0^i},\quad i=1,2,...,N-1.$$

C. Bosonization of level-one vertex operators

In order to construct the vertex operators of $U_q[\mathrm{sl}(\hat{N}|1)]$, we firstly consider the level-zero representations (i.e., the evaluation representations) of $U_q[\mathrm{sl}(\hat{N}|1)]$.

Let $E_{i,j}$ be the $(N+1)\times(N+1)$ matrix whose (i,j) element is unity and zero elsewhere. Let $\{v_1,v_2,\ldots,v_{N+1}\}$ be the basis vectors of the $(N+1)$-dimensional graded vector space V. The $\mathbf{Z}_2$-grading of these basis vectors is chosen to be $[v_i]=(\nu_i+1)/2$. The $(N+1)$-dimensional level-zero representation V_z of $U_q[\mathrm{sl}(\hat{N}|1)]$ is given by

$$\begin{aligned} &e_i=E_{i,i+1},\quad f_i=\nu_i E_{i+1,i},\quad t_i=q^{\nu_i E_{i,i}-\nu_{i+1}E_{i+1,i+1}},\\ &e_0=-zE_{N+1,1},\quad f_0=z^{-1}E_{1,N+1},\quad t_0=q^{-E_{1,1}-E_{N+1,N+1}}, \end{aligned} \tag{II.11}$$

where $i=1,...,N$. Let V_z^{*S} be the left dual module of V_z, defined by

$$\pi_{V_z^{*S}}(a)=\pi_{V_z}(S(a))^{\mathrm{st}},\quad \forall a\in U_q[\mathrm{sl}(\hat{N}|1)], \tag{II.12}$$

where st denotes the supertransposition.

Now, we study the level-one vertex operators[1] of $U_q[\mathrm{sl}(\hat{N}|1)]$. Let $V(\lambda)$ be the highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ module with the highest weight λ and the highest weight vector $|\lambda\rangle$. Consider the following intertwiners of $U_q[\mathrm{sl}(\hat{N}|1)]$ modules:[10]

$$\begin{aligned} &\Phi_\lambda^{\mu V}(z):V(\lambda)\to V(\mu)\otimes V_z,\quad \Phi_\lambda^{\mu V^*}(z):V(\lambda)\to V(\mu)\otimes V_z^{*S},\\ &\Psi_\lambda^{V\mu}(z):V(\lambda)\to V_z\otimes V(\mu),\quad \Psi_\lambda^{V^*\mu}(z):V(\lambda)\to V_z^{*S}\otimes V(\mu). \end{aligned} \tag{II.13}$$

They are intertwiners in the sense that for any $x\in U_q[\mathrm{sl}(\hat{N}|1)]$

$$\Xi(z)\cdot x=\Delta(x)\cdot\Xi(z),\quad \Xi(z)=\Phi_\lambda^{\mu V}(z),\Phi_\lambda^{\mu V^*}(z),\Psi_\lambda^{V\mu}(z),\Psi_\lambda^{V^*\mu}(z). \tag{II.14}$$

We expand the vertex operators as[10]

$$\begin{aligned} &\Phi_\lambda^{\mu V}(z)=\sum_{j=1}^{N}\Phi_{\lambda,j}^{\mu V}(z)\otimes v_j,\quad \Phi_\lambda^{\mu V^*}(z)=\sum_{j=1}^{N}\Phi_{\lambda,j}^{\mu V^*}(z)\otimes v_j^*,\\ &\Psi_\lambda^{V\mu}(z)=\sum_{j=1}^{N}v_j\otimes\Psi_{\lambda,j}^{V\mu}(z),\quad \Psi_\lambda^{V^*\mu}(z)=\sum_{j=1}^{N}v_j^*\otimes\Psi_{\lambda,j}^{V^*\mu}(z). \end{aligned} \tag{II.15}$$

The intertwiners are even, which implies $[\Phi_{\lambda,j}^{\mu V}(z)]=[\Phi_{\lambda,j}^{\mu V^*}(z)]=[\Psi_{\lambda,j}^{V\mu}(z)]=[\Psi_{\lambda,j}^{V^*\mu}(z)]=[v_j]=(\nu_j+1)/2$. According to Ref. 10, $\Phi_\lambda^{\mu V}(z)(\Phi_\lambda^{\mu V^*}(z))$ is called type I (dual) vertex operator and $\Psi_\lambda^{V\mu}(z)(\Psi_\lambda^{V^*\mu}(z))$ type II (dual) vertex operator.

Introduce the bosonic operators $\phi_j(z)$, $\phi_j^*(z)$, $\psi_j(z)$, and $\psi_j^*(z)$:[4]

$$\begin{aligned} &\phi_{N\pm1}(z)=:e^{-h_N^*(q^Nz;1/2)}e^{c(q^Nz;0)}(q^Nz)^{[(N-2)/(2N-1)]}:e^{\sqrt{-1}\pi\Sigma_{i=1}^N(1-i)/(N-1)a_0^i},\\ &\nu_l\phi_l(z)(-1)^{[f_l]([v_l]+[v_{l+1}])}=[\phi_{l+1}(z),\ f_l]_{q^{\nu_{l+1}}},\\ &\phi_1^*(z)=:e^{h_1^*(qz;1/2)}(q^Nz)^{[(N-2)/2(N-1)]}:e^{-\sqrt{-1}\pi\Sigma_{i=1}^N(1-i)/(N-1)a_0^i},\\ &-\nu_l q^{\nu_l}\phi_{l+1}^*(z)(-1)^{[f_l]([v_l]+[v_{l+1}])}=[\phi_l^*(z),f_l]_{q^{\nu_l}}, \end{aligned} \tag{II.16}$$

$$\psi_1(z) = :e^{-h_1^*(qz;-1/2)}(q^N z)^{[(N-2)/2(N-1)]}: e^{\sqrt{-1}\pi\Sigma_{i=1}^N (1-i)/(N-1) a_o^i},$$

$$\psi_{l+1}(z) = [\psi_l(z), e_l]_{q^{\nu_l}},$$

$$\psi_{N+1}^*(z) = :e^{h_N^*(q^{2-N}z;-1/2)}\partial_z\{e^{-c(q^{2-N}z;0)}\}(q^N z)^{[(N-2)/2(N-1)]}: e^{-\sqrt{-1}\pi\Sigma_{i=1}^N (1-i)/(N-1) a_0^i},$$

$$-\nu_l \nu_{l+1} q^{-\nu_l}\psi_l^*(z) = [\psi_{l+1}^*(z), e_l]_{q^{\nu_{l+1}}},$$

where

$$h_n^{*i} = \sum_{j=1}^N \frac{[\alpha_{ij}m]_q [b_{ij}m]_q}{[(N-1)m]_q [m]_q} h_n^j, \quad Q_{h^i}^* = \sum_{j=1}^N \frac{\alpha_{ij}\beta_{ij}}{N-1} Q_{h^i}, \quad h_0^{*i} = \sum_{j=1}^N \frac{\alpha_{ij}\beta_{ij}}{N-1} h^j,$$

with $\alpha_{ij} = \min(i,j)$, and $\beta_{ij} = N-1-\max(i,j)$. Define the even operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$, and $\psi^*(z)$ by $\phi(z) = \Sigma_{j=1}^{N+1}\phi_j(z)\otimes v_j$, $\phi^*(z) = \Sigma_{j=1}^{N+1}\phi_j^*(z)\otimes v_j^*$, $\psi(z) = \Sigma_{j=1}^{N+1} v_j \otimes \psi_j(z)$, and $\psi^*(z) = \Sigma_{j=1}^{N+1} v_j^* \otimes \psi_j^*(z)$. Then the vertex operators $\Phi_\lambda^{\mu V}(z)$, $\Phi_\lambda^{\mu V^*}(z)$, $\Psi_\lambda^{V\mu}(z)$, and $\Psi_\lambda^{V^*\mu}(z)$, if they exist, are bosonized by $\phi(z)$, $\phi^*(z)$, and $\psi(z)$, $\psi^*(z)$, respectively.[4] We remark that our vertex operators differ from those of Kimura *et al.*[4] by a scalar factor $(q^N z)[(N-2)/2(N-1)]$ which is needed in order for the vertex operators also satisfy (II.14) for the element $x = d$. $\phi(z)$, $\phi^*(z)$, $\psi(z)$, and $\psi^*(z)$ are referred to as the ''elementary q-vertex operators'' of $U_q[\mathrm{sl}(\hat{N}|1)]$.

