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$U_q[\widehat{sl(2|1)}]$ vertex operators, screen currents, and correlation functions at an arbitrary level

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Bosonized q-vertex operators related to the four-dimensional evaluation modules of the quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$ are constructed for arbitrary level $k=\alpha$, where $\alpha\neq 0,-1$ is a complex parameter appearing in the four-dimensional evaluation representations. They are intertwiners among the level-α highest weight Fock–Wakimoto modules. Screen currents which commute with the action of $U_q[\widehat{sl(2|1)}]$ up to total differences are presented. Integral formulas for N-point functions of type I and type II q-vertex operators are proposed. © *2000 American Institute of Physics.* [S0022-2488(00)00608-3]

I. INTRODUCTION

The notion of q-vertex operators as certain intertwiners of highest weight modules of quantum affine algebras was introduced by Frenkel and Reshetikhin[1] in their work on the q-deformation of the Wess–Zumino–Novikov–Witten model. These q-vertex operators give rise to q-analogs of the primary fields in conformal field theory.

Similar to the classical case, q-vertex operators are characterized by the intertwining property defined from the relevant quantum affine algebras. However it is nontrivial to obtain explicit expressions of them. A powerful tool for constructing such explicit formulas is the bosonization technique,[2–4] initiated by Wakimoto[5] in the theory of affine Lie algebras. This method enables one in principle to determine q-vertex operators in terms of certain free bosonic fields. So far, level-one bosonized q-vertex operators have been constructed for most quantum affine algebras[6–8] and the type I quantum affine superalgebras $U_q[\widehat{sl(M|N)}], M\neq N$ (Ref. 9) and $U_q[\widehat{gl(N|N)}]$.[10,11] In the case of arbitrary level, bosonized formulas have been known only for the type I q-vertex operators of $U_q[\widehat{sl(2)}]$ (Refs. 12–15) and $U_q[\widehat{sl(N)}]$.[4]

One of the central issues in conformal field theory and massive integrable models is the computation of correlation functions, which are matrix elements of certain products of vertex operators. The explicit bosonized expressions of vertex operators play an essential role. They enable one to compute correlators exactly in the form of integral representations. This was demonstrated by the Kyoto group and collaborators in their groundbreaking work on the diagonalization of the XXZ spin chain.[16,17] In Refs. 6, 18, 19, certain correlation functions of other quantum affine (super)algebras at level one were computed via the bosonization procedure, generalizing the work of the Kyoto group and collaborators.

The case of the arbitrary level is more complicated. Due to the existence of nontrivial background charges, the naive solutions to the intertwining relations in terms of free bosonic fields do not give rise to proper bosonizations of the q-vertex operators, which ensure the nonvanishing of correlation functions. As in conformal field theory, q-screen currents which balance the background charges are generally needed. Such q-screen currents are dimension 1 operators which (anti-)commute with the relevant quantum algebra generators up to total differences. Bosonized q-screen currents have been obtained for $U_q[\widehat{sl(N)}]$ (Refs. 12–15, 4), and been applied to com-

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pute the correlation functions of the type I $U_q[\widehat{sl(2)}]$ vertex operators.[12–15]

In this paper, by using the free field realization of $U_q[\widehat{sl(2|1)}]$ at an arbitrary level $k\neq 0,-1$ (Ref. 20) we investigate the bosonization of q-vertex operators related to the four-dimensional evaluation modules of $U_q[\widehat{sl(2|1)}]$. It is worth mentioning that our four-dimensional representation contains an extra complex parameter $\alpha\neq 0,-1$. For arbitrary level $k=\alpha$, the q-vertex operators are mappings of certain highest weight Fock–Wakimoto modules in a bosonic Fock space. Screen currents which (anti-)commute with the action of $U_q[\widehat{sl(2|1)}]$ are obtained and bosonized q-vertex operators dressed with the screen charges are proposed. This provides a natural way to write down an integral representation for correlation functions of the bosonized q-vertex operators.

The results obtained in this paper will be useful in analyzing the supersymmetric integrable model introduced in Ref. 21. This is a quantum spin chain model arising from the R-matrix for the four-dimensional $U_q[\widehat{sl(2|1)}]$ evaluation representation and can be interpreted as a model describing strongly correlated electrons.

II. PRELIMINARIES

A. Quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$

The simple roots of the affine superalgebra $\widehat{sl(2|1)}$ (Ref. 22) are

$$\alpha_0=\delta-\varepsilon_1+\delta_1,\quad \alpha_1-\varepsilon_1-\varepsilon_2,\quad \alpha_2=\varepsilon_2-\delta_1,$$

where δ is the null root and $\{\varepsilon_1,\varepsilon_2,\delta_1\}$ are orthonormal basis satisfying

$$(\delta,\delta)=(\delta,\varepsilon_i)=(\delta,\delta_1)=(\delta_1,\varepsilon_i)=0,\quad i=1,2,$$

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij},\quad (\delta_1,\delta_1)=-1.$$

The fundamental weights are

$$\Lambda_0,\quad \Lambda_1=\Lambda_0-\varepsilon_2+\delta_1,\quad \Lambda_2=\Lambda_0-\varepsilon_1-\varepsilon_2+2\delta_1,$$

where Λ_0 is the affine weight obeying $(\Lambda_0,\Lambda_0)=(\Lambda_0,\varepsilon_i)=0$, $i=1,2$ and $(\Lambda_0,\delta)=1$. The symmetric Cartan matrix (a_{ij}) of the affine Lie superalgebra $\widehat{sl(2|1)}$ has elements $a_{ij}=(\alpha_i,\alpha_j),i,j=0,1,2$. Explicitly,

$$(a_{ij})=\begin{pmatrix} 0 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 0\end{pmatrix}.$$

Quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$ is a q-analog of the universal enveloping algebra of $\widehat{sl(2|1)}$ generated by the Chevalley generators $\{e_i,f_i,q^{h_i},d|i=0,1,2\}$, where d is the usual derivation operator. The $\mathbf{Z}_2$-grading of the generators are $[e_0]=[f_0]=[e_2]=[f_2]=1$ and zero otherwise. The defining relations are[23]

$$\begin{gathered}
h_ih_j=h_jh_i,\quad h_id=dh_i,\quad [d,e_i]=\delta_{i,0}e_i,\quad [d,f_i]=-\delta_{i,0}f_i,\\
q^{h_i}e_jq^{-h_i}=q^{a_{ij}}e_j,\quad q^{h_i}f_jq^{-h_i}=q^{-a_{ij}}f_j,\quad [e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}},\\
[e_i,e_j]=[f_i,f_j]=0,\quad \text{for } a_{ij}=0,\\
[e_1,[e_1,e_l]_{q^{-1}}]_q=0,\quad [f_1,[f_1,f_l]_{q^{-1}}]_q=0,\quad l=0,2,
\end{gathered}\tag{II.1}$$

$$[e_0,[e_2,[e_0,[e_2,e_1]_q]]]_q=[e_2,[e_0,[e_2,[e_0,e_1]_q]]]_q,$$

$$[f_0,[f_2,[f_0,[f_2,f_1]_q]]]_q=[f_2,[f_0,[f_2,[f_0,f_1]_q]]]_q.$$

Here and throughout, $[X,Y]_\xi=XY-(-1)^{[X][Y]}\xi YX$ and $[X,Y]=[X,Y]_1$.

