# Solution of a two-leg spin ladder system

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A model for a spin-1/2 ladder system with two legs is introduced. It is demonstrated that this model is solvable via the Bethe ansatz method for arbitrary values of the rung coupling $J$. This is achieved by a suitable mapping from the Hubbard model with appropriate twisted boundary conditions. We determine that a phase transition between gapped and gapless spin excitations occurs at the critical value $J_c=1/2$ of the rung coupling.

Research in spin ladder systems continues to attract considerable attention, primarily motivated by the desire to understand the phenomenon of high temperature superconductivity observed in doped antiferromagnetic materials. In studying ladder materials, important insights are gained into the transition to two-dimensional systems from the one-dimensional scenario, where there exists a greater understanding of the physics from the theoretical perspective. Moreover, it is possible to experimentally study ladder materials and numerical simulations are easier to treat which facilitates a greater interaction between theory and phenomenology. For a review of these aspects we refer to Ref. 1.

In order to gain some results in the theory of spin ladder systems many authors have considered generalized models which incorporate biquadratic spin exchange interactions.[2–4] Doing this has lead to some results in relation to ground state structures and phases for the excitation spectra. Simultaneously, there has been an effort to apply the mathematically rich techniques of Bethe ansatz procedures, which have successfully been used in the study of one-dimensional quantum systems,[5] to obtain further results regarding the behavior of the ladder systems. In order to extend the standard one-dimensional approach of the Bethe ansatz to the case of ladders, a number of methods have thus far been proposed.

In the works of Refs. 6,7 a construction was developed for generalized zig-zag ladder systems where the extension from the one-dimensional system to the ladder was obtained by an algebra homomorphism. In this manner, the symmetry algebra of the ladder system remains the same as the original one-dimensional model. Closely related to this approach is that adopted by Muramoto and Takahashi[8] who employed the higher order conservation laws of the Heisenberg chain to define a two-leg system which generalizes the Majumdar-Ghosh model.[9]

Alternatively, the approach can be considered where the symmetry algebra is extended to describe the ladder model. This notion was promoted by Wang[10] who constructed a two-leg bipartite ladder system based on the symmetry algebra su(4) as opposed to the su(2) symmetry of the one-dimensional Heisenberg chain. Employing this method allows for the introduction of rung interactions by way of a chemical potential- (or external field) like term. Subsequently, this method was extended and generalized by a number of authors.[11–15] All of these examples on bipartite ladder lattices contain biquadratic spin exchange interaction terms in order to maintain solvability.

Our aim in this work is to obtain a solvable bipartite ladder system with arbitrarily coupled rung interactions and the absence of biquadratic spin exchange interactions. To achieve this end, we begin with the coupled spin formulation of the Hubbard model as introduced by Shastry,[16] on a closed lattice with twisted boundary conditions. The algebraic Bethe ansatz solution of this model has been studied by Martins and Ramos.[17] By means of carefully chosen transformations, we map this model to a spin ladder system with periodic boundary conditions. (Similar transformations have recently been discussed in Ref. 18 in a different context.) Remarkably, the resulting model assumes a simple form with three basic forms of interaction. The energy expression in terms of a Bethe ansatz solution is also obtained. In this case, the rung interactions are not simply of the chemical potential type referred to above. Rather, the rung interaction parameter appears explicitly in the Bethe ansatz equations, in contrast to all other integrable bipartite ladders that have appeared in the literature. So, it is reasonable to expect the behavior of this model to differ from the class of ladder models with chemical potential type rung interaction. We find the critical value $J_c=1/2$ for the rung interaction parameter indicating the transition between gapped and gapless phases.

We will show solvability of the following two-leg ladder Hamiltonian with an even number of rungs and periodic boundary conditions. Explicitly, the global Hamiltonian is of the form

$$H=\sum_{i=1}^{L-1} h_{i(i+1)}+h_{L1}, \tag{1}$$

where the local Hamiltonians read

$$h_{ij}=(\sigma_i^+\sigma_j^-+\sigma_i^-\sigma_j^+)(\tau_i^z\tau_j^z)^{i+1}+(\tau_i^+\tau_j^-+\tau_i^-\tau_j^+)\times(\sigma_i^z\sigma_j^z)^i+J/2(\vec{\sigma}_i\,.\,\vec{\tau}_i+\vec{\sigma}_j\,.\,\vec{\tau}_j). \tag{2}$$

Above, the coupling $J$ can take arbitrary values.