III. HIGHEST WEIGHT $U_q[\mathrm{sl}(\hat{N}|1)]$ MODULES

We begin by defining the Fock module. Denote by $F_{\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}}$ the bosonic Fock space generated by $a_{-m}^i, b_{-m}, c_{-m} (m>0)$ over the vector $|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle$:

$$F_{\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}} = \mathbf{C}[a_{-1}^i, a_{-2}^i,\ldots; b_{-1}, b_{-2},\ldots; c_{-1}, c_{-2}\ldots]|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle,$$

where

$$|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = e^{\Sigma_{i=1}^N \lambda_i Q_{a^i} + \lambda_{N+1}Q_b + \lambda_{N+2}Q_c}|0\rangle.$$

The vacuum vector $|0\rangle$ is defined by $a_m^i|0\rangle = b_m|0\rangle = c_m|0\rangle = 0$ for $i = 1,2,\ldots,N$, and $m \geqslant 0$. Obviously,

$$a_m^i|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = 0, \quad \text{for } i = 1,2,\ldots,N \text{ and } m>0,$$

$$b_m|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = c_m|\lambda_1,\lambda_2,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = 0, \quad \text{for } m>0.$$

To obtain the highest weight vectors of $U_q[\mathrm{sl}(\hat{N}|1)]$, we impose the conditions

$$e_i|\lambda_1,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = 0, \quad i = 0,1,2,\ldots,N,$$
$$h_i|\lambda_1,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle = \lambda^i|\lambda_1,\ldots,\lambda_{N+1};\lambda_{N+2}\rangle, \quad i = 0,1,2,\ldots,N. \tag{III.1}$$

Solving these equations, we obtain two classes of solutions:

(1)

$$(\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_{N+1};\lambda_{N+2}) = (\beta+1,\ldots,\underbrace{\beta+1,\beta}_{i,i+1},\ldots,\beta;0),$$

where $i = 1,\ldots,N$, and β is arbitrary. It follows that

$$(\lambda^0,\lambda^1,\ldots,\lambda^i,\lambda^{i+1}\ldots,\lambda^N) = (0,0,\ldots,\underbrace{0,1,0}_{i-1,i,i+1},\ldots,0)$$

and we have the identification

$$|\Lambda_i\rangle = |\beta+1,...,\underbrace{\beta+1,\beta,}_{i,i+1}\dots,\beta;0\rangle.$$

(2) $(\lambda_1,\dots,\lambda_N,\lambda_{N+1};\lambda_{N+2}) = (\beta,\dots,\beta,\beta-\alpha;-\alpha)$, where α, β are arbitrary. We have $(\lambda^0,\lambda^1,\dots,\lambda^{N-1},\lambda^N) = (1-\alpha,0,...,0,\alpha)$ and $|(1-\alpha)\Lambda_0+\alpha\Lambda_N\rangle = |\beta,\dots,\beta,\beta-\alpha;-\alpha\rangle$.

Associated to the above two classes of solutions are the following Fock spaces:

$$\mathcal{F}^m_\beta = \bigoplus_{\{i_1,\dots,i_N\}\in\mathbf{Z}} F_{\beta+1+i_1,\beta+1-i_1+i_2,\dots,\beta+1-i_{m-1}+i_m,\beta-i_m+i_{m+1},\dots,\beta-i_{N-1}+i_N,\beta+i_N;i_N},$$

$$\mathcal{F}_{(\alpha,\beta)} = \bigoplus_{\{i_1,\dots,i_N\}\in\mathbf{Z}} F_{\beta+i_1,\beta-i_1+i_2,\dots,\beta-i_{N-1}+i_N,\beta-\alpha+i_N;-\alpha+i_N},$$

where $m=1,2,...,\ N$, and it should be understood that $i_0\equiv 0$. However, it is easily seen that $\mathcal{F}^m_\beta = F_{(m;\beta)}$, $m=1,...,\ N$. Thus, it is sufficient to study the Fock space $\mathcal{F}_{(\alpha;\beta)}$. In the following we shall also restrict ourselves to the $\alpha\in\mathbf{Z}$ case.

It can be shown that the bosonized action of $U_q[\mathrm{sl}(\hat{N}|1)]$ (II.10) on $\mathcal{F}_{(\alpha;\beta)}$ is closed:

$$U_q[\mathrm{sl}(\hat{N}|1)]\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}.$$

Hence each Fock space $\mathcal{F}_{(\alpha;\beta)}$ constitutes a $U_q[\mathrm{sl}(\hat{N}|1)]$ module. However, these modules are not irreducible in general. To obtain irreducible subspaces, we introduce a pair of ghost fields[4]

$$\eta(z) = \sum_{n\in Z}\eta_n z^{-n-1} := e^{c(z)}:,\quad \xi(z) = \sum_{n\in Z}\xi_n z^{-n} =: e^{-c(z)}:.$$

The mode expansion of $\eta(z)$ and $\xi(z)$ is well defined on $\mathcal{F}_{(\alpha;\beta)}$ for $\alpha\in\mathbf{Z}$, and the modes satisfy the relations

$$\xi_m\xi_n+\xi_n\xi_m = \eta_m\eta_n+\eta_n\eta_m = 0,\quad \xi_0\eta_n+\eta_n\xi_m = \delta_{m+n,0}. \tag{III.2}$$

Since $\eta_0\xi_0$ and $\xi_0\eta_0$ qualify as projectors, we use them to decompose $\mathcal{F}_{(\alpha;\beta)}$ into a direct sum $\mathcal{F}_{(\alpha;\beta)} = \eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}\oplus\xi_0\eta_0\mathcal{F}_{(\alpha;\beta)}$ for $\alpha\in\mathbf{Z}$. $\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ is referred to as Ker_{η_0} and $\xi_0\eta_0\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}/\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ as Coker_{η_0}. Since η_0 commutes (or anticommutes) with the bosonized action of $U_q[\mathrm{sl}(\hat{N}|1)]$, Ker_{η_0} and Coker_{η_0} are both $U_q[\mathrm{sl}(\hat{N}|1)]$ modules for $\alpha\in\mathbf{Z}$.

A. Character and supercharacter

We want to determine the character and supercharacter formulas of the $U_q[\mathrm{sl}(\hat{N}|1)]$ modules constructed in the bosonic Fock space. We first of all bosonize the derivation operator d as

$$d = -\sum_{m\geqslant 1}\frac{m^2}{[m]_q^2}\left\{\sum_{i=1}^N h^i_{-m}h^{*i}_m + c_{-m}c_m\right\} - \frac{1}{2}\left\{\sum_{i=1}^N h^i_0 h^{*i}_0 + c_0(c_0+1)\right\}. \tag{III.3}$$

It obeys the commutation relations

$$[d,h_i]=0,\quad [d,h^i_m]=mh^i_m,\quad [d,X^{\pm,i}_m]=mX^{\pm,i}_m,\quad i=1,2,...,\ N,$$

as required. Moreover, $[d,\xi_0]=[d,\eta_0]=0$.

The character and supercharacter of a $U_q[\mathrm{sl}(\hat{N}|1)]$ module M are defined by

$$\mathrm{Ch}_M(q;x_1,x_2,\dots,x_N) = \mathrm{tr}_M(q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N}), \tag{III.4}$$

$$\mathrm{Sch}_M(q;x_1,x_2,\ldots,x_N)=\mathrm{Str}_M(q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N})=\mathrm{tr}_M((-1)^{N_f}q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N}),$$

respectively. The Fermi-number operator N_f can be bosonized as

$$N_f=\begin{cases}(N-1)b_0 & \text{if } N \text{ even, i.e., } N=2L,\\ L(\Sigma_{k=1}^N a_0^i-b_0)+c_0 & \text{if } N \text{ odd, i.e., } N=2L+1.\end{cases} \tag{III.5}$$

Indeed, N_f satisfies

$$(-1)^{N_f}\Theta(z)=(-1)^{[\Theta(z)]}\Theta(z)(-1)^{N_f},$$

where $\Theta(z)=X^{\pm,i}(z),\ \phi_i(z),\ \phi_i^*(z),\ \psi_i(z)$, and $\psi_i^*(z)$.