$U_q[sl(\widehat{2|1})]$ is a quasitriangular Hopf superalgebra endowed with the $\mathbf{Z}_2$-graded Hopf algebra structure,

$$\begin{gathered}\Delta(h_i)=h_i\otimes 1+1\otimes h_i,\quad \Delta(d)=d\otimes 1+1\otimes d,\\ \Delta(e_i)=e_i\otimes 1+q^{h_i}\otimes e_i,\quad \Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i,\\ \epsilon(h_i)=\epsilon(d)=\epsilon(e_i)=\epsilon(f_i)=0,\\ S(e_i)=-q^{-h_i}e_i,\quad S(f_i)=-f_iq^{h_i},\quad S(h_i)=-h_i,\quad S(d)=-d.\end{gathered}\tag{II.2}$$

Note the antipode S is a $\mathbf{Z}_2$-graded algebra anti-automorphism. Namely for homogeneous elements $a,b\in U_q[sl(\widehat{2|1})], S(ab)=(-1)^{[a][b]}S(b)S(a)$. The multiplication rule for the tensor product is $\mathbf{Z}_2$ graded and is defined for homogeneous elements $a,b,a',b'\in U_q[sl(\widehat{2|1})]$ by $(a\otimes b)(a'\otimes b')=(-1)^{[b][a']}(aa'\otimes bb')$, which extends to inhomogeneous elements through linearity.

$U_q[sl(\widehat{2|1})]$ can also be realized by the Drinfeld generators[24] $\{X_m^{\pm,i},h_n^i,q^{h_0^i},c,d|i=1,2,m\in\mathbf{Z},n\in\mathbf{Z}_{\neq 0}\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators are $[X_m^{\pm,2}]=1(m\in\mathbf{Z})$ and zero otherwise. The relations read[23,25]

$$\begin{gathered}c:\ \text{central element},\\ [h_0^i,h_m^j]=0,\quad [d,h_0^i]=0,\quad [d,h_m^j]=mh_m^j,\\ [h_m^i,h_n^j]=\delta_{m+n,0}\frac{[a_{ij}m]_q[nc]_q}{m},\\ q^{h_0^i}X_m^{\pm,j}q^{-h_0^i}=q^{\pm a_{ij}}X_m^{\pm,j},\quad [d,X_m^{\pm,j}]=mX_m^{\pm,j},\\ [h_m^i,X_n^{\pm,j}]=\pm\frac{[a_{ij}m]_q}{m}q^{\pm|m|c/2}X_{n+m}^{\pm,j},\\ [X_m^{+,i},X_n^{-,j}]=\frac{\delta_{i,j}}{q-q^{-1}}(q^{(m-n)c/2}\psi_{m+n}^{+,j}-q^{-(m-n)c/2}\psi_{m+n}^{-,j}),\\ [X_m^{\pm,2},X_n^{\pm,2}]=0,\\ [X_{m+1}^{\pm,i},X_n^{\pm,j}]_{q^{\pm a_{ij}}}+[X_{n+1}^{\pm,j},X_m^{\pm,i}]_{q^{\pm a_{ij}}}=0,\quad \text{for}\ a_{ij}\neq 0,\\ [X_{n_1}^{\pm,1},[X_{n_2}^{\pm,1},X_m^{\pm,2}]_{q^{-1}}]_q+(n_1\leftrightarrow n_2)=0,\end{gathered}\tag{II.3}$$

where $[m]_q=(q^m-q^{-m})/(q-q^{-1})$ and $\psi_n^{\pm,i}$ are defined by

$$\sum_{n\in\mathbf{Z}}\psi_n^{\pm,i}z^{-n}=q^{\pm h_0^i}\exp\left(\pm(q-q^{-1})\sum_{n>0}h_{\pm n}^i z^{\mp n}\right).$$

The Chevalley generators are related to the Drinfeld generators by the formulas,

$$h_i=h_0^i,\quad e_i=X_0^{+,i},\quad h_0=c-h_0^1-h_0^2,\quad f_i=X_0^{-,i},\quad i=1,2,$$
$$e_0=-[X_0^{-,2},X_1^{-,1}]_{q^{-1}}q^{-h_0^1-h_0^2},\quad f_0=q^{h_0^1+h_0^2}[X_{-1}^{+,1},X_0^{+,2}]_q. \tag{II.4}$$

B. Bosonization of $U_q[sl(\widehat{2|1})]$ at an arbitrary level k

In this subsection we briefly recall the free boson realization of $U_q[sl(\widehat{2|1})]$ at an arbitrary level k.[20] Let us introduce the bosonic q-oscillators $\{a_n^1,a_n^2,b_n^{ij},c_n,Q_{a^1},Q_{a^2},Q_{b^{ij}},Q_c|n\in\mathbf{Z},1\leqslant i<j\leqslant 3\}$ which satisfy the commutation relations

$$[a_m^i,a_n^j]=\delta_{m+n,0}\frac{[a_{ij}m]_q[(k+1)m]_q}{m},\quad [a_0^i,Q_{a^j}]=(k+1)a_{i,j},$$
$$[b_m^{ij},b_n^{i'j'}]=(-1)^{\delta_{j2}}\delta^{ii'}\delta^{jj'}\delta_{m+n,0}\frac{[m]_q^2}{m},\quad [b_0^{ij},Q_{b^{i'j'}}]=(-1)^{\delta_{j2}}\delta^{ii'}\delta^{jj'}, \tag{II.5}$$
$$[c_m,c_n]=\delta_{m+n,0}\frac{[m]_q^2}{m},\quad [c_0,Q_c]=1.$$

The remaining commutators vanish. Here and throughout $k\neq 0,-1$ is a complex parameter. For any pair (a_n,Q_a), we define

$$a(z;\kappa)=-\sum_{n\neq 0}\frac{a_n}{[n]_q}q^{-\kappa|n|}z^{-n}+Q_a+a_0\ln z,$$
$$a_\pm(z)=\pm(q-q^{-1})\sum_{n>0}a_{\pm n}z^{\mp n}\pm a_0\ln q. \tag{II.6}$$

We have

Theorem 1 (Ref. 20): *Define the fields* $X^{\pm,i}(z)$ *by*

$$X^{\pm,i}(z)=\sum_{n\in\mathbf{Z}}X_n^{\pm,i}z^{-n-1}.$$

Then at arbitrary level $k\neq 0,-1,U_q[sl(\widehat{2|1})]$ is realized by the free boson fields as follows:

$$c=k,\quad h_0^1=a_0^1+2b_0^{12}+b_0^{13}-b_0^{23},\quad h_0^2=a_0^2-b_0^{12}-b_0^{13},$$
$$h_m^1=a_m^1q^{-(|m|/2)}+b_m^{12}q^{-((k/2)+1)|m|}(q^{|m|}+q^{-|m|})-b_m^{13}q^{-((k/2)+2)|m|}-b_m^{23}q^{-((k/2)+1)|m|},$$
$$h_m^2=a_m^2q^{-(|m|/2)}-b_m^{12}q^{-((k/2)+1)|m|}-b_m^{13}q^{-((k/2)+1)|m|},$$
$$X^{+,1}(z)=-\frac{1}{(q-q^{-1})z}:e^{-b^{12}(z;-1)}(e^{-c(qz;0)}-e^{-c(q^{-1}z;0)}):e^{\sqrt{-1}\pi(c_0+b_0^{12})},$$
$$X^{+,2}(z)=-:e^{-b_+^{12}(qz)-b_+^{13}(qz)+b^{23}(qz;0)}:e^{\sqrt{-1}\pi(c_0+b_0^{12}+b_0^{13}+b_0^{23})}+:e^{b^{12}(z;0)+b^{13}(z;0)+c(z;0)}:,$$