 

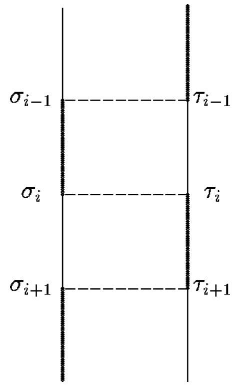


FIG. 1. The two-leg ladder lattice.

The ladder system is depicted graphically in Fig. 1 above. Across the rungs there is the usual Heisenberg $XXX$ interaction while along the legs the interactions alternate between pure and correlated $XX$ exchanges. Clearly the correlated exchange is a four body interaction depending on the spins of the opposing leg. In the $J=0$ limit the correlated exchanges have no real physical significance since for this case the model may be mapped back to two decoupled $XX$ (or free fermion) chains with twisted boundary conditions. This is in some contrast to the case of Ref. 10 where in the absence of rung interactions the model maintains nontrival biquadratic spin exchange interactions between the legs. In the thermodynamic limit, the boundary conditions become irrelevant and we conclude that this region is gapless. On the other hand for large $J$ the system approximates that of the two-leg Heisenberg ladder. In this limit the ground state consists of a product of rung singlets and the excitations are gapped.[1] Hence we expect there to exist a finite critical value of $J$ defining the phase transition.

A more detailed analysis of the model can be made using the fact that there exists an exact solution. The energy levels of this model take the form

$$E=4JN-3JL+\sum_{j=1}^{N} 2\cos k_j, \tag{3}$$

where the variables $k_j$ are solutions of the following Bethe ansatz equations:

$$\begin{aligned} -(-1)^N\exp(iLk_j)&=\prod_{l=1}^{M}\frac{\sin k_j-u_l+iJ}{\sin k_j-u_l-iJ},\\ \prod_{j=1}^{N}\frac{\sin k_j-u_l+iJ}{\sin k_j-u_l-iJ}&=-\prod_{k=1}^{M}\frac{u_l-u_k-2iJ}{u_l-u_k+2iJ}\end{aligned} \tag{4}$$

with $j=1,2,\ldots,N$ and $l=1,2,\ldots,M$. The states associated with solutions of the above equations are eigenstates of the total spin operator

$$\frac{1}{2}\sum_{i=1}(\sigma_i^z+\tau_i^z) \tag{5}$$

with eigenvalues $N-2M$.

The existence of the critical point is evident from the Bethe ansatz equations. Exact diagonalization of the two-site Hamiltonian shows that there is a unique ground state with energy $-6J$ for $J>1/2$ which is given by the product of the two rung singlets. For $J=1/2$ the ground state turns out to be threefold degenerate [two previous excitation states with energy $-2(1+J)$ ''collapse'' into the ground state] while for $J<1/2$ no singlet rung exists in the ground state configuration. For $L$ sites it then follows that when $J>1/2$ the ground state is still the product of rung singlets with energy $E=-3JL$. This is in fact the reference state used in the Bethe ansatz calculation and corresponds to the case $M=N=0$ for the Bethe ansatz equations. To describe an elementary excitation to a spin 1 state we take $N=1$, $M=0$ in Eq. (4) which yields real solutions for the variable $k$, viz.

$$k=\frac{2\pi r}{L},\quad r=0,1,2,\ldots,L-1.$$

It is then apparent from the energy expression (3) that for $J>1/2$ these elementary excitations are gapped. The choice $r=L/2$ (recalling that $L$ is assumed even) shows that for all $J>1/2$ there is a gap

$$\Delta=4J-2.$$

We therefore deduce that $J_c=1/2$ gives the critical point between the gapped and gapless phases of the elementary spin excitations alluded to earlier. It is clear that the gap $\Delta$ is independent of the system size $L$ and this result extends to the thermodynamic limit.

The model also exhibits elementary bound state excitations which we illustrate in the two-site case. For $L=2$, $N=2$, $M=1$ there is a solution of the Bethe ansatz equations given by

$$u=0,\quad k_1=-k_2=\arccos(-J)$$

which describes an excited state of energy $E=-2J$. From the eigenvalue expression for Eq. (5) we see that this state has zero spin. Such a state has the interpretation of the excitation of two bound quasiparticles of opposite spin.