We calculate the characters and supercharacters by using the BRST resolution.[7] Let us define the Fock spaces, for $l\in\mathbf{Z}$

$$\mathcal{F}_{(\alpha;\beta)}^{(i)}=\bigoplus_{\{i_1,\cdots,i_N\}\in\mathbf{Z}}F_{\beta+i_1,\beta-i_1+i_2,\ldots,,\beta-i_{N-1}+i_N,\beta-\alpha+i_N;-\alpha+i_N+l}.$$

We have $\mathcal{F}_{(\alpha;\beta)}^{(0)}=\mathcal{F}_{(\alpha;\beta)}$. It can be shown that η_0 and ξ_0 intertwine these Fock spaces as follows:

$$\eta_0:\mathcal{F}_{(\alpha;\beta)}^{(l)}\to\mathcal{F}_{(\alpha;\beta)}^{(l+1)},\quad \xi_0:\mathcal{F}_{(\alpha;\beta)}^{(l)}\to\mathcal{F}_{(\alpha;\beta)}^{(l-1)}.$$

We have the following BRST complexes:

$$\begin{array}{ccccccc}\cdots & \xrightarrow{Q_{l-1}=\eta_0} & \mathcal{F}_{(\alpha;\beta)}^{(l)} & \xrightarrow{Q_l=\eta_0} & \mathcal{F}_{(\alpha;\beta)}^{(l+1)} & \xrightarrow{Q_{l+1}=\eta_0} & \cdots\\ & & \big|\mathbf{O} & & \big|\mathbf{O} & & \\ \cdots & \xrightarrow{Q_{l-1}=\eta_0} & \mathcal{F}_{(\alpha;\beta)}^{(l)} & \xrightarrow{Q_l=\eta_0} & \mathcal{F}_{(\alpha;\beta)}^{(l+1)} & \xrightarrow{Q_{l+1}=\eta_0} & \cdots,\end{array} \tag{III.6}$$

where $\mathbf{O}$ is an operator such that $\mathcal{F}_{(\alpha;\beta)}^{(l)}\to\mathcal{F}_{(\alpha;\beta)}^{(l)}$. Noting the fact that $\eta_0\xi_0+\xi_0\eta_0=1$, and $\eta_0\xi_0(\xi_0\eta_0)$ is the projection operator from $\mathcal{F}_{(\alpha;\beta)}^{(l)}$ to $\mathrm{Ker}_{Q_l}(\mathrm{Coker}_{Q_l})$, we get

$$\mathrm{Ker}_{Q_l}=\mathrm{Im}_{Q_{l-1}}\quad\text{for any }l\in\mathbf{Z},$$
$$\mathrm{tr}(O)|_{\mathrm{Ker}_{Q_l}}=\mathrm{tr}(O)|_{\mathrm{Im}_{Q_{l-1}}}=\mathrm{tr}(\mathbf{O})|_{\mathrm{Coker}_{Q_{l-1}}}. \tag{III.7}$$

By the above results, we can write the trace over Ker or Coker as the sum of trace over $\mathcal{F}_{(\alpha;\beta)}^{(l)}$, and compute the latter by using the technique introduced in Ref. 26. The results are

$$\mathrm{Ch}_{\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=\frac{q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^\infty(1-q^n)^{N+1}}\sum_{l=1}^\infty(-1)^{l+1}q^{1/2\{l^2+l(2\alpha-1)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}q^{1/2\{i_N^2+i_N(1-2\alpha-2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},$$

$$\mathrm{Ch}_{\mathrm{Coker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=\frac{q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{1/2\{l^2+l(1-2\alpha)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in\mathbf{Z}}q^{1/2\{i_N^2+i_N(1-2\alpha+2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},$$

where $\Delta(i_1,\ldots,i_N)=\Sigma_{l,l'=1}^{N}(\alpha_{ll'}\beta_{ll'}/N-1)\lambda_{i_1,\ldots,i_N}^{l}\lambda_{i_1,\ldots,i_N}^{l'}$ and

$$\lambda_{i_1,\ldots,i_N}^{l}=2i_l-i_{l-1}-i_{l+1},\quad 2\leqslant l\leqslant N-1$$
$$\lambda_{i_1,\ldots,i_N}^{1}=2i_1-i_2,\quad \lambda_{i_1,\ldots,i_N}^{N}=\alpha-i_N \qquad \text{(III.8)}$$

Similarly, the supercharacters of $\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}$ and $\mathrm{Coker}_{\mathcal{F}_{(\alpha;\beta)}}$ are given by

(1) For $N=2L$:

$$\mathrm{Sch}_{\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=\frac{(-1)^{\alpha}q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{1/2\{l^2+l(2\alpha-1)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}(-1)^{i_N}q^{1/2\{i_N^2+i_N(1-2\alpha-2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},$$

$$\mathrm{Sch}_{\mathrm{Coker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=\frac{(-1)^{\alpha}q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{1/2\{l^2+l(l-2\alpha)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}(-1)^{i_N}q^{1/2\{i_N^2+i_N(1-2\alpha+2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N}.$$

(2) For $N=2L+1$:

$$\mathrm{Sch}_{\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=-\frac{(-1)^{(L+1)\alpha}q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}q^{1/2\{l^2+l(2\alpha-1)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}(-1)^{i_N}q^{1/2\{i_N^2+i_N(1-2\alpha-2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},$$

$$\mathrm{Sch}_{\mathrm{Coker}_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\ldots,x_N)=-\frac{(-1)^{(L+1)\alpha}q^{1/2\alpha(\alpha-1)}}{\Pi_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}q^{1/2\{l^2+l(1-2\alpha)\}}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}(-1)^{i_N}q^{1/2\{i_N^2+i_N(1-2\alpha+2l)\}}q^{1/2\Delta(i_1,\ldots,i_N)}$$
$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N}.$$

Since $\mathcal{F}^{(1)}_{(\alpha-(N-1);\beta+1)}=\mathcal{F}_{(\alpha;\beta)}$ and by (III.7), we have

$$\mathrm{Ch}_{\mathrm{Coker}_{\mathcal{F}_{(\alpha-(N-1);\beta+1)}}}=\mathrm{Ch}_{\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}},\quad \mathrm{Sch}_{\mathrm{Coker}_{\mathcal{F}_{(\alpha-(N-1);\beta+1)}}}=\mathrm{Sch}_{\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}}. \tag{III.9}$$

Relations (III.9) can also be checked by using the above explicit formulas of the (super) characters.

B. $U_q[\mathrm{sl}(\hat{N}|1)]$ module structure of $\mathcal{F}_{(\alpha;\beta-[1/(N-1)/]\alpha)}$

Set $\lambda_\alpha=(1-\alpha)\Lambda_0+\alpha\Lambda_N$ and

$$|\lambda_\alpha\rangle=|\beta,\dots,\beta,\beta-\alpha;-\alpha\rangle\in\mathcal{F}_{(\alpha;\beta)},\quad \alpha\in\mathbf{Z},$$

$$|\Lambda_m\rangle=|\beta+1,...,\beta+1,\beta,\dots,\beta;0\rangle\in\mathcal{F}_{(m;\beta)},\quad m=1,...,\ N,$$

The above vectors play the role of the highest weight vectors of $U_q[\mathrm{sl}(\hat{N}|1)]$ modules. One can check that

$$\eta_0|\lambda_\alpha\rangle=0,\quad \text{for}\ \ \alpha=0,-1,...$$

$$\eta_0|\Lambda_m\rangle=0,\quad \text{for}\ \ m=1,...,\ N \tag{III.10}$$

$$\eta_0|\lambda_\alpha\rangle\neq 0,\quad \text{for}\ \ \alpha=1,2,\dots.$$

It follows that the modules

$$\mathrm{Coker}_{\mathcal{F}_{(\alpha,\beta)}}(\alpha=1,2,...),\quad \mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta)}}(\alpha=0,-1,-2,...),$$

$$\mathrm{Ker}_{\mathcal{F}_{(\mathrm{m};\beta)}}(m=1,2,...,N),$$

are highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ modules. Denote them by $\overline{V}(\lambda_\alpha)$ and $\overline{V}(\Lambda_m)$, respectively. From (III.10) and (III.9), we have the following identifications of the highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$-modules:

$$\overline{V}(\lambda_\alpha)\cong\mathrm{Ker}_{\mathcal{F}_{(\alpha;\beta-1/N-1\alpha)}}\equiv\mathrm{Coker}_{\mathcal{F}_{(\alpha-(N-1);\beta-1/N-1\alpha+1)}}\quad \text{for}\ \ \alpha=0,-1,-2,...,$$

$$\cong\mathrm{Coker}_{\mathcal{F}_{(\alpha;\beta-1/N-1\alpha)}}\equiv\mathrm{Ker}_{\mathcal{F}_{(\alpha+(N-1);\beta-1/N-1\alpha-1)}}\quad \text{for}\ \ \alpha=1,2,...,\tag{III.11}$$

$$\overline{V}(\Lambda_m)\cong\mathrm{Ker}_{\mathcal{F}_{(m;\beta-1/N-1m)}}\equiv\mathrm{Coker}_{\mathcal{F}_{(m-(N-1);\beta-1/N-1m+1)}}\quad \text{for}\ \ m=1,...,N.\tag{III.12}$$