$$
\begin{aligned}
X^{-,1}(z)=\frac{1}{(q-q^{-1})z}&:(e^{a_+^1(q^{[(k+1)/2]}z)+b^{12}(q^{k+2}z;1)+b_+^{13}(q^{k+2}z)-b_+^{23}(q^{k+1}z)+c(q^{k+1}z;0)}\\
&-e^{a_-^1(q^{[(k+1)/2]}z)+b^{12}(q^{-k-2}z;1)+b_-^{13}(q^{-k-2}z)-b_-^{23}(q^{-k-1}z)+c(q^{-k-1}z;0)}):\\
&e^{-\sqrt{-1}\pi(c_0+b_0^{12})}+q^{k+1}:e^{a_+^1([(k+1)/2]z)-b^{13}(q^{k+1}z;0)+b^{23}(q^{k+1}z;-1)}:e^{\sqrt{-1}\pi(b_0^{13}+b_0^{23})},
\end{aligned}
$$

$$
\begin{aligned}
X^{-,2}(z)=\frac{1}{(q-q^{-1})z}&(q:(e^{a_+^2(q^{[(k+1)/2]}z)-b^{23}(q^{k+1}z;0)}-e^{a_-^2(q^{-[(k+1)/2]}z)-b^{23}(q^{-k-1}z;0)}):\\
&e^{-\sqrt{-1}\pi(c_0+b_0^{12}+b_0^{13}+b_0^{23})}-:e^{a_-^2(q^{-[(k+1)/2]}z)-b^{12}(q^{-k-1}z;1)-b^{13}(q^{-k-1}z;1)}\\
&\times(e^{-c(q^{-k}z;0)}-e^{-c(q^{-k-2}z;0)}):).
\end{aligned}
\tag{II.7}
$$

III. LEVEL-ZERO REPRESENTATIONS

We discuss level-zero representations of $U_q[\widehat{sl(2|1)}]$, which are needed in next section for the investigation of q-vertex operators.

Let V_α is the one parameter family of the four-dimensional typical irreducible representation of $U_q[\widehat{sl(2|1)}]$. Here and throughout, $\alpha\neq 0,-1$ is a complex parameter. We choose the basis vectors $\{v_1,v_2,v_3,v_4\}$ of V_α and assign them the $\mathbf{Z}_2$ gradings $[v_1]=[v_4]=0$, $[v_2]=[v_3]=1$. Let e_{ij} be the 4×4 matrices satisfying $(e_{ij})_{kl}=\delta_{ik}\delta_{jl}$. In the homogeneous gradation, the evaluation representation $V_{\alpha,z}$ of $U_q[\widehat{sl(2|1)}]$ is given by

$$
\begin{aligned}
&e_1=e_{23},\quad f_1=e_{32},\quad h_1=e_{22}-e_{33},\\
&e_2=\sqrt{[\alpha]_q}e_{12}+\sqrt{[\alpha+1]_q}e_{34},\\
&f_2=\sqrt{[\alpha]_q}e_{21}+\sqrt{[\alpha+1]_q}e_{43},\\
&h_2=\alpha(e_{11}+e_2)+(\alpha+1)(e_{33}+e_{44}),\\
&e_0=-z(-\sqrt{[\alpha]_q}e_{31}+\sqrt{[\alpha+1]_q}e_{42}),\\
&f_0=z^{-1}(-\sqrt{[\alpha]_q}e_{13}+\sqrt{[\alpha+1]_q}e_{24}),\\
&h_0=-\alpha(e_{11}+e_{33})-(\alpha+1)(e_{22}+e_{44}).
\end{aligned}
\tag{III.1}
$$

We define the dual module $V_{\alpha,z}^{*S}$ of $V_{\alpha,z}$ by $\pi_{V_{\alpha,z}^{*S}}(a)=\pi_{V_{\alpha,z}}(S(a))^{\mathrm{st}}$, $\forall a\in U_q[\widehat{sl(2|1)}]$, where st is the supertransposition operation. On $V_{\alpha,z}^{*S}$, the Chevalley generators are represented by

$$
\begin{aligned}
&e_1=-q^{-1}e_{32},\quad f_1=-qe_{23},\quad h_1=-e_{22}+e_{33},\\
&e_2=q^{-\alpha}\sqrt{[\alpha]_q}e_{21}-q^{-\alpha-1}\sqrt{[\alpha+1]_q}e_{43},\\
&f_2=-q^{\alpha}\sqrt{[\alpha]_q}e_{12}+q^{\alpha+1}\sqrt{[\alpha+1]_q}e_{34},\\
&h_2=-\alpha(e_{11}+e_{22})-(\alpha+1)(e_{33}+e_{44}),\\
&e_0=-z(q^{\alpha}\sqrt{[\alpha]_q}e_{13}+q^{\alpha+1}\sqrt{[\alpha+1]_q}e_{24}),\\
&f_0=-z^{-1}(q^{-\alpha}\sqrt{[\alpha]_q}e_{31}+q^{-\alpha-1}\sqrt{[\alpha+1]_q}e_{42}),\\
&h_0=\alpha(e_{11}+e_{33})+(\alpha+1)(e_{22}+e_{44}).
\end{aligned}
\tag{III.2}
$$

We state

Proposition 1: The Drinfeld generators are represented on $V_{\alpha,z}$ *by*

$$\begin{aligned}
&h_0^1=e_{22}-e_{33}, \quad h_0^2=\alpha(e_{11}+e_{22})+(\alpha+1)(e_{33}+e_{44}),\\
&X_m^{+,1}=(zq^{\alpha+1})^m e_{23}, \quad X_m^{-,1}=(zq^{\alpha+1})^m e_{32},\\
&X_m^{+,2}=(zq^{\alpha+1})^m(q^{-m}\sqrt{[\alpha]_q}e_{12}+q^m\sqrt{[\alpha+1]}e_{34}),\\
&X_m^{-,2}=(zq^{\alpha+1})^m(q^{-m}\sqrt{[\alpha]_q}e_{21}+q^m\sqrt{[\alpha+1]_q}e_{43}),\\
&h_m^1=(zq^{\alpha+1})^m\frac{[m]_q}{m}(q^{-m}e_{22}-q^m e_{33}),\\
&h_m^2=\frac{z^m}{m}([\alpha m]_q(e_{11}+e_{22})+q^m[(\alpha+1)m]_q(e_{33}+e_{44})),
\end{aligned} \tag{III.3}$$

and on $V_{\alpha,z}^{*S}$ *by*

$$\begin{aligned}
&h_0^1=-e_{22}+e_{33}, \quad h_0^2=-\alpha(e_{11}+e_{22})-(\alpha+1)(e_{33}+e_{44}),\\
&X_m^{+,1}=-z^m q^{-m\alpha-m-1}e_{32}, \quad X_m^{-,1}=-z^m q^{-m\alpha-m+1}e_{23},\\
&X_m^{+,2}=z^m q^{-(1+m)\alpha}(\sqrt{[\alpha]_q}e_{21}-q^{-2m-1}\sqrt{[\alpha+1]_q}e_{43}),\\
&X_m^{-,2}=z^m q^{(1-m)\alpha}(-\sqrt{[\alpha]_q}e_{12}+q^{-2m+1}\sqrt{[\alpha+1]_q}e_{34}),\\
&h_m^1=-(zq^{-\alpha-1})^m\frac{[m]_q}{m}(q^m e_{22}-q^{-m}e_{33}),\\
&h_m^2=-\frac{z^m}{m}([\alpha m]_q(e_{11}+e_{22})+q^{-m}[(\alpha+1)m]_q(e_{33}+e_{44})).
\end{aligned} \tag{III.4}$$