In order to obtain the solution of this model, we begin with the coupled spin version of the Hubbard model as introduced by Shastry[16] with the imposition of twisted boundary conditions. The local Hamiltonian has the form

$$\begin{aligned} h_{i(i+1)}=&-\sigma_i^+\sigma_{(i+1)}^- -\sigma_{(i+1)}^+\sigma_i^- -\tau_i^+\tau_{(i+1)}^- -\tau_{(i+1)}^+\tau_i^-\\ &-\frac{U}{8}[(\sigma_i^z+I)(\tau_i^z+I)+(\sigma_{(i+1)}^z+I)(\tau_{(i+1)}^z+I)]\\ &+\frac{U}{4},\end{aligned}$$

where $\{\sigma_i^{\pm},\sigma_i^z\}$ and $\{\tau_i^{\pm},\tau_i^z\}$ are two commuting sets of Pauli matrices acting on the site $i$. For our convenience an additional applied magnetic field term has been added and an overall factor of $-1$ included. For the twisted boundary term we take

$$\begin{aligned} h_{L1}=&-e^{-i\phi_1}\sigma_L^+\sigma_1^- -e^{i\phi_1}\sigma_1^+\sigma_L^- -e^{-i\phi_2}\tau_L^+\tau_1^- -e^{i\phi_2}\tau_1^+\tau_L^-\\ &-\frac{U}{8}[(\sigma_L^z+I)(\tau_L^z+I)+(\sigma_1^z+I)(\tau_1^z+I)]+\frac{U}{4}.\end{aligned}$$

The first step is to apply a nonlocal transformation given by

$$\theta(\sigma_i^{\pm})=\sigma_i^{\pm}\prod_{k=1}^{i-1}\tau_k^z,$$

$$\theta(\sigma_i^z)=\sigma_i^z,$$

$$\theta(\tau_i^{\pm})=\tau_i^{\pm}\prod_{k=i+1}^{L}\sigma_k^z,$$

$$\theta(\tau_i^z)=\tau_i^z.$$

Under the transformation $\theta$ we yield a new Hamiltonian of the form (2) where the bulk two-site operators now read

$$\begin{aligned}h_{i(i+1)}=&-\sigma_i^+\sigma_{(i+1)}^-\tau_i^z-\sigma_{(i+1)}^+\sigma_i^-\tau_i^z-\tau_i^+\tau_{(i+1)}^-\sigma_{(i+1)}^z\\&-\tau_{(i+1)}^+\tau_i^-\sigma_{(i+1)}^z-\frac{U}{8}[(\sigma_i^z+I)(\tau_i^z+I)\\&+(\sigma_{(i+1)}^z+I)(\tau_{(i+1)}^z+I)]+\frac{U}{4}\end{aligned}\tag{6}$$

and the boundary term is given by

$$\begin{aligned}h_{L1}=&-e^{-i\phi_1}\sigma_L^+\sigma_1^-\tau_L^z\prod_{k=1}^{L}\tau_k^z-e^{i\phi_1}\sigma_1^+\sigma_L^-\tau_L^z\prod_{k=1}^{L}\tau_k^z\\&-e^{-i\phi_2}\tau_L^+\tau_1^-\sigma_1^z\prod_{k=1}^{L}\sigma_k^z-e^{i\phi_2}\tau_1^+\tau_L^-\sigma_1^z\prod_{k=1}^{L}\sigma_k^z\\&-\frac{U}{8}[(\sigma_L^z+I)(\tau_L^z+I)+(\sigma_1^z+I)(\tau_1^z+I)]+\frac{U}{4}.\end{aligned}\tag{7}$$