It is easy to see that the vertex operators (II.16) also commute (or anticommute) with η_0. It follows from (III.11)–(III.12) that each Fock space $\mathcal{F}_{(\alpha;\beta-[1/(N-1)]\alpha)}$ is decomposed into a direct sum of the highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ modules:

$$\begin{array}{ccccc}
 & \mathrm{Ker} & & \mathrm{Coker} \\
\vdots & \vdots & & \vdots \\
F_{(-N;\beta+1+[1/(N-1)])}= & \overline{V}(\lambda_{-N}) & \oplus & \overline{V}(\lambda_{-1}) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(-N+1;\beta+1)}= & \overline{V}(\lambda_{-N+1}) & \oplus & \overline{V}(\Lambda_0) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(-N+2;\beta+1-[1/(N-1)])}= & \overline{V}(\lambda_{-N+2}) & \oplus & \overline{V}(\Lambda_1) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
\vdots & \vdots & & \vdots \\
F_{(-2;\beta+1-[(N-2)/(N-1)])}= & \overline{V}(\lambda_{-2}) & \oplus & \overline{V}(\Lambda_{N-3})
\end{array}$$

$$
\begin{array}{lccc}
F_{(-1;\beta+1[(N-2)/N-1)])}= & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
 & \overline{V}(\lambda_{-1}) & \oplus & \overline{V}(\Lambda_{N-2}) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(0;\beta)}= & \overline{V}(\Lambda_0) & \oplus & \overline{V}(\Lambda_{N-1}) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(1;\beta-[1/(N-1)])}= & \overline{V}(\Lambda_1) & \oplus & \overline{V}(\Lambda_N) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(2;\beta-[2/(N-1)])}= & \overline{V}(\Lambda_2) & \oplus & \overline{V}(\lambda_2) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
F_{(N-2;\beta-[(N-2)/(N-1)])}= & \overline{V}(\Lambda_{N-2}) & \oplus & \overline{V}(\lambda_{N-2}) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(N-1;\beta-1)}= & \overline{V}(\Lambda_{N-1}) & \oplus & \overline{V}(\Lambda_{M-1}) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) \\
F_{(N;\beta-1-[1/(N-1)])}= & \overline{V}(\Lambda_N) & \oplus & \overline{V}(\lambda_N) \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z).
\end{array}
\tag{III.13}
$$

It is expected that $\overline{V}(\lambda_\alpha)(\alpha\in Z)$ and $\overline{V}(\Lambda_m)$ $(m=1,2,\ldots,\ N-1)$ are irreducible highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ modules with the highest weights λ_α and Λ_m, respectively. Thus we conjecture that

$$\overline{V}(\lambda_\alpha)=V(\lambda_\alpha),\quad \overline{V}(\Lambda_m)=V(\Lambda_m). \tag{III.14}$$

IV. EXCHANGE RELATIONS OF VERTEX OPERATORS

In this section, we derive the exchange relations of the type I and type II bosonized vertex operators of $U_q[\mathrm{sl}(\hat{N}|1)]$. As expected, these vertex operators satisfy the graded Faddeev–Zamolodchikov algebra.

A. The *R* matrix

Throughout, we use the abbreviation

$$
\begin{aligned}
(z;x_1,\ldots,x_m)_\infty &= \prod_{\{n_1,\ldots,n_m\}=0}^{\infty}(1-zx_1^{n_1}\cdots x_m^{n_m}),\\
\{z\}_\infty &\overset{\mathrm{def}}{=} (z;q^{2(N-1)},q^{2(N-1)})_\infty .
\end{aligned}
\tag{IV.1}
$$

Let $\overline{R}(z)\in\mathrm{End}(V\otimes V)$ be the R matrix of $U_q[\mathrm{sl}(\hat{N}|1)]$,

$$\overline{R}(z)(v_i\otimes v_j)=\sum_{k,l=1}^{2N}\overline{R}_{kl}^{ij}(z)v_k\otimes v_l,\quad \forall v_i,v_j,v_k,v_l\in V, \tag{IV.2}$$

where the matrix elements of $\overline{R}(z)$ are given by

$$\overline{R}_{i,i}^{i,i}(z)=-1,\quad \overline{R}_{N+1,N+1}^{N+1,N+1}(z)=-\frac{zq^{-1}-q}{zq-q^{-1}},\quad i=1,2,\ldots,\ N,$$

$$\overline{R}_{ij}^{ij}(z)=\frac{z-1}{zq-q^{-1}},\quad i\neq j,$$

$$\overline{R}^{ji}_{ij}(z)=\frac{q-q^{-1}}{zq-q^{-1}}(-1)^{[i][j]},\quad i<j,$$

$$\overline{R}^{ji}_{ij}(z)=\frac{(q-q^{-1})z}{zq-q^{-1}}(-1)^{[i][j]},\quad i>j,$$

$$\overline{R}^{ij}_{kl}(z)=0,\quad \text{otherwise}.$$

Define the R matrices $R^{(\mathrm{I})}(z)$ and $R^{(\mathrm{II})}(z)$ by

$$R^{(\mathrm{I})}(z)=r(z)\overline{R}(z),\quad R^{(\mathrm{II})}(z)=\overline{r}(z)\overline{R}(z), \tag{IV.3}$$

where

$$r(z)=z^{[(2-N)/(N-1)]}\frac{(zq^2;q^{2(N-1)})_\infty(z^{-1}q^{2N-2};q^{2(N-1)})_\infty}{(z^{-1}q^2;q^{2(N-1)})_\infty(zq^{2N-2};q^{2(N-1)})_\infty},$$

$$\overline{r}(z)=-z-^{[1/(N-1)]}\frac{(zq^{2N-4};q^{2(N-1)})_\infty(z^{-1}q^{2N-2};q^{2(N-1)})_\infty}{(z^{-1}q^{2N-4};q^{2(N-1)})_\infty(zq^{2N-2};q^{2(N-1)})_\infty}.$$

These R matrices satisfy the graded Yang–Baxter equation on $V\otimes V\otimes V$:

$$R^{(i)}_{12}(z)R^{(i)}_{13}(zw)R^{(i)}_{23}(w)=R^{(i)}_{23}(w)R^{(i)}_{13}(zw)R^{(i)}_{12}(z),\quad i=\mathrm{I,II}.$$

Moreover, they enjoy (i) the initial condition $R^{(i)}(1)=P$, $i=\mathrm{I,II}$, where P is the graded permutation operator; (ii) the unitarity condition $R^{(i)}_{12}(z/w)R^{(i)}_{21}(w/z)=1$, $i=\mathrm{I,II}$, where $R^{(i)}_{21}(z)=PR^{(i)}_{12}\times(z)P$; (iii) the crossing unitarity

$$(R^{(i)})^{-1,\mathrm{st}_1}(z)((q^{-2\overline{\rho}}\otimes 1)R^{(i)}(zq^{2(1-N)})(q^{2\overline{\rho}}\otimes 1))^{\mathrm{st}_1}=1,\quad i=\mathrm{I,II},$$

where

$$q^{2\overline{\rho}}\equiv\mathrm{diag}(q^{2\rho_1},q^{2\rho_2},\ldots,q^{2\rho_N},q^{2\rho_{N+1}})=\mathrm{diag}(q^{N-2},q^{N-4},\ldots,q^{-N},q^{-N}).$$

The various supertranspositions of the R matrix are given by

$$(R^{\mathrm{st}_1}(z))^{kl}_{ij}=R^{il}_{kj}(z)(-1)^{[i]([i]+[k])},\quad (R^{\mathrm{st}_2}(z))^{kl}_{ij}=R^{kj}_{il}(z)(-1)^{[j]([l]+[j])},$$

$$(R^{\mathrm{st}_{12}}(z))^{kl}_{ij}=R^{ij}_{kl}(z)(-1)^{([i]+[j])([i]+[j]+[k]+[l])}=R^{ij}_{kl}(z).$$

B. The graded Faddeev–Zamolodchikov algebra

We now calculate the exchange relations of the type I and type II bosonic vertex operators of $U_q[\mathrm{sl}(\hat{N}|1)]$. Define

$$\oint dz\, f(z)=\mathrm{Res}(f)=f_{-1},\ \text{for a formal function}\ f(z)=\Sigma_{n\in\mathbf{z}}f_nz^n.$$

Then, the Chevalley generators of $U_q[\mathrm{sl}(\hat{N}|1)]$ can be expressed by the integrals

$$e_i=\oint dz\, X^{+,i}(z),\quad f_i=\oint dz\, X^{-,i}(z),\quad i=1,2,\ldots,\ N.$$