IV. VERTEX OPERATORS AT AN ARBITRARY LEVEL $k=\alpha$

Let $V(\lambda)$ be a level-k highest weight $U_q[\widehat{sl(2|1)}]$-module with highest weight λ and highest weight vector $|\lambda\rangle$. Consider the following intertwiners of $U_q[\widehat{sl(2|1)}]$-modules,

$$\begin{aligned}
&\Phi_\lambda^{\mu V}(z): V(\lambda)\to V(\mu)\otimes V_{\alpha,z}, \quad \Phi_\lambda^{\mu V^*}(z): V(\lambda)\to V(\mu)\otimes V_{\alpha,z}^{*S},\\
&\Psi_\lambda^{V\mu}(z): V(\lambda)\to V_{\alpha,z}\otimes V(\mu), \quad \Psi_\lambda^{V^*\mu}(z): V(\lambda)\to V_{\alpha,z}^{*S}\otimes V(\mu).
\end{aligned} \tag{IV.1}$$

They are intertwiners in the sense that for any $x\in U_q[\widehat{sl(2|1)}]$,

$$\Theta(z)\cdot x=\Delta(x)\cdot\Theta(z), \quad \Theta(z)=\Phi(z),\Phi^*(z),\Psi(z),\Psi^*(z). \tag{IV.2}$$

The intertwiners are even operators, that is their grading is $[\Theta(z)]=0$. $\Phi(z)(\Phi^*(z))$ is called type I (dual) vertex operator and $\Psi(z)(\Psi^*(z))$ type II (dual) vertex operator.

Expand these vertex operators in terms of their components,

$$\Phi(z)=\sum_{r=1}^{4}\Phi_r(z)\otimes v_r, \quad \Phi^*(z)=\sum_{r=1}^{4}\Phi_r^*(z)\otimes v_r^*, \tag{IV.3}$$

$$\Psi(z)=\sum_{r=1}^{4} v_r\otimes\Psi_r(z),\quad \Psi^*(z)=\sum_{r=1}^{4} v_r^*\otimes\Psi_r^*(z), \tag{IV.4}$$

where $v_r\in V_\alpha$ and $v_r^*\in V_\alpha^{*S}$. Then we have

Proposition 2: The operators $\Phi(z)$ *and* $\Psi(z)$ *with respect to* $V_{\alpha,z}$ *are determined by the components* $\Phi_4(z)$ *and* $\Psi_1(z)$*, respectively. More explicitly,*

$$\begin{gathered}\Phi_3(z)=-\frac{1}{\sqrt{\alpha+1}}[\Phi_4(z),f_2]_{q^{-\alpha-1}},\\ \Phi_2(z)=[\Phi_3(z),f_1]_q,\quad \Phi_1(z)=-\frac{1}{\sqrt{\alpha}}[\Phi_2(z),f_2]_{q^{-\alpha}},\\ \Psi_2(z)=\frac{1}{\sqrt{\alpha}}[\Psi_1(z),e_2]_{q^{\alpha}},\quad \Psi_3(z)=[\Psi_2(z),e_1]_q,\\ \Psi_4(z)=\frac{1}{\sqrt{\alpha+1}}[\Psi_3(z),e_2]_{q^{\alpha+1}}.\end{gathered} \tag{IV.5}$$

With respect to $V_{\alpha,z}^{*S}$*, the operators* $\Phi^*(z)$ *and* $\Psi^*(z)$ *are determined by* $\Phi_1^*(z)$ *and* $\Psi_4^*(z)$*, respectively,*

$$\begin{gathered}\Phi_2^*(z)=\frac{q^{-\alpha}}{\sqrt{\alpha}}[\Phi_1^*(z),f_2]_{q^{\alpha}},\quad \Phi_3^*(z)=-q^{-1}[\Phi_2^*(z),f_1]_q,\\ \Phi_4^*(z)=-\frac{q^{-\alpha-1}}{\sqrt{\alpha+1}}[\Phi_3^*(z),f_2]_{q^{\alpha+1}},\\ \Psi_3^*(z)=-\frac{q^{\alpha+1}}{\sqrt{\alpha+1}}[\Psi_4^*(z),e_2]_{q^{-\alpha-1}},\\ \Psi_2^*(z)=-q[\Psi_3^*(z),e_1]_q,\quad \Psi_1^*(z)=\frac{q^{\alpha}}{\sqrt{\alpha}}[\Psi_2^*(z),e_2]_{q^{-\alpha}}.\end{gathered} \tag{IV.6}$$

Next we determine the relations between the components $\Phi_4(z),\Phi_1^*(z),\Psi_1(z),\Psi_4^*(z)$ and the Drinfeld generators. We have

Proposition 3: For $\Phi(z)$ *associated with* $V_{\alpha,z}$*,*

$$\begin{gathered}[\Phi_4(z),X^{+,i}(w)]=0,\quad i=1,2,\\ q^{h_0^i}\Phi_4(z)q^{-h_0^i}=q^{-(\alpha+1)\delta_{i2}}\Phi_4(z),\\ [h_n^i,\Phi_4(z)]=-\delta_{i2}q^{(1+3/2\,k)n}\frac{[(\alpha+1)n]_q}{n}z^n\Phi_4(z),\\ [h_{-n}^i,\Phi_4(z)]=-\delta_{i2}q^{-(1+1/2\,k)n}\frac{[(\alpha+1)n]_q}{n}z^{-n}\Phi_4(z);\end{gathered} \tag{IV.7}$$

for $\Phi^*(z)$ *associated with* $V_{\alpha,z}^{*S}$*,*

$$
\begin{aligned}
&[\Phi_1^*(z),X^{+,i}(w)]=0, \quad i=1,2,\\
&q^{h_0^i}\Phi_1^*(z)q^{-h_0^i}=q^{\alpha\delta_{i2}}\Phi_1^*(z),\\
&[h_n^i,\Phi_1^*(z)]=\delta_{i2}q^{(3/2)kn}\frac{[\alpha n]_q}{n}z^n\Phi_1^*(z),\\
&[h_{-n}^i,\Phi_1^*(z)]=\delta_{i2}q^{-1/2\,kn}\frac{[\alpha n]_q}{n}z^{-n}\Phi_1^*(z);
\end{aligned}
\tag{IV.8}
$$

for $\Psi(z)$ *associated with* $V_{\alpha,z}$,

$$
\begin{aligned}
&[\Psi_1(z),X^{-,i}(w)]=0, \quad i=1,2,\\
&q^{h_0^i}\Psi_1(z)q^{-h_0^i}=q^{-\alpha\delta_{i2}}\Psi_1(z),\\
&[h_n^i,\Psi_1(z)]=-\delta_{i2}q^{1/2\,kn}\frac{[\alpha n]_q}{n}z^n\Psi_1(z),\\
&[h_{-n}^i,\Psi_1(z)]=-\delta_{i2}q^{-3/2\,kn}\frac{[\alpha n]_q}{n}z^{-n}\Psi_1(z);
\end{aligned}
\tag{IV.9}
$$

and for $\Psi^*(z)$ *associated with* $V_{\alpha,z}^{*S}$,

$$
\begin{aligned}
&[\Psi_4^*(z),X^{-,i}(w)]=0, \quad i=1,2,\\
&q^{h_0^i}\Psi_4^*(z)q^{-h_0^i}=q^{(\alpha+1)\delta_{i2}}\Psi_4^*(z),\\
&[h_n^i,\Psi_4^*(z)]=\delta_{i2}q^{(1/2\,k-1)n}\frac{[(\alpha+1)n]_q}{n}z^n\Psi_4^*(z),\\
&[h_{-n}^i,\Psi_4^*(z)]=\delta_{i2}q^{(-3/2\,k+1)n}\frac{[(\alpha+1)n]_q}{n}z^{-n}\Psi_4^*(z);
\end{aligned}
\tag{IV.10}
$$