An important observation to make is that the boundary term above has nonlocal terms. To accommodate for this, note that we may write

$$\prod_{k=1}^{L}\sigma_k^z=(-1)^{\mathcal{M}},$$

$$\prod_{k=1}^{L}\tau_k^z=(-1)^{\mathcal{N}-\mathcal{M}},$$

where $\mathcal{M}=\Sigma_{i-1}^{L}m_i$, $\mathcal{N}=\Sigma_{i=1}^{L}n_i$, and

$$m=\frac{1}{2}(I-\sigma^z),$$

$$n=I-\frac{1}{2}(\sigma^z+\tau^z).$$

Since the global operators $\mathcal{M}$, $\mathcal{N}$ are conserved quantities, we can treat the twisted boundary conditions in Eq. (7) in a sector dependent manner. Letting $M$ and $N$ denote the eigenvalues of $\mathcal{M}$, $\mathcal{N}$, respectively, we now choose

$$e^{i\phi_1}=(-1)^{(N-M)},\quad e^{i\phi_2}=(-1)^M.\tag{8}$$

The validity of making this choice without destroying the solvability stems from the fact that states with differing values of $M$ and $N$ are orthogonal *independent* of the values of $\phi_1$ and $\phi_2$. Hence we may choose different values of $\phi_1$ and $\phi_2$ for each of the subspaces corresponding to a fixed $M$ and $N$.

The next step is to now employ a local transformation on the Pauli matrices which has the form

$$\Phi(\sigma^{\pm})=\frac{1}{\sqrt{2}}(\sigma^{\pm}+\tau^{\pm}\sigma^z),$$

$$\Phi(\sigma^z)=-\sigma^+\tau^--\sigma^-\tau^++\frac{1}{2}(\sigma^z+\tau^z),$$

$$\Phi(\tau^{\pm})=\frac{1}{\sqrt{2}}(\tau^{\pm}-\sigma^{\pm}\tau^z),$$

$$\Phi(\tau^z)=-\sigma^+\tau^--\sigma^-\tau^+-\frac{1}{2}(\sigma^z+\tau^z).$$

It is worth noting that the above transformation can be expressed

$$\Phi(x)=TxT^{-1},$$

where

$$T=\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)\tau^x-i\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)\sigma^z\tau^y-\frac{i}{2\sqrt{2}}\sigma^y-\frac{1}{2\sqrt{2}}\sigma^x\tau^z$$

is a unitary operator. Applying this transformation to the local Hamiltonians (6),(7) gives us the local ladder Hamiltonians

$$\begin{aligned}h_{i(i+1)}=&\tau_i^+\sigma_{i+1}^-+\tau_i^-\sigma_{i+1}^++(\sigma_i^+\tau_{i+1}^-+\sigma_i^-\tau_{i+1}^+)\tau_i^z\sigma_{i+1}^z\\&+\frac{U}{8}(\vec{\sigma}_i.\vec{\tau}_i+\vec{\sigma}_{i+1}.\vec{\tau}_{i+1})\end{aligned}\tag{9}$$

and the global Hamiltonian has regular periodic boundary conditions.

The final step in obtaining Eq. (2) is to set $J=U/4$ and perform the transformation

$$\sigma_i=\sigma_i+1/2[1-(-1)^i](\tau_i-\sigma_i),$$

$$\tau_i=\tau_i+1/2[1-(-1)^i](\sigma_i-\tau_i),$$

which has the effect of interchanging the leg spaces on the odd rungs while leaving the even numbered rungs unchanged.

The energy expression for the Shastry model with twisted boundary conditions can be obtained through the Bethe ansatz. The result is

$$E=\frac{U(4N-3L)}{4}+\sum_{j=1}^{N}2\cos k_j$$

such that the $k_j$ satisfy the Bethe ansatz equations

$$(-1)^{M+N}e^{-i\phi_2}\exp(iLk_j)=-\prod_{l=1}^{M}\frac{\sin k_j-u_l+iU/4}{\sin k_j-u_l-iU/4},$$

$$\prod_{j=1}^{N} \frac{\sin k_j - u_l + iU/4}{\sin k_j - u_l - iU/4} = -(-1)^N e^{-i(\phi_1 - \phi_2)} \times \prod_{k=1}^{M} \frac{u_l - u_k - iU/2}{u_l - u_k + iU/2}.$$

An important point here is that the numbers $M$ and $N$ above have precisely the same meaning as the interpretation presented earlier, i.e., they are the eigenvalues of the conserved operators $\mathcal{M}$ and $\mathcal{N}$. Consequently, we need only substitute the values of Eq. (8) into the above energy expression and Bethe ansatz equations which gives us Eqs. (3),(4) with the parametrization $J = U/4$.

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