One can also get the integral expressions of the bosonic vertex operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$, and $\psi^*(z)$. Using these integral expressions and the relations given in Appendixes A and B, we find that the bosonic vertex operators defined in (II.16) satisfy the graded Faddeev–Zamolodchikov algebra

$$\phi_j(z_2)\phi_i(z_1)=\sum_{k,l=1}^{N+1} R^{(\mathrm{I})}\left(\frac{z_1}{z_2}\right)_{ij}^{kl}\phi_k(z_1)\phi_l(z_2)(-1)^{[i][j]},$$

$$\psi_i^*(z_1)\psi_j^*(z_2)=\sum_{k,l=1}^{N+1} R^{(\mathrm{II})}\left(\frac{z_1}{z_2}\right)_{kl}^{ij}\psi_l^*(z_2)\psi_k^*(z_1)(-1)^{[i][j]}, \tag{IV.4}$$

$$\psi_i^*(z_1)\phi_j(z_2)=\tau\left(\frac{z_1}{z_2}\right)\phi_j(z_2)\psi_i^*(z_1)(-1)^{[i][j]},$$

where

$$\tau(z)=-z^{[(2-N)/(N-1)]}\frac{(zq;q^{2(N-1)})_\infty(z^{-1}q^{2N-3};q^{2(N-1)})_\infty}{(z^{-1}q;q^{2(N-1)})_\infty(zq^{2N-3};q^{2(N-1)})_\infty}.$$

By

$$:e^{-h_N^*(zq^N;1/2)+h_1^*(zq;1/2)-h^1(zq^2;1/2)-h^2(zq^3;1/2)\cdots-h^N(zq^{N+1};1/2)}:=1,$$

we obtain the first invertibility relations

$$\phi_i(z)\phi_j^*(z)=g^{-1}(-1)^{[i]}\delta_{ij}, \quad \sum_{k=1}^{N+1}(-1)^{[k]}\phi_k^*(z)\phi_k(z)=g^{-1}, \tag{IV.5}$$

and the second invertibility relations

$$\phi_i^*(zq^{2(N-1)})\phi_j(z)=-g^{-1}q^{2\rho_i}\delta_{ij}, \quad \sum_{k=1}^{N+1} q^{-2\rho_k}\phi_k(z)\phi_k^*(zq^{2(N-1)})=-g^{-1}, \tag{IV.6}$$

where

$$g=e^{\sqrt{-1}\pi N/2(N-1)}\frac{(q^2;q^{2(N-1)})_\infty}{(q^{2(N-1)};q^{2(N-1)})_\infty}.$$

Using the fact that $\eta_0\xi_0$ is a projection operator, we can make the following identifications:

$$\begin{aligned}\Phi_i(z)&=\eta_0\xi_0\phi_i(z)\eta_0\xi_0, \quad \Phi_i^*(z)=\eta_0\xi_0\phi_i^*(z)\eta_0\xi_0,\\ \Psi_i(z)&=\eta_0\xi_0\psi_i(z)\eta_0\xi_0, \quad \Psi_i^*(z)=\eta_0\xi_0\psi_i^*(z)\eta_0\xi_0.\end{aligned} \tag{IV.7}$$

Set

$$\mu_\alpha=\begin{cases}\Lambda_\alpha & \alpha=0,1,...,N,\\ \lambda_{\alpha-(N-1)} & \text{for } \alpha>N,\\ \lambda_\alpha & \text{for } \alpha<0.\end{cases} \tag{IV.8}$$

It is easy to see that the vertex operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$, and $\psi^*(z)$ commute (or anti-commute) with the BRST charge η_0. It follows from (III.13) and (III.14) that the vertex operators (IV.7) intertwine all the level-one irreducible highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ modules $V(\mu_\alpha)$ ($\alpha \in \mathbf{Z}$) as follows:

$$\Phi(z):V(\mu_\alpha)\to V(\mu_{\alpha-1})\otimes V_z, \quad \Phi^*(z):V(\mu_\alpha)\to V(\mu_{\alpha+1})\otimes V_z^{*S}, \tag{IV.9}$$

$$\Psi(z):V(\mu_\alpha)\to V_z\otimes V(\mu_{\alpha-1}),\quad \Psi^*(z):V(\mu_\alpha)\to V_z^{*S}\otimes V(\mu_{\alpha+1}).$$

From (IV.4), we have

$$\Phi_j(z_2)\Phi_i(z_1)=\sum_{k,l=1}^{N+1} R^{(\mathrm{I})}\left(\frac{z_1}{z_2}\right)_{ij}^{kl}\Phi_k(z_1)\Phi_l(z_2)(-1)^{[i][j]},$$

$$\Psi_i^*(z_1)\Psi_j^*(z_2)=\sum_{k,l=1}^{N+1} R^{(\mathrm{II})}\left(\frac{z_1}{z_2}\right)_{kl}^{ij}\Psi_l^*(z_2)\Psi_k^*(z_1)(-1)^{[i][j]}, \tag{IV.10}$$

$$\Psi_i^*(z_1)\Phi_j(z_2)=\tau\left(\frac{z_1}{z_2}\right)\Phi_j(z_2)\Psi_i^*(z_1)(-1)^{[i][j]}.$$

Moreover, we have the following invertibility relations:

$$\Phi_i(z)\Phi_j^*(z)=g^{-1}(-1)^{[i]}\delta_{ij}\,\mathrm{id}_{V(\mu_\alpha)},$$

$$\sum_{k=1}^{N+1}(-1)^{[k]}\Phi_k^*(z)\Phi_k(z)=g^{-1}\,\mathrm{id}_{V(\mu_\alpha)},$$

$$\Phi_i^*(zq^{2(N-1)})\Phi_j(z)=-g^{-1}q^{2\rho_i}\delta_{ij}\,\mathrm{id}_{V(\mu_\alpha)}, \tag{IV.11}$$

$$\sum_{k=1}^{N+1}q^{-2\rho_k}\Phi_k(z)\Phi_k^*(zq^{2(N-1)})=-g^{-1}\,\mathrm{id}_{V(\mu_\alpha)}.$$

V. MULTICOMPONENT SUPER $t-J$ MODEL

In this section, we give a mathematical definition of the multicomponent super $t-J$ model on an infinite lattice.

A. Space of states

By means of the R-matrix (IV.2) of $U_q[\mathrm{sl}(\hat{N}|1)]$, one defines a spin chain model, referred to as the multicomponent super $t-J$ model, on the infinite lattice $\cdots\otimes V\otimes V\otimes V\cdots$. Let h be the operator on $V\otimes V$ such that

$$P\bar{R}\left(\frac{z_1}{z_2}\right)=1+uh+\cdots,\quad y\to 0,$$

$$P:\ \text{the graded permutation operator},\quad e^u\equiv z_1/z_2.$$

The Hamiltonian H of this model is given by

$$H=\sum_{l\in Z}h_{l+1,l}. \tag{V.1}$$

H acts formally on the infinite tensor product,

$$\cdots V\otimes V\otimes V\cdots. \tag{V.2}$$

It can be easily checked that

$$[U_q'(\mathrm{sl}(N|1)),\ H]=0,$$

where $U'_q[\mathrm{sl}(N|1)]$ is the subalgebra of $U_q[\mathrm{sl}(\hat{N}|1)]$ with the derivation operator d being dropped. So $U'_q[\mathrm{sl}(N|1)]$ plays the role of infinite dimensional *non-Abelian* symmetry of the multicomponent super $t-J$ model on the infinite lattice.