To obtain bosonized expressions of the intertwining operators, we introduce the combinations of bosonic oscillators for $m\in\mathbf{Z}$,

$$
\begin{aligned}
&A_m^*=-\left(a_m^1+\frac{[2m]_q}{[m]_q}a_m^2\right)q^{|m|/2},\\
&B_m^*=-\frac{[\alpha m]_q}{[(\alpha+1)m]_q}\left(a_m^1+\frac{[2m]_q}{[m]_q}a_m^2\right)q^{|m|/2},\\
&\tilde{B}_m^*=-\left(a_m^1+\frac{[2m]_q}{[m]_q}a_m^2\right)q^{-|m|/2}+(b_m^{13}+q^{-|m|}b_m^{23})q^{-(\alpha/2)|m|},\\
&\tilde{A}_m^*=-\frac{[\alpha m]_q}{[(\alpha+1)m]_q}\left(a_m^1+\frac{[2m]_q}{[m]_q}a_m^2\right)q^{-|m|/2}-(q^{|m|}b_m^{13}+b_m^{23})q^{(3\alpha/2)|m|},
\end{aligned}
\tag{IV.11}
$$

$$
Q_{A*}=-Q_{a^1}-2Q_{a^2}, \quad Q_{B*}=-\frac{\alpha}{\alpha+1}(Q_{a^1}+2Q_{a^2}),
$$

$$Q_{\tilde{B}*}=-Q_{a^1}-2Q_{a^2}+Q_{b^{13}}+Q_{b^{23}},$$

$$Q_{\tilde{A}*}=-\frac{\alpha}{\alpha+1}(Q_{a^1}+2Q_{a^2})-Q_{b^{13}}-Q_{b^{23}}.$$

For $k=\alpha$, these operators obey the commutation relations, among others,

$$[A^*_m,h^i_n]=\delta_{i2}\delta_{m+n,0}\frac{[m]_q[(\alpha+1)m]_q}{m}=[\tilde{A}^*_m,h^i_n],$$
$$[B^*_m,h^i_n]=\delta_{i2}\delta_{m+n,0}\frac{[m]_q[\alpha m]_q}{m}=[\tilde{B}^*_m,h^i_n]. \tag{IV.12}$$

Then

Theorem 2: *For $k=\alpha$, the bosonized forms $\phi_4(z)$, $\phi^*_1(z)$, $\psi_1(z)$, and $\psi^*_4(z)$ of the vertex operator components $\Phi_4(z)$, $\Phi^*_1(z)$, $\Psi_1(z)$, and $\Psi^*_4(z)$ are given by*

$$\phi_4(z)=:e^{-A^*(q^{\alpha+1}z;-(\alpha/2))}:,$$
$$\phi^*_1(z)=:e^{B^*(q^{\alpha}z;-(\alpha/2))}:,$$
$$\psi_1(z)=:e^{-\tilde{B}^*(q^{\alpha}z;(\alpha/2))}:, \tag{IV.13}$$
$$\psi^*_4(z)=:e^{\tilde{A}^*(q^{\alpha-1}z;(\alpha/2))}:e^{\sqrt{-1}\pi(b^{13}_0+b^{23}_0)}.$$

The other components $\phi_r(z)$, $\phi^*_r(z)$, $\psi_r(z)$, and $\psi^*_r(z)$ are represented by multiple contour integrals of the Drinfeld currents (cf. Proposition 2).

Vertex operators (IV.13) are referred to as ''elementary q-vertex operators'' and are determined solely from their commutation relations with the bosonized $U_q[\widehat{sl(2|1)}]$ generators. The construction is completely independent of which infinite dimensional modules the vertex operators intertwine. In next section, we shall clarify on which space these bosonized vertex operators act.

V. FOCK SPACE AND FOCK–WAKIMOTO MODULES

In this section we study bosonic Fock space on which the $U_q[\widehat{sl(2|1)}]$ generators and the bosonized vertex operators act. As we will see, all highest weight modules of $U_q[\widehat{sl(2|1)}]$ can be embedded in the bosonic Fock space. Note that $k=\alpha\neq 0,-1$.

Let $|0\rangle$ be the vacuum vector, which is defined by $a^i_n|0\rangle=b^{12}_n|0\rangle=b^{13}_n|0\rangle=b^{23}_n|0\rangle=c_n|0\rangle=0$ for $n\geqslant 0$. Introduce the vector

$$|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle=e^{[1/(\alpha+1)]\lambda_{a^1}Q_{a^1}+[2/(\alpha+1)]\lambda_{a^2}Q_{a^2}+\lambda_{b^{12}}Q_{b^{12}}+\lambda_{b^{13}}Q_{b^{13}}+\lambda_{b^{23}}Q_{b^{23}}+\lambda_cQ_c}|0\rangle, \tag{V.1}$$

which carries the weight $(\lambda_{a^1}/(\alpha+1),2\lambda_{a^2}/(\alpha+1),\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)\in\mathbf{C}^6$. Denote by

$$F_{[1/(\alpha+1)]\lambda_{a^1},[2/(\alpha+1)]\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c}$$

the module generated by the creation operators a^1_n, a^2_n, b^{12}_n, b^{13}_n, b^{23}_n, and $c_n(n<0)$ over the vector $|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle$. Introduce the bosonic Fock space

$$F_{(\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)}=\bigoplus_{i,j,l\in\mathbf{Z}}F_{[1/(\alpha+1)]\lambda_{a^1},[2/(\alpha+1)]\lambda_{a^2},\lambda_{b^{12}}+i+j,\lambda_{b^{13}}+j,\lambda_{b^{23}}+l,\lambda_c+i}$$

It can be shown that the action of $U_q[\widehat{sl(2|1)}]$ on this space is closed, i.e., $U_q[\widehat{sl(2|1)}]F_* = F_*$ for $*=(\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)$. Hence the Fock space F_* constitutes a $U_q[\widehat{sl(2|1)}]$-module. The elementary q-vertex operators are maps of the following Fock spaces:

$$\begin{aligned}\phi_r(z),\psi_r(z) \quad &: \quad F_{(\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)} \to F_{(\lambda_{a^1}+\alpha+1,\lambda_{a^2}+\alpha+1,\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c},\\ \phi_r^*(z),\psi_r^*(z) \quad &: \quad F_{(\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)} \to F_{(\lambda_{a^1}-\alpha,\lambda_{a^2}-\alpha,\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c)},\end{aligned} \tag{V.3}$$

for all $r=1,2,3,4$.

Let us now discuss the emdedding of the highest weight module $V(\lambda)$ in the bosonic Fock space F_*. We impose the highest weight conditions on the vector $|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle$,

$$\begin{aligned}e_i|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle&=0,\\ h_i|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle&=\lambda_i|\lambda_{a^1},\lambda_{a^2},\lambda_{b^{12}},\lambda_{b^{13}},\lambda_{b^{23}},\lambda_c\rangle\end{aligned} \tag{V.4}$$

for all $i=0,1,2$. Solving these conditions, we obtain the highest weight vector $|\beta,\gamma,0,0,0,0\rangle$, where β and γ are arbitrary complex parameters. The corresponding highest weight is $\lambda_{\beta,\gamma}=(\alpha-\beta+2\gamma)\Lambda_0+2(\beta-\gamma)\Lambda_1-\beta\Lambda_2$. Thus we have the identification

$$|\lambda_{\beta,\gamma}\rangle=|\beta,\gamma,0,0,0,0\rangle. \tag{V.5}$$

Denote by

$$F_{(\beta,\gamma)}=\bigoplus_{i,j,l\in\mathbf{Z}} F_{[1/(\alpha+1)]\beta,[2/(\alpha+1)]\gamma,i+j,j,l,i+j} \tag{V.6}$$

the Fock space associated to this highest weight vector. It is easy to see that the $U_q[\widehat{sl(2|1)}]$ action on the subspace $F_{(\beta,\gamma)}$ is still closed and therefore $F_{(\beta,\gamma)}$ is a $U_q[\widehat{sl(2|1)}]$-module. Using the highest weight vector $|\lambda_{\beta,\gamma}\rangle$, we construct the level-α highest weight module of $U_q[\widehat{sl(2|1)}]$,

$$V(\lambda_{\beta,\gamma})=U_q[\widehat{sl(2|1)}]|\lambda_{\beta,\gamma}\rangle. \tag{V.7}$$

This module is not irreducible in general, but contains a maximal proper submodule $M(\lambda_{\beta,\gamma})$ such that $V(\lambda_{\beta,\gamma})/M(\lambda_{\beta,\gamma})$ yields an irreducible $U_q[\widehat{sl(2|1)}]$ module. It is clear that the module $V(\lambda_{\beta,\gamma})$ can be embedded in the bosonic Fock space $F_{(\beta,\gamma)}$. Moreover, from (V.3) the elementary q-vertex operators are mappings of the Fock spaces,

$$\begin{aligned}\phi_r(z),\quad \psi_r(z) \quad &: \quad F_{(\beta,\gamma)} \to F_{(\beta+\alpha+1,\gamma+\alpha+1)},\\ \phi_r^*(z),\quad \psi_r^*(z) \quad &: \quad F_{(\beta,\gamma)} \to F_{(\beta-\alpha,\gamma-\alpha)}.\end{aligned} \tag{V.8}$$

However, the Fock space $F_{(\beta,\gamma)}$ contains some redundancies arising from the free bosonic field $c(z;0)$. To see this, we define the fermionic ghost system (η,ξ) of dimension (1, 0),

$$\eta(z)=\sum_{n\in\mathbf{Z}}\eta_n z^{-n-1}=:e^{c(z;0)}:,\quad \xi(z)=\sum_{n\in\mathbf{Z}}\xi_n z^{-n}=:e^{-c(z;0)}: \tag{V.9}$$

The mode expansion of $\eta(z)$ and $\xi(z)$ is well defined on $F_{(\beta,\gamma)}$, and the modes satisfy the relations

$$\xi_m\xi_n+\xi_n\xi_m=0=\eta_m\eta_n+\eta_n\eta_m,\quad \xi_m\eta_n+\eta_n\xi_m=\delta_{m+n,0}. \tag{V.10}$$

Obviously, $\eta_0\xi_0$ and $\xi_0\eta_0$ qualify as projectors and so we use them to decompose $F_{(\beta,\gamma)}$ into a direct sum of subspaces

$$F_{(\beta,\gamma)}=\eta_0\xi_0F_{(\beta,\gamma)}\oplus\xi_0\eta_0F_{(\beta,\gamma)}. \tag{V.11}$$

$\eta_0\xi_0F_{(\beta,\gamma)}$ is referred to as Ker_{η_0} and $\xi_0\eta_0F_{(\beta,\gamma)}=F_{(\beta,\gamma)}/\eta_0\xi_0F_{(\beta,\gamma)}$ as Coker_{η_0}.

Proposition 4: η_0 commutes (or anticommutes) with the action of $U_q[\widehat{sl(2|1)}]$. Thus Ker_{η_0} *and* Coker_{η_0} *are both $U_q[\widehat{sl(2|1)}]$-modules.*

We are now in a position to consider a restriction of the Fock space $F_{(\beta,\gamma)}$ to a smaller space $\mathcal{F}_{(\beta,\gamma)}$, referred to as the Fock–Wakimoto space.

Proposition 5: The restricted Fock space

$$\mathcal{F}_{(\beta,\gamma)}\equiv\mathrm{Ker}_{\eta_0}F_{(\beta,\gamma)}=\eta_0\xi_0F_{(\beta,\gamma)} \tag{V.12}$$

constitutes a Fock–Wakimoto module of $U_q[\widehat{sl(2|1)}]$.

One can check that $\eta_0|\lambda_{\beta,\gamma}\rangle=0$ for any $\beta,\ \gamma\in\mathbf{C}$. Thus $|\lambda_{\beta,\gamma}\rangle$ is a $U_q[\widehat{sl(2|1)}]$ highest weight vector belonging to the smaller space $\mathrm{Ker}_{\eta_0}F_{(\beta,\gamma)}$. It follows that

Proposition 6: The Fock–Wakimoto module $\mathcal{F}_{(\beta,\gamma)}$ is a highest weight $U_q[\widehat{sl(2|1)}]$-module with highest weight vector $|\lambda_{\beta,\gamma}\rangle$ and highest weight $\lambda_{\beta,\gamma}$.

Using the projection operator $\eta_0\xi_0$, we define the ''projected q-vertex operators'' $\tilde{\phi}_r(z)$, $\tilde{\phi}_r^*(z)$, $\tilde{\psi}_r(z)$, and $\tilde{\psi}_r^*(z)$ as follows:

$$\tilde{\Theta}(z)=\eta_0\xi_0\Theta(z)\eta_0\xi_0,\quad \Theta(z)=\phi_r(z),\ \ \phi_r^*(z),\ \ \psi_r(z)\ \ \text{or}\ \ \psi_r^*(z). \tag{V.13}$$

Since η_0 commutes with the elementary q-vertex operators, we can deduce from (V.8) that the projected q-vertex operators are mappings of the highest weight Fock–Wakimoto modules:

$$\begin{aligned}
\tilde{\phi}_r(z)\ &:\ \ \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta+\alpha+1,\gamma+\alpha+1)},\\
\tilde{\psi}_r(z)\ &:\ \ \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta+\alpha+1,\gamma+\alpha+1)},\\
\tilde{\phi}_r^*(z)\ &:\ \ \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta-\alpha,\gamma-\alpha)},\\
\tilde{\psi}_r^*(z)\ &:\ \ \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta-\alpha,\gamma-\alpha)}.
\end{aligned} \tag{V.14}$$

VI. SCREEN CURRENTS AND CORRELATION FUNCTIONS

Due to the existence of backgound charges, the projected q-vertex operators are not yet the proper bosonizations of the q-vertex operators (IV.1). In this section we construct q-screen currents which balance the background charges and thus ensure the nonvanishing of correlation functions of the bosonized q-vertex operators.

Let us introduce the oscillators

$$a_m^{*,i}=\frac{[m]_q}{[(k+1)m]_q}a_m^i,\quad Q_{a^{*,i}}=\frac{1}{k+1}Q_{a^i},\quad i=1,2 \tag{VI.1}$$

and define the corresponding currents $S^i(z)$ by

$$S^i(z)=:e^{-a^{*,i}(z;(k+1)/2)}\tilde{S}^i(z), \tag{VI.2}$$

$$\widetilde{S}^1(z)=:e^{-b^{12}(z;0)-b_-^{12}(q^{-1}z)-b_-^{13}(q^{-1}z)+b_-^{23}(z)}\,{}_1\partial_z e^{-c(q^{-1}z;0)}:e^{\sqrt{-1}\pi(c_0+b_0^{12})}$$
$$+q:e^{b^{13}(z;0)-b^{23}(qz;0)+b_+^{23}(z)}:e^{-\sqrt{-1}\pi(b_0^{13}+b_0^{23})}, \tag{VI.3}$$

$$\widetilde{S}^2(z)=-q^{-1}:e^{b^{23}(z;0)}:e^{-\sqrt{-1}\pi(c_0+b_0^{12}+b_0^{13}+b_0^{23})}. \tag{VI.4}$$

Here we have used the notation

$${}_k\partial_z f(z)=\frac{f(q^kz)-f(q^{-k}z)}{(q-q^{-1})z}. \tag{VI.5}$$

Then we can verify

Theorem 3: *The currents $S^i(z)$ satisfy the following commutation relations with the $U_q[sl\widehat{(2|}1)]$ generators,*

$$[h_n^i,S^j(w)]=0,\quad n\in\mathbf{Z},$$
$$[X^{+,i}(z),S^j(w)]=0, \tag{VI.6}$$
$$[X^{-,i}(z),S^j(w)]=\delta^{ij}\,{}_{k+1}\partial_w\left(-z^{-1}\cdot\delta\left(\frac{w}{z}\right):e^{-a^{*,i}(w;-(k+1)/2)}:\right).$$

That is, the currents $S^i(z)$ (anti-)commute with the action of $U_q[sl\widehat{(2|}1)]$ up to total differences. The currents $S^i(z)$ are referred to as the q-screen currents of $U_q[sl\widehat{(2|}1)]$.