From the intertwining relation (IV.9), one has the following composition of the type I vertex operators:

$$V(\mu_\alpha) \xrightarrow{\Phi(1)} V(\mu_{\alpha-1})\otimes V \xrightarrow{\Phi(1)\otimes \mathrm{id}} V(\mu_{\alpha-1})\otimes V\otimes V \xrightarrow{\Phi(1)\otimes \mathrm{id}\otimes \mathrm{id}} \cdots \to W_t, \tag{V.3}$$

where

$$W_l \overset{\mathrm{def}}{=} \cdots \otimes V\otimes V,$$

i.e., the left half-infinite tensor product. We conjecture that such a composition converges to a map:

$$i: V(\mu_\alpha)\to W_l.$$

Such a map i satisfies $i(xv)=\Delta^{(\infty)}(x)i(v)$, $x\in U_q[\mathrm{sl}(\hat{N}|1)]$ and $v\in V(\mu_\alpha)$. Following Ref. 9, we could replace the infinite tensor product (V.2) by the level-zero $U_q[\mathrm{sl}(\hat{N}|1)]$ module,

$$F_{\alpha\alpha'}=\mathrm{Hom}(V(\mu_\alpha),V(\mu_{\alpha'}))\cong V(\mu_\alpha)\otimes V(\mu_{\alpha'})^*,$$

where $V(\mu_\alpha)$ is level-one irreducible highest weight $U_q[\mathrm{sl}(\hat{N}|1)]$ module and $V(\mu_{\alpha'})^*$ is the dual module of $V(\mu_{\alpha'})$. By (III.13), this homomorphism can be realized by applying the type I vertex operators repeatedly. So we shall make the (hypothetical) identification:

$$\text{``the space of physical states''}=\bigoplus_{\alpha\alpha'\in\mathbf{Z}} V(\mu_\alpha)\otimes V(\mu_{\alpha'})^*.$$

Namely, we take

$$F\equiv \mathrm{End}\Big(\bigoplus_{\alpha\in\mathbf{Z}} V(\mu_\alpha)\Big)\cong \bigoplus_{\alpha,\alpha'\in\mathbf{Z}} F_{\alpha\alpha'}$$

as the space of states of the multicomponent super $t-J$ model on the infinite lattice. The left action of $U_q[\mathrm{sl}(\hat{N}|1)]$ on F is defined by

$$x\cdot f=\sum x_{(1)}\circ f\circ S(x_{(2)})(-1)^{[f][x_{(2)}]}, \quad \forall x\in U_q[\mathrm{sl}(\hat{N}|1)], f\in F,$$

where we have used notation $\Delta(x)=\Sigma x_{(1)}\otimes x_{(2)}$. Note that $F_{\alpha\alpha}$ has the unique canonical element $\mathrm{id}_{V(\mu_\alpha)}$. We call it the vacuum[10] and denote it by $|\mathrm{vac}\rangle_\alpha$.

B. Local structure and local operators

Following Jimbo *et al.*,[10] we use the type I vertex operators and their variants to incorporate the local structure into the space of physical states F, that is to formulate the action of local operators of the multicomponent super $t-J$ model on the infinite tensor product (V.2) in terms of their actions on $F_{\alpha\alpha'}$.

Using the isomorphisms

$$\begin{aligned} \Phi(1) \quad &: \quad V(\mu_\alpha)\to V(\mu_{\alpha-1})\otimes V, \\ \Phi^{*,\mathrm{st}}(q^{2(N-1)}) \quad &: \quad V\otimes V(\mu_\alpha)^*\to V(\mu_{\alpha-1})^*, \end{aligned} \tag{V.4}$$

where st is the supertransposition on the quantum space, we have the following identification:

$$V(\mu_\alpha)\otimes V(\mu_{\alpha'})^* \overset{\sim}{\to} V(\mu_{\alpha-1})\otimes V\otimes V(\mu_{\alpha'})^* \overset{\sim}{\to} V(\mu_{\alpha-1})\otimes V(\mu_{\alpha'-1})^*.$$

The resulting isomorphism can be identified with the super translation (or shift) operator defined by

$$T=-g\sum_i \Phi_i(1)\otimes\Phi_i^{*,\mathrm{st}}(q^{2(N-1)})(-1)^{[i]}q^{-2p_i}.$$

The inverse is given by

$$T^{-1}=g\sum_i \Phi_i^*(1)\otimes\Phi_i^{\mathrm{st}}(1).$$

Thus we can define the local operators on V as operators on $F_{\alpha\alpha'}$.[10] Let us label the tensor components from the middle as 1, 2,... for the left half and as 0, −1, −2,... for the right half. The operators acting on the site 1 are defined by

$$E_{ij}\overset{\mathrm{def}}{=}E_{ij}^{(1)}=g\Phi_i^*(1)\Phi_j(1)(-1)^{[j]}\otimes\mathrm{id}. \tag{V.5}$$

More generally we set

$$E_{ij}^{(n)}=T^{-(n-1)}E_{ij}T^{n-1}\quad (n\in Z). \tag{V.6}$$

Then, from the invertibility relations of the type I vertex operators of $U_q[\mathrm{sl}(\hat{N}|1)]$, we can show that the local operators $E_{ij}^{(n)}$ acting on $F_{\alpha\alpha'}$ satisfy the following relations:

$$E_{ij}^{(m)}E_{kl}^{(n)}=\begin{cases}\delta_{jk}E_{il}^{(n)} & \text{if } m=n,\\ (-1)^{([i]+[j])([k]+[l])}E_{kl}^{(n)}E_{ij}^{(m)} & \text{if } m\neq n.\end{cases}$$

This result implies that the local operators $E_{ij}^{(n)}$ are nothing but the $U_q[\mathrm{sl}(N|1)]$ generators acting on the nth component of $\cdots\otimes V\otimes V\otimes\cdots$. They include all the local operators in the multicomponent super $t-J$ model.[10]

As is expected from the physical point of view, the vacuum vectors $|\mathrm{vac}\rangle_\alpha$ are supertranslationally invariant and singlets (i.e., they belong to the trivial representation of $U_q[\mathrm{sl}(\hat{N}|1)]$):

$$T|\mathrm{vac}\rangle_\alpha=|\mathrm{vac}\rangle_{\alpha-1},\quad x.|\mathrm{vac}\rangle_\alpha=\epsilon(x)|\mathrm{vac}\rangle_\alpha,\quad \forall x\in U_q[\mathrm{sl}(\hat{N}|1)].$$

This is proved as follows. Let $u_l^{(\alpha)}(u_l^{*(\alpha)})$ be a basis vectors of $V(\mu_\alpha)(V(\mu_\alpha)^*)$ and

$$|\mathrm{vac}\rangle_\alpha\overset{\mathrm{def}}{=}\mathrm{id}_{V(\mu_a)}=\sum_l u_l^{(\alpha)}\otimes u_l^{*(\alpha)}.$$

Then

$$T|\mathrm{vac}\rangle_\alpha=-g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}\otimes\Phi_m^{*,\mathrm{st}}(q^{2(N-1)})u_l^{*(\alpha)}(-1)^{[m]+[l][m]}.$$

We want to show $T|\mathrm{vac}\rangle_\alpha=|\mathrm{vac}\rangle_{\alpha-1}$. This is equivalent to proving

$$-g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}\Phi_m^{*,\mathrm{st}}(q^{2(N-1)})\cdot u_l^{*(\alpha)}(v)(-1)^{[m]+[l][m]}=v,\quad \forall v\in V(\mu_{\alpha-1}).$$

Now

$$\mathrm{lhs} = -g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^q u_l^{*(\alpha)}(\Phi_m^*(q^{2(N-1)})^{\mathrm{st}})^{\mathrm{st}}v)(-1)^{[m]}$$

$$= -g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}u_l^{*(\alpha)}(\Phi_m^*(q^{2(N-1)})v)$$

$$= -g\sum_{m} q^{-2\rho_m}\Phi_m(1)\Phi_m^*(q^{2(N-1)})v = v,$$

where we have used $(\Phi_m^*(z)^{\mathrm{st}})^{\mathrm{st}} = \Phi_m^*(z)(-1)^{[m]}$ and (IV.11). As to the second equation, we have

$$\begin{aligned} x\cdot|\mathrm{vac}\rangle_\alpha &= \Sigma x_{(1)}u_l^{(\alpha)}\otimes x_{(2)}u_l^{*(\alpha)}(-1)^{[l][x_{(2)}]} \\ &= \Sigma x_{(1)}u_l^{(\alpha)}\otimes \pi_{V(\mu_\alpha)^*}(x_{(2)})_{ml}u_m^{*(\alpha)}(-1)^{[l][x_{(2)}]} \\ &= \Sigma x_{(1)}u_l^{(\alpha)}\otimes \pi_{V(\mu_a)}(S(x_{(2)}))_{lm}u_m^{*(\alpha)} \\ &= \Sigma x_{(1)}\pi_{V(\mu_\alpha)}(S(x_{(2)}))_{lm}u_l^{(\alpha)}\otimes u_m^{*(\alpha)} \\ &= \Sigma x_{(1)}S(x_{(2)})u_m^{(\alpha)}\otimes u_m^{*(\alpha)} = \epsilon(x)|\mathrm{vac}\rangle_\alpha . \end{aligned}$$

This completes the proof.

For any local operator $O\in F$, its vacuum expectation value is defined by

$$_\alpha\langle\mathrm{vac}|O|\mathrm{vac}\rangle_\alpha \overset{\mathrm{def}}{=} \frac{\mathrm{tr}_{V(\mu_\alpha)}(q^{-2\rho}O)}{\mathrm{tr}_{V(\mu_\alpha)}(q^{-2\rho})} = \frac{\mathrm{tr}_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar{\rho}}}O)}{\mathrm{tr}_{V(\mu_a)}(q^{-2(N-1)d-2h_{\bar{\rho}}})}, \tag{V.7}$$

where

$$2h_{\bar{\rho}} = \sum_{l=1}^{N} l(N-1-l)h_l .$$

We shall denote the correlator $_\alpha\langle\mathrm{vac}|O|\mathrm{vac}\rangle_\alpha$ by $\langle O\rangle_\alpha$.

VI. CORRELATION FUNCTIONS

The aim of this section is to calculate $\langle E_{mn}\rangle_\alpha$. The generalization to the calculation of the multipoint functions is straightforward.

Set

$$P_n^m(z_1,z_2|q|\alpha) = \frac{\mathrm{tr}_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar{\rho}}}\Phi_m^*(z_1)\Phi_n(z_2))}{\mathrm{tr}_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar{\rho}}})},$$

then $\langle E_{mn}\rangle_\alpha = P_n^m(z,z|q|\alpha)$. By (IV.8), it is sufficient to calculate

$$F_{mn}^{(\alpha)}(z_1,z_2) = \frac{\mathrm{tr}_{F_{(\alpha;\beta-\alpha)}}(q^{-2(N-1)d-2h_{\bar{\rho}}}\Phi_m^*(z_1)\,\phi_n(z_2)\,\eta_0\xi_0)}{\mathrm{tr}_{F_{(\alpha;\beta-\alpha)}}(q^{-2(N-1)d-2h_{\bar{\rho}}}\eta_0\xi_0)}. \tag{VI.1}$$

Using the Clavelli–Shapiro technique,[26] we get

$$F_{mn}^{(\alpha)}(z_1,z_2) = \frac{\delta_{mn}}{\chi\alpha}F_m^{(\alpha)}(z_1,z_2) \equiv \frac{\delta_{mn}}{\chi\alpha}\sum_{l=1}^{\infty} (-1)^{l+1}F_{m_i-l}^{(\alpha)}(z_1,z_2),$$

where

$$\chi\alpha=\mathrm{Ch}_{\mathrm{Ker}\mathcal{F}_{(\alpha;\beta)}}(q^{2(N-1)};q^{-(N-2)},\ldots,q^{-l(N-1-l)},\ldots,q^{N}),$$

$$F_{m,l}^{(\alpha)}(z_1,z_2)=-e^{[\sqrt{-1}\pi N/2(N-1)]}C_1^{*}C_N^{*}(C_1)^{N-1}(C_{N+1})^2(z_1q)^{[1/(N-1)]}$$
$$\times\frac{\left\{\frac{z_1}{z_2}q^{2(N-1)}\right\}_\infty\left\{\frac{z_2}{z_1}q^{2(N-1)}\right\}_\infty}{\left\{\frac{z_1}{z_2}q^{2N}\right\}_\infty\left\{\frac{z_2}{z_1}q^{2N}\right\}_\infty}\oint dw_1\cdots\oint dw_N$$
$$\times\left\{\prod_{k=1}^{m-1}\frac{(1-q^2)}{qw_{k-1}\left(\frac{w_k}{w_{k-1}}q;q^{2(N-1)}\right)_\infty\left(\frac{w_{k-1}}{w_k}q;q^{2(N-1)}\right)_\infty}\right\}$$
$$\times\frac{1}{w_{m-1}\left(\frac{w_m}{w_{m-1}}q;q^{2(N-1)}\right)_\infty\left(\frac{w_{m-1}}{w_m}q^{2N-1};q^{2(N-1)}\right)_\infty}$$
$$\times\left\{\prod_{k=m+1}^{N}\frac{(1-q^2)}{w_k\left(\frac{w_k}{w_{k-1}}q;q^{2(N-1)}\right)_\infty\left(\frac{w_{k-1}}{w_k}q;q^{2(N-1)}\right)_\infty}\right\}$$
$$\times\sum_{\{i_1,\ldots,i_N\}\in Z}I_{i_1,\ldots,i_N}^{(a,l)}(z_1,z_2|w_1,\ldots,w_N)$$
$$\times\left\{\frac{\left(\frac{z_2}{w_N}q^{N-1}\right)^{l-\alpha+i_N}}{w_Nq\left(\frac{z_2}{w_N}q^{N-1};q^{2(N-1)}\right)_\infty\left(\frac{w_N}{z_2}q^{N-1};q^{2(N-1)}\right)_\infty}\right.$$
$$\left.+\frac{\left(\frac{z_2}{w_N}q^{N+1}\right)^{l-\alpha+i_N}}{z_2q^N\left(\frac{z_2}{w_N}q^{3N-1};q^{2(N-1)}\right)_\infty\left(\frac{w_N}{z_2}q^{-N-1};q^{2(N-1)}\right)_\infty}\right\},$$

for $m=1,\ldots,N,$

$$F_{N+1,l}^{(\alpha)}(z_1,z_2)=e^{[\sqrt{-1}\pi N/2(N-1)]}C_1^{*}C_N^{*}(C_1)^{N}(C_{N+1})^2(z_1q)^{1/N-1}\frac{\left\{\frac{z_1}{z_2}q^{2(N-1)}\right\}_\infty\left\{\frac{z_2}{z_1}q^{2(N-1)}\right\}_\infty}{\left\{\frac{z_1}{z_2}q^{2N}\right\}_\infty\left\{\frac{z_2}{z_1}q^{2N}\right\}_\infty}$$
$$\times\oint dw_1\cdots\oint dw_N\left\{\prod_{k=1}^{N}\frac{(1-q^2)}{qw_{k-1}\left(\frac{w_k}{w_{k-1}}q;q^{2(N-1)}\right)_\infty\left(\frac{w_{k-1}}{w_k}q;q^{2(N-1)}\right)_\infty}\right\}$$
$$\times\frac{1}{w_N\left(\frac{z_2}{w_N}q^{N+1};q^{2(N-1)}\right)_\infty\left(\frac{w_N}{z_2}q^{N-1};q^{2(N-1)}\right)_\infty}$$

$$\times \sum_{\{i_1,\ldots,i_N\}\in Z} I^{(a,l)}_{i_1,\ldots,i_N}(z_1,z_2|w_1,\ldots,w_N)\times\partial_{w_N}\left\{\frac{\left(\frac{z_2}{w_N}q^N\right)^{l-\alpha+i_N}}{w_N\left(\frac{z_2}{w_N}q^N;q^{2(N-1)}\right)_\infty\left(\frac{w_N}{z_2}q^{N-2};q^{2(N-1)}\right)_\infty}\right\}.$$

In the above equations, $w_0\equiv z_1 q$, and

$$I^{(a,l)}_{i_1,\ldots,i_N}(z_1,z_2|w_1,\ldots,w_N)=q^{(N-1)\alpha(\alpha-1)}(z_1q)^{i_1-[\alpha/(N-1)]}(z_2q^N)^{[N/(N-1)]\alpha-i_N}$$
$$\times q^{(N-1)\{l^2+l(1-2\alpha)+i_N^2+i_N(1-2\alpha+2l)+\Delta(i_1,\ldots,i_N)\}}$$
$$\times\prod_{k=1}^{N}(w_kq^{k(N-1-k)})^{-\lambda^k_{i_1,\ldots,i_N}},$$

$$C_1^*=\frac{\{q^{2N}\}_\infty}{\{q^{4N-4}\}_\infty},\quad C_N^*=\frac{\{q^{4N-2}\}_\infty}{\{q^{2(N-1)}\}_\infty},$$

$$C_1=(q^{2(N-1)};q^{2(N-1)})_\infty(q^{2N};q^{2(N-1)})_\infty,\quad C_{N+1}=(q^{2(N-1)};\;q^{2(N-1)})_\infty.$$

We now derive the difference equations satisfied by these one-point functions. Noticing that

$$x^d\phi_i(z)x^{-d}=\phi_i(zx^{-1}),\quad x^d\phi_i^*(z)x^{-d}=\phi_i^*(zx^{-1}),$$

$$x^d\psi_i(z)x^{-d}=\psi_i(zx^{-1}),\quad x^d\psi_i^*(z)x^{-d}=\psi_i^*(zx^{-1}),$$

$$x^d\eta_0x^{-d}=\eta_0,\quad x^d\xi_0x^{-d}=\xi_0,$$

we get the difference equations

$$F_m^{(\alpha)}(z_1,z_2q^{2(N-1)})=q^{-2\rho_m}\sum_k R(z_2,z_1)^{km}_{mk}F_k^{(\alpha-1)}(z_1,z_2)(-1)^{[m]+[k]+[m][k]}.$$

Since $\alpha\in\mathbf{Z}$, it is easily seen that this is a set of infinite number of difference equations.