For $p\in\mathbf{C}$, $|p|<1$ and $s\in\mathbf{C}-\{0\}$, one defines the Jackson integral

$$\int_0^{s\infty}f(z)d_pz=s(1-p)\sum_{m\in\mathbf{Z}}f(sp^m)p^m. \tag{VI.7}$$

The Jackson integral enjoys the following property, among others,

$$\int_0^{s\infty}f(z)d_pz=\int_0^{s\infty}pf(pz)d_pz, \tag{VI.8}$$

which implies that for $p=q^{2k}$,

$$\int_0^{s\infty}{}_k\partial_z f(z)d_pz=0. \tag{VI.9}$$

Note that the right-hand side of (VI.6) is a total $p=q^{2(k+1)}$ difference. We have

Corollary 1: The screen charges

$$Q^i=\int_0^{s\infty}S^i(z)d_pz,\quad p=q^{2(k+1)}, \tag{VI.10}$$

assuming that the Jackson integrals are convergent, (anti-)commute with all the generators of $U_q[sl\widehat{(2|}1)]$.

Since η_0 commutes with $S^i(z), i=1,2$, the screen charges with $k=\alpha$ give rise to the following mappings of the Fock–Wakimoto modules,

$$Q^1\ :\quad \mathcal{F}_{(\beta,\gamma)}\rightarrow\mathcal{F}_{(\beta-1,\gamma)}, \tag{VI.11}$$

$$Q^2\ :\quad \mathcal{F}_{(\beta,\gamma)}\rightarrow\mathcal{F}_{(\beta,\gamma-1/2)}. \tag{VI.12}$$

Introduce the screened q-vertex operators,

$$\begin{aligned}\tilde{\phi}_r^{(x_1,\tilde{x}_1)}(z)&=(Q^1)^{x_1}(Q^2)^{\tilde{x}_1}\tilde{\phi}_r(z),\\ \tilde{\phi}_r^{*(y_1,\tilde{y}_1)}(z)&=(Q^1)^{y_1}(Q^2)^{\tilde{y}_1}\tilde{\phi}_r^*(z),\\ \tilde{\psi}_r^{(x_1',\tilde{x}_1')}(z)&=(Q^1)^{x_1'}(Q^2)^{\tilde{x}_1'}\tilde{\psi}_r(z),\\ \tilde{\psi}_r^{*(y_1',\tilde{y}_1')}(z)&=(Q^1)^{y_1'}(Q^2)^{\tilde{y}_1'}\tilde{\psi}_r^*(z).\end{aligned} \tag{VI.13}$$

We are now in a position to state

Theorem 4: *The q-vertex operators (IV.1) are bosonized as*

$$\begin{aligned}\tilde{\Phi}_{\lambda_{\beta,\gamma}}^{\lambda_{\beta_+^1(x),\gamma_+^1(\tilde{x})}V}(z)&=\sum_{r=1}^4 \tilde{\phi}_r^{(x_1,\tilde{x}_1)}(z)\otimes v_r,\\ \tilde{\Phi}_{\lambda_{\beta,\gamma}}^{\lambda_{\beta_-^1(y),\gamma_-^1(\tilde{y})}V^*}(z)&=\sum_{r=1}^4 \tilde{\phi}_r^{*(y_1,\tilde{y}_1)}(z)\otimes v_r^*,\\ \tilde{\Psi}_{\lambda_{\beta,\gamma}}^{V\lambda_{\beta_+^1(x'),\gamma_+^1(\tilde{x}')}}(z)&=\sum_{r=1}^4 v_r\otimes\tilde{\psi}_r^{(x_1',\tilde{x}_1')}(z),\\ \tilde{\Psi}_{\lambda_{\beta,\gamma}}^{V^*\lambda_{\beta_-^1(y'),\gamma_-^1(\tilde{y}')}}(z)&=\sum_{r=1}^4 v_r^*\otimes\tilde{\psi}_r^{*(y_1',\tilde{y}_1')}(z),\end{aligned} \tag{VI.14}$$

where

$$\begin{aligned}\beta_+^1(x)&=\beta+\alpha+1-x_1,\quad \gamma_+^1(\tilde{x})=\gamma+\alpha+1-\tfrac{1}{2}\tilde{x}_1,\\ \beta_-^1(y)&=\beta-\alpha-y_1,\quad \gamma_-^1(\tilde{y})=\gamma-\alpha-\tfrac{1}{2}\tilde{y}_1\end{aligned} \tag{VI.15}$$

for certain choices of non-negative integers x_1, $\tilde{x}_1$, y_1, and $\tilde{y}_1$. These operators are intertwiners of the highest weight $U_q[\widehat{sl(2|1)}]$-modules,

$$\begin{aligned}\tilde{\Phi}_{\lambda_{\beta,\gamma}}^{\lambda_{\beta_+^1(x),\gamma_+^1(\tilde{x})}V}(z)\ &:\quad \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta_+^1(x),\gamma_+^1(\tilde{x}))}\otimes V_{\alpha,z},\\ \tilde{\Phi}_{\lambda_{\beta,\gamma}}^{\lambda_{\beta_-^1(y),\gamma_-^1(\tilde{y})}V^*}(z)\ &:\quad \mathcal{F}_{(\beta,\gamma)}\to\mathcal{F}_{(\beta_-^1(y),\gamma_-^1(\tilde{y}))}\otimes V_{\alpha,z}^{*S},\\ \tilde{\Psi}_{\lambda_{\beta,\gamma}}^{V\lambda_{\beta_+^1(x'),\gamma_+^1(\tilde{x}')}}(z)\ &:\quad \mathcal{F}_{(\beta,\gamma)}\to V_{\alpha,z}\otimes\mathcal{F}_{(\beta_+^1(x'),\gamma_+^1(\tilde{x}'))},\\ \tilde{\Psi}_{\lambda_{\beta,\gamma}}^{V^*\lambda_{\beta_-^1(y'),\gamma_-^1(\tilde{y}')}}(z)\ &:\quad \mathcal{F}_{(\beta,\gamma)}\to V_{\alpha,z}^{*S}\otimes\mathcal{F}_{(\beta_-^1(y'),\gamma_-^1(\tilde{y}'))}.\end{aligned} \tag{VI.16}$$

In the following we compute N-point correlation function which is defined to be the trace of the bosonized q-vertex operators over the $U_q[\widehat{sl(2|1)}]$-module $\mathcal{F}_{(\beta,\gamma)}$, that is

$$\mathrm{Tr}_{\mathcal{F}_{(\beta,\gamma)}}(q^{L_0}\Theta_{r_N}(z_N)\cdots\Theta_{r_1}(z_1)). \tag{VI.17}$$