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APPENDIX A: NORMAL-ORDERED RELATIONS OF FUNDAMENTAL BOSONIC FIELDS

In this appendix, we give the normal ordered relations of the fundamental bosonic fields:

$$:e^{h^i(z;\beta_1)}::e^{h^j(w,\beta_2)}:=z^{a_{ij}}\left(1-\frac{w}{z}q^{\beta_1+\beta_2}\right)^{a_{ij}}:e^{h^i(z;\beta_1)+h^j(w;\beta_2)}:,\quad i\neq j,$$

$$:e^{h^i(z;\beta_1)}::e^{h^i(w;\beta_2)}:=z^2\left(1-\frac{w}{z}q^{\beta_1+\beta_2-1}\right)\left(1-\frac{w}{z}q^{\beta_1+\beta_2+1}\right):e^{h^i(z;\beta_1)+h^i(w;\beta_2)}:,\quad i\neq N,$$

$$:e^{h^N(z;\beta_1)}::e^{h^N(w,\beta_2)}:=:e^{h^N(z;\beta_1)+h^N(w;\beta_2)}:,$$

$$:e^{h^i(z;\beta_1)}::e^{h_j^*(w;\beta_2)}:=z^{\delta_{ij}}\left(1-\frac{w}{z}q^{\beta_1+\beta_2}\right)^{\delta_{ij}}:e^{h^i(z;\beta_1)+h_j^*(w;\beta_2)}:,$$

$$:e^{h_i^*(z;\beta_1)}::e^{h^j(w;\beta_2)}:=z^{\delta_{ij}}\left(1-\frac{w}{z}q^{\beta_1+\beta_2}\right)^{\delta_{ij}}:e^{h_i^*(z;\beta_1)+h^j(w;\beta_2)}:,$$

$$:e^{h_N^*(z;\beta_1)}::e^{h_N^*(w;\beta_2)}:=z^{-N/N-1}\frac{\left(\frac{w}{z}q^{\beta_1+\beta_2+2N-1};q^{2(N-1)}\right)}{\left(\frac{w}{z}q^{\beta_1+\beta_2-1};q^{2(N-1)}\right)}:e^{h_N^*(z;\beta_1)+h_N^*(w;\beta_2)}:,$$

$$:e^{h_1^*(z;\beta_1)}::e^{h_1^*(w;\beta_2)}:=z^{N-2/N-1}\frac{\left(\frac{w}{z}q^{\beta_1+\beta_2+1};q^{2(N-1)}\right)}{\left(\frac{w}{z}q^{\beta_1+\beta_2+2N-3};q^{2(N-1)}\right)}:e^{h_1^*(z;\beta_1)+h_1^*(w;\beta_2)}:,$$

$$:e^{h_1^*(z;\beta_1)}::e^{h_N^*(w;\beta_2)}:=z^{-1/N-1}\frac{\left(\frac{w}{z}q^{\beta_1+\beta_2+N};q^{2(N-1)}\right)}{\left(\frac{w}{z}q^{\beta_1+\beta_2+N-2};q^{2(N-1)}\right)}:e^{h_1^*(z;\beta_1)+h_N^*(w;\beta_2)}:,$$

$$:e^{h_N^*(z;\beta_1)}::e^{h_1^*(w;\beta_2)}:=z^{-1/N-1}\frac{\left(\frac{w}{z}q^{\beta_1+\beta_2+N};q^{2(N-1)}\right)}{\left(\frac{w}{z}q^{\beta_1+\beta_2+N-2};q^{2(N-1)}\right)}:e^{h_N^*(z;\beta_1)+h_1^*(w;\beta_2)}:,$$

$$:e^{c(z;\beta_1)}::e^{c(w;\beta_2)}:=z\left(1-\frac{w}{z}q^{\beta_1+\beta_2}\right):e^{c(z;\beta_1)+c(w;\beta_2)}:,$$

where a_{ij} is the Cartan matrix of $\mathrm{sl}(\hat{N}|1)$ and $i,j=1,2,...,N$.

APPENDIX B: COMMUTATION RELATIONS OF VERTEX OPERATORS

By means of the bosonic realization (II.10) of $U_q[\mathrm{sl}(\hat{N}|1)]$, the integral expressions of the bosonized vertex operators (II.16) and the technique given in Ref. 18, one can check the following relations.

For the type I vertex operators:

$$[\phi_k(z),f_l]=0 \quad \text{if} \quad k\neq l,l+1, \quad [\phi_{l+1}(z),f_l]_{q^{\nu_{l+1}}}=\nu_l\phi_l(z)(-1)^{[f_l]([v_l]+[v_{l+1}])},$$

$$[\phi_l(z),f_l]_{q^{-\nu_l}}=0, \quad [\phi_l(z),e_l]=q^{h_l}\phi_{l+1}(z)(-1)^{[e_l]([v_l]+[v_{l+1}])},$$

$$[\phi_k(z),e_l]=0 \quad \text{if} \quad k\neq l, \quad q^{h_1}\phi_l(z)q^{-h_l}=q^{-\nu_l}\phi_l(z),$$

$$q^{h_l}\phi_k(z)q^{-h_l}=\phi_k(z) \quad \text{if} \quad k\neq l,l+1, \quad q^{h_l}\phi_{l+1}(z)q^{-h_l}=q^{\nu_{l+1}}\phi_{l+1}(z),$$

$$[\phi_k^*(z),f_l]=0 \quad \text{if} \quad k\neq l,l+1, \quad [\phi_{l+1}^*(z),f_l]_{q^{-\nu_{l+1}}}=0,$$

$$[\phi_k^*(z),e_l]=0 \quad \text{if} \quad k\neq l+1, \quad [\phi_{l+1}^*(z),e_l]=-\nu_l\nu_{l+1}q^{h_l-\nu_l}\phi_l^*(z)(-1)^{[e_l]([v_l]+[v_{l+1}])},$$

$$[\phi_l^*(z),f_l]_{q^{\nu_l}}=-\nu_l q^{\nu_l}\phi_{l+1}^*(z)(-1)^{[f_l]([v_l]+[v_{l+1}])}, \quad q^{h_l}\phi_l^*(z)q^{-h_l}=q^{\nu_l}\phi_l^*(z),$$

$$q^{h_l}\phi_k^*(z)q^{-h_l}=\phi_k^*(z) \text{ if } k\neq l,l+1, \quad q^{h_1}\phi_{l+1}^*(z)q^{-h_1}=q^{-\nu_{l+1}}\phi_{l+1}^*(z).$$

For the type II vertex operators:

$$[\psi_k(z),e_l]=0 \text{ if } k\neq l,l+1, \quad [\psi_{l+1}(z),e_l]_{q^{-\nu_{l+1}}}=0, \quad [\psi_l(z),e_l]_{q^{\nu_l}}=\psi_{l+1}(z),$$

$$[\psi_k(z),f_1]=0 \text{ if } k\neq l+1, \quad [\psi_{l+1}(z),f_l]=\nu_l q^{-h_l}\psi_l(z),$$

$$q^{h_1}\psi_l(z)q^{-h_l}=q^{-\nu_l}\psi_l(z), \quad q^{h_l}\psi_{l+1}(z)q^{-h_l}=q^{\nu_{l+1}}\psi_{l+1}(z),$$

$$q^{h_l}\psi_k(z)q^{-h_1}=\psi_k(z) \text{ if } k\neq l, \ l+1,$$

$$[\psi_k^*(z),e_l]=0 \text{ if } k\neq l,l+1, \quad [\psi_l^*(z),e_l]_{q^{-\nu_l}}=0,$$

$$[\psi_k^*(z),f_l]=0 \text{ if } k\neq l, \quad [\psi_l^*(z),f_l]=-\nu_l q^{-h_l+\nu_l}\psi_{l+1}^*(z),$$

$$[\psi_{l+1}^*(z),e_l]_{q^{\nu_{l+1}}}=-\nu_l\nu_{l+1}q^{-\nu_1}\psi_l^*(z), \quad q^{h_1}\psi_l^*(z)q^{-h_l}=q^{\nu_l}\psi_l^*(z),$$

$$q^{h_l}\psi_k^*(z)q^{-h_l}=\psi_k^*(z) \text{ if } k\neq l,l+1, \quad q^{h_l}\psi_{l+1}^*(z)q^{-h_l}=q^{-\nu_{l+1}}\psi_{l+1}^*(z).$$

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