Here $\Theta_{r_l}(z_l)$ stands for the type I q-vertex operators $\tilde{\phi}_{r_l}^{(x_l,\tilde{x}_l)}(z_l),\tilde{\phi}_{r_l}^{*(y_l,\tilde{y}_l)}(z_l)$ or the type II q-vertex operators $\tilde{\psi}_{r_l}^{(x'_l,\tilde{x}'_l)}(z_l)$, $\tilde{\psi}_{r_l}^{*(y'_l,\tilde{y}'_l)}(z_l)$; $L_0\equiv -d$ is the q-Virasoro operator which is bosonized as (for $k=\alpha\neq 0,-1$),

$$-L_0=\sum_{n>0}\left(\frac{n^2}{[n]_q[(\alpha+1)n]_q}(a^1_{-n}a^2_n+a^2_{-n}a^1_n+(q^n+q^{-n})a^2_{-n}a^2_n)\right.$$
$$\left.+\frac{n^2}{[n]_q^2}(b^{12}_{-n}b^{12}_n-b^{13}_{-n}b^{13}_n-b^{23}_{-n}b^{23}_n-c_{-n}c_n)\right)+\frac{1}{\alpha+1}(a^1_0a^2_0+(a^2_0)^2+a^1_0+3a^2_0)$$
$$+\tfrac{1}{2}((b^{12}_0)^2-b^{13}_0(b^{13}_0+1)-b^{23}_0(b^{23}_0+1)-(c_0)^2). \tag{VI.18}$$

The zero mode part of the a^1_n,a^2_n oscillators is added to the L_0 operator so that its eigenvalue on $|\lambda_{\beta,\gamma}\rangle$ is $1/2(a+1)(\lambda_{\beta,\gamma},\lambda_{\beta,\gamma}+2p)$, where $\rho=\Lambda_0+\Lambda_1+\Lambda_2$.

Let us define the Fock spaces for $s\in\mathbf{Z}$,

$$F^{(s)}_{(\beta,\gamma)}=\bigoplus_{i,j,l\in\mathbf{Z}}F_{[1/(\alpha+1)]\beta,[2/(\alpha+1)]\gamma,i+j,j,l,i+j+s}. \tag{VI.19}$$

We have $F^{(0)}_{(\beta,\gamma)}=F_{(\beta,\gamma)}$. It can be shown that η_0,ξ_0 intertwine various Fock spaces

$$\eta_0\ :\ F^{(s)}_{(\beta,\gamma)}\to F^{(s+1)}_{(\beta,\gamma)},\quad \xi_0\ :\ F^{(s)}_{(\beta,\gamma)}\to F^{(s-1)}_{(\beta,\gamma)}.$$

Since $\eta_0^2=0$, we obtain the following BRST complex:

$$\cdots\xrightarrow{Q_{s-1}=\eta_0}F^{(s)}_{(\beta,\gamma)}\xrightarrow{Q_s=\eta_0}F^{(s+1)}_{(\beta,\gamma)}\xrightarrow{Q_{s+1}=\eta_0}\cdots. \tag{VI.20}$$

It follows from $\eta_0\xi_0+\xi_0\eta_0=1$, that $\mathrm{Ker}_{Q_s}=\mathrm{Im}_{Q_{s-1}}$ for any $s\in\mathbf{Z}$. We have

Proposition 7: The N-point correlation function of the type I vertex operators,

$$\mathrm{Tr}_{\mathcal{F}_{(\beta,\gamma)}}(q^{L_0}\tilde{\phi}_{r_N}^{(x_N,\tilde{x}_N)}(z_N)\cdots\tilde{\phi}_{r_1}^{(x_1,\tilde{x}_1)}(z_1))\neq 0$$

iff $\alpha\in\mathbf{N}$ *and* $\Sigma_{i=1}^N x_i=\frac{1}{2}\Sigma_{i=1}^N\tilde{x}_i=N(\alpha+1)$. *For such* α *and* x_i, $\tilde{x}_i$, *the above trace is given by*

$$\sum_{s=1}^{\infty}(-1)^{s+1}\mathrm{Tr}_{F^{(-s)}_{(\beta,\gamma)}}(q^{L_0}(Q^1)^{x_N}(Q^2)^{\tilde{x}_N}\phi_{r_N}(z_N)\cdots(Q^1)^{x_1}(Q^2)^{\tilde{x}_1}\phi_{r_1}(z_1)). \tag{VI.21}$$

Similarly, the N-point correlator of the type II vertex operators,

$$\mathrm{Tr}_{\mathcal{F}_{(\beta,\gamma)}}(q^{L_0}\tilde{\psi}_{r_N}^{(x'_N,\tilde{x}'_N)}(z_N)\cdots\tilde{\psi}_{r_1}^{(x'_1,\tilde{x}'_1)}(z_1))$$
$$=\sum_{s=1}^{\infty}(-1)^{s+1}\mathrm{Tr}_{F^{(-s)}_{(\beta,\gamma)}}(q^{L_0}(Q^1)^{x'_N}(Q^2)^{\tilde{x}'_N}\psi_{r_N}(z_N)\cdots(Q^1)^{x'_1}(Q^2)^{\tilde{x}'_1}\psi_{r_1}(z_1)) \tag{VI.22}$$

is nonvanishing iff $\alpha\in\mathbf{N}$ and $\Sigma_{i=1}^N x'_i=1/2\Sigma_{i=1}^N\tilde{x}'_i=N(\alpha+1)$.

We now consider the N-point correlation function involving also dual vertex operators,

$$\mathrm{Tr}_{\mathcal{F}_{(\beta,\gamma)}}(q^{L_0}\tilde{\phi}^{*(y_N,\tilde{y}_N)}_{r_N}(z_N)\cdots\tilde{\phi}^{*(y_{l+1},\tilde{y}_{l+1})}_{r_{l+1}}(z_{l+1})\tilde{\phi}^{(x_l,\tilde{x}_l)}_{r_l}(z_l)\cdots\tilde{\phi}^{(x_1,\tilde{x}_1)}_{r_1}(z_1)). \tag{VI.23}$$

Then we have

Proposition 8: For $\alpha\in\mathbf{N}$, *(VI.23) is nonzero iff* $\Sigma_{i=1}^{l}x_i+\Sigma_{i=l+1}^{N}\ y_i=\frac{1}{2}(\Sigma_{i=l}^{l}\tilde{x}_i+\Sigma_{i=l+1}^{N}\tilde{y}_i)=(2l-N)\alpha+l$. *And for* $\alpha\notin\mathbf{N}$ *it is nonvanishing iff N is even, i.e.,* $N=2L$, *and* $l=L=\Sigma_{i=1}^{L}x_i+\Sigma_{i=L+1}^{N}y_i=\frac{1}{2}(\Sigma_{i=1}^{L}\tilde{x}_i+\Sigma_{i=L+1}^{N}\tilde{y}_i)$. *In both cases, the trace (VI.23) can be written as the following unified formula:*

$$(VI.23)=\sum_{s=1}^{\infty}(-1)^{s+1}\mathrm{Tr}_{F_{(\beta,\gamma)}^{(-s)}}(q^{L_0}(Q^1)^{y_N}(Q^2)^{\tilde{y}_N}\phi_{r_N}^{*}(z_N)\cdots(Q^1)^{y_{l+1}}(Q^2)^{\tilde{y}_{l+1}}\phi_{r_{l+1}}^{*}(z_{l+1})$$

$$\times(Q^1)^{x_l}(Q^2)^{\tilde{x}_l}\phi_{r_l}(z_l)\cdots(Q^1)^{x_1}(Q^2)^{\tilde{x}_1}\phi_{r_1}(z_1)). \quad \text{(VI.24)}$$

An integral formula for the N-point functions of type II (dual) vertex operators can be written down in a similar way, which we omit.

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